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Wayne Lewis
University of Hawaii

Peter Loth
Sacred Heart University, lothp@sacredheart.edu

Adolf Mader
University of Hawaii

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Free subgroups with torsion quotients and profinite subgroups with torus quotients

Wayne Lewis (*) – Peter Loth (**) – Adolf Mader (***)

Dedicated to László Fuchs on his 95th birthday

Abstract – Here "group" means abelian group. Compact connected groups contain δ-subgroups, that is, compact totally disconnected subgroups with torus quotients, which are essential ingredients in the important Resolution Theorem, a description of compact groups. Dually, full free subgroups of discrete torsion-free groups of finite rank are studied in order to obtain a comprehensive picture of the abundance of δ-subgroups of a protorus. Associated concepts are also considered.

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Keywords. Torsion-free abelian group, finite rank, full free subgroup, Pontryagin Duality, compact abelian group, totally disconnected, profinite, torus quotient, Resolution Theorem.

(*) *Indirizzo dell'A.*: University of Hawaii, Honolulu Community College, 874 Dillingham Blvd., Honolulu, HI 96817, USA
E-mail: waynel@math.hawaii.edu

(**) *Indirizzo dell'A.*: Department of Mathematics, Sacred Heart University, 5401 Park Avenue, Fairfield, CT 06825, USA
E-mail: LothP@sacredheart.edu

(***) *Indirizzo dell'A.*: Department of Mathematics, University of Hawaii at Manoa, 2565 McCarthy Mall, Honolulu, HI 96922, USA
E-mail: adolf@math.hawaii.edu

1. Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A *torus* is a topological group isomorphic with a power $\mathbb{T}^{\mathfrak{m}}$. A *protorus* is a compact, connected, finite dimensional topological group. In particular, a protorus is divisible as an abelian group ([9, Corollary 8.5, p. 377]). [9, Proposition 8.15, p. 383] deals with the existence of compact totally disconnected subgroups Δ of a protorus G such that G/Δ is a torus. These *δ-subgroups* enter into the Resolution Theorem for compact abelian groups [9, Theorem 8.20, p. 387]). The Pontryagin duals of protori are the torsion-free groups of finite rank and the duals of short exact sequences $\Delta \rightarrowtail G \twoheadrightarrow T$ where G is a protorus, Δ is a δ-subgroup of G and thus T is a torus, are exact sequences $F \rightarrowtail A \twoheadrightarrow D$ where A is a discrete torsion-free group of finite rank, F is a free subgroup of A and D is a torsion group. This suggests to study the *full free subgroups* F of A, i.e., the free subgroups of A with torsion quotient. Let $\mathcal{F}(A)$ denote the set of all full free subgroups of A and let $\mathcal{D}(G)$ denote the set of all δ-subgroups of the protorus G. In Theorem 3.5 a comprehensive description of $\mathcal{F}(A)$ is established, and by duality a similarly comprehensive description of $\mathcal{D}(G)$ is obtained (Theorem 4.2, Theorem 4.3). In fact, there is an inclusion reversing lattice isomorphism $\delta\colon \mathcal{F}(A) \to \mathcal{D}(G)$ where $G = A^{\vee}$ (Theorem 4.2). In Theorem 3.5(11) and Proposition 3.7 the structure of the quotients A/F of A is determined, and in Theorem 4.3(10) and Proposition 4.10 the structure of the subgroups Δ of G is described. The families $\mathcal{F}(A)$ and $\mathcal{D}(G)$ taken as neighborhood bases of open sets at 0 define linear topologies on A and G respectively. The completion of A with the $\mathcal{F}(A)$-topology can be described to a high degree (Proposition 5.9). The canonical subgroup $\mathbf{\Delta} := \bigcup \mathcal{D}(G)$ of G has interesting properties (Theorem 4.3(2,8,9), Corollary 4.9) and leads to a “canonical” resolution theorem (Theorem 4.11). Also G and $\mathbf{\Delta}$ with the $\mathcal{D}(G)$-topology turn out to be locally compact and therefore complete (Theorem 6.2). This leads to a second canonical form for the resolution theorem (Theorem 6.6).

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2. Notation and background

As a rule $A, B, C, D, E, \ldots$ denote discrete groups and $G, H, K, L, \ldots$ are used to denote topological groups. Unless otherwise stated, p denotes an arbitrary prime number. TFFR is the category of torsion-free abelian groups of finite rank, and as usual LCA is the category of locally compact abelian groups. $(\cdot)^{\vee}$ denotes the

Pontryagin dual, while $\widehat{(\cdot)}$ is reserved for completions such as $\widehat{\mathbb{Z}}$, the completion of $\mathbb{Z}$ in the n-adic topology, $\widehat{\mathbb{Z}}_p$ for the completion of $\mathbb{Z}$ in the p-adic topology, $\widehat{A}_{\mathcal{F}}$ is the completion of A in the free topology (Definition 5.2). We will use $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{P}$ denotes the set of all prime numbers.

Let $\boldsymbol{C}$ be a category. The symbol $A \cong_{\boldsymbol{C}} B$ will say that A, B are isomorphic in the category $\boldsymbol{C}$. We will deal particularly with the category Ab of (discrete) abelian groups and the category topAb of topological abelian groups. The symbol $\cong$ without subscript is used to indicate algebraic isomorphism.

For a discrete abelian group A, $\mathrm{rk}_p(A) = \dim_{\mathbb{Z}/p\mathbb{Z}}(A/pA)$ denotes the *p-rank* of A and $\mathrm{rk}(A) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes A)$ is the *torsion-free rank* of A. We also will need $\dim(A[p])$ which is just the vector space dimension of the p-socle $A[p] = \{a \in A \mid pa = 0\}$ as a $\mathbb{Z}/p\mathbb{Z}$-vector space.

In the literature the dimension of a compact abelian group is defined in several ways that coincide. Let G be any compact abelian group. The cardinal

$$\mathfrak{m} = \sup\{\kappa \colon G/H \cong_{\mathrm{topAb}} \mathbb{T}^{\kappa} \text{ and } H \text{ is a closed subgroup of } G\}$$

is the *dimension of* G, $\dim(G) = \mathfrak{m}$ ([10]). If $\mathfrak{m}$ is finite, then $\mathfrak{m}$ is exactly the dimension of G as defined in [8] and coincides with $\dim_{\mathbb{R}}(\mathfrak{L}(G))$ ([9, Theorem 8.22, p. 390]).

Lemma 2.1 ([10] and [9]). *Let G be a compact abelian group. Then,*

1. *G is totally disconnected if and only if* $\dim(G) = 0$;
2. $\dim(G) = \mathrm{rk}(G^{\vee})$.

We also recall that *profinite groups* are the limits of surjective inverse systems of finite groups.

Theorem 2.2 ([17, Theorem 1.1.12, p. 10]). *A group is profinite if and only if it is compact, totally disconnected, and Hausdorff.*

We remark that all topological groups encountered in this paper are Hausdorff so that any compact totally disconnected group is profinite.

3. The lattice of full free subgroups

We will use Proposition 3.1 in proving one of the properties of $\mathcal{F}(A)$.

Proposition 3.1. *Let $A \in$ TFFR of rank n and let F be a full free subgroup of A. Then* $\dim((A/F)[p]) \leq n$. *Consequently* $(A/F)_p = \bigoplus_{i=1}^{n} \mathbb{Z}(p^{n_{pi}})$ *where* $0 \leq n_{pi} \leq \infty$.

Proof. Write $(A/F)[p] = \bigoplus_{i\in I}\langle x_i + F\rangle$. Let J be a finite subset of I. We will show that $\{x_j \mid j \in J\}$ is linearly independent in A and hence $|J| \le \mathrm{rk}(A) = n$. It then follows that $|I| \le n$.

Suppose that $\sum_{j\in J} m_j x_j = 0$. Then $\sum_{j\in J} m_j(x_j + F) = 0$ and it follows that $p \mid m_j$ for all $j \in J$. Hence $\sum_{j\in J}(m_j/p)x_j = 0$ and by induction it follows that $p^k \mid m_j$ for all k. This is possible only if $m_j = 0$, and we have established that $\{x_j \mid j \in J\}$ is a linearly independent set in A.

The structure of the p-primary component $(A/F)_p$ can be seen by considering a basic subgroup B of $(A/F)_p$ ([6, Theorem 5.2, p. 167]). The subgroup B must be finite because $(A/F)_p[p]$ is finite, hence $B[p]$ is finite, and B is a direct sum of cyclic groups. Also B is pure in $(A/F)_p$, hence a direct summand ([6, Theorem 2.5, p. 156]. Consequently $(A/F)_p$ is the direct sum of a finite group and finitely many copies of $\mathbb{Z}(p^\infty)$. □

Corollary 3.2. *Let* $A \in$ TFFR *and* $F \in \mathcal{F}(A)$.

1. *If* A/F *is bounded, then* A/F *is finite and* A *is free.*
2. *If* A *is not free, then* A/F *is unbounded, and hence either there exists* $p \in \mathbb{P}$ *such that* $(A/F)_p$ *has a summand isomorphic to* $\mathbb{Z}(p^\infty)$ *or* $(A/F)_p \neq 0$ *for infinitely many primes* p.

Two (discrete) torsion groups T_1, T_2 are *quasi-isomorphic*, $T_1 \cong_{\mathrm{qu}} T_2$, if there exist homomorphisms $f\colon T_1 \to T_2$ and $g\colon T_2 \to T_1$ such that $T_2/f(T_1)$ and $T_1/g(T_2)$ are bounded ([1, pp. 11–12]).

Lemma 3.3 ([1, Proposition 1.9, p. 12]). *Let* $A \in$ TFFR. *If* $F_1, F_2 \in \mathcal{F}(A)$, *then* $A/F_1 \cong_{\mathrm{qu}} A/F_2$.

The quasi-isomorphism class $[A/F]_{\mathrm{qu}}$ is the *Richman type* of A, $\mathrm{RT}(A)$, an invariant of the group A by Lemma 3.3.

We say analogously, that two compact totally disconnected groups Δ_1, Δ_2 are *isogenous*, $\Delta_1 \cong_{\mathrm{isog}} \Delta_2$, if there exist morphisms $f\colon \Delta_1 \to \Delta_2$ and $g\colon \Delta_2 \to \Delta_1$ such that $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$ are bounded.

Lemma 3.4. *If* T_1, T_2 *are torsion groups and* $T_1 \cong_{\mathrm{qu}} T_2$, *then the* $D_i := T_i^\vee$ *are compact totally disconnected groups and* $D_1 \cong_{\mathrm{isog}} D_2$.

Proof. Immediate by duality. □

For a group X, a subgroup Y and $m \in \mathbb{N}$, define

$$m_X^{-1}Y = \{x \in X \mid mx \in Y\}.$$

For future use we observe that for torsion-free A with divisible hull $\mathbb{Q}A$ and a subgroup F of A, the following formula is true.

$$F \subset m_A^{-1} F = A \cap \mu_{m^{-1}}^{\mathbb{Q}A}(F), \tag{1}$$

where $\mu_{m^{-1}}^{\mathbb{Q}X}$ denotes multiplication by m^{-1} in $\mathbb{Q}X$.

Theorem 3.5. *For $A \in$ TFFR of rank n, the family $\mathfrak{F} := \mathfrak{F}(A)$ has the following properties.*

1. $\mathrm{rk}(F) = \mathrm{rk}(A)$ *for all $F \in \mathfrak{F}$; in particular, every $F \in \mathfrak{F}$ is finitely generated and $\mathbb{Q}F = \mathbb{Q}A$.*
2. *$\mathfrak{F}$ is closed under finite intersections.*
3. *$\mathfrak{F}$ is closed under finite sums.*
4. *For any $m \in \mathbb{N}$, and $F \in \mathfrak{F}$, $m_A^{-1} F \in \mathfrak{F}$. In particular, $\mathfrak{F}$ is closed under finite extensions.*
5. *$\mathfrak{F}$ is closed under subgroups of finite index.*
6. *If $F \in \mathfrak{F}$, then $mF \in \mathfrak{F}$ and $\bigcap_m mF = 0$ for all $m \in \mathbb{N}$.*
7. $\bigcap \mathfrak{F} = 0$, *and* $\bigcup \mathfrak{F} = A$.
8. *$\mathfrak{F}$ is a lattice with meet $= \cap$ and join $= +$.*
9. *If $F_1, F_2 \in \mathfrak{F}$, then $A/F_1 \cong_{\mathrm{qu}} A/F_2$.*
10. *Let $F, F' \in \mathfrak{F}$. Then there exists $k \in \mathbb{N}$ such that $kF' \subset F$.*
11. *Let $F \in \mathfrak{F}$. Then, for the primary decomposition $A/F = \bigoplus_{p\in\mathbb{P}}(A/F)_p$, we get $\dim((A/F)[p]) \leq n$ and $(A/F)_p \cong_{\mathrm{Ab}} \bigoplus_{i=1}^{n} \mathbb{Z}(p^{n_{pi}})$ for all $p \in \mathbb{P}$, where $0 \leq n_{pi} \leq \infty$.*

Proof. (1) Obvious and well known.

(2) Certainly $F_1 \cap F_2$ is free. The map $A/(F_1 \cap F_2) \to A/F_1 \oplus A/F_2$, $a + (F_1 \cap F_2) \mapsto (a + F_1, a + F_2)$ is well defined and injective. Hence $A/(F_1 \cap F_2)$ is torsion.

(3) Certainly, $A/(F_1 + F_2)$ is a torsion group as an epimorphic image of A/F_1. Also $F_1 + F_2$ is finitely generated and torsion-free, so it is free.

(4) Let $m \in \mathbb{N}$ and $F \in \mathfrak{F}$. Then $\mu_{m^{-1}}^{\mathbb{Q}A}(F) \cong F$ is a free group and hence so is $m_A^{-1} F = A \cap \mu_{m^{-1}}^{\mathbb{Q}A}(F)$. As $m_A^{-1} F$ contains F it is a full free subgroup of A. Finite extensions of F are a special case of what has been considered.

(5) Let $F \in \mathfrak{F}$ and F' a subgroup of F with F/F' finite. Then F' is free and $F/F' \rightarrowtail A/F' \twoheadrightarrow A/F$ is exact with torsion ends, hence A/F' is torsion.

(6) Clear as F is free.

(7) By (6) $mF \in \mathcal{F}$ and $\bigcap \mathcal{F} \subset \bigcap_m mF = 0$ as F is free. Every element of A is contained in some full free subgroup, so $\bigcup \mathcal{F} = A$.

(8) follows from (2) and (3).

(9) Lemma 3.3.

(10) $(F' + F)/F \subset A/F$ is a finitely generated torsion group, hence bounded (actually finite), so there is $k \in \mathbb{N}$ such that $kF' \subset F$.

(11) Proposition 3.1. □

Theorem 3.5(11) shows that not every torsion group can appear as a quotient A/F. In fact, it shows the following:

Corollary 3.6. *Let $A \in$ TFFR, $\mathrm{rk}(A) = n$, and $F \in \mathcal{F}(A)$. Then there is a monomorphism $A/F \to \left(\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)\right)^n$.*

Proposition 3.7 shows that the restrictions of Theorem 3.5(11) are the only ones necessary.

Proposition 3.7. *Let $T = \bigoplus_{p\in\mathbb{P}} T_p$ be a torsion group with p-primary components T_p such that $\dim(T[p]) \leq n$ for a fixed $n \in \mathbb{N}$ for all $p \in \mathbb{P}$. Then there exists a group $A \in$ TFFR with $\mathrm{rk}(A) = n$ and $F \in \mathcal{F}(A)$ such that $A/F \cong_{\mathrm{Ab}} T$. In fact, A may be taken to be completely decomposable.*

Proof. By Proposition 3.1 $T = \bigoplus_{i=1}^n \left(\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{n_{pi}})\right)$ where $0 \leq n_{pi} \leq \infty$. Let $F = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$, the free group of rank n, and let $V = \mathbb{Q}v_1 \oplus \cdots \oplus \mathbb{Q}v_n$ be the divisible hull of F. For each i a summand $\mathbb{Z}(p^{n_{pi}}) \neq 0$ may be finite cyclic or the Prüfer group $\mathbb{Z}(p^\infty)$. In the first case let $A_{pi} = \mathbb{Z}v_i + \mathbb{Z}\frac{1}{p^{n_{pi}}}v_i$ and if $n_{pi} = \infty$, then let $A_{pi} = \mathbb{Z}v_i + \sum_{j=1}^\infty \mathbb{Z}\frac{1}{p^j}v_i$, and finally let $A_i = \sum_{p\in\mathbb{P}} A_{pi}$. Then A_i is a rank 1 group containing $\mathbb{Z}v_i$ and clearly $A_i/\mathbb{Z}v_i \cong_{\mathrm{Ab}} \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{n_{pi}})$. Finally, let $A = \bigoplus_{i=1}^n A_i$, and note that $A/F \cong_{\mathrm{Ab}} T$. □

4. $\mathcal{F}(A)$ and $\mathcal{D}(G)$

We now establish the connection of discrete groups and full free subgroups with compact groups and δ-subgroups.

As an essential tool we will use annihilators. For $G \in$ LCA, we have the pairing $G^\vee \times G \to \mathbb{R}/\mathbb{Z}$: $(\chi, g) \mapsto \chi(g)$. For a subset X of G, we define the *annihilator of X in $G^\vee$* by $X^\perp = (G^\vee, X) = \{\chi \in G^\vee \mid \chi(X) = 0\}$. We assume familiarity with the basic properties of annihilators [9, pp. 314–325], in particular see [9, Theorem 7.64, p. 359]; alternatively see [10, pp. 270–275].

Theorem 4.1 ([9, Theorem 7.64(v), p. 360]). *Let $G \in$ LCA. Then $\sigma\colon H \mapsto H^{\perp} = (G^{\vee}, H)$ is a lattice antiisomorphism between the lattice of all closed subgroups of G and the lattice of all closed subgroups of the* LCA *group $G^{\vee}$. In particular, $H \subset K$ if and only if $\sigma(K) \subset \sigma(H)$.*

Let G be a protorus. Without loss of generality assume that $G = A^{\vee}$ for some $A \in$ TFFR. Let $F \in \mathcal{F}(A)$. Then $F \overset{\text{ins}}{\rightarrowtail} A \overset{\alpha}{\twoheadrightarrow} A/F$ is exact where α is the natural epimorphism. Therefore

$$(2) \qquad (A/F)^{\vee} \overset{\alpha^{\vee}}{\rightarrowtail} G \overset{\text{restr.}}{\twoheadrightarrow} F^{\vee}$$

is exact, where $F^{\vee}$ is a torus $\mathbb{T}^{\mathrm{rk}(A)}$ and $\alpha^{\vee}((A/F)^{\vee})$ is a compact totally disconnected subgroup of G. Hence $\alpha^{\vee}((A/F)^{\vee}) \in \mathcal{D}(G)$. Define

$$(3) \qquad \delta\colon \mathcal{F}(A) \longrightarrow \mathcal{D}(G), \quad \delta(F) = \alpha^{\vee}((A/F)^{\vee}).$$

Theorem 4.2. *Let $A \in$ TFFR and $G = A^{\vee}$. Then, for all $F \in \mathcal{F}(A)$, $(G, F) = \alpha^{\vee}((A/F)^{\vee})$, $\delta(F) = (G, F) = F^{\perp}$, and $\delta\colon \mathcal{F}(A) \to \mathcal{D}(G)$ is a containment reversing bijective map satisfying $\delta(F_1 + F_2) = \delta(F_1) \cap \delta(F_2)$ and $\delta(F_1 \cap F_2) = \delta(F_1) + \delta(F_2)$. In particular, $\mathcal{D}(G)$ is a lattice with join $+$ and meet $\cap$ that is antiisomorphic with $\mathcal{F}(A)$.*

Proof. (a) We first show that $\alpha^{\vee}((A/F)^{\vee})$ coincides with $(A^{\vee}, F)$, the annihilator of F in $A^{\vee} = G$. In fact, suppose first that $\chi \in (G, F)$. Then $\chi(F) = 0$, so $\chi\restriction_F = 0$. Hence χ is in the kernel of the restriction map in (2), i.e., $\chi \in \mathrm{Ker(restr.)} = \alpha^{\vee}((A/F)^{\vee})$. Conversely, if $\chi \in \mathrm{Ker(restr.)}$, then $\chi(F) = 0$ and $\chi \in (G, F)$.

(b) By Theorem 4.1, $H \mapsto H^{\perp}$ establishes a containment reversing lattice isomorphism between the lattice of closed subgroups of G and the lattice of closed subgroups of $G^{\vee}$. Applying this result to A and $G = A^{\vee}$, we need only observe that $(G, F) \in \mathcal{D}(G)$ for all $F \in \mathcal{F}(A)$ and $\Delta = (G, F)$ for some $F \in \mathcal{F}(A)$ for all $\Delta \in \mathcal{D}(G)$. □

We now establish, for a protorus $G = A^{\vee}$, the properties of $\mathcal{D}(G)$ corresponding to the properties of $\mathcal{F}(A)$. Recall that for any $m \in \mathbb{N}$ and any subgroup Y of X, we have $m_X^{-1}Y = \{x \in X \mid mx \in Y\}$. Furthermore, set

$$\boldsymbol{\Delta} := \sum \mathcal{D}(G).$$

THEOREM 4.3. *Let G be a protorus. The family $\mathcal{D} := \mathcal{D}(G)$ has the following properties.*

1. *$\mathcal{D}$ is a lattice with join $+$ and meet $\cap$. Hence $\mathbf{\Delta} = \bigcup \mathcal{D}$.*
2. *$\bigcap \mathcal{D} = 0$ and $\mathbf{\Delta} = \bigcup \mathcal{D}$ is dense in G.*
3. *If $\Delta_1, \Delta_2 \in \mathcal{D}$, then $\Delta_1 \cong_{\text{isog}} \Delta_2$.*
4. *Let $\Delta = \delta(F)$ and $\Delta' = \delta(F')$ and assume that $\Delta \subset \Delta'$. Then $F' \subset F$ and $[\Delta' : \Delta] = [F : F'] < \infty$.*
5. *Let $\Delta, \Delta' \in \mathcal{D}$. Then there exists $k \in \mathbb{N}$ such that $k\Delta' \subset \Delta$.*
6. *If $\Delta \in \mathcal{D}$, then $m_G^{-1}\Delta \in \mathcal{D}$ for any $m \in \mathbb{N}$. Hence $\operatorname{tor}(G) \subset \mathbf{\Delta}$ and $\mathbf{\Delta} = \sum_{m\in\mathbb{N}} m_G^{-1}\Delta$.*
7. *Let $\Delta \in \mathcal{D}$ and $m \in \mathbb{N}$. Then $m\Delta \in \mathcal{D}$.*
8. *$\mathbf{\Delta}/\Delta = \operatorname{tor}(G/\Delta)$ for all $\Delta \in \mathcal{D}$, and hence $G/\mathbf{\Delta}$ is torsion-free.*
9. *$\mathbf{\Delta}$ is divisible.*
10. *Let $\Delta \in \mathcal{D}$ and $n = \dim(G)$. Then $\Delta \cong_{\text{topAb}} \prod_{p\in\mathbb{P}} \Delta_p$, $\operatorname{rk}_p(\Delta_p) \le n$, and $\Delta_p \cong_{\text{topAb}} \prod_{i=1}^{n} \Delta_{pi}$, where Δ_{pi} is either a cyclic p-group or else $\Delta_{pi} = \widehat{\mathbb{Z}}_p$, the group of p-adic integers ($i = 1, \dots, n$).*

PROOF. Without loss of generality $G = A^\vee$ for $A \in \text{TFFR}$.

(1) Theorem 4.2 establishes the lattice property. As $\mathcal{D}$ is closed under finite sums, we have $\sum \mathcal{D} = \bigcup \mathcal{D}$.

(2) By [9, Theorem 7.64(vii), p. 360], $\bigcap \mathcal{D} = \big(G, \bigcup \mathcal{F}(A)\big) = (G, A) = 0$ and $\overline{\bigcup \mathcal{D}} = \big(G, \bigcap \mathcal{F}(A)\big) = (G, 0) = G$.

(3) There exist $F_1, F_2 \in \mathcal{F}(A)$ such that $\Delta_i = (G, F_i)$. By Theorem 3.5(9) we conclude that $A/F_1 \cong_{\text{qu}} A/F_2$ and hence by Lemma 3.4 $(A/F_1)^\vee \cong_{\text{isog}} (A/F_2)^\vee$.

(4) By Theorem 4.2 $F' \subset F$ and as F/F' is both finitely generated and torsion, it is finite. We now employ [9, Theorem 7.64(ii), p. 360] to conclude that $\Delta'/\Delta \cong_{\text{topAb}} (F/F')^\vee \cong_{\text{topAb}} F/F'$ and the claim follows.

(5) By (4), $[\Delta + \Delta' : \Delta]$ is finite. Hence there is $k \in \mathbb{N}$ such that $k\Delta' \subset \Delta$.

(6) Let $\Delta = (G, F) \in \mathcal{D}$ with $F \in \mathcal{F}(A)$. If $m \in \mathbb{N}$, then $m_G^{-1}\Delta = m_G^{-1}(G, F) = (G, mF)$ (cf. [10, Lemma 6.4.14, p. 274]), so since $mF \in \mathcal{F}(A)$ we have $\Delta \subset m_G^{-1}\Delta \in \mathcal{D}$. Therefore $\mathbf{\Delta} = \sum \mathcal{D} = \sum_m m_G^{-1}\Delta$ by (5).

(7) Without loss of generality $\Delta = \delta(F)$ for some $F \in \mathcal{F}(A)$. By Theorem 3.5(4) we know that $m_A^{-1}F \in \mathcal{F}(A)$. Using [10, Lemma 6.4.14, p. 274] we obtain $m\Delta = \delta(m_A^{-1}F) \in \mathcal{D}$.

(8) This follows from (5) and (6).

(9) The protorus G is divisible and by (8) $\boldsymbol{\Delta}$ is pure in G hence also divisible.

(10) Dual of Theorem 3.5(11). □

Remark 4.4. Our results include the case that A is free of finite rank, i.e., $G = A^\vee$ is a finite-dimensional torus. In this case the quotients A/F, $F \in \mathcal{F}(A)$, are all finite, so are the $\Delta \in \mathcal{D}(G)$ and $G/\Delta \cong_{\mathrm{topAb}} G$.

Corollary 4.5. *Let G be a protorus of dimension n and $\Delta \in \mathcal{D}(G)$. Then there is the exact sequence*

$$\frac{\boldsymbol{\Delta}}{\Delta} \rightarrowtail \frac{G}{\Delta} \twoheadrightarrow \frac{G}{\boldsymbol{\Delta}},$$

where $\frac{G}{\Delta} \cong_{\mathrm{topAb}} \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^n$ *is a torus,* $\frac{\boldsymbol{\Delta}}{\Delta} = \mathrm{tor}\left(\frac{G}{\Delta}\right) \cong \left(\frac{\mathbb{Q}}{\mathbb{Z}}\right)^n$ *is dense in* $\frac{G}{\Delta}$, *and* $\frac{G}{\boldsymbol{\Delta}} \cong \left(\frac{\mathbb{R}}{\mathbb{Q}}\right)^n$.

Proof. By Theorem 4.3(2) $\boldsymbol{\Delta}$ is dense in G, hence so is $\frac{\boldsymbol{\Delta}}{\Delta}$ in $\frac{G}{\Delta}$. The quotient $\frac{G}{\Delta}$ is a torus by definition of Δ, and the rest follows. □

A topological group G is *finitely generated* if there is a finite subset S of G such that $G = \overline{\langle S \rangle}$. If $G = \overline{\langle S \rangle}$ for some singleton S, then G is called *monothetic*. For example, the compact group $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \widehat{\mathbb{Z}}_p$ is monothetic (see [8, Theorem 25.16, p. 408]).

Dually to Corollary 3.6 we have:

Corollary 4.6. *Let G be an n-dimensional protorus and $\Delta \in \mathcal{D}(G)$. Then there is a continuous epimorphism* $\left(\prod_{p \in \mathbb{P}} \widehat{\mathbb{Z}}_p\right)^n \to \Delta$.

Lemma 4.7. *Let G and H be topological groups. Then,*

1. *if $G = G_1 \times \ldots \times G_n$ and $G_1, \ldots, G_n$ are finitely generated, then so is G;*
2. *if G is finitely generated and $\phi\colon G \to H$ is a continuous epimorphism, then H is finitely generated.*

Proof. (1) Write $G_i = \overline{\langle S_i \rangle}$ with S_i a finite subset of G_i $(i = 1, \ldots, n)$. Then $G = \overline{\langle S_1 \rangle} \times \ldots \times \overline{\langle S_n \rangle} = \overline{\langle S_1 \rangle \times \ldots \times \langle S_n \rangle} = \overline{\langle S_1 \times \ldots \times S_n \rangle}$.

(2) Notice that $\phi(\overline{E}) \subset \overline{\phi(E)}$ for each $E \subset G$ since ϕ is a continuous map. If $G = \overline{\langle S \rangle}$, then $H = \phi(G) = \phi(\overline{\langle S \rangle}) \subset \overline{\phi(\langle S \rangle)} = \overline{\langle \phi(S) \rangle}$. Thus $H = \overline{\langle \phi(S) \rangle}$ and the claim follows. □

By Corollary 4.6 and Lemma 4.7 we have:

Corollary 4.8. *Let G be a protorus. Then each $\Delta \in \mathfrak{D}(G)$ is finitely generated.*

Corollary 4.9. *Let G be a nontrivial protorus. Then $G \neq \boldsymbol{\Delta}$ and hence $\boldsymbol{\Delta}$ is not a locally compact subgroup of G.*

Proof. Let $\Delta \in \mathfrak{D}(G)$. Then G/Δ is a torus and contains torsion-free elements, while $\boldsymbol{\Delta}/\Delta \subset G/\Delta$ is a torsion group. Hence $\boldsymbol{\Delta}/\Delta$ is properly contained in G/Δ and $G \neq \boldsymbol{\Delta}$. The subgroup $\boldsymbol{\Delta}$ is dense in G and different from G. Locally compact subgroups are closed, hence $\boldsymbol{\Delta}$ cannot be locally compact. □

Recall that a compact totally disconnected group Δ is topologically isomorphic to $\prod_{p\in\mathbb{P}} \Delta_p$ with p-primary components Δ_p (see for instance [10, Proposition 6.6.9, p. 281]). Dually to Proposition 3.7 we have:

Proposition 4.10. *Let Δ be a compact totally disconnected group with p-primary components Δ_p such that $\mathrm{rk}_p(\Delta_p) \leq n$ for a fixed $n \in \mathbb{N}$ for all $p \in \mathbb{P}$. Then there exists a protorus G with $\dim(G) = n$ and $\Delta' \in \mathfrak{D}(G)$ such that $\Delta' \cong_{\mathrm{topAb}} \Delta$. In fact, G may be taken to be completely factorable, i.e., the product of protori of dimension 1.*

We also obtain a "canonical resolution." This is just the Resolution Theorem [9, Theorem 8.20, p. 387] for protori where the arbitrary $\Delta \in \mathfrak{D}(G)$ is replaced by the canonical subgroup $\boldsymbol{\Delta}$. We remind the reader that the "Lie algebra" of G, $\mathfrak{L}(G)$, defined as $\mathfrak{L}(G) = \mathrm{cHom}(\mathbb{R}, G)$, the group of continuous homomorphisms $\mathbb{R} \to G$, is a real topological vector space via the stipulation $(rf)(x) := f(rx)$ where $f \in \mathfrak{L}(G)$ and $r, x \in \mathbb{R}$, and carries the topology of uniform convergence on compact sets [9, Definition 5.7, p. 117, Proposition 7.36, p. 340]. Furthermore $\exp\colon \mathfrak{L}(G) \to G$, $\exp(\chi) = \chi(1)$ ([9, p. 340]). [9, Theorem 7.66, p. 362] contains results on the exponential morphism.

Theorem 4.11. *Let G be a protorus and $\boldsymbol{\Delta} = \bigcup \mathfrak{D}(G)$. Then*

$$\frac{\boldsymbol{\Delta} \times \mathfrak{L}(G)}{\Gamma} \cong_{\mathrm{topAb}} G,$$

the isomorphism being induced by the map $\varphi\colon \boldsymbol{\Delta} \times \mathfrak{L}(G) \to G$, $\varphi((a, \chi)) = a + \exp(\chi) = a + \chi(1)$ and $\Gamma = \{(\exp(a), -a) \mid a \in \exp^{-1}[\boldsymbol{\Delta}]\}$.

Proof. We have to show that $\varphi\colon \mathbf{\Delta} \times \mathfrak{L}(G) \to G$, $\varphi((d,\chi)) = d + \exp(\chi)$ is continuous and open and then compute its kernel. The map is continuous as it is the composite of the continuous map $\mathrm{id} \times \exp\colon \mathbf{\Delta} \times \mathfrak{L}(G) \to \mathbf{\Delta} \times \exp(\mathfrak{L}(G))$ with addition.

To show that φ is open we use the fact that by [9, Theorem 8.20, p. 387] for every $\Delta \in \mathcal{D}(G)$ the maps $\varphi_\Delta\colon \Delta \times \mathfrak{L}(G) \to G$, $\varphi_\Delta((a,\chi)) = a + \exp(\chi)$ are open maps. The φ_Δ are just the restrictions of φ. An open set in $\mathbf{\Delta}$ is of the form $U \cap \mathbf{\Delta}$ where U is an open set of G. Then for any $\Delta \in \mathcal{D}(G)$, the intersection $U \cap \Delta$ is open in Δ and $U \cap \mathbf{\Delta} = \bigcup\{U \cap \Delta \mid \Delta \in \mathcal{D}(G)\}$. By [9, Theorem 8.20, p. 387] the sets $\varphi(U \cap \Delta) = \varphi_\Delta(U \cap \Delta)$ are open in G, hence so is $\varphi(U \cap \mathbf{\Delta}) = \bigcup\{\varphi(U \cap \Delta) \mid \Delta \in \mathcal{D}(G)\}$. Of course, φ is surjective as any φ_Δ is surjective.

Finally,

$$\Gamma = \mathrm{Ker}(\varphi) = \{(a,\chi) \mid a + \exp(\chi) = 0\} = \{(\exp(a), -a) \mid a \in \exp^{-1}(\mathbf{\Delta})\}. \qquad \square$$

One may ask whether $\mathcal{F}(A)$ is a distributive lattice. As this is a side issue without direct bearing on our study, we state the answer without proof.

Proposition 4.12. *Let $A \in$ TFFR. If $\mathrm{rk}(A) = 1$, then $\mathcal{F}(A)$ is distributive. For larger ranks $\mathcal{F}(A)$ largely fails to be distributive.*

The dimension 1 case is settled by a celebrated theorem.

Theorem 4.13 ([16, Theorem 4, p. 267]). *The lattice of subgroups of a group G is distributive if and only if G is locally cyclic.*

5. The $\mathcal{F}$-topology of A

Let $A \in$ TFFR. Recall that $\mathcal{F}(A)$ is a lattice with lattice operations $\cap$ and $+$.

Remark 5.1. Let $A \in$ TFFR. Then one has $A = \varinjlim\{F \mid F \in \mathcal{F}(A)\}$ with, for $F \subset F' \in \mathcal{F}(A)$, the map $f_{FF'}\colon F \to F'$ being the insertion. Consequently, $A^\vee = \varprojlim\{F^\vee \mid F \in \mathcal{F}(A)\}$ with, for $F \subset F' \in \mathcal{F}(A)$, the map $f^\vee_{FF'}\colon (F')^\vee \to F^\vee$ being the restriction. As $F^\vee$ is a torus, we obtain that $A^\vee$ is an inverse limit of tori. Hence the name "protorus" in analogy of "profinite."

For a general reference on topology, topological groups, and completion see [11], [8], and [3].

Let A be a topological group. The topology on A is determined by a neighborhood basis $\mathcal{U}$ at $0 \in A$ consisting of open sets and called *local basis* for short, or by a basis $\mathcal{O}$ of the topology itself. We will write $A[\mathcal{U}]$ respectively $A[\mathcal{O}]$ to indicate that A is a topological group with topology given by $\mathcal{U}$ respectively $\mathcal{O}$. We denote the completions of $A[\mathcal{U}] = A[\mathcal{O}]$ by $\widehat{A}_{\mathcal{U}} = \widehat{A}_{\mathcal{O}}$.

Definition 5.2. The family $\mathcal{F}(A)$ for $A \in \mathrm{TFFR}$ is directed downward (filtered below) and taken as a local basis defines a linear Hausdorff topology on A, *the free or $\mathcal{F}$-topology of A*. Let $A[\mathcal{F}]$ denote A as a topological group with the $\mathcal{F}$-topology and let $\widehat{A}_{\mathcal{F}}$ denote the completion of A in the free topology.

Remark 5.3. If $A \in \mathrm{TFFR}$ and A/U is torsion for a subgroup U of A, then U contains some $F \in \mathcal{F}(A)$. This means that the $\mathcal{F}$-topology on $A \in \mathrm{TFFR}$ coincides with the minimal functorial topology determined by the discrete class $\mathcal{T}$ consisting of all torsion abelian groups as we shall explain.

A *functorial topology* is a functor T on the category Ab of all abelian groups to the category topAb of topological abelian groups having the property that $\mathrm{T}(f) = f : \mathrm{T}(A) \to \mathrm{T}(B)$ for all $f \in \mathrm{Hom}(A, B)$, which means that every homomorphism f is continuous. The concept is due to B. Charles ([5]). Relevant references are the survey [13] and the more specialized papers [4], [12], [14], and [15].

Associated with a functorial topology T on Ab, there is the *discrete class* $\mathcal{C}(\mathrm{T})$ of T consisting of all groups $\mathrm{T}(A)$ with the discrete topology. A discrete class is always closed under embedded groups and finite direct sums. Conversely, given a *discrete class* $\mathcal{C}$, i.e., a class of groups that is closed under embedded groups and finite direct sums, then we obtain a *minimal functorial topology* $\mathrm{T}_{\mathcal{C}}$ by assigning $\mathrm{T}(A)$ the smallest topology that makes every homomorphism $f : A \to C$ continuous where $C \in \mathcal{C}$. This amounts to saying that the subgroups U of A with $A/U \in \mathcal{C}$ form a local basis for the topology $\mathrm{T}_{\mathcal{C}}$ ([4]).

The discrete class $\mathcal{T}$ consisting of all torsion groups is closed under embedded groups, arbitrary direct sums, epimorphic images, and extensions. We will write $A[\mathcal{T}]$ for a group with the minimal functorial topology with discrete class $\mathcal{T}$ and refer to it as the *$\mathcal{T}$-topology* or *torsion topology*. We observed in Remark 5.3 that the free topology on $A \in \mathrm{TFFR}$ coincides with the torsion topology on A, $A[\mathcal{T}] = A[\mathcal{F}]$.

At this point we need a lemma.

Lemma 5.4. *Let A be an abelian group and $0 \neq a \in A$. Then there exists a subgroup U of A with torsion quotient A/U and $a \notin U$. Consequently every group $A[\mathcal{T}]$ is Hausdorff.*

Proof. Let V be maximal disjoint from $\langle a\rangle$. Then $V \oplus \langle a\rangle$ is essential in A, in particular $A/(V \oplus \langle a\rangle)$ is a torsion group. If $\langle a\rangle$ is finite, then for $U = V$ the quotient A/U is a torsion group and $a \notin U$. If $\langle a\rangle$ is infinite, $U = V \oplus \langle 2a\rangle$ does not contain a and A/U is a torsion group. □

The $\mathcal{T}$-topology has special desirable features due to the closure properties of $\mathcal{T}$.

Lemma 5.5. *Let $A, B \in$ TFFR. Every $f \in \mathrm{Hom}(A, B)$ is a continuous map $f: A[\mathcal{F}] \to B[\mathcal{F}]$ and if f is surjective, then $f: A[\mathcal{F}] \to B[\mathcal{F}]$ is an open map. Every subgroup of $A[\mathcal{F}]$ is closed.*

Proof. The first two claims are general facts and easily verified directly.

Let C be a subgroup of A. By Lemma 5.4 $(A/C)[\mathcal{T}]$ is Hausdorff, so C is closed in A. □

Our definition of the $\mathcal{F}$-topology amounts to saying that we found a rather special local basis for the $\mathcal{T}$-topology on A.

Lemma 5.6. *Let $A \in$ TFFR. Fix $F \in \mathcal{F}(A)$. Then $\{k!F \mid k \in \mathbb{N}\}$ is a local basis for $A[\mathcal{T}] = A[\mathcal{F}]$.*

Proof. First $k!F \in \mathcal{F}(A)$, hence is an open subgroup. By Theorem 3.5(10) given any $F' \in \mathcal{F}(A)$ there is $k \in \mathbb{N}$ such that $k!F \subset F'$. □

It is well known that

$$\begin{aligned}\widehat{A}_{\mathcal{F}} &\cong_{\mathrm{topAb}} \varprojlim\{A/k!F \mid k \in \mathbb{N}\}\\ &= \{(\dots, a_k + k!F, \dots) \mid \text{for all } k \le \ell \in \mathbb{N}, a_k - a_\ell \in k!F\}\end{aligned}$$

([3, III, § 7.3, Corollary 2, p. 290] and [2, III, Exercise 14, p. 236]). We identify $\widehat{A}_{\mathcal{F}}$ and $\varprojlim\{A/k!F \mid k \in \mathbb{N}\}$, so now $A = \{(\dots, a + k!F, \dots) \mid a \in A\}$. Let $A[\mathcal{N}]$ be the topology of A (any abelian group) with $\{nA \mid n \in \mathbb{N}\}$ as a local basis. This $\mathcal{N}$-*topology* is well known as the $\mathbb{Z}$-adic or n-adic topology and has $\{k!A \mid k \in \mathbb{N}\}$ as a special local basis. This is the minimal functorial topology with the discrete class consisting of all bounded groups.

Theorem 5.7. *Let $A \in$ TFFR, let $\widehat{A}_{\mathcal{F}}$ be the completion of $A[\mathcal{F}]$ and let $\widehat{F}_{\mathcal{N}}$ be the completion of $F[\mathcal{N}]$. Then, for any fixed $F \in \mathcal{F}(A)$,*

$$\widehat{A}_{\mathcal{F}} \cong_{\mathrm{topAb}} \frac{A \times \widehat{F}_{\mathcal{N}}}{\Gamma} \quad \textit{where } \Gamma = \{(x, x) \mid x \in F\}. \tag{4}$$

Proof. Let $x = (a_1 + 1!F, a_2 + 2!F, \dots, a_k + k!F, \dots) \in \widehat{A}_{\mathcal{F}}$. Then $a_k - a_\ell \in \ell!F$, for all $\ell \le k$. In particular, for all k, $a_k - a_1 \in F$ and so $a_k = a_1 + b_k$ with $b_k \in F$. Note that $a_k - a_\ell \in \ell!F$ if and only if $b_k - b_\ell \in \ell!F$. Hence

$$\begin{aligned} x &= (a_1 + b_1 + 1!F, a_1 + b_2 + 2!F, \dots, a_1 + b_k + k!F, \dots) \\ &= a_1 + (b_1 + 1!F, b_2 + 2!F, \dots, b_k + k!F, \dots), \end{aligned}$$

where $a_1 \in A$ and $(b_1 + 1!F, b_2 + 2!F, \dots, b_k + k!F, \dots) \in \widehat{F}_{\mathcal{N}}$. We define

$$\varphi\colon A \times \widehat{F}_{\mathcal{N}} \longrightarrow \widehat{A}_{\mathcal{F}}, \quad \varphi((a, y)) = a - y.$$

This is clearly an algebraic epimorphism, and topologically a continuous map because $\widehat{A}_{\mathcal{F}} \times \widehat{A}_{\mathcal{F}} \to \widehat{A}_{\mathcal{F}}\colon (x, y) \mapsto x - y$ is continuous and so is its restriction φ to a subspace. Further $\varphi((a, x)) = 0$ if and only if $a = x \in F$. We will show that φ is open which then establishes (4) by [8, Theorem 5.27, p. 41].

A local basis for the product topology on $\prod_{k\in\mathbb{N}} A/k!F$ consists of subsets of the form $U_K = \prod_{k>K} A/k!F$ where $K \in \mathbb{N}$. Hence the sets $U_K \cap \widehat{A}_{\mathcal{F}}$ form a local basis for $\widehat{A}_{\mathcal{F}}$. Let $U_K \cap \widehat{A}_{\mathcal{F}}$ be given and let $x \in U_K \cap \widehat{A}_{\mathcal{F}}$. Then

$$\begin{aligned} x &= (0, \dots, 0, a' + b_{K+1} + (K + 1)!F, \dots, a' + b_i + i!F, \dots) \\ &= (0, \dots, 0, a' + (K + 1)!F, \dots, a' + i!F, \dots) \\ &\quad + (0, \dots, 0, b_{K+1} + (K + 1)!F, \dots, b_i + i!F, \dots), \end{aligned}$$

where $a' \in A$ and $b_i \in F$. Thus $a := (0, \dots, 0, a' + (K+1)!F, \dots, a' + i!F, \dots) \in U_K \cap A$, $y := -(0, \dots, 0, b_{K+1} + (K + 1)!F, \dots, b_i + i!F, \dots) \in (U_K \cap \widehat{F}_{\mathcal{N}})$ and $(a, y) \in (U_K \cap A) \times (U_K \cap \widehat{F}_{\mathcal{N}})$. Now $(U_K \cap A) \times (U_K \cap \widehat{F}_{\mathcal{N}})$ is an open subgroup of $A \times \widehat{F}_{\mathcal{N}}$ that is mapped onto the open set $U_K \cap \widehat{A}_{\mathcal{F}}$ by φ because $\varphi((a, y)) = a - y = x$. □

In the following $F \in \mathcal{F}(A)$ is fixed and we employ the local basis $\{k!F \mid k \in \mathbb{N}\}$. It is routine to check that the short sequence of inverse systems is exact, all groups carrying the discrete topology,

$$(5) \qquad \Big\{\frac{F}{k!F} \,\Big|\, k \in \mathbb{N}\Big\} \rightarrowtail \Big\{\frac{A}{k!F} \,\Big|\, k \in \mathbb{N}\Big\} \twoheadrightarrow \Big\{\frac{A}{F} \,\Big|\, k \in \mathbb{N}\Big\},$$

Lemma 5.8. *$A[\mathcal{F}]/F$ is discrete and* $\varprojlim\{A/F \mid k \in \mathbb{N}\} \cong_{\mathrm{topAb}} A[\mathcal{F}]/F$.

Proof. The subgroup F is open in $A[\mathcal{F}]$, so A/F is discrete. Every $x \in \varprojlim_k A/F$ has the form $x = (a + F, \dots, a + F, \dots), a \in A$ and the inverse limit topology is the discrete topology. The assignment $(\dots, a + F, \dots) \mapsto a + F$ is the desired isomorphism of topological groups. □

Proposition 5.9. *Let $A \in$ TFFR of rank n. Fix $F \in \mathcal{F}(A)$. Then $A[\mathcal{F}]/F$ is a discrete torsion group and there is a sequence of completions exact in* topAb

$$\widehat{F}_{\mathcal{N}} \overset{\alpha}{\rightarrowtail} \widehat{A}_{\mathcal{F}} \overset{\beta}{\twoheadrightarrow} A[\mathcal{F}]/F,$$

where α is the insertion map and β is continuous and open. Furthermore, $\widehat{F}_{\mathcal{N}} \cong_{\text{topAb}} \widehat{\mathbb{Z}}^n$, and $\widehat{A}_{\mathcal{F}}$ is a torsion-free group that is an essential extension of $\widehat{F}_{\mathcal{N}}$.

Proof. (a) Taking inverse limits is a process that is left exact. Therefore (5) implies that

$$\widehat{F}_{\mathcal{N}} \overset{\alpha}{\rightarrowtail} \widehat{A}_{\mathcal{F}} \overset{\beta}{\twoheadrightarrow} A[\mathcal{F}]/F,$$

is exact. Here, α is the insertion and thus continuous and open onto its image. By [17, p. 5] the maps β are continuous and we used Lemma 5.8. Evidently, $\widehat{A}_{\mathcal{F}} \to A/F$ is surjective as $(\ldots, a + k!F, \ldots) \in \widehat{A}_{\mathcal{F}}$ maps to $a + F \in A/F$ and open as A/F is discrete. Finally, it is well known that the $\mathcal{N}$-completion of $F \cong \mathbb{Z}^n$ is isomorphic in topAb to $\widehat{\mathbb{Z}}^n$.

(b) As a preliminary step we show the following fact. Let

$$x = (\ldots, a_k + k!F, \ldots) \in \widehat{A}_{\mathcal{F}}$$

and assume that infinitely many $a_k \in F$. Then $x \in \widehat{F}_{\mathcal{N}}$. Indeed, given $a_k + k!F$, there exists $\ell > k$ such that $a_\ell \in F$. Then $a_\ell - a_k \in k!F$ and hence $a_k \in F$, showing that $x \in \widehat{F}_{\mathcal{N}}$.

(c) By way of contradiction, suppose that $x = (\ldots, a_k + k!F, \ldots) \in \widehat{A}_{\mathcal{F}}$ and $mx = 0$ for some nonzero integer m. Then $ma_k \in k!F$, for all k. Hence for every $k \geq m$, we have that $a_k \in (k!/m)F \subset F$. By (b) $x \in \widehat{F}_{\mathcal{N}}$, a torsion-free group, contradiction. □

6. The $\mathfrak{D}$-topology of G

Let G be a protorus. By Theorem 4.3(1,2) the family $\mathfrak{D} = \mathfrak{D}(G)$ serves as a local basis for a linear Hausdorff topology of G, the $\mathfrak{D}$-*topology* of G. Then $G[\mathfrak{D}]$ is a 0-dimensional topological group ([8, (4.21)(a), p. 25]), thus $G[\mathfrak{D}]$ is totally disconnected.

Proposition 6.1. *The topology of $\mathbf{\Delta}[\mathfrak{D}]$ as a subgroup of $G[\mathfrak{D}]$ has the local basis $\mathfrak{D}$. The quotient topology on $G[\mathfrak{D}]/\mathbf{\Delta}$ is the discrete topology and*

$$G[\mathfrak{D}] \cong_{\text{topAb}} \mathbf{\Delta}[\mathfrak{D}] \times G[\mathfrak{D}]/\mathbf{\Delta}.$$

The group $G[\mathfrak{D}]/\mathbf{\Delta}$ is torsion-free (Theorem 4.3(8)) and divisible, so algebraically a direct sum of copies of $\mathbb{Q}$.

Proof. The quotient topology of $G[\mathcal{D}]/\mathbf{\Delta}$ is discrete because $\varphi^{-1}[0+\mathbf{\Delta}] = \mathbf{\Delta}$ is open in $G[\mathcal{D}]$. It is evident that the sequence

$$\mathbf{\Delta}[\mathcal{D}] \overset{\text{ins.}}{\rightarrowtail} G[\mathcal{D}] \overset{\varphi}{\twoheadrightarrow} G[\mathcal{D}]/\mathbf{\Delta}$$

is exact and the maps are continuous and open. The group $\mathbf{\Delta}[\mathcal{D}]$ is an open divisible subgroup of $G[\mathcal{D}]$. Now use [8, (6.22)(b), p. 59]. □

In contrast to Corollary 4.9 $\mathbf{\Delta}[\mathcal{D}]$ is locally compact as we will show next.

Theorem 6.2. *Let G be a protorus. Then the topological group $\mathbf{\Delta}[\mathcal{D}]$ is the union of open compact subgroups $\Delta \in \mathcal{D}(G)$, hence in particular is locally compact.*

For later use we single out a part of the proof as a lemma.

Lemma 6.3. *Let G be a protorus and $\Delta \in \mathcal{D}(G)$. Then $\Delta[\mathcal{D}] = \Delta$ as topological groups, i.e. the topologies of Δ as a subspace of $\mathbf{\Delta}[\mathcal{D}]$ and as a subspace of G coincide.*

Proof. Since Δ is a compact totally disconnected subgroup of G, it has a local basis $\mathcal{U}$ consisting of compact subgroups ([8, Theorem 7.7, p. 62]). On the other hand, $\Delta[\mathcal{D}]$ has a local basis $\mathcal{V}$ consisting of all $\Gamma \in \mathcal{D}$ with $\Gamma \subset \Delta$. We will show that the topologies of Δ and $\Delta[\mathcal{D}]$ coincide. Let $C \in \mathcal{U}$. Then Δ/C is compact and discrete, hence finite. Thus $m\Delta \subset C$ for some positive integer m, and by Theorem 4.3(7) $m\Delta \in \mathcal{D}$. Therefore C, as a finite union of translates of $m\Delta$, is an open subgroup of $\Delta[\mathcal{D}]$. Now let $\Gamma \in \mathcal{V}$, i.e., $\Gamma \in \mathcal{D}$ and $\Gamma \subset \Delta$. Then Γ is compact and hence closed in G and in its subgroup Δ. By Theorem 4.3(4), Δ/Γ is finite and therefore discrete, thus Γ is open in Δ. Therefore the topologies of Δ and $\Delta[\mathcal{D}]$ coincide. □

Ploof of Theorem 6.2. Fix $\Delta \in \mathcal{D} = \mathcal{D}(G)$. Since Δ is compact as a subgroup of G, this implies that $\Delta[\mathcal{D}]$ is also compact. Then the assertion follows since $\Delta[\mathcal{D}]$ is an open subgroup of $\mathbf{\Delta}[\mathcal{D}]$. □

Recall that a topological group G is called *periodic* if G is totally disconnected and $\overline{\langle g\rangle}$ is compact for all $g \in G$. Periodic groups are studied extensively in [7].

Corollary 6.4. *Let G be a nontrivial protorus. Then both groups $G[\mathcal{D}]$ and $\mathbf{\Delta}[\mathcal{D}]$ are locally compact and totally disconnected. The group $\mathbf{\Delta}[\mathcal{D}]$ is periodic while $G[\mathcal{D}]$ is not.*

Proof. $G[\mathcal{D}]$ is locally compact since $\mathbf{\Delta}[\mathcal{D}]$ is, and $\mathbf{\Delta}[\mathcal{D}]$ is totally disconnected since $G[\mathcal{D}]$ is. By Corollary 4.9 and Proposition 6.1, $G[\mathcal{D}]$ is not periodic. □

We are now in a position to improve on the resolution theorem Theorem 4.11 by replacing $\mathbf{\Delta}$ by the simpler periodic group $\mathbf{\Delta}[\mathcal{D}]$. We begin with a lemma that is interesting in itself.

Lemma 6.5. *Let G be a protorus. Fix $\Delta \in \mathcal{D}(G)$. Then $\Delta[\mathcal{D}]$, by definition, has the local basis $\mathcal{B}_\Delta := \{\Gamma \in \mathcal{D}(G) \mid \Gamma \subset \Delta\}$. Let $\mathfrak{U}$ be a local basis of $\mathfrak{L}(G)$. Then*

$$\mathcal{B} := \{\Gamma + \exp(U) \mid \Gamma \in \mathcal{B}_\Delta, U \in \mathfrak{U}\}$$

is a local basis of G. Even more, $\mathcal{B}^* := \{\Delta + \exp(U) \mid \Delta \in \mathcal{D}(G), U \in \mathfrak{U}\}$ is a local basis of G.

Proof. By [9, Theorem 8.20, p. 387], for any $\Delta \in \mathcal{D}(G)$ the homomorphism

$$\varphi_\Delta \colon \Delta \times \mathfrak{L}(G) \longrightarrow G, \quad \varphi_\Delta((d,\chi)) = d + \exp(\chi)$$

is surjective, continuous and open. With the notation of Lemma 6.5 the product $\Delta \times \mathfrak{L}(G)$ has the local basis $\mathcal{B}_\Delta \times \mathfrak{U}$. Let $\Gamma \times U \in \mathcal{B}_\Delta \times \mathfrak{U}$. Then $\Gamma + \exp(U) = \varphi_\Delta(\Gamma \times U)$ is an open neighborhood of $0 \in G$. On the other hand, let V be an open neighborhood of $0 \in G$. Then $\varphi_\Delta^{-1}[V]$ is open in $\Delta \times \mathfrak{L}(G)$. Hence there is $\Gamma \in \mathcal{B}_\Delta$ and $U \in \mathfrak{U}$ such that $\Gamma \times U \subset \varphi_\Delta^{-1}[V]$. It follows that $\Gamma + \exp(U) = \varphi_\Delta(\Gamma \times U) \subset V$. □

Theorem 6.6. *Let G be a protorus and $\mathbf{\Delta} = \bigcup \mathcal{D}(G)$. Then*

$$\frac{\mathbf{\Delta}[\mathcal{D}] \times \mathfrak{L}(G)}{K} \cong_{\mathrm{topAb}} G,$$

the isomorphism being induced by the map $\psi \colon \mathbf{\Delta} \times \mathfrak{L}(G) \to G$, $\psi((d,\chi)) = d + \exp(\chi)$ and $K = \{(\exp(d), -d) \mid d \in \exp^{-1}[\mathbf{\Delta}]\}$.

Proof. Clearly ψ is surjective and the kernel is as stated. The identity morphism $\mathrm{id} \colon \mathbf{\Delta}[\mathcal{D}] \to \mathbf{\Delta}$ is continuous because $\mathbf{\Delta}$ has the local basis $\mathcal{B}^* \cap \mathbf{\Delta}$ (Lemma 6.5) and $\Delta \subset \Delta + \exp U$. So $\varphi \colon \mathbf{\Delta}[\mathcal{D}] \times \mathfrak{L}(G) \xrightarrow{\mathrm{id}} \mathbf{\Delta} \times \mathfrak{L}(G) \xrightarrow{\psi} G$ is continuous. Finally, an open neighborhood $\Delta \times U \in \mathcal{D} \times \mathfrak{U}$ maps to $\Delta + U$ which is open in G. It follows that φ is an open map. □

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