Marco Amabili
Dipartimento di Meccanica
Università di Ancona
I-60131 Ancona, Italy
e-mail: amabili@mehp1.cineca.it

Free Vibration of a Fluid-Filled Circular Cylindrical Shell with Lumped Masses Attached, Using the Receptance Method

The receptance method is applied to the analytical study of the free vibrations of a simply supported circular cylindrical shell that is either empty or filled with an inviscid, incompressible fluid and with lumped masses attached at arbitrary positions. The receptance of the fluid-filled shell is obtained using the added virtual mass approach to model the fluid–structure interaction. The starting data for the computations is the modal properties of the cylinder that can be obtained using any theory of shells. Numerical results are obtained as roots of the frequency equation and also by considering the trivial solution. They are compared to data obtained by experimental modal analysis performed on a stainless steel tank, empty, or filled with water, with a lead mass attached. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

Vibrations of structures made by connecting simple elements together are studied by numerous authors using different techniques. The interest in this topic is surely due to the wide engineering applications of simple components that are joined to construct a system of practical interest. The techniques used in these studies can be divided into numerical, like the finite element method, and analytical. Among the analytical methods, the receptance method, first introduced by Bishop and Johnson (1960), has recently attained great success. In particular, the important work of Professor Soedel and his school has given a fundamental stimulus to the diffusion of this technique. The receptance (reciprocal mobility) method is illustrated in the book of Soedel (1993) with some applications. It predicts the response and frequencies of a combined system in terms of the responses and frequencies of its subsystems. The receptance method was applied, for example, to ring stiffened cylindrical shells by Wilken and Soedel (1976a,b), to rectangular plates by Azimi et al. (1984), to circular plates by Azimi (1988a,b), to ring and tires by Allaei et al. (1986, 1987, 1988), and to plates welded to a cylindrical shell by Huang and Soedel (1993a–c). A great advantage of this technique is that it allows a good understanding of the physics of combined structures.

In this article the receptance method is applied to a simply supported circular cylindrical shell filled with fluid and with lumped masses attached at arbitrary positions. To simulate the inertial effect of fluid, the added virtual mass approach, introduced by Berry and Reissner (1958) and

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Lindholm et al. (1962), was used. The aim of this work was the study of a problem of practical interest and the extension of the receptance method to structures in contact with a fluid. In fact, combined structures like tanks are often used to store fluids. In this work, the effect of lumped masses, which simulate the effect of an apparatus connected to the shell, on the free vibration of the shell was investigated either when the cylinder was empty or filled with fluid. A comparison between the numerical data and the results of experimental modal tests performed on an AISI 304 stainless steel tank, empty or filled with water, with a lead mass attached, is given.

THEORY

Two masses attached to a simply supported, circular cylindrical shell at arbitrary axial and angular positions were considered first. The constraints imposed at the shell ends are

$$N_x = M_x = v = w = 0 \quad \text{at } x = 0 \text{ and } x = L, \quad (1)$$

where u, v, and w are the displacements of a generic point of the shell in the longitudinal, circumferential, and radial directions, respectively; N_x is the normal force and M_x the bending moment, both acting upon the unit length of the surface element; and L is the shell length. The constraint given by Eq. (1) is called "simply support" by some authors (Soedel, 1993) and by others "shear diaphragm" (Leissa, 1973). The shell is considered thin and made of linear elastic, isotropic, and homogeneous material. A cylindrical polar coordinate system (O, r, θ, x) is introduced with the pole at the center of a shell end. The angle φ indicates the angular distance between the two masses and is given by [see Fig. 1(a)]

$$\theta_2 - \theta_1 = \varphi, \quad (2)$$

where (θ_1, x_1) and (θ_2, x_2) are the angular and axial coordinates of the two masses. The two masses B and C attached to the shell A in Fig. 1(a) are connected, respectively, to the two point 1 and 2 of the same figure. If one considers the shell without the masses, it is necessary to introduce the two forces F_{A1} and F_{A2} at the mass locations [Fig. 1(b)]. The displacement X_{A1} of the shell at point 1, where mass B is attached, is expressed by the following

$$X_{A1} = \alpha_{11}F_{A1} + \alpha_{12}F_{A2}. \quad (3)$$

F_{A1} and F_{A2} are the amplitudes of the forces applied by joints at points 1 and 2, respectively; in the case of undamped free vibrations, the coupling forces F_{Ai} are considered to be harmonic. The definition of the receptance α_{ij} of the system A (cylinder) is given as the ratio of a displacement response at a certain point i to a harmonic force input at the point j. For the Maxwell's reciprocity theorem $\alpha_{ij} = \alpha_{ji}$. The displacement X_{A2} of the shell at point 2, where mass C is connected, is given by

$$X_{A2} = \alpha_{21}F_{A1} + \alpha_{22}F_{A2}. \quad (4)$$

For mass B the displacement X_{B1} is

$$X_{B1} = \beta_{11}F_{B1}, \quad (5)$$

where β_{11} is the receptance of mass B. Then, the displacement X_{C2} of mass C is

$$X_{C2} = \delta_{22}F_{C2}, \quad (6)$$

where δ_{22} is the receptance of mass C. Due to the connection between A and B at point 1 and between A and C at point 2, we obtain the following equations [see Fig. 1(b)]:

$$F_{A1} = -F_{B1}, \quad (7)$$

$$F_{A2} = -F_{C2}, \quad (8)$$

$$X_{A1} = X_{B1}, \quad (9)$$

$$X_{A2} = X_{C2}. \quad (10)$$

Substituting Eqs. (5)–(10) in Eqs. (3) and (4) results in

$$(\alpha_{11} + \beta_{11})F_{A1} + \alpha_{12}F_{A2} = 0, \quad (11)$$

$$\alpha_{21}F_{A1} + (\alpha_{22} + \delta_{22})F_{A2} = 0. \quad (12)$$

Equations (11) and (12) are a system of two homogeneous linear equations; the following determinant must be equal to zero to obtain a nontrivial solution

$$\begin{vmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} + \delta_{22} \end{vmatrix} = 0. \quad (13)$$

From Eq. (13) we obtain the frequency equation

$$(\alpha_{11} + \beta_{11})(\alpha_{22} + \delta_{22}) - \alpha_{12}^2 = 0. \quad (14)$$

The solution of the problem is obtained by the roots of Eq. (14); this equation is singular at each

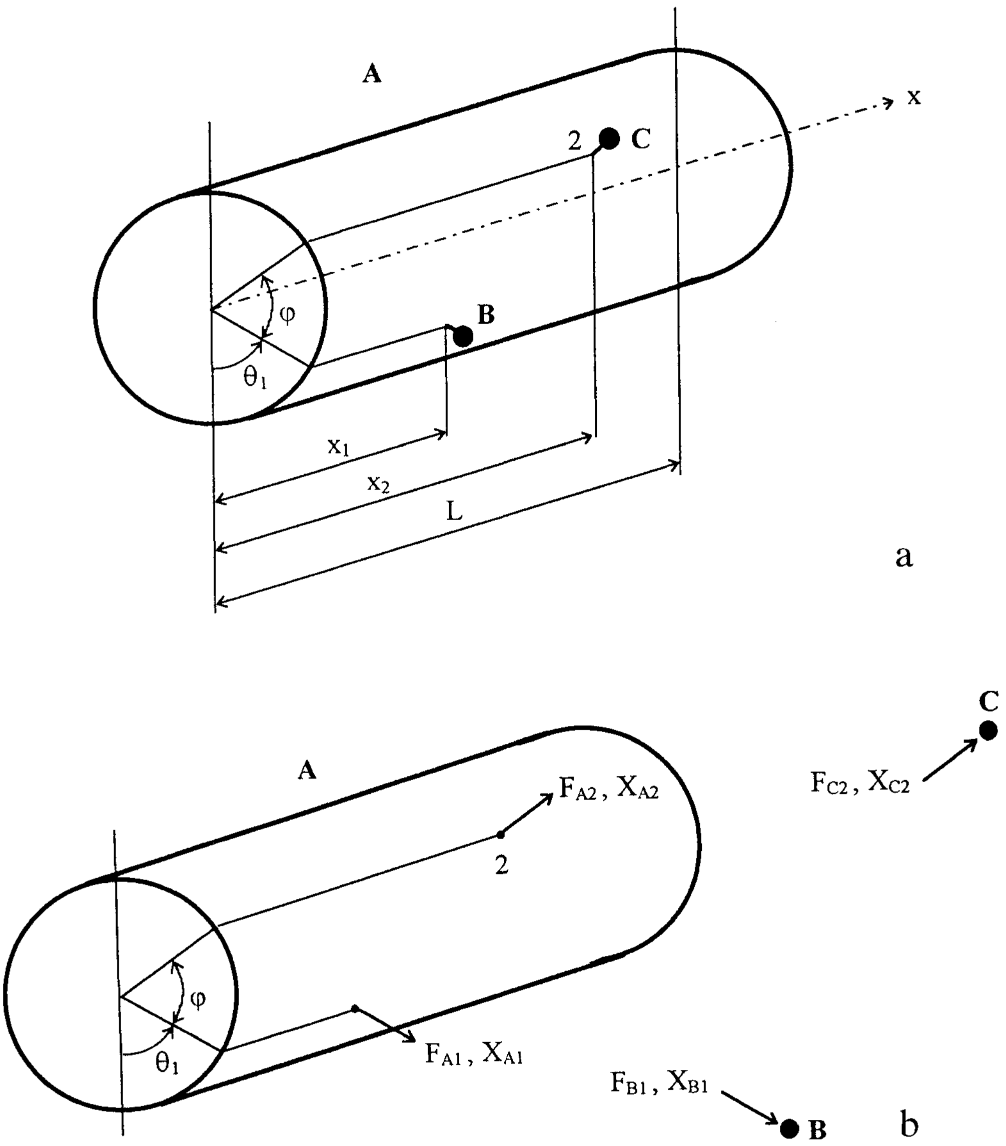

FIGURE 1 Shell-masses system. (a) Shell (*A*) with lumped masses (*B* and *C*) attached. (b) Forces and responses: used symbols.

natural frequency of the shell and presents the value "zero" between the singularities. For this purpose, all the receptances involved in this equation are evaluated. The natural circular frequency ω of the combined system and the two mass values M_1 and M_2 of the lumped masses B and C are introduced. The equation of motion of the lumped mass B is

$$M_1\ddot{X}_{B1} = F_{B1}e^{i\omega t}, \tag{15}$$

where i is now the imaginary unit. Then, for undamped free vibrations the equation of motion is transformed into

$$-M_1\omega^2 X_{B1} = F_{B1}. \tag{16}$$

By using Eq. (5), the receptance β_{11} of the mass C can be written as

$$\beta_{11} = -\frac{1}{M_1\omega^2}. \tag{17}$$

Similarly the receptance δ_{22} for the lumped mass C is obtained

$$\delta_{22} = -\frac{1}{M_2\omega^2}. \tag{18}$$

Focusing the attention on the shell, due to a harmonic radial force F applied at (θ^*, x*), the harmonic radial response w at the point of coordinates (θ, x) is given by (Soedel, 1993)

$$w(\theta, x, t) = \frac{2Fe^{i\omega t}}{hLR\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x^*/L)\sin(m\pi x/L)\cos[n(\theta - \theta^*)]}{\rho_{mn}(\omega_{mn}^2 - \omega^2)}, \tag{19}$$

h, L, and R being the shell thickness, length, and radius, respectively; m is the number of longitudinal half-waves and n the number of circumferential waves; ρ_{mn} is the virtual density factor; and ω_{mn} is the natural circular frequency of the shell without masses attached. In Eq. (19) only the shell modes with greater radial displacements (flexural modes) are considered and the influence of the axisymmetric modes ($n = 0$) are neglected in this expansion. In fact, the natural frequencies of modes with greater circumferential and longitudinal displacements (twisting and extension–compression modes) are so high that their contribution to the sum in Eq. (19) can be neglected in most engineering applications. This consideration can also be applied to axisymmetric modes.

The shell is studied either vibrating in a vacuum or filled with fluid. The circular frequencies ω_{mn} must be calculated in the vacuum or in the fluid-filled configuration. They are obtained by the first root (flexural mode) of the following characteristic bicubic equation (e.g., Amabili and Dalpiaz, 1995) for any given couple of m and n

$$\Omega^6 - K_2\Omega^4 + K_1\Omega^2 - K_0 = 0, \tag{20}$$

where the frequency parameter Ω is defined by

$$\Omega^2 = (\rho_S/E)(1 - \nu^2)\omega_{mn}^2 R^2. \tag{21}$$

In Eq. (21) E is the Young modulus and ρ_S is the mass density of the shell. Using the Donnell theory of shells (Leissa, 1973), the coefficients K_i are given by

$$K_2 = \frac{1}{\gamma_{mn}} + \frac{3-\nu}{2}(n^2 + \tilde{\lambda}^2) + \frac{k}{\gamma_{mn}}(n^2 + \tilde{\lambda}^2)^2, \tag{22}$$

$$K_1 = \frac{1-\nu}{2\gamma_{mn}}\left[(3 + 2\nu)\tilde{\lambda}^2 + n^2 + \gamma_{mn}(n^2 + \tilde{\lambda}^2)^2 + \frac{3-\nu}{1-\nu}k(n^2 + \tilde{\lambda}^2)^3\right], \tag{23}$$

$$K_0 = \frac{1-\nu}{2\gamma_{mn}}[(1 - \nu^2)\tilde{\lambda}^4 + k(n^2 + \tilde{\lambda}^2)^4], \tag{24}$$

where ν is the Poisson's ratio, $\tilde{\lambda} = m\pi R/L$, and $k = 1/12(h/R)^2$. The virtual density factor ρ_{mn} of the shell is introduced and is given by (Amabili, 1996; Lindholm et al., 1962)

$$\rho_{mn} = \begin{cases} \rho_S, & \text{vacuum} \\ \rho_S + \dfrac{\rho_F L}{m\pi h}\dfrac{I_n\left(\frac{m\pi}{L}R\right)}{I_n'\left(\frac{m\pi}{L}R\right)}, & \text{fluid filled.} \end{cases} \tag{25}$$

In Eq. (25) ρ_F is the mass density of the fluid, I_n the modified Bessel function, and I_n' its derivative with respect to the argument. The dimensionless virtual mass factor γ_{mn} used in Eqs. (22)–(24) is defined as $\gamma_{mn} = \rho_{mn}/\rho_S$. In Eqs. (24) and (25) the fluid is considered incompressible and inviscid and both the shell ends are assumed to be open. The added virtual mass approach is used to model the fluid–structure interaction; this technique is largely applied to the study of the vibration of single components, such as plates and shells, and structures in contact with fluids. By using Eq. (19), the receptances α_{ij} of the shell can be calculated; they are given by

$$\alpha_{11} = \frac{2}{hLR\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\rho_{mn}(\omega_{mn}^2 - \omega^2)} \sin^2\frac{m\pi x_1}{L}, \tag{26}$$

$$\alpha_{22} = \frac{2}{hLR\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\rho_{mn}(\omega_{mn}^2 - \omega^2)} \sin^2\frac{m\pi x_2}{L}, \tag{27}$$

$$\alpha_{21} = \frac{2}{hLR\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\rho_{mn}(\omega_{mn}^2 - \omega^2)} \sin\frac{m\pi x_1}{L} \cdot \sin\frac{m\pi x_2}{L}\cos(n\varphi). \tag{28}$$

The shell mode shapes are computed by the following expression, where the responses due to the harmonic forces F_{A1} and F_{A2} are combined, and the ratio F_{A2}/F_{A1} is obtained by one of the linear dependent Eqs. (11) and (12)

$$w(x, \theta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{m\pi x}{L}}{\rho_{mn}(\omega_{mn}^2 - \omega^2)} \cdot \left\{\sin\frac{m\pi x_1}{L}\cos[n(\theta - \theta_1)] + \frac{F_{A2}}{F_{A1}}\sin\frac{m\pi x_2}{L}\cos[n(\theta - \theta_2)]\right\}, \tag{29}$$

where ω is the natural circular frequency of the coupled shell-masses system. With the proposed method, the vibration of the coupled system can be studied; the circular frequencies are given by the roots of Eq. (14) and mode shapes by Eq. (29).

It is important to remember that also the trivial solution of the linear system given by Eqs. (11) and (12) exists and is obviously given by $F_{A1} = F_{A2} = 0$. In this case, no force couples the masses and the shell, therefore the natural circular frequencies ω_{mn} of the shell modes that present nodes at both the locations of the masses are unchanged. Therefore, together with the shell modes resulting from combination with masses, there is the family of typical modes of the simply supported circular cylindrical shell.

In the case of a single mass B attached to the cylinder, Eq. (14) is simplified to

$$\alpha_{11} + \beta_{11} = 0. \tag{30}$$

Mode shapes of the shell with one mass attached at $x = x_1$ and $\theta = \theta_1$ are given by

$$w(\theta, x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\rho_{mn}(\omega_{mn}^2 - \omega^2)} \sin\frac{m\pi x}{L} \cdot \sin\frac{m\pi x_1}{L} \cos[n(\theta - \theta_1)]. \tag{31}$$

In this case, all the natural frequencies of the simply supported cylinder without any mass attached are also solutions of the shell-mass system; in fact, due to the axial symmetry of the cylinder, the trivial solution $F_{A1} = 0$ is possible for all these modes. Corresponding modes present a nodal line at $\theta = \theta_1$.

By using the same method, N masses can be considered joined to the shell. In this case the frequency equation is gained by the following determinant

$$\begin{vmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} + \delta_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} + \eta_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0, \tag{32}$$

η_{33} representing the receptance of the mass D connected to the shell at point 3.

NUMERICAL AND EXPERIMENTAL RESULTS

Numerical computations and experimental modal tests were performed on a circular cylindrical shell having a diameter of 175 mm, a thickness of 1 mm, and a length of 664 mm. The material of the cylinder was AISI 304 stainless steel with Young's modulus $E = 206$ GPa, Poisson's ratio $\nu = 0.3$, and density $\rho_S = 7.7 \cdot 10^3$ kg m^{-3}. Two steel plates of 0.4-mm thickness were welded to both ends of the cylinder. The boundary conditions imposed on the cylindrical shell by the thin plates at both ends can be considered very close to the case of a simply supported shell; moreover, the open-ended fluid boundary condition is also approximated by flexible plates. Two small pipe fittings were welded to one of the plates in such a position to not affect the shell vibrations; one was used for supplying water, the other for mounting a liquid level indicator. The tank was hung with its axis kept horizontal by a pliant suspension, using low stiffness cables connected to the end plates.

The cylinder was tested both empty and filled with water, $\rho_F = 10^3$ kg m^{-3}, and with a 0.0967-kg lead mass attached at a distance of $x_1 = 262$ mm from the shell's end. Numerical results were computed using the software Mathematica (Wolfram, 1988) and experimental data was obtained using the software CADA-X by LMS working at a workstation.

The experimental modal tests were performed using an impact excitation. A hammer with a weight of 284 g and a plastic tip was used to excite the structure. The frequency response functions (FRFs) between 84 excitation points and one response point were measured. The excitation was applied to a grid with six equidistant points in the longitudinal direction and 14 positions around the circumference; this allows the detection of modal shapes possessing up to seven circumferential waves. Both the excitation force and measured response were in the radial direction. The non-contact sensor used to measure the shell velocity during tests was a laser Doppler vibrometer Polytech OFV 1102 in order to not alter the mass of the studied system. During tests, the frequency resolution of the FRFs was 0.5 Hz; exponential windows and eight averages were used. The modal parameters were evaluated using the frequency domain direct parameter estimation method.

In Table 1 the computed and measured natural frequencies of the cylinder with the attached mass

Table 1. Theoretical (Th.) and Experimental (Exp.) Natural Frequencies

Empty Shell				Water-Filled Shell			
Eq. (30)		ω_{mn}		Eq. (30)		ω_{mn}	
Th.	Exp.	Th.	Exp.	Th.	Exp.	Th.	Exp.
208.9	207.9	221.5	219.8	89.5	90.4	91.0	91.8
226.0	224.6	229.0	228.2	101.5	101.7	102.8	103.4
285.3	280.1	295.3	290.1	115.6	114.8	117.2	113.1
311.8	299.9	317.9	306.7	139.8	140.7	141.4	144.2

Empty and water-filled shells with a mass of 96.7 g attached. Data relative to the two families of modes obtained by the frequency equation, Eq. (30), and by the trivial solution $F_{A1} = 0$, ω_{mn}, are shown in different columns.

are compared; the mean error is 1.4%. Data in this table are presented for both the empty and the water-filled cylinder with the attached mass, and modes obtained by the trivial solution are also considered. Solutions of the frequency equation, Eq. (30), are given graphically in Fig. 2 for the empty shell and in Fig. 3 for the water-filled shell; 20 terms are considered in the sum that gives the receptances α_{11}. The computed and experimentally detected mode shapes in a cross section at $x = x_1$ for the empty and the water-filled shell with the attached mass are shown in Figs. 4 and 5, respectively. The connection between the results of the theoretical and experimental data is shown. It is also interesting to see that the presence of the attached mass has a greater influence on natural frequencies and mode shapes in the case of the empty cylinder. In fact, the water-filled cylinder presents a modal mass much greater than the empty one. In Fig. 2, therefore, the intersections between α_{11} and β_{11} are closer to the natural frequencies of the cylinder without the attached mass (Fig. 3). In Fig. 6, an experimental FRF (amplitude) of the water-filled shell with the mass attached is given with its coherence function. The plotted FRF is obtained as a ratio of velocity response to force input and therefore is a mobility. The coherence of the measured FRF is "good" (that is near the value one) in all the plotted frequency range, excluding the antiresonances, where the signal to noise ratio is low. It is interesting to note (in Fig. 6) the high modal

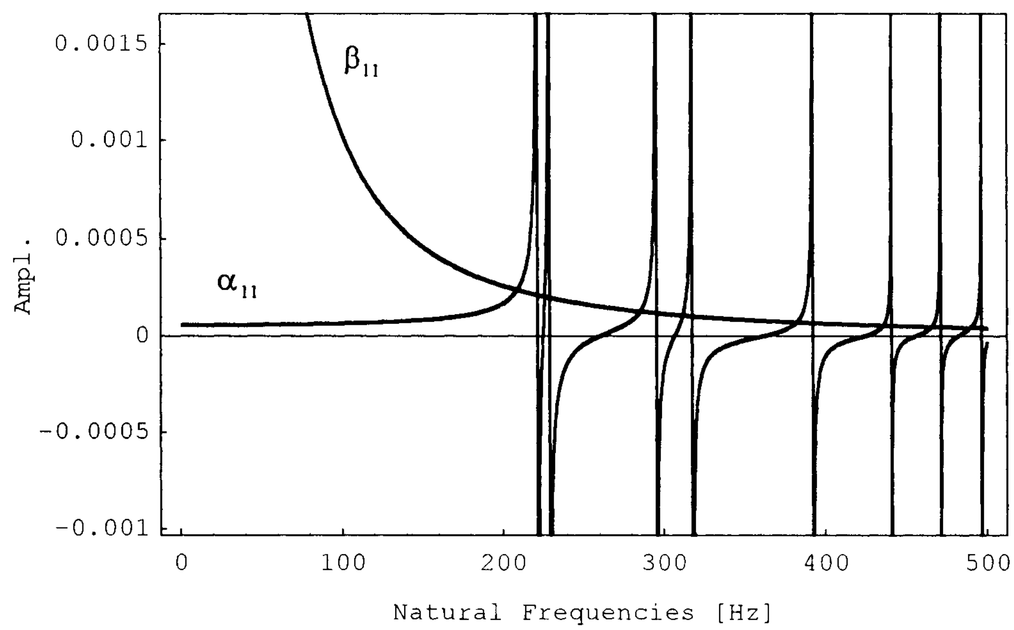

FIGURE 2 Graph of the shell and attached mass receptances, Eq. (30); empty shell with attached mass of 0.0976 kg.

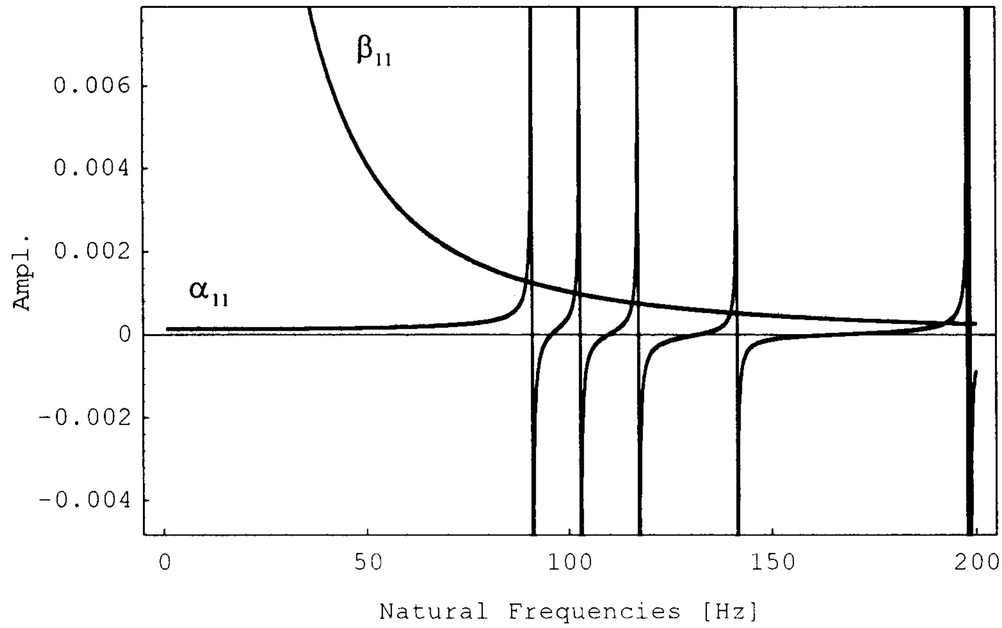

FIGURE 3 Graph of the shell and attached mass receptances, Eq. (30); water-filled shell with attached mass of 0.0976 kg.

density of the shell and the very low frequency (first peak of the FRF) of the system of suspension of the tank used in the test.

CONCLUSION

The receptance method, already successfully applied to the theoretical study of vibrations of structures made by connecting simple components, can also be used to study circular cylindrical shells in contact with a fluid and with lumped masses attached, by using the added virtual mass approach to evaluate the inertial effect of the fluid. The starting data of numerical computation are the modal properties of the shell, empty or fluid-filled, that can be quickly obtained by the characteristic bicubic equation. Then, natural frequencies of the combined system are given by the roots of the frequency equation. The com-

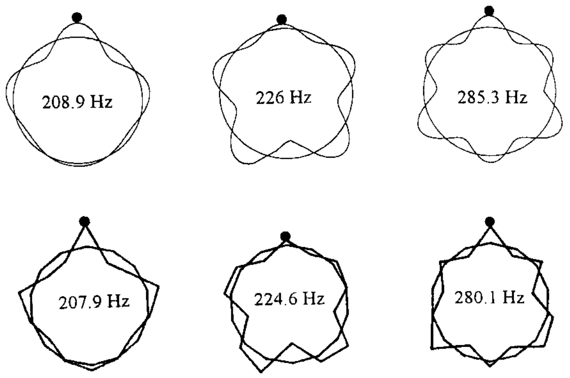

FIGURE 4 First three mode shapes of the empty shell obtained by roots of the frequency equation, Eq. (30). Above: computed mode shapes. Below: measured mode shapes.

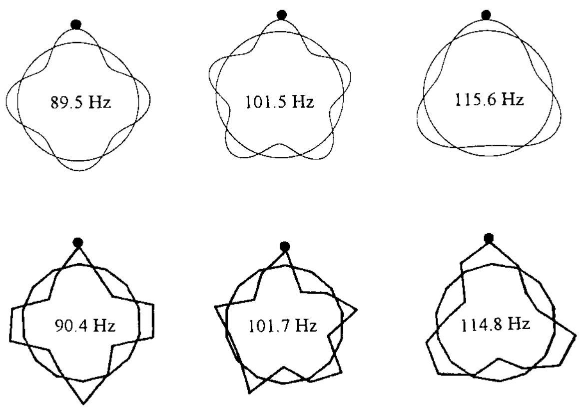

FIGURE 5 First three mode shapes of the water-filled shell obtained by roots of the frequency equation, Eq. (30). Above: computed mode shapes. Below: measured mode shapes.

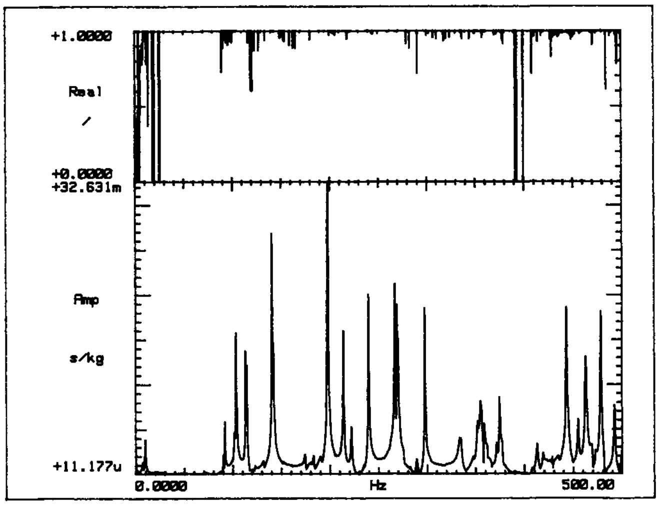

FIGURE 6 Experimental FRF (amplitude) and its coherence measured on the water-filled shell with the lumped mass attached.

puted data is in accordance with the experimental results obtained by tests on an AISI 304 stainless steel tank that validate the analytical approach.

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