Springer

Journal of Mechanical Science and Technology 22 (2008) 2062~2075

Journal of Mechanical Science and Technology

www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-008-0747-4

Free vibration of multi-layered circular cylindrical shell with cross-ply walls, including shear deformation by using spline function method †

K. K.Viswanathan*, Kyung Su Kim, Jang Hyun Lee, Hee Seung Koh and Jae Beom Lee

School of Mechanical Engineering, Department of Naval Architecture and Ocean Engineering, Inha University, Incheon 402 751, Korea South

(Manuscript Received March 20, 2008; Revised July 7, 2008; Accepted July 27, 2008)

Abstract

The spline function technique is used to analyze the vibration of multi-layered circular cylindrical shells with cross-ply walls including first-order shear deformation theory. Both antisymmetric and symmetric cross-ply laminations are considered in this analysis. The governing equilibrium equations are obtained in terms of displacement and rotational functions. A system of coupled ordinary differential equations in terms of displacement and rotational functions are obtained by assuming the solution in a separable form. These functions are approximated by using Bickley-type splines of suitable order to obtain the generalized eigenvalue problem by applying point collocation techniques with appropriate boundary conditions. Parametric studies are performed to analyze the frequency response of the shell with reference to the material properties, number of layers, fiber orientation, thickness to radius ratio, length to radius ratio and circumferential node number. Reasonable agreement is found with existing results obtained by FEM and other methods. Valuable results are presented as graphs and discussed.

Keywords: Cylindrical shell; Cross-ply; Free vibration; Spline method; Shear deformation

1. Introduction

The design of cylindrical shell structures is widely used in the fields of aerospace, automobile, chemical industries and ship building. Present day engineers are mainly focusing on the design of composite structures due to their higher specific stiffness, better damping and shock absorbing characteristics. The natural frequencies depend highly on ply orientations, boundary conditions and geometric parameters [1-7]. In the case of composite laminated shells there is an advantage for adjusting the natural frequency, which is by designing a suitable lamination scheme [8, 9]. Several types of theories and methods of analysis have been developed and applied for free vibration studies of composite elements. Many of these theories were developed originally for thin shells and are based on the Kirchhoff-Love hypothesis, i.e., transverse shear deformation is neglected. Reddy [10] presented the bending of laminated anisotropic composite shells by a shear deformation theory using a finite element method. An extension of Sander's shell theory for doubly curved isotropic shells to a shear deformation theory of laminated shells is studied by Reddy [11]. The study dealt with the development of exact solutions for simply supported cross-ply laminated shells. Lam and Loy [12] analyzed the laminated cylindrical shells by using four common thin shell theories, namely Donnell's, Flugge's, Love's and Sander's. Only simply supported conditions were used in this analysis for finding the frequencies using three-layered cross-ply lamination. A Finite element method was implemented by Sun *et al.* [13] to investigate the fundamental frequencies of

† This paper was recommended for publication in revised form by Associate Editor Maenghyo Cho

*Corresponding author. Tel.: +82 32 860 8486, Fax.: +82 32 864 5850
E-mail address: visu20@yahoo.com

circular cylinders including the shear deformation. Hu and Tsai [14] presented work on maximizing the frequencies with respect to fiber orientation using a golden section method, while in paper [15] a wave propagation approach was used for cross-ply composite cylindrical shells in which transverse shear deformation and rotary inertia were neglected. Analytical calculation methods were adopted by Hufenbach *et al*. [16] to discuss the problem regarding vibration and damping behavior of multi-layered composite cylindrical shells. In the works of Viswanathan and Navaneethakrishnan [17, 18] on cylindrical and conical shells, only a specially orthotropic shell wall is considered and rotary inertia and shear deformation were totally neglected. More recently, Viswanathan and Lee [19] considered the vibration of cross-ply laminates including rotary inertia and shear deformation, but only of a plate.

In the present paper, the free vibration of a multi-layered circular cylindrical shell with cross-ply walls including shear deformation is analyzed by using the spline method. The choice of this method is due to the possibility that a chain of lower order approximation, as used here, can yield greater accuracy than a global higher order approximation. Bickley [20] successfully tested the spline collocation method over a two point boundary value problem with a cubic spline. Viswanathan and Lee successfully applied this in their work [19]. These splines are simple and clear for analytical processes and have a significant computational advantage.

In this work the problem is formulated by using a first order shear deformation theory from which we obtained a system of coupled differential equations on a set of assumed displacement and rotational functions which are functions of space coordinates. These functions are approximated by Bickley-type splines, which are cubic. Collocation with these splines yields a set of field equations which, along with the equations of boundary conditions, reduce to a system of homogeneous simultaneous algebraic equations on the assumed spline coefficients. Then the problem is solved for a frequency parameter, using an eigensolution technique, to obtain as many frequencies as required, starting from the least. The mode shapes can be constructed from the spline coefficients, which are computed from the eigenvectors.

Parametric studies have been made for the frequency parameters with respect to the thickness to radius ratio, length to radius ratio with two-, three- and four-layered shells. Two types of layered materials are considered. Numerical results are presented in terms of graphs and tables and discussed.

2. Formulation of the problem

Consider a thin circular cylindrical shell of length ℓ, thickness h and radius r. The shell consists of a finite number of layers which are perfectly bonded together. The reference surface of the shell is taken to be at its middle surface when an orthogonal coordinate system (x,θ,z) is fixed. The x coordinate is taken in the axial direction, whereas θ and z are taken in the circumferential and transverse directions, respectively. Following Toorani and Lakis [21] and Hufenbach et al. [16], the displacement components are assumed to be:

$$\begin{aligned}
u(x,\theta,z,t) &= u_0(x,\theta,t) + z\psi_x(x,\theta,t)\\
v(x,\theta,z,t) &= v_0(x,\theta,t) + z\psi_\theta(x,\theta,t)\\
w(x,\theta,z,t) &= w_0(x,\theta,t)
\end{aligned} \tag{1}$$

where u, v and w are the displacement components in the x, θ and z directions, respectively, u_0 and v_0 are the in-plane displacements of the shell in the mid-plane, and ψ_x and ψ_θ are the shear rotations of any point on the middle surface of the shell. The strain-displacement relations of linear elasticity for cylindrical shell of radius r can be written as:

$$\begin{aligned}
&\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \psi_x}{\partial x},\quad \varepsilon_\theta = \frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{w}{r} + \frac{z}{r}\frac{\partial \psi_\theta}{\partial \theta},\\
&\gamma_{x\theta} = \frac{1}{r}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + z\left(\frac{1}{r}\frac{\partial \psi_x}{\partial \theta} + \frac{\partial \psi_\theta}{\partial x}\right)\\
&\gamma_{xz} = \psi_x + \frac{\partial w}{\partial x} \quad \text{and} \quad \gamma_{\theta z} = \psi_y + \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v_0}{r}
\end{aligned} \tag{2}$$

The stress-strain relations of the k-th layer by neglecting the transverse normal strain and stress, are of the form:

$$\begin{pmatrix} \sigma_x^{(k)} \\ \sigma_\theta^{(k)} \\ \tau_{x\theta}^{(k)} \\ \tau_{\theta z}^{(k)} \\ \tau_{xz}^{(k)} \end{pmatrix} = \begin{pmatrix} C_{11}^{(k)} & C_{12}^{(k)} & 0 & 0 & 0 \\ C_{21}^{(k)} & C_{22}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & C_{66}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(k)} & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(k)} \\ \varepsilon_\theta^{(k)} \\ \gamma_{x\theta}^{(k)} \\ \gamma_{\theta z}^{(k)} \\ \gamma_{xz}^{(k)} \end{pmatrix} \tag{3}$$

When the materials are oriented at an angle α with the x -axis, the transformed stress-strain relations are:

$$\begin{pmatrix} \sigma_x^{(k)} \\ \sigma_\theta^{(k)} \\ \tau_{x\theta}^{(k)} \\ \tau_{\theta z}^{(k)} \\ \tau_{xz}^{(k)} \end{pmatrix} = \begin{pmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} & 0 & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & Q_{26}^{(k)} & 0 & 0 \\ Q_{16}^{(k)} & Q_{26}^{(k)} & Q_{66}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} \\ 0 & 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(k)} \\ \varepsilon_\theta^{(k)} \\ \gamma_{x\theta}^{(k)} \\ \gamma_{\theta z}^{(k)} \\ \gamma_{xz}^{(k)} \end{pmatrix} \quad (4)$$

where

$$\left[Q^{(k)}\right] = [T]^{-1}\left[C^{(k)}\right][T] \quad (5)$$

The matrix $[T]$ is the transformation matrix (Toorani amd Lakis [21]). Using the transformation given in Eq. (5) one can obtain the quantities $Q_{ij}^{(k)}$ as functions of $C_{ij}^{(k)}$ and α given in Appendix A.

The stress resultants and stress couples are given by:

$$\left(N_x, N_\theta, N_{x\theta}, Q_x, Q_\theta\right) = \int_z \left(\sigma_x, \sigma_\theta, \tau_{x\theta}, \tau_{xz}, \tau_{\theta z}\right) dz$$

$$\left(M_x, M_\theta, M_{x\theta}\right) = \int_z \left(\sigma_x, \sigma_\theta, \tau_{x\theta}\right) z\, dz \quad (6)$$

Applying Eq. (2) into Eq. (4) and then substituting into Eq. (6) leads to the equations of stress-resultants and moment resultant as:

$$\begin{pmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{pmatrix} \begin{pmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{1}{r}\dfrac{\partial v_0}{\partial \theta} \\ \dfrac{\partial v_0}{\partial x} + \dfrac{1}{r}\dfrac{\partial u_0}{\partial \theta} \\ \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{1}{r}\dfrac{\partial \psi_\theta}{\partial \theta} \\ \dfrac{\partial \psi_\theta}{\partial x} + \dfrac{1}{r}\dfrac{\partial \psi_x}{\partial \theta} \end{pmatrix}$$

and

$$\begin{pmatrix} Q_\theta \\ Q_x \end{pmatrix} = K \begin{pmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{pmatrix} \begin{pmatrix} \psi_\theta + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} - \dfrac{v_0}{r} \\ \psi_x + \dfrac{\partial w}{\partial x} \end{pmatrix} \quad (7)$$

in which A_{ij}, B_{ij} and D_{ij} are the laminate stiffnesses defined by

$$A_{ij} = \sum_k Q_{ij}^{(k)}(z_k - z_{k-1}),\ B_{ij} = \frac{1}{2}\sum_k Q_{ij}^{(k)}(z^2{}_k - z^2{}_{k-1}),$$

$$D_{ij} = \frac{1}{3}\sum_k Q_{ij}^{(k)}(z^3{}_k - z^3{}_{k-1}) \quad (i, j = 1, 2, 6) \quad (8)$$

and K is the shear correction factor, z_{k-1} and z_k are boundaries of the k -th layer. The value of K for a general laminate depends on lamina properties and lamination scheme and may be calculated by various static and dynamic methods (Whitney and Sun [22], Whitney [23], Bert [24] and Perngjin [25]).

On Substitution of Eq. (7) into the equations of motion of anisotropic circular cylindrical shells by taking into account the shear deformation and rotary inertia effects (given in Appendix B), the following differential equations are obtained in terms of displacements and rotational functions in the following form:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w \\ \psi_x \\ \psi_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

where

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + A_{66}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - I_1\frac{\partial^2}{\partial t^2}$$

$$L_{12} = A_{16}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66})\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + A_{26}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$L_{13} = A_{12}\frac{1}{r}\frac{\partial}{\partial x} + A_{26}\frac{1}{r^2}\frac{\partial}{\partial \theta}$$

$$L_{14} = B_{11}\frac{\partial^2}{\partial x^2} + 2B_{16}\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + B_{66}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$L_{15} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66})\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + B_{26}\frac{1}{r}\frac{\partial^2}{\partial \theta^2}$$

$$L_{21} = A_{16}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66})\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + A_{26}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$L_{22} = A_{66}\frac{\partial^2}{\partial x^2} + 2A_{26}\frac{1}{r}\frac{\partial^2}{\partial x \partial \theta} + A_{22}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - K\,A_{44}\frac{1}{r^2} - I_1\frac{\partial^2}{\partial t^2}$$

$$L_{23} = (A_{26} + K\,A_{45})\frac{1}{r}\frac{\partial}{\partial x} + (A_{22} + K\,A_{44})\frac{1}{r^2}\frac{\partial}{\partial \theta}$$

$$L_{24} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12}+B_{66})\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{26}\frac{1}{r}\frac{\partial^2}{\partial\theta^2} + K\,A_{45}\frac{1}{r}$$

$$L_{25} = B_{66}\frac{\partial^2}{\partial x^2} + 2B_{26}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{22}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + KA_{44}\frac{1}{r}$$

$$L_{31} = -A_{12}\frac{1}{r}\frac{\partial}{\partial x} - A_{26}\frac{1}{r^2}\frac{\partial}{\partial\theta}$$

$$L_{32} = -(A_{26}+K\,A_{45})\frac{1}{r}\frac{\partial}{\partial x} - (A_{22}+K\,A_{44})\frac{1}{r^2}\frac{\partial}{\partial\theta}$$

$$L_{33} = K\,A_{55}\frac{\partial^2}{\partial x^2} + 2K\,A_{45}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + K\,A_{44}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - A_{22}\frac{1}{r^2} - I_1\frac{\partial^2}{\partial t^2}$$

$$L_{34} = (K\,A_{55} - B_{12}\frac{1}{r})\frac{\partial}{\partial x} + (K\,A_{45}\frac{1}{r} - B_{26}\frac{1}{r^2})\frac{\partial}{\partial\theta}$$

$$L_{35} = (K\,A_{45} - B_{26}\frac{1}{r})\frac{\partial}{\partial x} + (K\,A_{44}\frac{1}{r} - B_{22}\frac{1}{r^2})\frac{\partial}{\partial\theta}$$

$$L_{41} = B_{11}\frac{\partial^2}{\partial x^2} + 2B_{16}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{66}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$L_{42} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12}+B_{66})\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{26}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + K\,A_{45}\frac{1}{r}$$

$$L_{43} = (B_{12}\frac{1}{r} - K\,A_{55})\frac{\partial}{\partial x} + (B_{26}\frac{1}{r^2} - K\,A_{45}\frac{1}{r})\frac{\partial}{\partial\theta}$$

$$L_{44} = D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + D_{66}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - K\,A_{55} - I_3\frac{\partial^2}{\partial t^2}$$

$$L_{45} = D_{16}\frac{\partial^2}{\partial x^2} + (D_{12}+D_{66})\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + D_{26}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - K\,A_{45}$$

$$L_{51} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12}+B_{66})\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{26}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$L_{52} = B_{66}\frac{\partial^2}{\partial x^2} + 2B_{26}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + B_{22}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + K\,A_{44}\frac{1}{r}$$

$$L_{53} = (B_{26}\frac{1}{r} - K\,A_{45})\frac{\partial}{\partial x} + (B_{22}\frac{1}{r^2} - K\,A_{44}\frac{1}{r})\frac{\partial}{\partial\theta}$$

$$L_{54} = D_{16}\frac{\partial^2}{\partial x^2} + (D_{12}+D_{66})\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + D_{26}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - K\,A_{45}$$

$$L_{55} = D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{1}{r}\frac{\partial^2}{\partial x\partial\theta} + D_{22}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - K\,A_{44} - I_3\frac{\partial^2}{\partial t^2} \tag{10}$$

The coefficients $A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$ and A_{45} are identically zero for cross-ply laminates (see [11]).

The displacement components u_0, v_0, w and shear rotations ψ_x, ψ_θ are assumed in the form

$$\begin{aligned}
u_0(x,\theta,t) &= U(x)\cos n\theta\, e^{i\omega t}\\
v_0(x,\theta,t) &= V(x)\sin n\theta\, e^{i\omega t}\\
w(x,\theta,t) &= W(x)\cos n\theta\, e^{i\omega t}\\
\psi_x(x,\theta,t) &= \Psi_x(x)\cos n\theta\, e^{i\omega t}\\
\psi_\theta(x,\theta,t) &= \Psi_\theta(x)\sin n\theta\, e^{i\omega t}
\end{aligned} \tag{11}$$

where ω is the angular frequency of vibration and t is the time and n is the circumferential node number.

The non-dimensional parameters are introduced as follows:

$\lambda = \omega\ell\sqrt{\frac{I_1}{A_{11}}}$, a frequency parameter

$X = \frac{x}{\ell}$, a distance co-ordinate

$H = \frac{h}{r}$, thickness parameter

$L = \frac{\ell}{r}$, a length parameter

$\delta_k = \frac{h_k}{h}$, relative layer thickness of the k -th layer

$R = \frac{r}{\ell}$, a radius parameter. (12)

Substituting Eq. (11) into Eq. (9) to get the differential equations depending on a single variable x and then applying the non-dimensional parameters given in Eq. (12), we get the modified governing differential equations as follows:

$$\begin{bmatrix} L_{11}^* & L_{12}^* & L_{13}^* & L_{14}^* & L_{15}^* \\ L_{21}^* & L_{22}^* & L_{23}^* & L_{24}^* & L_{25}^* \\ L_{31}^* & L_{32}^* & L_{33}^* & L_{34}^* & L_{35}^* \\ L_{41}^* & L_{42}^* & L_{43}^* & L_{44}^* & L_{45}^* \\ L_{51}^* & L_{52}^* & L_{53}^* & L_{54}^* & L_{55}^* \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \\ \ell\Psi_X \\ \ell\Psi_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{13}$$

where

$$L_{11}^* = \frac{d^2}{dX^2} - S_{10}\frac{n^2}{R^2} + \lambda^2 ,$$

$$L_{12}^* = (S_{10} + S_2)\frac{n}{R}\frac{d}{dX} = -L_{21}^* ,$$

$$L_{13}^* = S_2\frac{1}{R}\frac{d}{dX} = -L_{31}^* , \quad L_{14}^* = S_4\frac{d^2}{dX^2} - S_{11}\frac{n^2}{R^2} = L_{41}^* ,$$

$$L_{15}^* = (S_{11} + S_5)\frac{n}{R}\frac{d}{dX} = -L_{51}^* ,$$

$$L_{22}^* = S_{10}\frac{d^2}{dX^2} - (S_3\frac{n^2}{R^2} + K\,S_{13}\frac{1}{R^2}) + \lambda^2 ,$$

$$L_{23}^* = -(S_3 + K\,S_{13})\frac{n}{R^2} = L_{32}^* ,$$

$$L_{24}^* = -(S_{11} + S_5)\frac{n}{R}\frac{d}{dX} = -L_{42}^* ,$$

$$L_{25}^* = S_{11}\frac{d^2}{dX^2} + (K\,S_{13}\frac{1}{R^2} - S_6\frac{n^2}{R^2}) = L_{52}^* ,$$

$$L_{33}^* = K\,S_{14}\frac{d^2}{dX^2} - (S_3\frac{1}{R^2} + K\,S_{13}\frac{n^2}{R^2}) + \lambda^2 ,$$

$$L_{34}^* = (K\,S_{14} - S_5\frac{1}{R})\frac{d}{dX} = -L_{43}^* ,$$

$$L_{35}^* = K\,S_{13}\frac{n}{R} - S_6\frac{n}{R^2} = L_{53}^* ,$$

$$L_{44}^* = S_7\frac{d^2}{dX^2} - (S_{12}\frac{n^2}{R^2} + K\,S_{14}) + \left(\frac{I_3}{I_1\ell^2}\right)\lambda^2 ,$$

$$L_{45}^* = (S_{12} + S_8)\frac{n}{R}\frac{d}{dX} = -L_{54}^* ,$$

$$L_{55}^* = S_{12}\frac{d^2}{dX^2} - (S_9\frac{n^2}{R^2} + K\,S_{13}) + \left(\frac{I_3}{I_1\ell^2}\right)\lambda^2 \tag{14}$$

The quantities $S_i\,(i = 1,2,....,14)$ are defined by

$$S_2 = \frac{A_{12}}{A_{11}},\ S_3 = \frac{A_{22}}{A_{11}},\ S_4 = \frac{B_{11}}{\ell A_{11}},\ S_5 = \frac{B_{12}}{\ell A_{11}},$$
$$S_6 = \frac{B_{22}}{\ell A_{11}},\ S_7 = \frac{D_{11}}{\ell^2 A_{11}},\ S_8 = \frac{D_{12}}{\ell^2 A_{11}},$$
$$S_9 = \frac{D_{22}}{\ell^2 A_{11}},\ S_{10} = \frac{A_{66}}{A_{11}},\ S_{11} = \frac{B_{66}}{\ell A_{11}},\ S_{12} = \frac{D_{66}}{\ell^2 A_{11}},$$
$$S_{13} = \frac{A_{44}}{A_{11}},\ S_{14} = \frac{A_{55}}{A_{11}}. \tag{15}$$

3. Spline collocation procedure

The differential equations given in Eq. (13) are to be solved. The numerical method of solution is resorted to, since no closed-form solution exists for these problems in general. The spline technique is used since it is relatively simple and elegant and uses a series of lower order approximations rather than a global higher order approximation, affording fast convergence and high accuracy.

The displacement functions $U(X)$, $V(X)$, $W(X)$ and rotational functions $\ell\Psi_X(X)$, $\ell\Psi_\theta(X)$ are approximated by the cubic spline functions $U^*(X)$, $V^*(X)$, $W^*(X)$, $\Psi_X^*(X)$ and $\Psi_\theta^*(X)$ as stated below:

$$\begin{aligned}
U^*(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j) \\
V^*(X) &= \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 H(X - X_j) \\
W^*(X) &= \sum_{i=0}^{2} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^3 H(X - X_j) \\
\Psi_X^*(X) &= \sum_{i=0}^{2} g_i X^i + \sum_{j=0}^{N-1} p_j (X - X_j)^3 H(X - X_j) \\
\Psi_\theta^*(X) &= \sum_{i=0}^{2} l_i X^i + \sum_{j=0}^{N-1} q_j (X - X_j)^3 H(X - X_j)
\end{aligned} \tag{16}$$

Here N is the number of intervals into which the range [0, 1] of X is divided and $H(X - X_j)$ is the Heaviside step function. The points of division $X = X_s = s/N$, $(s = 0,1,....,N)$ are chosen as the knots of the splines, as well as the collocation points. Imposing the condition that the differential equations given by Eq. (13) are satisfied by these splines at these points, a set of $5N+5$ homogeneous equations in $5N+15$ unknown spline coefficients $a_i, c_i, e_i, g_i, l_i, b_j, d_j, f_j, p_j, q_j$ $(i = 0,\ 1,\ 2; j = 0,\ 1,\ 2,.....,\ N-1)$ are obtained.

The following boundary conditions are used to analyze the problem.

Clamped-Clamped (C-C) (both the ends are clamped)

Simply supported-Simply supported (S-S) (both the ends are simply supported)

Clamped-Simply supported (C-S) (one end is clamped and the other end is Simply supported)

Each pair of the boundary conditions imposed gives 10 equations on spline coefficients. Combining them with those obtained earlier, we get $5N+15$ homogeneous equations in the same number of unknowns. Thus, we have a generalized eigenvalue problem in the form

$$[M]\{q\} = \lambda^2 [P]\{q\} \tag{17}$$

where $[M]$ and $[P]$ are square matrices, $\{q\}$ is the column matrix of the spline coefficients and λ is the eigenfrequency parameter .

4. Results and discussion

4.1 Convergence and comparative studies

A convergence study of the frequency parameter values is carried out for several cases of parametric values, material combinations under different boundary conditions, for values of $N = 4$ onwards. It can be seen that the choice of $N = 16$ is more adequate since for the next value of N, the percentage changes in values of λ_1 are very low, the maximum being 0.17%. Table 1 compares the fundamental frequency parameter values obtained for a cross-ply laminated cylindrical shell for $R = 0.5L$, using S-S boundary conditions with the FEM results of [13]. The results of Toorani and Lakis [26] on a simply supported anisotropic cross-ply cylinder $(0^0/90^0/90^0/0^0)$ including shear deformation and rotary inertia of different radii ratio (R/t), the circumferential node number (n) and the corresponding results presently obtained are shown in Table 2. The differences are not significant. Table 3 shows the comparison results of frequency values of an orthotropic homogeneous cylindrical shell for various circumferential node numbers (n) and axial modes (m) with

Table 1. Comparative study of fundamental frequency $\bar{\omega} = \omega(R^2/h)\sqrt{(\rho/E_2)}$
R : radius of the cylinder, h : total thickness, ρ : density

Thickness h	Lamination	(m,n)	$\bar{\omega}$		Difference
			$\bar{\omega}^p$	$\bar{\omega}^s$	
0.04 R	0º/90º	(1,3)	8.823	9.526	-7.967
	90º/0º/90º/0º	(1,3)	12.301	11.331	7.885
	90º/0º/0º/90º	(1,3)	13.127	12.652	3.618
0.1 R	0º/90º	(1,2)	4.830	5.142	-6.459
	90º/0º/90º/0º	(1,2)	6.543	5.924	9.460
	90º/0º/0º/90º	(1,2)	6.335	6.019	4.988

[p]Present analysis; [s]Sun et al. [13]

Table 2. Comparison of the fundamental frequency $\Omega = \omega R^2(\rho/E_2)^{1/2}/t$ of simply supported anisotropic cross-ply cylinder $(0^0/90^0/90^0/0^0)$ including shear deformation and rotary inertia.

n	R/t=100		R/t=50	
	Present	Ref.[26] (approximately)	Present	Ref.[26] (approximately)
1	30.414	30	15.224	14
2	17.861	18	9.073	8
3	12.416	13	7.343	7
4	11.697	11	6.851	6.5
5	13.987	11.5	9.882	8

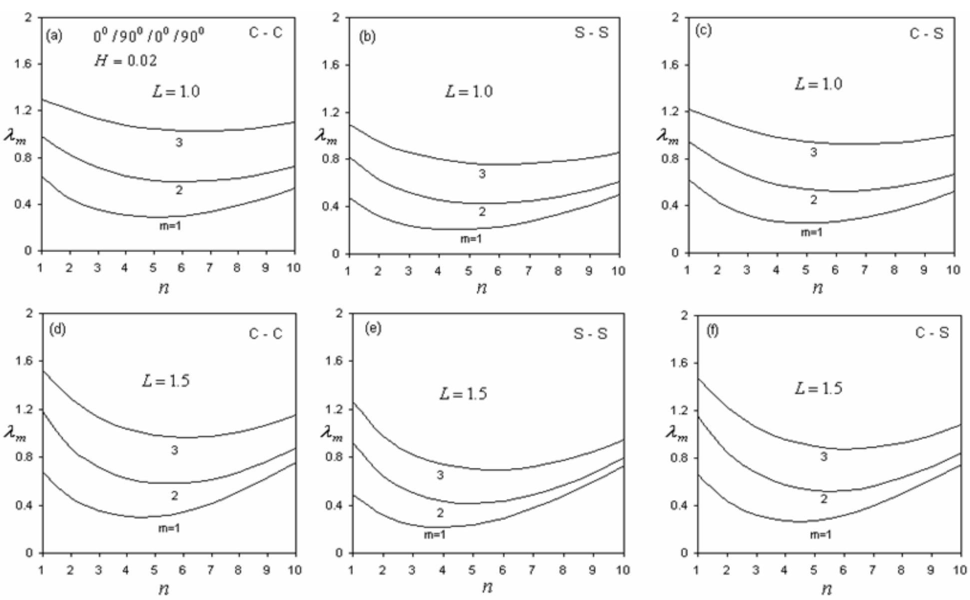


Fig. 1. Variation of frequency parameter with circumferential node number of four-layered antisymmetric cross-ply shells under C-C, S-S and C-S boundary conditions. Layer materials: KGE- AGE-AGE-KGE.

Table 3. Comparison of the frequency parameter $\Omega = \omega\ell\sqrt{\rho/E_2}$ of simply supported orthotropic cylindrical shells ($r/\ell = 1, h/\ell = 0.1$).

Circumferential Node number (n)	Axial mode (m)			
		1	2	3
1	Present	1.50734	4.29029	6.37845
	Ref[28][a]	1.32416	3.42856	5.65598
	Ref[28][b]	1.33745	3.43487	5.68570
	Ref[28][c]	1.29858	3.19538	5.13383
	Ref[28][d]	1.57257	5.26048	7.34788
2	Present	1.89851	4.45693	6.48173
	Ref[28][a]	1.65929	3.59532	5.77045
	Ref[28][b]	1.69031	3.60790	5.80137
	Ref[28][c]	1.63768	3.36074	5.24850
	Ref[28][d]	1.99961	5.85093	9.08880
3	Present	2.12132	4.51845	7.31001
	Ref[28][a]	2.50573	4.08989	6.10449
	Ref[28][b]	2.53918	4.10523	6.13436
	Ref[28][c]	2.44409	3.83876	5.57354
	Ref[28][d]	3.09808	6.45360	11.36729
4	Present	2.82843	4.60743	7.44014
	Ref[28][a]	3.64582	4.89994	6.68608
	Ref[28][b]	3.67091	4.91181	6.71169
	Ref[28][c]	3.49515	4.59461	6.12261.
	Ref[28][d]	4.77389	7.53134	13.18665
5	Present	3.53848	5.01318	7.45914
	Ref[28][a]	4.92951	5.93701	7.48804
	Ref[28][b]	4.94282	5.94121	7.50705
	Ref[28][c]	4.65406	5.53630	6.85888
	Ref[28][d]	6.89750	9.13471	14.23784
6	Present	4.21136	5.84827	7.98317
	Ref[28][a]	6.28273	7.11414	8.45781
	Ref[28][b]	6.28481	7.11024	8.47035
	Ref[28][c]	5.85751	6.58546	7.73038
	Ref[28][d]	9.36898	11.20323	15.62888
7	Present	6.73013	7.50782	8.35424
	Ref[28][a]	7.66909	8.37330	9.54541
	Ref[28][b]	7.66367	8.36430	9.55452
	Ref[28][c]	7.07696	7.69351	8.69288
	Ref[28][d]	12.10934	13.64076	17.42526
8	Present	7.48689	8.00767	9.43707
	Ref[28][a]	9.07057	9.68036	10.03301
	Ref[28][b]	9.06401	9.67213	10.72479
	Ref[28][c]	8.29955	8.83331	9.71513
	Ref[28][d]	15.05423	16.35665	19.58434

[a]3-D analysis; [b]Parabolic shear deformation theory; [c]Constant shear deformation theory; [d]Thin shell theory

Table 4. Comparison of the fundamental frequency ω^* (Ye and Soldatos[29]) of simply supported cylindrical shells with symmetric cross-ply laminates.
$r/\ell = 1,\ E_x/E_\theta = 40, G_{x\theta} = 0.6E_\theta, G_{xz} = G_{\theta z} = 0.5E_\theta, \nu_{x\theta} = 0.25$

h/r	n	0º/90º/90º/0º		90º/0º/0º/90º	
		Present	Ye and Soldatos[29]	Present	Ye and Soldatos[29]
0.1	2	0.082831	0.079277	0.062899	0.070738
	4	0.076309	0.066335	0.051555	0.052748
	6	0.074118	0.064600	0.054261	0.059130
0.2	2	0.177462	0.175188	0.152241	0.150651
	4	0.170696	0.162844	0.121318	0.130168
	6	0.176990	0.170868	0.163189	0.158886
0.3	2	0.280720	0.272860	0.244864	0.236385
	4	0.275497	0.263048	0.231831	0.218779
	6	0.294626	0.283798	0.284595	0.268258

$r/\ell = 1$ and $h/\ell = 0.1$. The present values are com pared with the results obtained by 3-D analysis, parabolic shear deformation, constant shear deformation and thin shell theory [28]. The comparison of the fundamental frequency $\omega^* = \omega h(\rho/\pi^2 C_{66})^{1/2}$ for various thickness ratios (h/ℓ) and circumferential node numbers (n) with those results using 3-D analysis obtained by Ye and Soldatos [29], for simply supported cylindrical shells with symmetric cross-ply laminates is presented in Table 4. The properties for the comparison are $r/\ell = 0.1$, $E_x/E_\theta = 40$, $G_{x\theta} = 0.6E_\theta$, $G_{xz} = G_{\theta z} = 0.5E_\theta$ and $\nu_{x\theta} = 0.25$. The agreement correlated with the previously published results given in the tables, which indicates that the present analysis is accurate. The presence of transverse shear deformation effects is more significant, which reduces the frequency parameters predicted by the classical shell theory.

4.2 Discussion

The effects of material anisotropy, transverse shear deformation, thickness-to-radius ratio, length-to-radius ratio, coupling between bending and stretching and the number of layers on the first three frequencies are investigated. Both symmetric and antisymmetric cross-ply laminations are considered. The shear correction factor K is taken as 5/6 for all the cases (see [25]). Two types of materials, namely Kevler-49 ep-

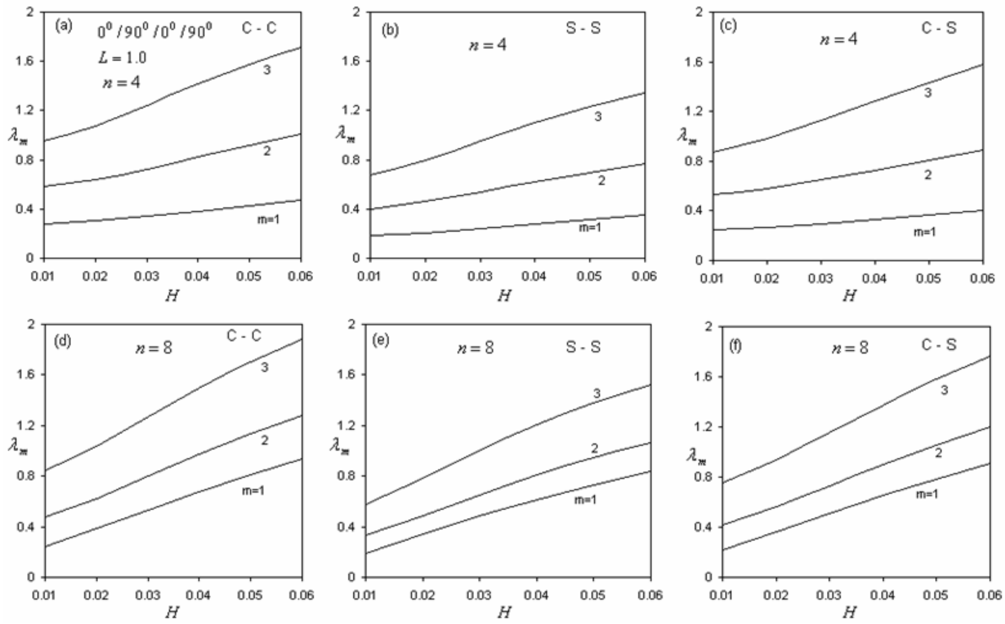


Fig. 2. Effect of thickness parameter and boundary conditions on the frequency parameters of four-layered antisymmetric cross-ply shells. Layer materials: KGE- AGE-AGE-KGE.

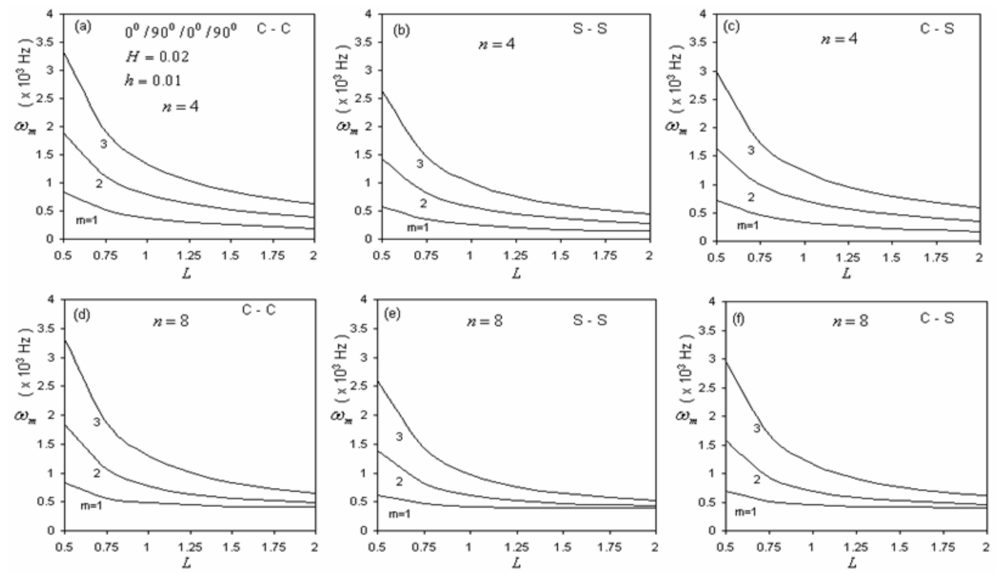


Fig. 3. Effect of length of the four-layered antisymmetric cross-ply shell and boundary conditions on frequencies. Layer materials: KGE-AGE-AGE-KGE.

oxy (KGE) and AS4/3501-6 Graphite/epoxy (AGE) (see [27]) are used to analyze the problem.

Figs. 1-3 depict the manner of frequencies of four-layered antisymmetric cross-ply shells made of KGE and AGE materials with three different boundary conditions, namely clamped-clamped (C-C), simply supported-simply supported (S-S) and clamped-simply supported (C-S) with arranging the layers in the order of KGE-AGE-AGE-KGE materials. All the layers considered are of equal thickness. Fig. 1((a)-(f)) shows the variation of frequency parameter $\lambda_m (m = 1, 2, 3)$ with reference to the circumferential node number n fixing $H = 0.02$. Figs. (a), (b) and (c) correspond to the boundary conditions C-C, S-S

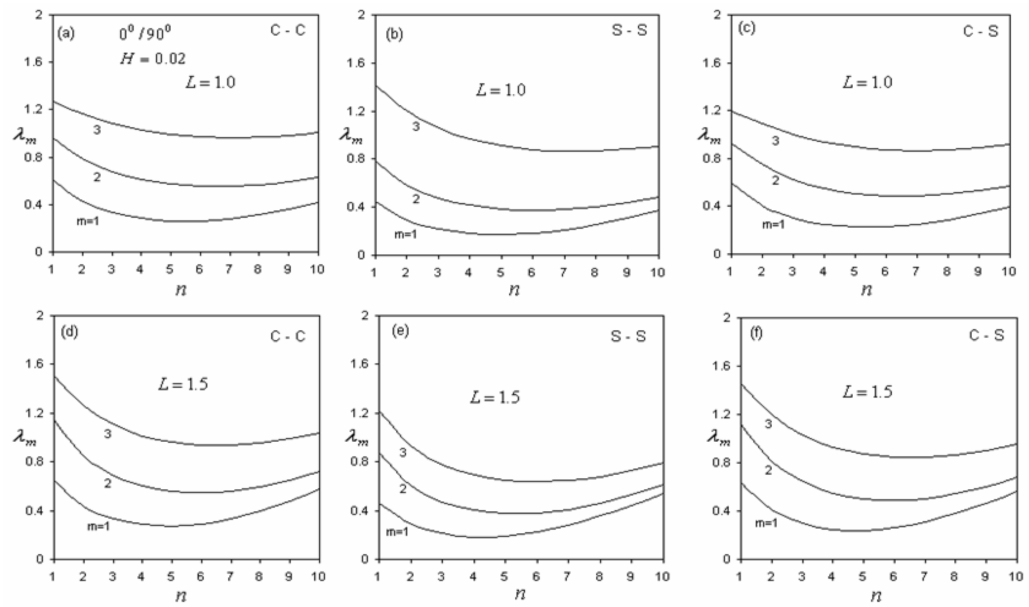


Fig. 4. Variation of frequency parameter with circumferential node number of two-layered antisymmetric cross-ply shells. Layer material: KGE-KGE.

and C-S, respectively, by fixing the length parameter $L = 1.0$, whereas Figs. (d), (e) and (f) correspond to the boundary conditions C-C, S-S and C-S, respectively, with the length parameter $L = 1.5$. It is seen from the figures that all the frequency parameter values $(\lambda_m, m = 1,2,3)$ decrease up to $n = 4$ and then increase for higher values of n . The value of the fundamental frequency $(m = 1)$ for $L = 1.0$ is higher than the fundamental frequency for $L = 1.5$ in all the cases. One may conclude from the results that the fundamental frequency is higher for shorter shells. Analyzing with reference to the types of boundary conditions, it is seen, as expected, the frequencies are higher for C-C conditions, lowest for S-S conditions and in-between the two for C-S conditions.

Fig. 2(a)-(f) shows the variation of frequency parameter λ with respect to the thickness parameter H for four-layered antisymmetric cross-ply shells. Figs. (a), (b), (c)) correspond to the value of $n = 4$ and Figs. (d), (e), (f) correspond to $n = 8$ with fixed value of $L = 1.0$. The value of λ increases as H increases and it seems to be almost linear. The rate of increase rises with the value of the mode number. The frequencies are higher for C-C conditions, lowest for S-S conditions and in-between the two for C-S conditions, as described earlier.

Fig. 3(a)-(f) describes how the length parameter L affects ω (in 10^4 Hz) for four-layered $(0^0/90^0/0^0/90^0)$ antisymmetric cross-ply shells under C-C, S-S and C-S boundary conditions, with $H = 0.02$ and $n = 4$ or 8. For the influence of the length of the cylinder on its vibrational behavior, the actual frequency ω is considered since the frequency parameter λ is explicitly a function of the length ℓ of the cylinder and then specific length is to be prescribed; therefore, the thickness of the shell h is taken to be 1cm. The value of ω decreases in general as L increases. The decrease is fast for very short shells (for $0.5 < L < 0.75$), the rate of decrease increasing with higher modes. The fundamental frequencies are almost constant for $L > 0.85$.

Figs. 4-6 relate to studies on two-layered $(0^0/90^0)$ antisymmetric cross-ply shells made of the single material KGE with different boundary conditions. The influence of the circumferential node number n on the frequency parameter λ is illustrated in Fig. 4((a)-(f)) for the cases $L = 1.0$ and 1.5. The other parameters are fixed. Qualitatively, the vibrational behavior of these shells is similar to that of the corresponding case of Fig. 1((a)-(f)). In Fig. 5, the influence of the thickness parameter H on the frequency parameter λ is analyzed by fixing $L = 1.0$ and $n = 4$ under C-C, S-S and C-S conditions. Fig. 6 shows the influence of the length parameter L on the frequencies ω_m . The pattern of the influence is similar in nature to the corresponding cases of four-

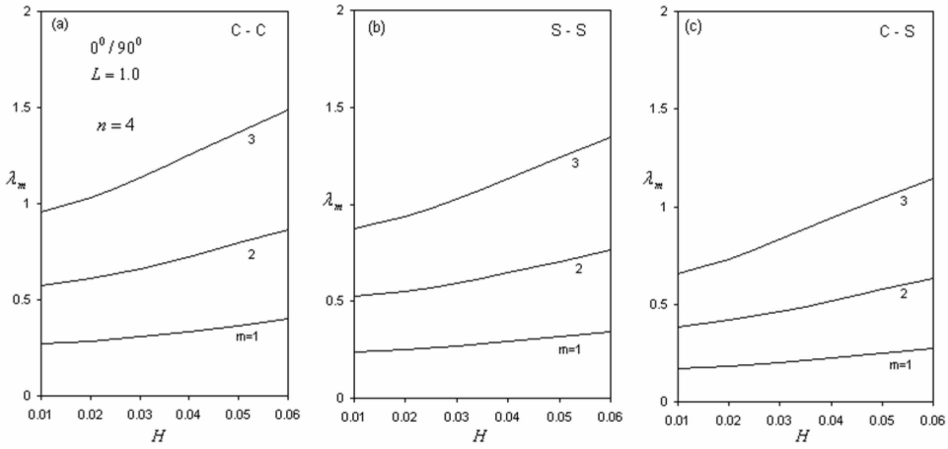


Fig. 5. Effect of thickness parameter and boundary conditions on frequency parameter of two-layered antisymmetric cross-ply shells. Layer material: KGE- KGE.

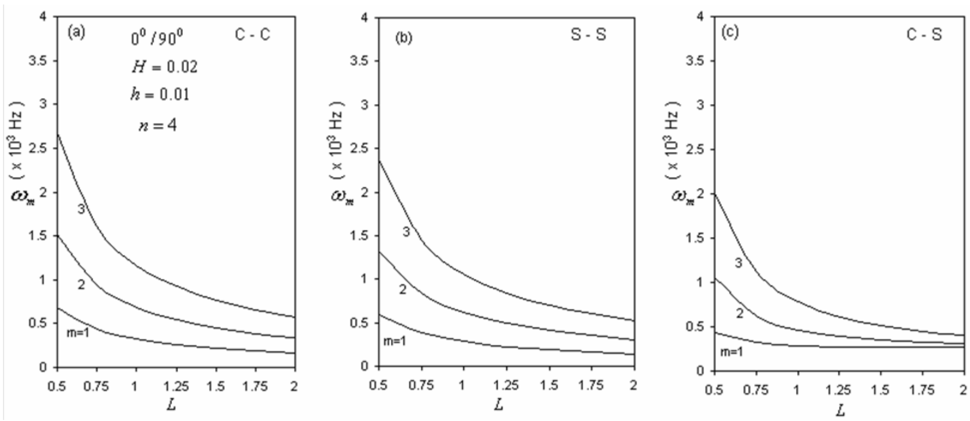


Fig. 6. Effect of length of the two-layered antisymmetric cross-ply shell and boundary conditions on frequencies. Layer materials: KGE-KGE.

layered antisymmetric cross-ply shells and differs only in magnitude.

Figs. 7-9 relate the study on three-layered symmetric cross-ply laminated shells in the form of $0^0/90^0/0^0$ angles. The layered materials are arranged in the order of KGE-AGE-KGE. Fig. 7((a)-(f)) shows the variation of frequency parameter with respect to the circumferential node number n for two fixed values of L with $H = 0.02$ under C-C, S-S and C-S boundary conditions. The curvature of the λ_m is higher for C-C and S-S conditions compared to the C-S conditions. The frequency values are lower in C-S conditions than the other two conditions. But in the case of antisymmetric cross-ply laminates, the values are vice versa. Fig. 8 describes the variation of frequency parameter with the thickness parameter H. It is seen that when increasing H, for any fixed L and n, the value of λ_m increases. Fig. 9 describes the manner of variation of the values of frequencies ω_m with respect to the length parameter L of three-layered symmetric cross-ply shells made of KGE and AGE materials in the order of KGE-AGE-KGE under three different boundary conditions, C-C, S-S and C-S. Clearly, the vibrational pattern is similar in all the cases of boundary conditions and suffers only parallel shifts. The same phenomenon prevails for two- and four-layered antisymmetric cross-ply shells. From the above discussions, we come to know that designers may be clear in choosing the respective materials for suitable designs in the field of ship building and aviation etc., which can be symmetric or antisymmetric laminated shells along with the necessary boundary conditions and number of layers.

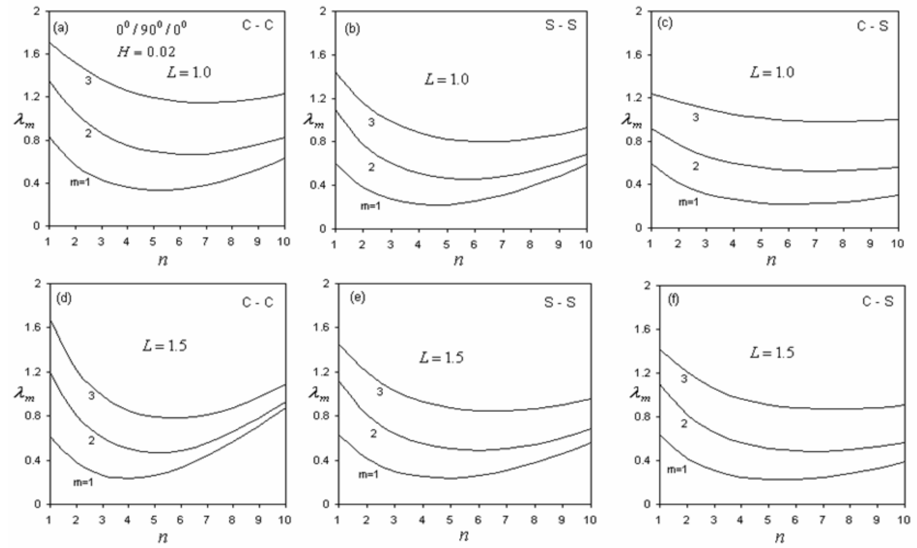


Fig. 7. Variation of frequency parameter with circumferential node number of three-layered symmetric cross-ply shells. Layer material: KGE-AGE-KGE.

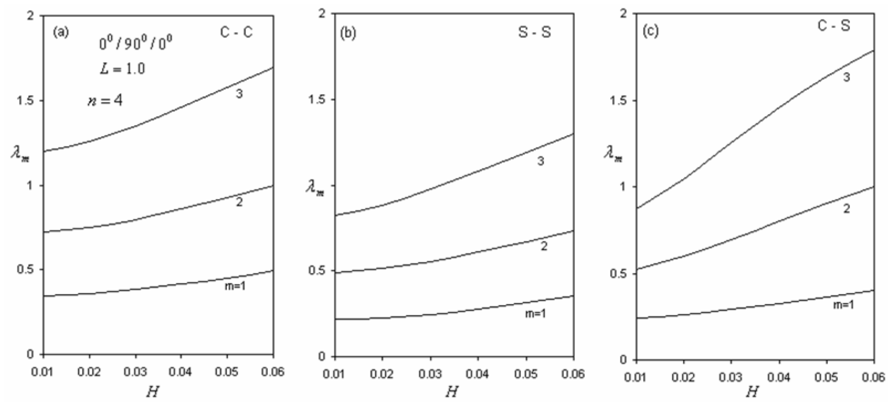


Fig. 8. Effect of thickness parameter and boundary conditions on the frequency parameters of three-layered symmetric cross-ply shells. Layer material: KGE- AGE-KGE.

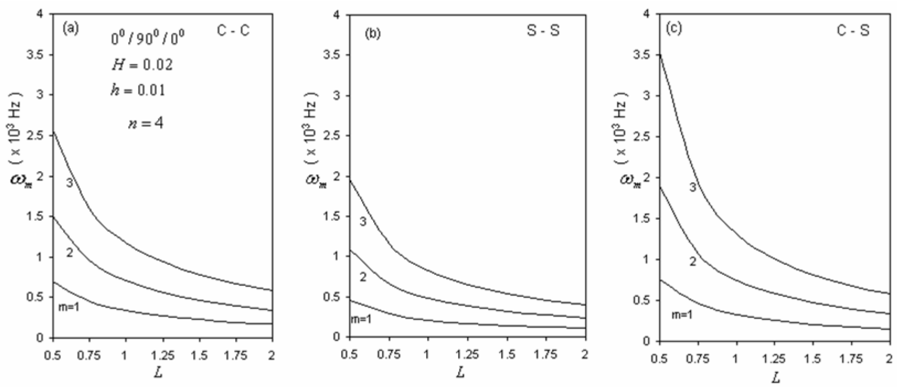


Fig. 9. Effect of length of the three-layered symmetric cross-ply shell and boundary conditions on frequencies. Layer materials: KGE-AGE-KGE.

5. Conclusion

The manner of variation of eigenfrequencies with respect to circumferential node number, thickness parameter, length parameter, cross-ply angles and the types of boundary conditions are investigated. Both symmetric and antisymmetric shells including shear deformation theory with three different numbers of layers having two types of materials on the vibrational behavior are studied and presented. In general, the frequency values decrease as the length parameter increases. The variation is fast for short shells and then it is slow when the length is increases. In the case of the thickness of the shell, the variation of the frequency parameter increases as the thickness increases. The increment is higher for higher modes and is almost linear.

The effect of transverse shear deformation is more significant, which yields lower values on the frequency parameters when we compare the values predicted by classical shell theory. The elegance and usefulness of spline function approximations by applying the collocation procedure for boundary value problems is clearly brought out in this study.

Acknowledgment

The authors gratefully acknowledge the financial support from the Brain Korea 21 (2007) project and INHA UNIVERSITY Research Grant, South Korea.

References

[1] C. W. Bert, J. L. Baker and D. M. Egle, Free vibration of multilayered anisotropic cylindrical shells, *J. Comp. Mat.,* 3 (1969) 480-499.

[2] J. B. Greenberg and Y. Stavsky, Buckling and vibration of orthotropic composite cylindrical shells, *Acta Mech.,* 36 (1980) 15-29.

[3] M. E. Vanderpool and C. W. Bert, Vibration of a materially monoclinic thick-wall circular cylindrical shell, *AIAA J.,* 19 (1981) 634-641.

[4] N. Alam and N. T. Asnani, Vibration and damping of multilayered cylindrical shell, Part II: numerical results, *AIAA J.* 22 (1984) 975-981.

[5] C.B. Sharma and M. Darvizeh, Free vibration of specially orthotropic, multilayered, thin cylindrical shells with various end conditions, *Comp. Struc.,* 7 (1987) 123-138.

[6] N. Alam and N. T. Asnani, Vibration and damping analysis of a fiber reinforced composite material cylindrical shell, *J. Comp. Mat.,* 21 (1987) 348-361.

[7] R. R. Kumar and Y. V. K. S. Rao, Free vibration of multilayered thick composite shells, *Comp. Struc.,* 28 (1992) 717-722.

[8] R. A. Raouf, Tailoring the dynamic characteristics of composite panels using fiber orientation, *Comp. Struc.* 29 (1994) 259-267.

[9] S. Abrate, Optimal design of laminated plates and shells, *Comp. Struc.,* 29 (1994) 269-286.

[10] J. N. Reddy, Bending of laminated anisotropic shells by a shear deformable finite element, *Fiber Sci. Tech.* 17 (1982) 9-24.

[11] J. N. Reddy, Exact solution of moderately thick laminated shells, *ASCE J. Engg . Mech. Div.,* 110 (1984) 794-809.

[12] K. Y. Lam and C. T. Loy, Analysis of rotating laminated cylindrical shells by different thin shell theories, *J. Sound and Vib.* 186 (1995) 23-35.

[13] G. Sun and P.N. Bennett, F.W. Williams, An investigation of fundamental frequencies of laminated circular cylinders given by shear deformable finite elements, *J. Sound and Vib.,* 205 (1997) 265-273.

[14] H.-T. Hu and J.-Y. Tsai, Maximization of the fundamental frequencies of laminated cylindrical shells with respect to fiber orientation, *J. Sound and Vib.,* 225 (1999) 723-740.

[15] X. M. Zhang, Vibration analysis of cross-ply laminated composite cylindrical shells using the wave propagation approach, *Appl. Acous.* 62 (2001) 1221-1228.

[16] Werner Hufenbach, Carsten Holst and Lothar Kroll. Vibration and damping behaviour of multilayered composite cylindrical shells, *Comp. Struc.* 58 (2002) 165-174.

[17] K. K. Viswanathan and P. V. Navaneethakrishnan, Free vibration study of layered cylindrical shells by collocation with splines, *J. Sound and Vib.,* 260 (2003) 807-827.

[18] K. K. Viswanathan and P. V. Navaneethakrishnan, Free vibration study of layered truncated conical shell frusta of differently varying thickness by the method of collocation with cubic and quintic splines, *Int. J. Solids and Struc.,* 42 (2005) 1129-1150.

[19] K. K. Viswanathan and S. K. Lee, Free vibration of laminated cross-ply plates including shear deformation by spline method, *Int. J. Mech. Sci.,* 49 (2007) 352-363.

[20] W. G. Bickley, Piecewise cubic interpolation and two point boundary problems, *Computer J.,* 11

(1968) 206-208.
[21] M. H. Toorani and A. A. Lakis, General equations of anisotropic plates and shells including transverse shear deformations, rotary inertia and initial curvature effects, *J. Sound and Vib.,* 237 (2000) 561-615.
[22] J. M. Whitney and Ct. Sun, A higher order theory for extensional motion of laminated composites, *J. Sound and Vib.,* 30 (1973) 85-97.
[23] J. M. Whitney, Shear correction factors for orthotropic laminates under static loads, *J. Appl. Mech.,* 40 (1973) 302-304.
[24] C. W. Bert, Simplified analysis of static shear factors for beams of nonhomogeneous cross section . *J. Comp. Mat.,* 7 (1973) 525-529.
[25] Perngjin, F. Pai, A new look at the shear correction factors and warping functions of anisotropic laminates. *Int. J. Solids and Struc.,* 32 (1995) 2295-22313.
[26] M. H. Toorani and A. A. Lakis, Free vibrations of non-uniform composite cylindrical shells. *Nucl. Engg. Design.,* 236 (2006) 1748-1758.
[27] A. Bhimaraddi, Large amplitude vibrations of imperfect antisymmetric angle-ply laminated plates, *J. Sound and Vib.,* 162 (1993) 457-470.
[28] A. Bhimaraddi, Free vibration analysis of doubly curved shallow shells on rectangular planform using three-dimensional elasticity theory, *Int. J. Solids and Struc.,* 27 (1991) 897-913.
[29] J. Ye and K. P. Soldatos, Three-dimensional vibration of laminated cylinders and cylindrical panels with symmetric or antisymmetric cross-ply lay-up, Comp. Engg., 4 (1994) 429-444.

Appendix. A

The quantities $Q_{ij}^{(k)}(i,j=1,2,4,5,6)$ appearing in Eqs.(4) are defined by

$$Q_{11}^{(k)} = C_{11}^{(k)}\cos^4\alpha + C_{22}^{(k)}\sin^4\alpha + 2(C_{12}^{(k)} + 2C_{66}^{(k)})\sin^2\alpha\cos^2\alpha \tag{A.1}$$

$$Q_{22}^{(k)} = C_{11}^{(k)}\sin^4\alpha + C_{22}^{(k)}\cos^4\alpha + 2(C_{12}^{(k)} + 2C_{66}^{(k)})\sin^2\alpha\cos^2\alpha \tag{A.2}$$

$$Q_{12}^{(k)} = (C_{11}^{(k)} + C_{22}^{(k)} - 4C_{66}^{(k)})\sin^2\alpha\cos^2\alpha + C_{12}^{(k)}(\cos^4\alpha + \sin^4\alpha) \tag{A.3}$$

$$Q_{16}^{(k)} = (C_{11}^{(k)} - C_{12}^{(k)} - 2C_{66}^{(k)})\cos^3\alpha\sin\alpha - (C_{22}^{(k)} - C_{12}^{(k)} - 2C_{66}^{(k)})\sin^3\alpha\cos\alpha \tag{A.4}$$

$$Q_{26}^{(k)} = (C_{11}^{(k)} - C_{12}^{(k)} - 2C_{66}^{(k)})\cos\alpha\sin^3\alpha - (C_{22}^{(k)} - C_{12}^{(k)} - 2C_{66}^{(k)})\sin\alpha\cos^3\alpha \tag{A.5}$$

$$Q_{66}^{(k)} = (C_{11}^{(k)} + C_{22}^{(k)} - 2C_{12}^{(k)} - 2C_{66}^{(k)})\cos^2\alpha\sin^2\alpha + C_{66}^{(k)}(\sin^4\alpha + \cos^4\alpha) \tag{A.6}$$

$$Q_{44}^{(k)} = C_{55}^{(k)}\sin^2\alpha + C_{44}^{(k)}\cos^2\alpha \tag{A.7}$$

$$Q_{55}^{(k)} = C_{55}^{(k)}\cos^2\alpha + C_{44}^{(k)}\sin^2\alpha \tag{A.8}$$

$$Q_{45}^{(k)} = (C_{55}^{(k)} - C_{44}^{(k)})\cos\alpha\sin\alpha \tag{A.9}$$

where

$$\begin{aligned} &C_{11}^{(k)} = \frac{E_x^{(k)}}{1-\upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)}}, \\ &C_{12}^{(k)} = \frac{\upsilon_{x\theta}^{(k)}E_\theta^{(k)}}{1-\upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)}} = \frac{\upsilon_{\theta x}^{(k)}E_x^{(k)}}{1-\upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)}} \\ &C_{22}^{(k)} = \frac{E_\theta^{(k)}}{1-\upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)}},\ C_{66}^{(k)} = G_{x\theta}^{(k)}, \\ &C_{44}^{(k)} = G_{\theta z}^{(k)},\ C_{55}^{(k)} = G_{xz}^{(k)} \end{aligned} \tag{A.10}$$

Appendix. B

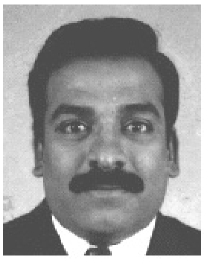
The equilibrium equations of circular cylindrical shells including shear deformation and rotary inertia are given by

$$\begin{aligned} &\frac{\partial N_{xx}}{\partial x} + \frac{1}{r}\frac{\partial N_{\theta x}}{\partial\theta} = I_1\frac{\partial^2 u}{\partial t^2} \\ &\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial\theta} + \frac{Q_{\theta\theta}}{r} = I_1\frac{\partial^2 v}{\partial t^2} \\ &\frac{\partial Q_{xx}}{\partial x} + \frac{1}{r}\frac{\partial Q_{\theta\theta}}{\partial\theta} - \frac{N_{\theta\theta}}{r} = I_1\frac{\partial^2 w}{\partial t^2} \\ &\frac{\partial M_{xx}}{\partial x} + \frac{1}{r}\frac{\partial M_{\theta x}}{\partial\theta} - Q_{xx} = I_3\frac{\partial^2\psi_x}{\partial t^2} \\ &\frac{\partial M_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta} - Q_{\theta\theta} = I_1\frac{\partial^2\psi_\theta}{\partial t^2} \end{aligned} \tag{B.1}$$

Here x and θ are the curvilinear co-ordinates of the cylindrical shells, and I_1 and I_3 are the normal and rotary inertia coefficients given by

$$(I_1, I_3) = \int \rho^{(k)}(1, z^2)\,dz \tag{B.2}$$

in which $\rho^{(k)}$ is the material density of the k -th layer of the shell.



Dr. K. K. Viswanathan was born in 1962 in Vellore District, India. He received his B.Sc. in Mathematics from University of Madras and M.Sc. in 1992 and Ph.D. in 1999 from Anna University, India. Later he was a Project Associate in Indian Institute of Science, Bangalore. He served as lecturer in Crescent Engg. College and as Asst. Professor in SRM University, India. He did his post doctoral research in Korea for three years. At present he serves as Professor in the Dept. of Naval Architecture, Inha University, Incheon, Korea. His research areas of interest includes vibration of plates, shells and the application of numerical techniques in Engineering problems.

Dr. Kyung Su Kim was born in Korea in 1954. He is a professor in Naval Architecture and Ocean Engineering at Inha University, Korea. He obtained his B.Sc. degree in Naval Architecture and Ocean Engineering from Seoul National University, Korea, in 1981. He worked for KR (Korean Register of Shipping) from 1981 to 1983. He obtained M.Sc. degree in Naval Architecture and Ocean Engineering in 1986, and Ph.D. degree in Structural Mechanics in 1991 from Rheinisch - Westfaelische Technische Hoch-schule Aachen, Germany. From 1986 to 1992, he was a Post Doctoral Research Engineer of Engineering Research Institute at Rheinisch - Westfaelische Technische Hochschule Aachen. He was appointed as a professor of Inha University, Korea, in 1994. His major area of study is Impact and Fatigue Fracture.

Dr. Jang Hyun Lee was born in Korea in 1969. Currently, he is an Assistant professor of the Department of Naval Architecture and Ocean Engineering at Inha University, Korea. He obtained his B.Sc., M.Sc. and Ph.D. degrees in Naval Architecture and Ocean Engineering from Seoul National University, Korea, in 1993, 1995 and 1999 respectively. From 1999 to 2002, he was a Post Doctoral Research Engineer of Engineering Research Institute at Seoul National University. He joined the Inha University in 2005 after holding the Chief Technology Officer at Xinnos for four years. His research interests include press forming of thick plates and shells, computational welding mechanics and Product Lifecycle Management.