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# Free vibration of nanorings/arches based on nonlocal elasticity — Source link

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# Free vibration of nanorings/arches based on nonlocal elasticity

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This paper deals with the free vibration problem of nanorings/arches. The problem is formulated on the basis of Eringen's nonlocal theory of elasticity in order to allow for the small length scale effect. Exact vibration frequencies are derived for the nanorings/arches and the effects of small length scale, defects, and elastic boundary conditions are investigated. The small length scale effect lowers the vibration frequencies. The defects and the use of elastic boundary conditions (instead of fixed restraints) also significantly reduce the frequencies and alter the vibration mode shapes of circular rings/arches. The results presented should be useful to engineers who are designing nanorings/arches for microelectromechanical and nanoelectromechanical devices. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2951642]

## I. INTRODUCTION

One-dimensional nanostructures, such as nanotube,[1] nanowires,[2] nanobelts,[3] nanosprings,[4] nanorings,[5] nanobows,[6] and nanohelices[7] have attracted a great deal of researchers' attention and spurred extensive studies. Studies have shown that nanostructures, such as zinc oxide nanowires among others, exhibit both semiconducting and piezoelectric properties that can form the basis for electromechanically coupled sensors and transducers.[8] In addition, zinc oxide nanowires are relatively biosafe and biocompatible, and thus they can be used for biomedical applications with little toxicity. Hence, one of the most important applications of nanostructures is likely to take advantage of their exceptionally mechanical, electrical, and chemical properties to be used as sensors,[9] resonators,[10] and transducers[11] for nanoelectronic and biotechnology applications.

Recently, vibrations of nanorings/arches have been the subject of some experimental and molecular dynamics (MD) simulations.[12–17] As experiments at the nanoscale are extremely difficult and atomistic computations remain prohibitively expensive for large size atomic systems, continuum models continue to play an essential role in the study of nanostructures.[18–24] However, there are strong evidences[25–29] that the small length scale effect (i.e., nonlocal effect) has a significant influence on the mechanical behavior of nanostructures. Therefore, classical structural theories need to be modified to account for the small length scale effect if they are to be used.

In this paper, an elastic ring/arch model is presented for free vibration analysis of nanorings/arches. In order to account for small length scale effect, Eringen[30] nonlocal elasticity theory is adopted. His nonlocal theory has been widely used to derive bending solutions, buckling loads, vibration frequencies, and phase velocities of micro- and nanobeams, rods and tubes (for example, see papers by Peddieson *et al.*,[31], Sudak,[32] Wang *et al.*,[33] Zhang *et al.*,[34] Wang and Varadan,[35] Wang and Liew,[36] Lu *et al.*,[37,38] Lim *et al.*,[39] Xu,[40] Wang *et al.*,[41] Wang,[42] Duan *et al.*,[43,44] Wang *et al.*,[45] and Reddy[46]). The effect of defects represented by a hinge and rotational restraint and elastic boundary conditions on vibrations of nanorings/arches are also considered. The exact nonlocal solutions for free vibration of nanorings/arches derived herein should be useful to engineer scientists who are designing microelectromechanical system (MEMS) and nanoelectromechanical system (NEMS) devices that make use of nanorings/arches.

## II. GOVERNING EQUATION FOR FLEXURAL VIBRATIONS OF NONLOCAL CIRCULAR SEGMENT

In this section, we develop a generic governing equation for the free vibration of a circular segment with allowance for the small length scale effect. The governing equation will be used to solve the following vibration problems: (a) a circular ring with constant cross section, (b) a circular ring containing a defect, and (c) a circular arch embedded in an elastic medium. It is assumed that the cross sectional dimensions of the ring/arch are small in comparison with the radius of the center line of the ring/arch.

Let us consider the in-plane flexural vibration of the ring as shown in Fig. 1. We denote the radius of the circle formed by the center line as $r$. Let $\theta$ be the angle between the radius drawn from the center of the circle to a point on it and a chosen radius. We take the positive displacement $\bar{w}$ to be directed radially outwards. The displacement $\bar{v}$ is directed along the tangent of the circle in the sense in which $\theta$ increases. We shall denote the moment of inertia $I_z$ of the cross section with respect to a principal axis that is parallel to the $z$ axis.

Owing to the displacements $\bar{w}$ and $\bar{v}$, the strain $\varepsilon$ of the center line of the ring at any point is given by

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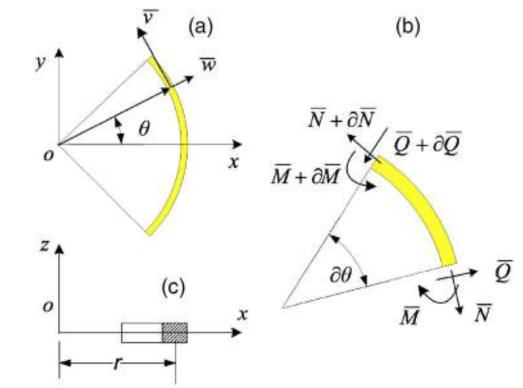


FIG. 1. (Color online) Geometry of circular arch (a) plan view, (b) side view, and (c) free body diagram of an elemental segment.

$$\varepsilon = \frac{\bar{w}}{r} + \frac{\partial \bar{v}}{r\partial\theta}. \tag{1}$$

Considering flexural vibration without extension, we have $\varepsilon=0$, and thus Eq. (1) gives

$$\bar{w} = -\frac{\partial \bar{v}}{\partial\theta}. \tag{2}$$

By taking equilibrium conditions of normal forces, tangential forces as well as the moment about one end of the elemental segment, as shown in Fig. 1, one obtains the following equilibrium equations:[47,48]

$$\frac{\partial \bar{N}}{\partial\theta} - \bar{Q} = mr\frac{\partial^2 \bar{v}}{\partial t^2}, \tag{3}$$

$$\frac{\partial \bar{Q}}{\partial\theta} + \bar{N} = -mr\frac{\partial^2 \bar{w}}{\partial t^2}, \tag{4}$$

$$\frac{\partial \bar{M}}{\partial\theta} + \bar{Q}r = 0, \tag{5}$$

where $\bar{N}$ is the tension, $\bar{Q}$ is the shear force, $\bar{M}$ is the bending moment, and $m$ is the mass of the ring per unit length.

Unlike most classical continuum theories that are based on hyperelastic constitutive relations, which assumes the stress at a point in a body is a function of the strain at that point, Eringen[30] proposed the nonlocal elasticity theory that allows for the stress at a point to be dependent on strains at all points in the body. This theory introduces the small length scale effect through a spatial integral constitutive relation. For beams, rods, and tubes, the complicated spatial integral constitutive relation may be reduced to a simple ordinary differential equation of the form given by[30]

$$\sigma - (e_0 a)\frac{d^2\sigma}{dx^2} = E\varepsilon, \tag{6}$$

where $\sigma$ is the normal stress, $\varepsilon$ is the normal strain, $E$ is the Young's modulus, $x$ is the coordinate along the beam axis direction, $a$ is the characteristic length (e.g., lattice length or bond length between atoms), and $e_0$ is the calibration constant which may be obtained by matching against experiments and MD simulations. Based on Eringen's nonlocal constitutive relation given in Eq. (6), one can derive the nonlocal moment-curvature relationship as follows:

$$\bar{M} - \left(\frac{e_0 a}{r}\right)^2 \frac{\partial^2 \bar{M}}{\partial\theta^2} = -\frac{EI_z}{r^2}\left(\frac{\partial^2 \bar{w}}{\partial\theta^2} + \bar{w}\right). \tag{7}$$

By substituting Eq. (14) into Eq. (7) and in view of Eq. (2), one obtains the governing equation for flexural vibration of a circular ring segment as

$$\frac{\partial^6 \bar{v}}{\partial\theta^6} + 2\frac{\partial^4 \bar{v}}{\partial\theta^4} + \frac{\partial^2 \bar{v}}{\partial\theta^2} = \frac{mr^4}{EI_z}\frac{\partial^2}{\partial t^2}\left(\alpha\frac{\partial^4 \bar{v}}{\partial\theta^4} - (1+\alpha)\frac{\partial^2 \bar{v}}{\partial\theta^2} + \bar{v}\right), \tag{8}$$

in which $\alpha = e_0 a/r$ is the nondimensional small length scale parameter.

Assume the displacement takes on the separable form

$$\bar{v} = \frac{v}{r}\cos(\omega t + \delta), \tag{9}$$

where $\omega$ is the resonant frequency of the ring and $\delta$ is the phase angle. The substitution of Eq. (9) into Eq. (8) leads to a homogeneous linear ordinary differential equation:

$$\frac{d^6 v}{d\theta^6} + (2+\alpha\Omega)\frac{d^4 v}{d\theta^4} + (1-\Omega-\alpha\Omega)\frac{d^2 v}{d\theta^2} + \Omega v = 0, \tag{10}$$

in which $\Omega = mr^4\omega^2/EI_z$ is the nondimensional frequency. Using the following nondimensional terms:

$$\bar{Q} = \frac{r^2 Q}{EI_z}, \quad \bar{N} = \frac{r^2 N}{EI_z}, \quad \text{and } \bar{M} = \frac{rM}{EI_z}, \tag{11}$$

Eqs. (2)–(5) and (7) may be decoupled to furnish explicit expressions for the tension $N$, the shear force $Q$, and the bending moment $M$ as shown below:

$$Q = -\frac{1}{1+\alpha^2}\left[\frac{\partial^4 v}{\partial\theta^4} + (1+\alpha^2\Omega)\frac{\partial^2 v}{\partial\theta^2} - \alpha^2\Omega v\right], \tag{12}$$

$$N = \frac{1}{1+\alpha^2}\left[\frac{\partial^5 v}{\partial\theta^5} + (1+\alpha^2\Omega)\frac{\partial^3 v}{\partial\theta^3} - (1+2\alpha^2)\Omega\frac{\partial v}{\partial\theta}\right], \tag{13}$$

$$M = \frac{1}{1+\alpha^2}\left[\alpha^2\frac{\partial^5 v}{\partial\theta^5} + (1+2\alpha^2+\alpha^4\Omega)\frac{\partial^3 v}{\partial\theta^3} + (1+\alpha^2-\alpha^4\Omega)\frac{\partial v}{\partial\theta}\right]. \tag{14}$$

Note that by setting $\alpha=0$ (i.e., neglecting the small length scale effect), Eqs. (10) and (12)–(14) reduce to the equations as derived by Love.[47] In the subsequent sections, Eq. (10) is applied to determine the resonant circular frequency and the corresponding mode shape of nanorings/arches with the allowance for the small length scale effect.

## III. SMALL LENGTH SCALE EFFECT AND VIBRATIONS OF NONLOCAL CIRCULAR RING

In the investigation of the small length scale effect using the nonlocal rings/arches model, it is crucial to know the magnitude of the parameter $e_0$ (i.e., $\alpha$). So far, no experi-

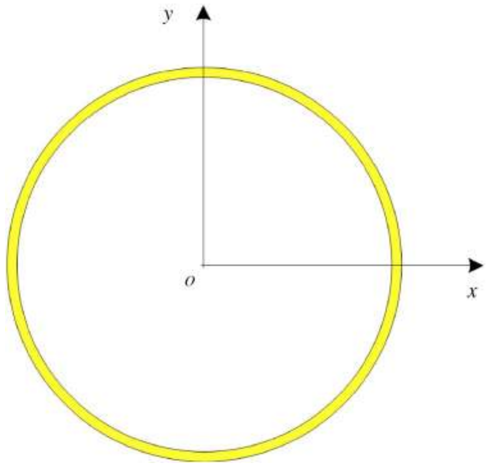


FIG. 2. (Color online) Geometry of circular ring.

ments have been conducted to predict the magnitude of $e_0$ for nanostructures. In the open literature, Wang and Hu,[49] who adopted the second-order strain gradient constitutive relation, proposed $e_0=0.288$ for the flexural wave propagation study in a single walled carbon nanotube through the use of the nonlocal Timoshenko beam model and MD simulations. Eringen[50] himself proposed $e_0$ as 0.39 based on the matching of the dispersion curves via nonlocal theory for plane wave and Born–Karman model of lattice dynamics at the end of the Brillouin zone, $ka=\pi$, where $a$ is the distance between atoms and $k$ is the wave number in the phonon analysis. On the other hand, Eringen proposed $e_0=0.31$ by comparing the Rayleigh surface wave via nonlocal continuum mechanics and lattice dynamics. Zhang *et al.*[34] estimated $e_0 \approx 0.82$ from the buckling analysis of single walled carbon nanotubes via the Donnell shell theory and MD simulations. The varied values of $e_0$ prompted Wang[42] to state that the adopted value of the scaling parameter $e_0$ depends on the crystal structure in lattice dynamics and the nature of physics under investigation. He also estimated that the scale coefficient $e_0 a < 2.0$ nm for a single walled carbon nanotubes if the measured wave propagation frequency value is assessed to be greater than 10 THz. By calibrating the small scaling parameter $e_0$ in the nonlocal Timoshenko beam theory using MD simulations results, Duan and Wang[43] found that the values of $e_0$ vary between 0 and 19 depending on the length-to-diameter ratio, boundary conditions, and mode shapes of nanostructures instead of a fixed value. In the present work, we adopt the same scale effect parameter $\alpha$, i.e., $\alpha=0$ to $\alpha=0.4$ as in Ref. 41 in the investigation of the small length scale effect on the vibration behavior of nanorings/arches.

Using the abovementioned small scale effect parameter, the free vibration of a full circular nanoring, as shown in Fig. 2, is investigated. First of all, the general solutions of Eq. (10) can be assumed to be of the form,

$$v = \sum_{k=1-3} (A_k \cos n_k\theta + B_k \sin n_k\theta), \tag{15}$$

where $n_1$, $n_2$, and $n_3$ are the roots of the equation

TABLE I. First five frequency parameters $\Omega$ for full circular ring and circular ring with a defect with various small scale parameters $\alpha$.

| Mode number | $\alpha=0$ | $\alpha=0.2$ | $\alpha=0.4$ |
|---|---|---|---|
| | Full circular ring | | |
| 1 | 7.2 | 6.2 | 4.4 |
| 2 | 57.6 | 42.4 | 23.6 |
| 3 | 211.7 | 129.1 | 59.5 |
| 4 | 553.8 | 276.9 | 110.8 |
| 5 | 1191.8 | 488.5 | 176.3 |
| | Circular ring with a defect ($K=5$) | | |
| 1 | 6.8 | 5.9 | 4.2 |
| 2 | 54.6 | 40.8 | 23.1 |
| 3 | 201.2 | 125.0 | 58.6 |
| 4 | 527.5 | 270.0 | 109.6 |
| 5 | 1137.6 | 478.6 | 174.9 |

$$n^2(n^2-1)^2 = (n^2+1)(\alpha^2 n^2+1)\Omega. \tag{16}$$

If the ring is complete and defect-free, $n$ must be an integer, and hence the frequency is given by

$$\Omega = \frac{n^2(n^2-1)^2}{(n^2+1)(\alpha^2 n^2+1)}. \tag{17}$$

It can be clearly seen that $n=1$ yields a trivial resonant frequency value. Thus the resonant frequencies $\Omega$ are associated with $n \geqslant 2$. In Table I, the first five frequency parameters $\Omega$ obtained from Eq. (17) are presented for various scaling effect parameters $\alpha=0$, 0.2, and 0.4. Note that the results associated with $\alpha=0$ correspond to those of the local theory where the small scale effect is ignored. The fundamental resonant frequency $\Omega=7.2$ is the same as the value given by Timoshenko[48] and Love.[47] The nonlocal results are smaller than the corresponding local results and the percentage differences are more significant for higher vibration modes. For example, for the fundamental mode and the fifth mode, the percentage differences $(\Omega_{\text{local}} - \Omega_{\text{nonlocal}}/\Omega_{\text{local}}) \times 100\%$ in the frequency parameters are 39% and 85% with $\alpha=0.4$, respectively. The vibration mode shape of the nonlocal circular nanoring with two full waves ($n=2$) is shown in Fig. 3. It can be seen from Eq. (15) that the vibration modes do not

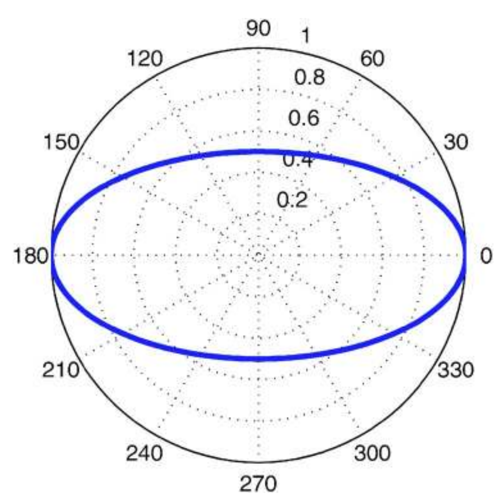


FIG. 3. (Color online) Mode shape of free vibration of a circular nanoring.

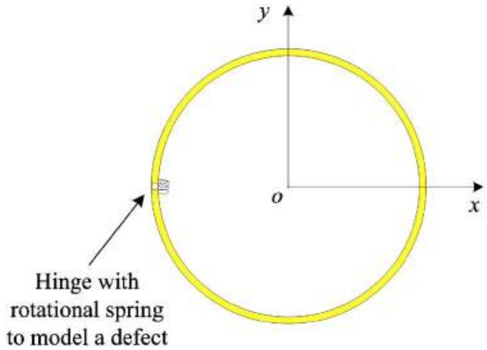


FIG. 4. (Color online) Mechanical model of circular nanoring with a defect.

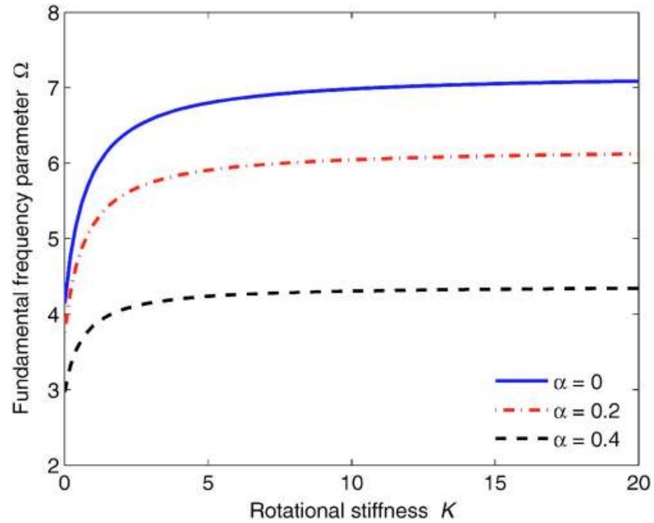


FIG. 5. (Color online) Resonant frequency of circular nanoring with a defect.

include any scale effect parameter. Thus, the vibration mode shapes for nonlocal circular ring are the same as the corresponding mode shapes for local circular ring, but their frequencies are affected by the small length scale effect parameter as shown in Eq. (17).

## IV. VIBRATION OF NONLOCAL CIRCULAR NANORINGS WITH A DEFECT

In this section, the effect of defects on vibration properties of nanorings/arches accounting for the small scale effect is investigated. From the viewpoint of continuum modeling, it is reasonable to model the defects due to the reduction of cross section of nanostructures as a hinge with a rotational restraint. The mechanical model for the nonlocal circular ring with a defect is shown in Fig. 4. The governing equation for the free vibration of the defected nanorings/arches is given by Eq. (10). Considering symmetric vibration mode shapes, its general solution is given by

$$v = \sum_{k=1-3} A_k \sin n_k \theta, \tag{18}$$

where $n_1$, $n_2$, and $n_3$ are the roots of Eq. (15). There are three integration constants $A_1$, $A_2$, and $A_3$ in Eq. (18). Therefore we need three conditions to solve this boundary value problem. The first two conditions are that there are no vertical displacements $v$ and shear force $Q$ at $\theta=\pi$. These conditions require

$$v = 0 \text{ and } Q = 0 \text{ at } \theta = \pi. \tag{19}$$

The third condition is that the moment at the hinge with a rotational restraint is equal to the product of the rotational spring constant $K$ and the change in angle, i.e.,

$$M|_{\theta=\pi} = K\left( \left.\frac{d^2 v}{d\theta^2}\right|_{\theta=-\pi} - \left.\frac{d^2 v}{d\theta^2}\right|_{\theta=\pi} \right), \tag{20}$$

where the stiffness of rotational restraint $K$ represents the scale of defects, which has to be calibrated by experiments. In this study, $K=0$ to $K=20$ is adopted arising from the observed mode shapes. By substituting Eqs. (12), (14), and (18) into Eqs. (19) and (20), one obtains the following characteristic equation:

$$|A|_{3\times 3} = 0, \tag{21}$$

where

$$\left.\begin{aligned} A_{1i} &= n_i \\ A_{2i} &= [n_i^2(n_i^2-1) - (n_i^2+1)\Omega\alpha^2]n_i \\ A_{3i} &= [n_i^2(n_i^2-1) - (n_i^2+1)\Omega\alpha^2]\tan^{-1} n_i\pi + 2(1+\alpha^2)Kn_i^3 \end{aligned}\right\} i = 1,2,3. \tag{22}$$

The roots $n_1$, $n_2$, and $n_3$ of Eq. (15) are functions of the resonant frequency $\Omega$. By substituting these roots into Eq. (21), one obtains an equation to determine $\Omega$, and hence the frequency $\omega$. Sample frequency parameters obtained from Eq. (21) are given in Table I. All frequency parameters are slightly smaller than the corresponding results for the case of the full circular ring. For example, the fundamental resonant frequencies for the case of the circular ring with a defect, i.e., 6.8 ($\alpha$=0), 5.9 ($\alpha$=0.2), and 4.2 ($\alpha$=0.4), are slightly smaller than those parameters for the case of the full circular ring, i.e., 7.2, 6.2, and 4.4, respectively. This is because the defect (due to the loss of cross sectional area) reduces the stiffness of the nanorings. In addition, the nonlocal results are smaller than the corresponding local results and the percentage differences are more significant for higher vibration modes. For example, for the fundamental mode and the fifth mode, the percentage differences $(\Omega_{\text{local}} - \Omega_{\text{nonlocal}}/\Omega_{\text{local}}) \times 100\%$ in the frequency parameters are 38% and 85% with

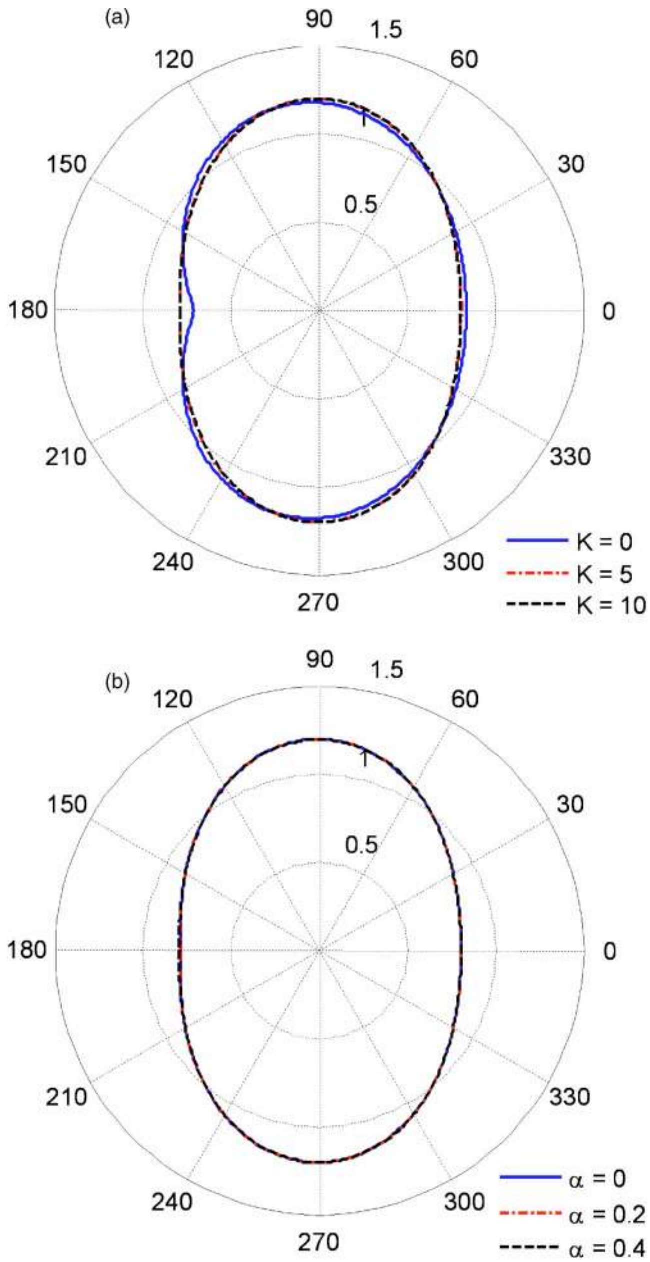


FIG. 6. (Color online) Vibration mode shapes for a nonlocal circular nanoring with a defect (a) variation of spring stiffness $K$ and (b) variation of small length scale parameter.

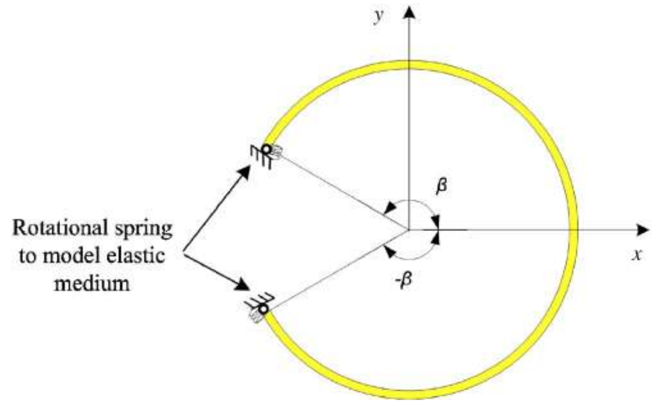


FIG. 7. (Color online) Geometry of circular nanoring embedded in elastic medium.

$\alpha=0.4$, respectively.

Figure 5 shows the variation of the fundamental resonant frequency parameter $\Omega$ with respect to the rotational stiffness $K$ for various small length scale parameters $\alpha$. For $\alpha=0$, the problem reduces to the free vibration problem of a local nanoring, i.e., no small effect and the results correspond to those obtained by Love.[47] The resonant frequency $\Omega$ increases from 4.1 to 7.1 as the rotational stiffness increases from $K=0$ to $K=20$. For $k\rightarrow\infty$, the characteristic Eq. (21) reduces to Eq. (17), i.e., the nanoring becomes a complete ring without a hinge. It is seen that the defects can reduce the value of resonant frequency by as much as 43% from 7.2 ($K\rightarrow\infty$, no defect) to 4.1 ($K=0$, defect is modeled as a hinge without rotational constraint). Moreover, it can be seen from Fig. 5 that the resonant frequencies based on the nonlocal theory is lower than their local counterparts and they decrease as the small length scale parameter increases. For $K=20$, the resonant frequency $\Omega$ decreases from 7.1 to 4.3 as $\alpha$ increases from 0 to 0.4. It is reasonable from Eq. (6) that the small length scale effect lowers the stress to achieve the same strain as compared to those associated with the local theory.

Figure 6(a) compares the vibration mode shapes associated with the fundamental resonant frequency of a nanoring containing a hinge with various rotational stiffnesses, where the small length scale parameter is set to be $\alpha=0$. When the rotational stiffness is relatively small (i.e., $K<1$), the mode shape of the ring has a kink at the hinge location. The changes in the mode shapes can be clearly observed as the rotational stiffness changes. The mode shapes, however, are almost identical at large values of rotational stiffness (i.e., $K\geqslant 10$). In these cases, the kink disappears and the nanoring vibrates in a similar manner as a complete ring (i.e., without a hinge). Figure 6(b) compares the vibration mode shapes of a nanoring containing a hinge with various $\alpha$ values, where the stiffness of rotational constraint is set to be $K=5$. When the small length scale parameter is varied from $\alpha=0$ to $\alpha=0.4$, the vibration mode shapes are almost identical. Hence unlike the resonant frequency $\Omega$, the vibration mode shape is not influenced by the small length scale effect.

## V. VIBRATION OF NONLOCAL CIRCULAR NANOARCHES EMBEDDED IN ELASTIC MEDIUM

In the research on nanostructures, the assumption that the boundary conditions are clamped at the ends of nanorings/arches may not be always true especially under the case of relatively soft surrounding elastic matrix for embedded nanostructures. Hence it is necessary to consider elastic boundary conditions. The embedded ends will be modeled as a hinge with rotational restraint instead of clamped ends. The rotational restraints at both ends are assumed herein to be identical, as shown in Fig. 7. The general solution of this problem remains the same as in Eq. (15). The nanoarch may vibrate either symmetrically (sinusoidal function) or asymmetrically (cosinusoid function). Thus, three conditions are needed to solve for the vibration problem. The symmetrical rotational restraints at both ends require that there are no displacements $v$, $w$ at $\theta=\beta$, i.e.,

$$v = 0 \text{ and } w = 0 \text{ at } \theta = \beta, \tag{23}$$

where $2\beta$ is the opening angle of nanoarches. The third condition is that the moment at the hinge with a rotational restraint is equal to the product of the rotational spring constant $K$ and the change in angle, i.e.,

$$M|_{\theta=\beta} = -K \left. \frac{d^2v}{d\theta^2} \right|_{\theta=\beta}. \tag{24}$$

By substituting Eqs. (12), (14), and (15) into Eqs. (23) and (24), one obtains the following characteristic equation.

- Symmetric mode

$$|B|_{3\times 3} = 0, \tag{25}$$

where

$$\left.\begin{aligned} B_{1i} &= n_i \\ B_{2i} &= n_i^2 \tan^{-1} n_i\beta \\ B_{3i} &= [n_i^2(n_i^2-1) - \Omega\alpha^2(n_i^2+1)]\tan^{-1} n_i\beta + (1+\alpha^2)Kn_i^3 \end{aligned}\right\} i = 1,2,3. \tag{26}$$

- Asymmetric mode

$$|C|_{3\times 3} = 0, \tag{27}$$

where

$$\left.\begin{aligned} C_{1i} &= n_i \\ C_{2i} &= n_2^2 \tan n_2\beta \\ C_{3i} &= [n_i^2(n_i^2-1) - \Omega\alpha^2(n_i^2+1)]\tan n_i\beta - (1+\alpha^2)Kn_i^3 \end{aligned}\right\} i = 1,2,3. \tag{28}$$

The roots $n_1$, $n_2$, and $n_3$ of Eq. (15) are functions of the resonant frequency $\Omega$. The substitution of these roots into Eqs. (25) and (27) gives the equation for determining $\Omega$, and hence the frequency $\omega$. The frequency parameters obtained from Eqs. (25) and (27) are presented in Table II. Again the nonlocal results are smaller than the corresponding local results and the percentage differences are more significant for higher vibration modes. For example, for the fundamental mode and the third asymmetric mode, the percentage differences $(\Omega_{\text{local}}-\Omega_{\text{nonlocal}}/\Omega_{\text{local}})\times 100\%$ in the frequency parameters are 21% and 75% with $\alpha=0.4$, respectively, while for the fundamental mode and the third symmetric mode, the percentage differences are 42% and 81%, respectively.

Consider a nanoarch with an opening angle $2\beta=2\pi/3$. Figure 8 shows the relationship between the first resonant frequencies $\Omega$ for symmetric and asymmetric modes and the rotational stiffness $K$ for various small length scale parameters $\alpha$. As it can be seen, the results are rather similar to those shown in Fig. 5 for a nanoring with a rotational constraint at the hinge location. The elastic end stiffness as well as the small length scale effect can significantly alter the value of resonant frequency. For $\alpha=0$, the resonant frequency $\Omega$ increases from 0.8 to 3.8 for asymmetric mode and from 11.0 to 21.1 for the symmetric mode as the rotational stiffness increases from $K=0$ to $K=20$. Moreover, the resonant frequencies based on the nonlocal theory are lower than their local counterparts and they decrease as the small scale parameter increases. For $K=20$, the resonant frequency $\Omega$ for the asymmetric mode decreases from 3.8 to 2.9 as $\alpha$ increases from 0 to 0.4 while the resonant frequency $\Omega$ for the symmetric mode decreases from 21.1 to 12.2.

The variations of the first resonant frequencies $\Omega$ with respect to the opening angle $2\beta$ are depicted in Fig. 9 for various small scale parameters $\alpha$, where the stiffness of rotational constraint is set as $K=5$. The resonant frequencies $\Omega$ decreases sharply as the opening angle increases for both the symmetric mode and the asymmetric mode. The nanoarch with an opening angle $2\beta=2\pi$ is virtually a circular nanoring containing a rotational spring at the hinge location discussed in the previous section. Again, it is observed that the resonant frequencies $\Omega$ is lowered by the small length scale effect. Moreover, the alteration of the resonant frequency due to small length scale effect is larger for relatively small opening angles than for larger opening angles. For example, for

TABLE II. First three frequency parameters $\Omega$ for circular arch ($2\beta=4\pi/3$ and $K=5$) embedded in elastic medium with various small scale parameters $\alpha$. asym denotes asymmetric mode while sym denotes symmetric mode.

| | $\alpha=0$ | | $\alpha=0.2$ | | $\alpha=0.4$ | |
|---|---|---|---|---|---|---|
| Mode number | asym | sym | asym | sym | asym | sym |
| 1 | 2.9 | 17.5 | 2.7 | 14.8 | 2.3 | 10.1 |
| 2 | 66.0 | 172.0 | 49.8 | 113.0 | 28.7 | 55.3 |
| 3 | 371.5 | 703.9 | 211.6 | 345.0 | 92.0 | 135.5 |

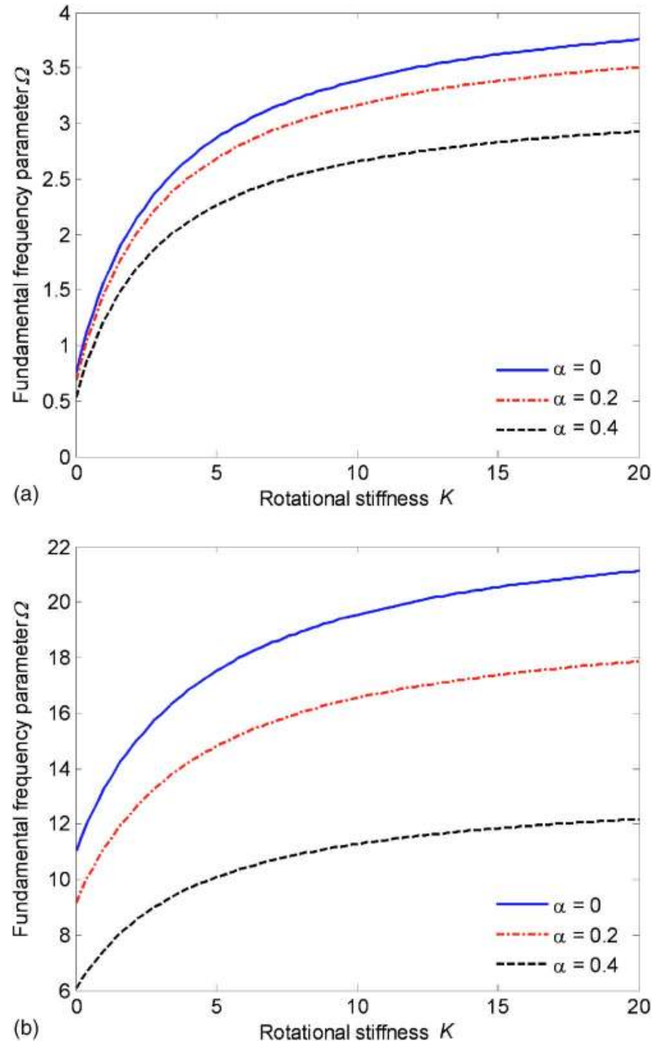


FIG. 8. (Color online) Fundamental frequency vs spring stiffness curves for a nonlocal nanoarch embedded in elastic medium (a) asymmetric mode and (b) symmetric mode.

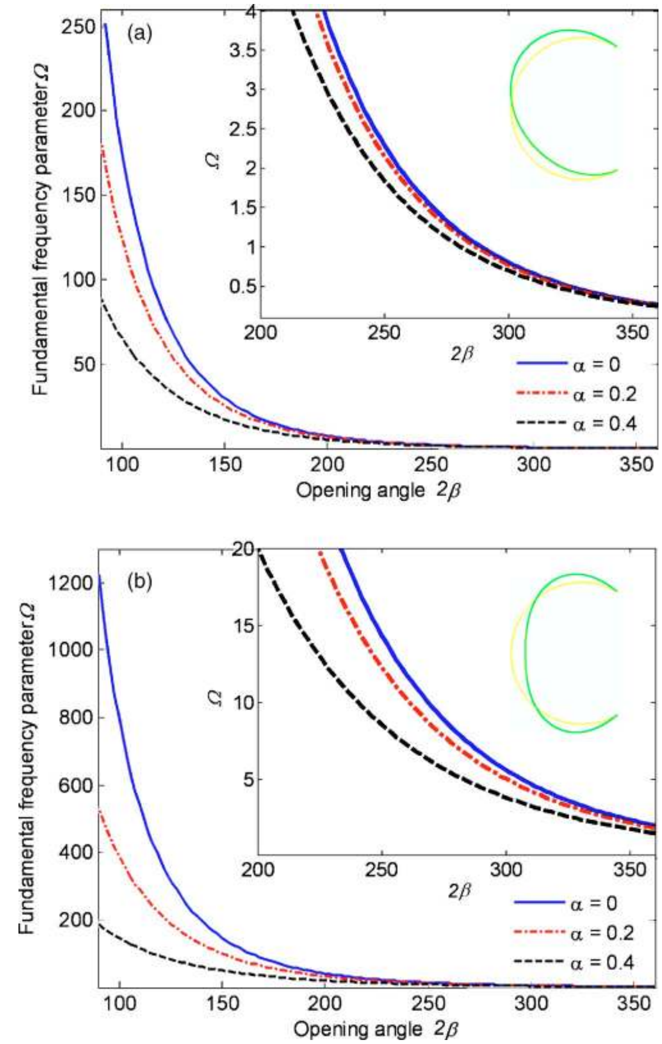


FIG. 9. (Color online) Fundamental frequency versus opening angle curves for a nonlocal nano-arch embedded in elastic medium (a) asymmetric mode and (b) symmetric mode.

$2\beta = 5\pi/3$ the resonant frequencies of the asymmetric and the symmetric modes decrease by 14% and 32%, respectively, as $\alpha$ increases from 0 to 0.4 while for $2\beta = 4\pi/3$ the resonant frequencies of the asymmetric and the symmetric modes decrease by 21% and 42%, respectively.

## VI. CONCLUDING REMARKS

Derived herein are the governing equations for the free vibration of circular nanorings/arches based on Eringen's nonlocal theory of elasticity. In addition to the small length scale effect, the effect of the defects due to reduction of cross section of nanostructures and elastic boundary conditions on the frequencies and vibration mode shapes are investigated. The nonlocal free vibration equations for a vibrating circular arch segment are specialized for the cases of a full circular ring, a circular ring with a defect and a circular arch embedded in elastic medium. It can be seen that the defects and elastic boundary will alter the resonant frequency values significantly as well as the vibration modes shapes. The small length scale effect lowers the resonant frequency but it does not affect the vibration modes shape. Work is underway to calibrate the small scale coefficient $e_0$ by using MD simulations in view to develop simple frequency formulas for engineers who are designing nanorings and nanoarches for applications in MEMS and NEMS devices.


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