Research Article **Open Access**

J. Seidi and S. Kamarian*

Free vibrations of non-uniform CNT/fiber/polymer nanocomposite beams

DOI 10.1515/cls-2017-0003
Received Jun 26, 2016; accepted Sep 28, 2016

Abstract: In this paper, free vibrations of non-uniform multi-scale nanocomposite beams reinforced by carbon nanotubes (CNTs) are studied. Mori-Tanaka (MT) technique is employed to estimate the effective mechanical properties of three-phase CNT/fiber/polymer composite (CNTFPC) beam. In order to obtain the natural frequencies of structure, the governing equation is solved by means of Generalized Differential Quadrature (GDQ) approach. The accuracy and efficiency of the applied methods are studied and compared with some experimental data reported in previous published works. The influences of volume fraction and agglomeration of nanotubes, volume fraction of long fibers, and different laminate lay-ups on the natural frequency response of structure are examined.

Keywords: Non-uniform beam; multi-scale nanocomposites; Carbon nanotube; Agglomeration effect; free vibration

1 Introduction

The outstanding characteristics of CNTs have motivated many researchers to study the behavior of CNT-reinforced composite (CNTRC) structures such as beams, shells and plates [1–10]. Investigations have shown that using CNTs in polymer matrix can significantly enhance the properties of composites [11–14]. However, one of the inevitable and destructive phenomena in CNTRCs is agglomeration of nanotubes which can extremely influence the properties of composites and their performance [15–17]. Hence, mechanical characteristics of CNTRC structures like Young modulus should be predicted as accurately as possible. For this reason, MT, as a powerful approach, has been implemented by many researchers to estimate the mechanical properties of composite structures reinforced by agglomerated nanotubes [18–20].

Due to light weight and high strength-to-mass ratio of three-phase CNTFPC structures, they have been attracted by a large number of researchers [21–25]. Refiee *et al.* [26] investigated nonlinear free vibration of simply supported CNTFPC plates with piezoelectric layers. The nanotubes were assumed to be uniformly distributed and randomly oriented in the matrix, and Halpin-Tsai model was employed to predict the material properties of nanocomposite plate. Recently, Kamarian *et al.* [27] presented natural frequency analysis and optimization of three-phase nanocomposite plates using GDQ technique, MT approach and firefly algorithm. In their work, first, a parametric study was carried out to investigate the influences of effective parameters on the natural frequencies of CNTFPC plates. Then, the fiber orientations of layers were optimized for natural frequency maximization of nanocomposite plates.

Non-uniform structures are being widely utilized in engineering applications especially in aerospace industry to minimize weigh of components or satisfy the geometrical constraints. Vibrational behavior of non-uniform engineering structures cannot be analyzed straightforwardly through closed-form or exact solutions. Therefore, many researchers have applied different numerical methods to investigate natural frequencies of structures with non-uniform cross section or inertia [28–34].

The main goal of this paper is to examine natural frequencies of non-uniform CNTFPC beams. The effects of aggregation and volume fraction of nanotubes, volume fraction of E-glass fibers, and various stacking sequences of layers and boundary conditions on the vibrational behavior of structure are studied in details. Here, MT and GDQ methods are used to obtain the mechanical properties of materials and solve the governing equation, respectively.

J. Seidi: Department of Mechanical Engineering, Ilam Branch, Islamic Azad University, Ilam, Iran
***Corresponding Author: S. Kamarian:** Department of Mechanical Engineering, Ilam Branch, Islamic Azad University, Ilam, Iran; Email: kamarian.saeed@yahoo.com

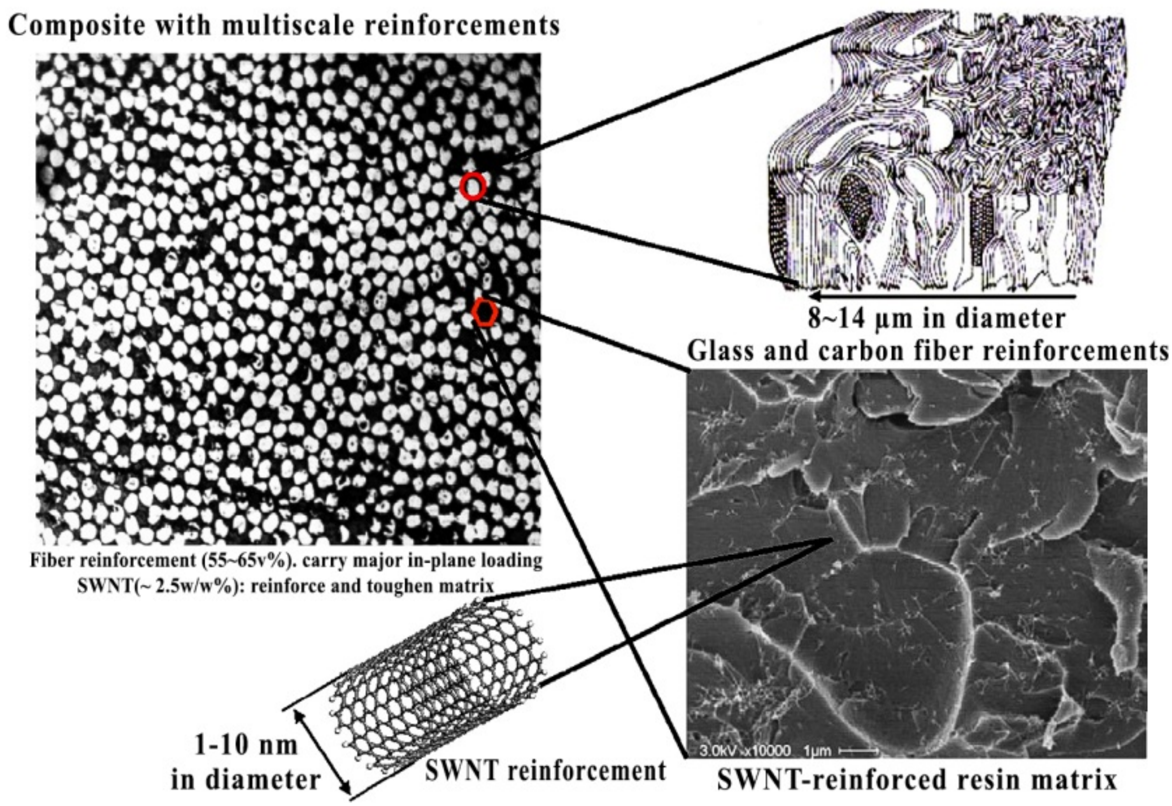

Figure 1: Concept of multi-scale reinforcement composites [26].

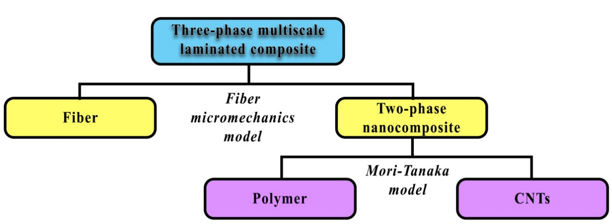

Figure 2: Hierarchy of the three-phase CNTFPC multi-scale composites.

2 Multi-scale composite material model

2.1 Fiber micromechanics model

Consider a three-phase multi-scale composites, as it is shown in Fig. 1. The flowchart for estimation of material properties of CNTFPCs is provided in Fig. 2. Micromechanics approach yields [26].

$$E_1 = V^F E_1^F + V^{NCM} E^{NCM} \tag{1}$$

$$\frac{1}{E_2} = \frac{V^F}{E_2^F} + \frac{V^{NCM}}{E^{NCM}} - V^{NCM} V^F \cdot \frac{\left(\upsilon^F\right)^2 E^{NCM}/E_2^F + \left(\upsilon^{NCM}\right)^2 E_2^F/E^{NCM} - 2\upsilon^F \upsilon^{NCM}}{V^F E_2^F + V^{NCM} E^{NCM}} \tag{2}$$

$$\frac{1}{G_{12}} = \frac{V^F}{G_{12}^F} + \frac{V^{NCM}}{G_{12}^{NCM}} \tag{3}$$

$$\upsilon_{12} = V^F \upsilon^F + V^{NCM} \upsilon^{NCM} \tag{4}$$

$$\rho = V^F \rho^F + V^{NCM} \rho^{NCM} \tag{5}$$

where F and NCM stand for E-glass fibers and CNT-reinforced epoxy materials. Also, E, G_{12}, υ, ρ and V indicate the Young's moduli, shear modulus, Poisson's ratio, mass density and volume fraction of materials, respectively. Now, in section 2.2, mechanical properties of the epoxy reinforced by nanotubes are modeled employing MT method.

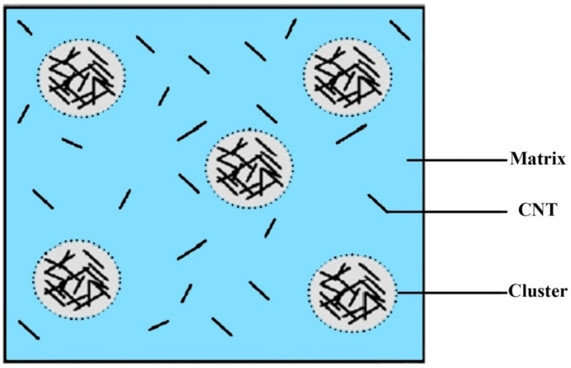

Figure 3: Representative Volume Element (RVE) with Eshelby cluster model of agglomeration of CNTs.

2.2 Material properties of CNT-reinforced matrix

Here, the equivalent material properties of epoxy resin reinforced by agglomerated CNTs are calculated based on Ref. [35]. In Fig. 3, it is observed that a number of nanotubes are uniformly dispersed in the epoxy and the other CNTs are agglomerated in the clusters. The following parameters are introduced

$$\mu = \frac{V_{cluster}}{V^*} \quad 0 \le \mu \le 1 \tag{6}$$

$$\eta = \frac{V^r_{cluster}}{V^r} \quad 0 \le \eta \le 1 \tag{7}$$

Where V^* denotes volume of RVE, V^r shows the total volume of nanotubes, $V_{cluster}$ represents volume of all clusters, and $V^r_{cluster}$ indicates the volume of those CNTs which are included by clusters. According to Eqs. (6, 7), $\mu = \eta = 1$ shows the case that nanotubs are uniformly distributed through the epoxy resin. The bulk and shear modulus of clusters (K_{in} and G_{in}), and the bulk and shear modulus of CNT-reinforced resin outside the clusters (K_{out} and G_{out}) can be obtained using Eqs. (8–11).

$$K_{in} = K_m + \frac{V^r \eta (\delta_r - 3K_m \alpha_r)}{3(\mu - V^r \eta + V^r \eta \alpha_r)} \tag{8}$$

$$K_{out} = K_m + \frac{V^r (1-\eta)(\delta_r - 3K_m \alpha_r)}{3[1 - \mu - V^r (1-\eta) + V^r (1-\eta)\alpha_r]} \tag{9}$$

$$G_{in} = G_m + \frac{V^r \eta (\eta_r - 2G_m \beta_r)}{2(\mu - V^r \eta + V^r \eta \beta_r)} \tag{10}$$

$$G_{out} = G_m + \frac{V^r (1-\eta)(\eta_r - 2G_m \beta_r)}{2[1 - \mu - V^r (1-\eta) + V^r (1-\eta)\beta_r]} \tag{11}$$

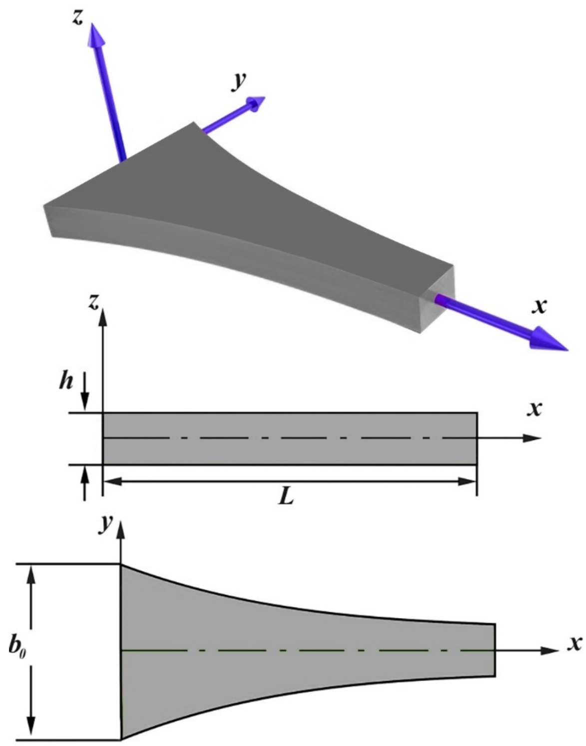

Figure 4: Geometry of a non-uniform CNTFPC beam.

in which the subscripts m and r stand for the quantities of the resin and nanotubes, and the other parameters can be defined in Ref. [36]. Using MT model, the bulk and shear modulus of nanocmposite matrix can be obtained as

$$K^{NCM} = K_{out}\left[1 + \frac{\mu\left(\frac{K_{in}}{K_{out}} - 1\right)}{1 + \alpha(1-\mu)\left(\frac{K_{in}}{K_{out}} - 1\right)}\right] \tag{12}$$

$$G^{NCM} = G_{out}\left[1 + \frac{\mu\left(\frac{G_{in}}{G_{out}} - 1\right)}{1 + \beta(1-\mu)\left(\frac{G_{in}}{G_{out}} - 1\right)}\right] \tag{13}$$

where α and β are defined in Ref. [36]. After calculating the bulk and shear modulus, the final Young's modulus and Poisson's ratio of the CNT-reinforced resin are obtained using the following relations

$$E^{NCM} = \frac{9K^{NCM} G^{NCM}}{3K^{NCM} + G^{NCM}} \tag{14}$$

$$\upsilon^{NCM} = \frac{3K^{NCM} - 2G^{NCM}}{6K^{NCM} + 2G^{NCM}} \tag{15}$$

3 Governing equations

Consider a non-uniform CNTFPC beam, as it is observed from Fig. 4. In this figure L and h denote the length and thickness of structure, respectively. Parameter δ, which is called non-uniformity parameter, is defined here to indicate the variations of width of beam in the length direction. In the present work, the width of beam is presumed to vary exponentially versus parameter δ according to $b(x) = b_0 e^{\delta x}$ and the non-uniformity parameter can vary from −2 to +2, in this paper. It is obvious that the case $\delta = 0$ refers to a beam with uniform section. Based on Euler-Bernoulli beam theory, the governing equation of motion is derived by applying Hamilton's principle

$$\delta \star \int_0^t (T - \Pi) dt = 0 \tag{16}$$

where $\delta\star$ indicates variational symbol, and T and Π are defined as kinetic energy and potential energy of the CNTFPC beam respectively

$$\Pi = \frac{1}{2} \int_0^L \int_{-h/2}^{h/2} b(x) \sigma_x \varepsilon_x \, dz dx \tag{17a}$$

$$T = \frac{1}{2} \int_0^L \int_{-h/2}^{h/2} b(x) \rho(z) \left(\frac{\partial w}{\partial t} \right)^2 dz dx \tag{17b}$$

in which, $\sigma_x = Q_{11} \varepsilon_x$ and $\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(-z \frac{\partial w}{\partial x}) = -z \frac{\partial^2 w}{\partial x^2}$ represent axial stress and axial strain, respectively. By combining Eqs. (16) and (17), after some simplifications, the governing equation is obtained as follow:

$$D_{11}(x) \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial D_{11}(x)}{\partial x} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 D_{11}(x)}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + m(x) \frac{\partial^2 w}{\partial t^2} = 0 \tag{18}$$

where $D_{11}(x)$ is the transformed bending stiffness coefficient, defined as

$$D_{11}(x) = \int_{-h/2}^{h/2} b(x) \bar{Q}_{11}^k z^2 dz = b_0 \, e^{\delta x} \sum_{k=1}^{N_L} \int_{z_{k-1}}^{z_k} \bar{Q}_{11}^k z^2 dz \tag{19}$$

where $\bar{Q}_{ij}^k$ are defined as the plane stress-reduced stiffness and N_L is the number of layers. In order to attain the fundamental frequencies of nanocomposite beam, Eq. (18) is formulated as an eigenvalue problem by using the periodic function$w(x, t) = W(x) \, e^{-i\omega t}$, in which ω and $W(x)$ are natural frequency and mode shape of the transverse motion of the CNTFPC beam. Hence, the eigenvalue problem is derived as:

$$D_{11}(x) \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial D_{11}(x)}{\partial x} \frac{\partial^3 W}{\partial x^3} + \frac{\partial^2 D_{11}(x)}{\partial x^2} \frac{\partial^2 W}{\partial x^2} + (-m\,(x)\,\omega^2)\, W = 0 \tag{20}$$

4 GDQ method

GDQ technique, as an efficient numerical technique, is implemented for solving partial differential equations, especially in the field of solid mechanics [37–41]. Here, this method is utilized to obtain the natural frequencies of non-uniform CNTFPC beam. In this approach, the spatial derivative of a function of given grid point is estimated as a weighted linear sum of all the functional value at all grid point in the whole domain [37]:

$$\frac{\partial f^{n(x_i, z)}}{\partial x^n} = \sum_{k=1}^{N} c_{ik}^n f(x_{ik}, z) \tag{21}$$

$$(i = 1, \ 2, \ N \ \ , \ \ n = 1, \ldots, \ n - 1)$$

In which N denotes the number of grid points, and c_{ij}^n is the x_i dependent weight coefficients. The fundamental principles of GDQ method can be found in Refs. [37, 38].

5 Results and discussion

To verify the efficiency of employed techniques (GDQ and MT), two examples are carried out for comparisons. In the first one, Young modulus of CNT-reinforced epoxy is predicted using MT model and compared with the results reported in Ref. [17] through experiments. For this comparison, the material properties of applied epoxy and CNT are provided in Table 1. Experimental Young modulus of the nanocomposite versus different CN volume fraction at the state $\eta = 1$ and $\mu = 0.4$ is depicted in Fig. 5. The predicted Young modulus by MT method is also shown in this figure for different values of CNT volume fraction, η and μ. The comparison shows that the just a little difference between experimental data and MT results exist.

As the second example, a comparison is made between the frequency parameters of a symmetrical twenty-

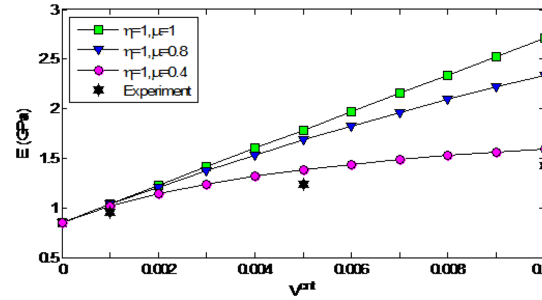

Figure 5: Comparison of the Young's moduli of CNTRCs at different degree of agglomeration with the experimental data [17].

Table 1: Parameters used in the calculation of effective elastic properties of CNT-reinforced polymer composite for comparison of the employed MT model and the experimental data.

Mechanical properties	Matrix phase [42]	CNT [43]
Isotropic Young's modulus (E)	0.85 GPa	–
Poisson's ratio (υ)	0.4	–
Longitudinal Young's modulus (E_{11})	–	1.06 TPa
Transverse bulk modulus (K_{23})	–	271 GPa
Transverse shear modulus (G_{23})	–	17 GPa
In plane shear modulus (G_{12})	–	442 GPa
In plane Poisson's ratio (υ_{12})	–	0.162

Table 2: Material properties of glass/epoxy composite beam [44].

Material	properties	value
Glass	Elasticity modulus	$E_1^F = E_2^F = 76\,GPa$
Fiber	Density	$\rho^F = 2056 kg/m^3$
	Poisson's coefficient	$\upsilon^F = 0.22$
	Fiber volume fraction	$V^F = 60\%$
Epoxy	Elasticity modulus	$E^m = 4\,GPa$
Resin	Density	$\rho^m = 1300 kg/m^3$
	Poisson's coefficient	$\upsilon^m = 0.4$
	Resin volume fraction	$V^m = 40\%$

layer $[0°/90°/0°/90°/0°/90°/0°/90°/0°/90°]_s$ laminate beams obtained from present study (by setting $V^r = 0$ and $\delta = 0$) and the experimental data available in Ref. [44]. The mechanical properties of materials are shown in Table 2. Results in Table 3 represent that GDQ can obtain the natural frequency of composite beams with high accuracy.

Now, the natural frequency response of non-uniform CNTFPC beams is characterized. The mechanical properties of beams are presented in Table 4 and unless otherwise stated, the length and height of beam are assumed to be *1m* and *0.01m*, respectively and b_0 is taken *0.05m*. For the analysis, a non-dimensional natural frequency is defined as $\Omega = \omega\sqrt{\frac{\rho^m}{E^m h^2}}$ which E^m and ρ^m denote Young's modulus and density ratio of epoxy resin, respectively. It is also mentioned that simply supported, clamped and free boundary conditions are specified by S, C and F letters symbols, respectively.

Table 5 illustrates the influence of CNT volume fraction (V^r) and E-glass fiber volume fraction $\left(V^F\right)$ on the non-dimensional natural frequency of a $[90°/0°/0°/90°]$ non-uniform CNTFPC beam for the case $\eta = \mu = 1$. It is clear that with increase of V^r or V^F, fundamental frequency parameter of nanocomposite beam increases. Table 5 also reveals that the influence of nanotubes on natural frequencies becomes lower in composite structures with higher fiber volume fraction. For example, it is evident that adding 5% CNT in the resin improves the natural frequency parameters of simply supported non-uniform CNTFPC beams about 15.3% when $V^F = 50\%$ while this improvement is 12.4% if $V^F = 70\%$.

The effect of stacking sequence of layers on the free vibration of non-uniform nanocomposite beams is shown in Table 6 for various CNT volume fraction and boundary conditions. By focusing on the results of this table, it can be inferred that orientation of fibers has an effective role in free vibrations of CNTFPC beams. It is resulted that the influence of CNT volume fraction is more significant in composite beams with more 90° layers. This is due to the fact that in the layers with angle 0°, CNTs do not lead to serious variations in $D_{11}(x)$ in Eq. (20), as the stiffness of structure. In other words, in 0° layers, the effect of long fibers on $D_{11}(x)$ is more dominant compared to the effect of CNTs.

Here, the role of agglomeration parameters (μ and η) in vibrational behavior of non-uniform CNTFPC beams is investigated. The fundamental frequencies of beam are tabulated in Table 7 for different nanotube volume fraction, aggregation states and boundary conditions. Table 7 demonstrates that the agglomeration influence of CNTs on natural frequencies of CNTFPC beams is more prominent at high values of nanotube volume fraction. Table 7 and Fig. 6 also confirms that the effect of volume fraction of nanotubes on the natural frequency of structure is reduced if more discrepancy between μ and η is observed.

In Table 8 and Fig. 7, the effects of non-uniformity parameters δ and boundary conditions on the first three frequency parameters of non-uniform nanocomposite beams are illustrated for the case $\eta = \mu = 1$. Results indicate that the edge conditions and non-uniformity parameter δ have crucial roles in the vibrational characteristics of the structure. As it is obvious from Table 8, increase in δ has different effects on the vibrational behavior of CNTFPC beams. It is observed that with increase of δ, natural fre-

Table 3: Comparison of natural frequencies of a twenty-layer composite beam $\left(\Omega = \omega\sqrt{\frac{E_2 I}{\rho A L^4}}\right)$.

Mode	ANSYS [44]	Experimental [44]	Present
1st flexural mode	5.5	5	5.7889
2nd flexural mode	34.7	28	36.2781
Vibration (plane 1-2)	91.0(*)	-	-
3rd flexural mode	97.1	84	101.5798
1st Torsional mode	108.0	-	-
4th flexural mode	190.0	155	199.559
5th flexural mode	313.7	272	329.0537
2nd Torsional mode	327.0	-	-

Table 4: Material Properties of CNT/fibereopxy beam [27].

Material	Properties	Value
CNT	Longitudinal Young's modulus (GPa)	649.12
	Transverse Young's modulus (GPa)	11.27
	Longitudinal Shear modulus (GPa)	5.13
	Poisson's ratio	0.284
	Density (kg/m^3)	1400
Epoxy Resin	Isotropic Young's modulus	10
	Poisson's ratio	0.3
	Density (kg/m^3)	1150
Glass Fiber	Elasticity modulus (GPa)	69
	Density (kg/m^3)	1200
	Poisson's coefficient	0.2

Table 5: Fundamental frequency parameters of a four-layer [90°/0°/0°/90°] non-uniform CNTFPC beams with various CNT volume fractions and fiber volume fractions ($\eta = 1$ $\mu = 1$ $\delta = -2$).

Boundary conditions	V^F	V^r			
		0%	1%	2%	5%
SS	50%	3.938	4.074	4.201	4.541
	60%	4.248	4.384	4.510	4.842
	70%	4.626	4.760	4.882	5.199
SC	50%	5.539	5.730	5.909	6.387
	60%	5.975	6.166	6.344	6.811
	70%	6.506	6.694	6.867	7.313
CS	50%	7.356	7.609	7.846	8.481
	60%	7.935	8.188	8.424	9.045
	70%	8.640	8.890	9.119	9.711
CC	50%	9.522	9.850	10.157	10.979
	60%	10.271	10.599	10.904	11.708
	70%	11.184	11.507	11.804	12.570
CF	50%	2.600	2.689	2.773	2.997
	60%	2.804	2.894	2.977	3.197
	70%	3.054	3.142	3.223	3.432

Table 6: Fundamental natural frequencies of non-uniform CNTFPC beam versus different stacking sequence of layers ($\delta = -2$ $V^F = 60\%$ $\mu = \eta = 1$).

Stacking sequence	V^r	Boundary conditions				
		SS	SC	CS	CC	CF
[90°/90°/90°/90°]	0%	3.986	5.606	7.444	9.636	2.631
	1%	4.145	5.830	7.742	10.022	2.736
	2%	4.292	6.037	8.017	10.377	2.833
	5%	4.675	6.575	8.731	11.302	3.086
[90°/0°/0°/90°]	0%	4.248	5.975	7.935	10.271	2.804
	1%	4.384	6.166	8.188	10.599	2.894
	2%	4.510	6.344	8.424	10.904	2.977
	5%	4.842	6.811	9.045	11.708	3.197
[0°/90°/90°/0°]	0%	5.570	7.834	10.404	13.467	3.677
	1%	5.607	7.887	10.473	13.557	3.701
	2%	5.644	7.938	10.541	13.645	3.725
	5%	5.748	8.085	10.736	13.897	3.794
[0°/0°/0°/0°]	0%	5.761	8.103	10.760	13.928	3.803
	1%	5.786	8.138	10.807	13.989	3.819
	2%	5.811	8.173	10.854	14.050	3.836
	5%	5.885	8.278	10.993	14.229	3.885

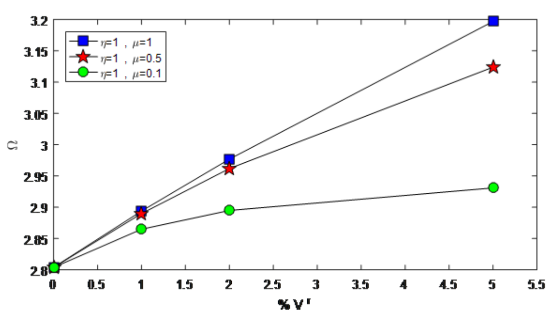

Figure 6: Variations of fundamental frequency of [90°/0°/0°/90°] cantilever nanocomposite beam versus CNT volume fraction at different degree of agglomeration ($V^F = 60\%$ $\delta = -2$).

quencies of the C-S and C-F beams decrease while they increase for S-C boundary conditions. For S-S and C-C beam, due to the symmetry of boundary conditions, the variations of frequencies are different and symmetrical behaviors are expected. Fig 7 and Table 8 reveal that with increase of δ, fundamental natural frequency of the structure with C-C boundary conditions first decrease to a minimum value (at $\delta = 0$ where the cross-section profile is uniform) and then increase. Similar behaviors can be observed for the second and third natural frequencies. For the CNTFPC beam with simply supported boundary conditions, it is apparent that the uniform cross section has maximum frequency parameters at the first mode and minimum values at higher modes. According those mentioned about Table 8 and Fig. 7, it can be concluded that using suitable non-uniformity parameter can be considered as an effective parameter for engineering designs. From Fig. 7, another important conclusion can be made regarding the non-uniformity of beam. Results represent that the role of parameter δ is more effective at first modes compared to other modes. Take C-C boundary conditions in this figure for example. Variations of δ from −2 to 0, leads to variations of first natural frequency from 10.904 to 10.636 (difference between the frequencies is about 2.46%) while the variations of third natural frequency of structure is less than 0.68% (from 57.865 to 57.476).

6 Conclusion

In this research work, natural frequency response of non-uniform multi-scale CNTFPC beams was studied taking aggregation of nanotubes into account and using Hamilton' principle, GDQ approach and MT model. From the present work, some conclusions can be made:

- Results show that CNT volume fraction does not have any significant influences on the natural frequencies of orthotropic beams with 0° layers.
- Variations of natural frequencies against non-uniformity parameter δ at different modes for var-

Table 7: Fundamental natural frequencies of [90°/0°/0°/90°] non-uniform CNTFPC beams with various values of CNT volume fractions, agglomeration parameters and boundary conditions ($\delta = -2$ $V^F = 60\%$).

V^r		Boundary Conditions				
		SS	CS	SC	CC	CF
0%	-	4.248	5.975	7.935	10.271	2.804
1%	$\eta = 1$ $\mu = 1^*$	4.384	6.166	8.188	10.599	2.894
	$\eta = 1$ $\mu = 0.5$	4.377	6.157	8.176	10.583	2.890
	$\eta = 1$ $\mu = 0.1$	4.341	6.106	8.108	10.495	2.865
2%	$\eta = 1$ $\mu = 1^*$	4.510	6.344	8.424	10.904	2.977
	$\eta = 1$ $\mu = 0.5$	4.486	6.310	8.380	10.847	2.962
	$\eta = 1$ $\mu = 0.1$	4.386	6.169	8.192	10.603	2.895
5%	$\eta = 1$ $\mu = 1^*$	4.842	6.811	9.045	11.708	3.197
	$\eta = 1$ $\mu = 0.5$	4.732	6.656	8.838	11.441	3.124
	$\eta = 1$ $\mu = 0.1$	4.440	6.244	8.292	10.734	2.931

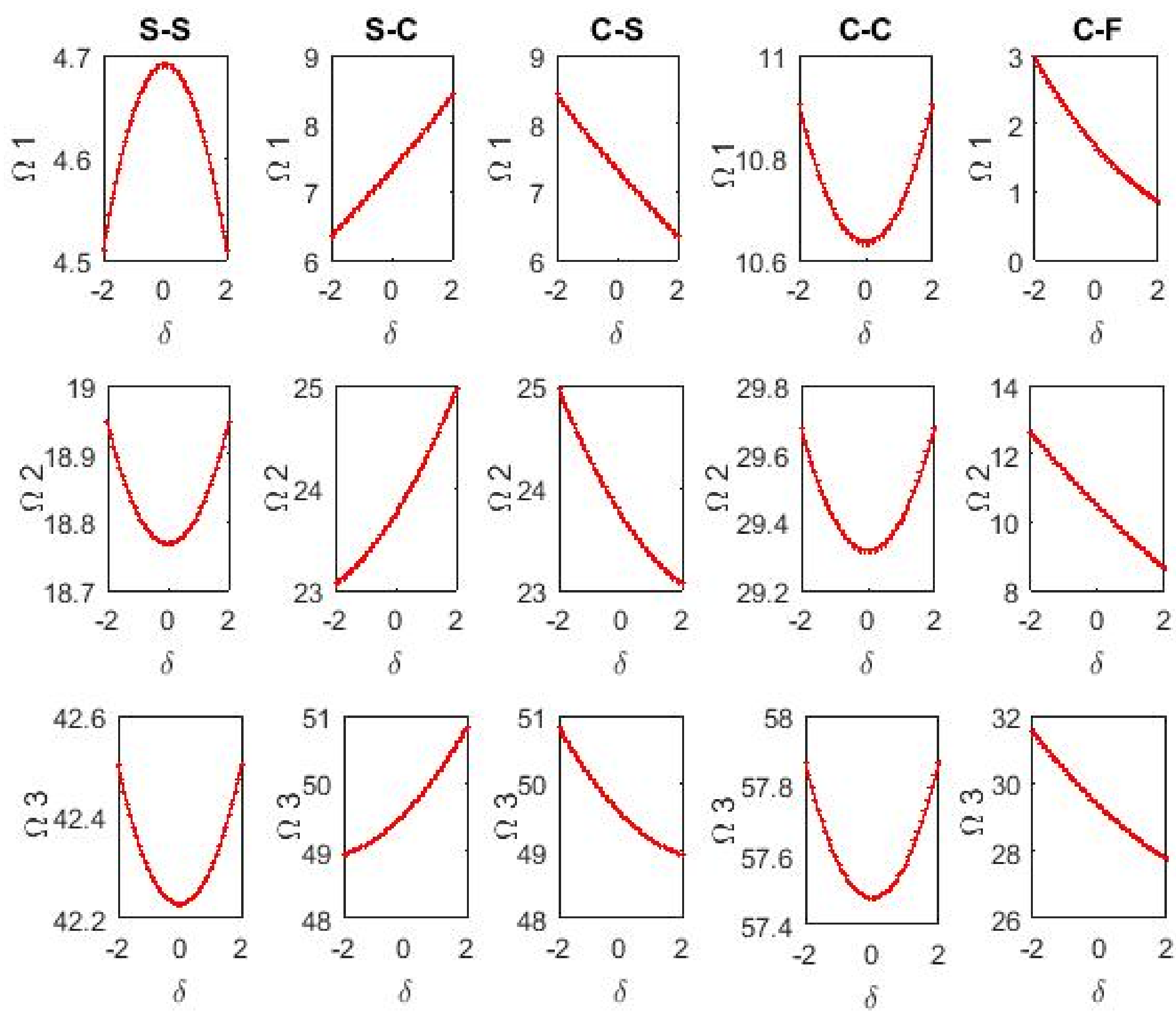

Figure 7: The first three natural frequencies of non-uniform [90°/0°/0°/90°] CNTRC beams with various non-uniformity parameters and boundary conditions ($V^F = 60\%$ $V^r = 2\%$ $\eta = 1$ $\mu = 1$).

Table 8: The first three natural frequencies of non-uniform [90°/0°/0°/90°] CNTRC beams with various non-uniformity parameters and boundary conditions (V^F = 60% V^r = 2% η = 1 μ = 1).

δ	Mode No.	Boundary Conditions				
		S-S	S-C	C-S	C-C	C-F
−2	1	4.510	6.344	8.424	10.904	2.977
	2	18.945	23.069	24.970	29.675	12.637
	3	42.502	48.940	50.841	57.865	31.553
−1	1	4.646	6.835	7.849	10.702	2.251
	2	18.811	23.344	24.293	29.407	11.505
	3	42.295	49.165	50.116	57.573	30.360
0	1	4.692	7.330	7.330	10.636	1.671
	2	18.767	23.753	23.753	29.318	10.475
	3	42.227	49.558	49.558	57.476	29.330
1	1	4.646	7.849	6.835	10.702	1.220
	2	18.811	24.293	23.344	29.407	9.526
	3	42.295	50.116	49.165	57.573	28.462
2	1	4.510	8.424	6.344	10.904	0.875
	2	18.945	24.970	23.069	29.675	8.639
	3	42.502	50.841	48.940	57.865	27.757

ious boundary conditions can be considered as an important factor for structures design.

- Due to the agglomeration of nanotubes in resin, using only low percent of CNTs (about 2%) leads to improvement in the stiffness of structure.
- The effects of CNTs on vibrational behavior of CNTFPC beams are more prominent for structures with less E-glass fibers.

References

[1] F. Tornabene, N. Fantuzzi, M. Bacciocchi, E. Viola, "Effect of agglomeration on the natural frequencies of functionally graded carbon nanotube-reinforced laminated composite doubly-curved shellsOriginal', Composites Part B: Engineering, 2016, 89: 187-218

[2] K. Mehar, S. Panda, "Geometrical nonlinear free vibration analysis of FG-CNT reinforced composite flat panel under uniform thermal field", Composite Structures, 2016, 143: 336–346.

[3] Z.X. Lei, L.W. Zhang, K.M. Liew, "Vibration of FG-CNT reinforced composite thick quadrilateral plates resting on Pasternak foundations", Engineering Analysis with Boundary Elements, 2016, 64: 1–11.

[4] S. Kamarian, M. Salim, R. Dimitri, F. Tornabene, "Free Vibration Analysis of Conical Shells Reinforced with Agglomerated Carbon Nanotubes" International Journal of Mechanical Sciences, 2016, 108 (1), 157-165.

[5] Z.X. Lei, K.M. Liew, J.L. Yu, "Buckling analysis of functionally graded carbon nanotube-reinforced composite plates using the element-free kp-Ritz method", Composite Structures, 2013, 98: 160–168

[6] L.W. Zhang, K.M. Liew, "Postbuckling analysis of axially compressed CNT reinforced functionally graded composite plates resting on Pasternak foundations using an element-free approach", Composite Structures, 2016, 138: 40–51.

[7] L.W. Zhang, K.M. Liew, J.N. Reddy, "Postbuckling of carbon nanotube reinforced functionally graded plates with edges elastically restrained against translation and rotation under axial compression", Comput. Methods Appl. Mech. Engrg., 2016, 298: 1–28.

[8] M. Mohammadimehr, M. Salemi, B. Rousta Navi, "Bending, buckling, and free vibration analysis of MSGT microcomposite Reddy plate reinforced by FG-SWCNTs with temperature-dependent material properties under hydro-thermo-mechanical loadings using DQM", Composite Structures, 2016, 138: 361–380.

[9] K. Mayandi, P. Jeyaraj, "Bending, buckling and free vibration characteristics of FG-CNT-reinforced polymer composite beam under non-uniform thermal load", Journal of Materials: Design and Applications, 2015, 229(1): 13–28.

[10] F. Tornabene, N. Fantuzzi, M. Bacciocchi "Linear static response of nanocomposite plates and shells reinforced by agglomerated carbon nanotubes" Composites Part B: Engineering, 20016, Available online.

[11] H. Dai, "Carbon nanotubes: opportunities and challenges", Surface Science, 2002, 500: 218-241.

[12] T. Lau, C. Gu, D. Hui, "A critical review on nanotube and nanotube/nanoclay related polymer composite materials", Composites Part B, 2006, 37: 425-436.

[13] I. Kang, Y. Heung, J. Kim, J. Lee, R. Gollapudi, S. Subramaniam, S. Narasimhadevara, D. Hurd, G. Kirkera, V. Shanov, M. Schulz, D. Shi, J. Boerio, S. Mall, D. Ruggles-Wren, "Introduction to carbon nanotube and nanofiber smart materials", Composites Part B, 2006, 37: 382-394.

[14] M. Shokrieh, R. Rafiee, "Prediction of mechanical properties of an embedded carbon nanotube in polymer matrix based on developing an equivalent long fiber". Mechanics Research Communications, 2010, 37: 235-240.

[15] A. Montazeri, J. Javadpour, A. Khavandi, A. Mohajeri, A. Tcharkhtch, "Mechanical properties of multi-walled carbon nanotube/epoxy composites", Materials & Design, 2010, 31(9): 4202-4208.

[16] Q.S. Yang, X.Q. He, X. Liu, F.F. Leng, Y.W. Ma, "The effective properties and local agglomeration effect of CNT/SMP composites", Composites Part B, 2012, 43(1): 33-38.

[17] P. Barai, G.J. Weng, "A theory of plasticity for carbon nanotube reinforced composites", International Journal of Plasticity, 2011, 27(4): 539-559.

[18] S. Kamarian, A. Pourasghar, M. H. Yas, "Eshelby-Mori-Tanaka approach for vibrational behavior of functionally graded carbon nanotube-reinforced plate resting on elastic foundation" Journal of Mechanical Science and Technology, 2013, 27 (11), 3395-3401.

[19] M. Heshmati, M. H. Yas, "Free vibration analysis of functionally graded CNT-reinforced nanocomposite beam using Eshelby-Mori-Tanaka approach", Journal of Mechanical Science and Technology, 2013, 27 (11): 3403-3408.

[20] A. Pourasghar, M.H. Yas, S. Kamarian, "Local agglomeration effect of CNT on the vibrational behavior of four-parameter continuous graded nanotube-reinforced cylindrical panels", Polymer Composites, 2013, 34(5): 707-721.

[21] E. Bekyarova, E.T. Thostenson, A. Yu, H. Kim, J. Gao, J. Tang, H.T. Hahn, T.W. Chou, M.E. Itkis, R.C. Haddon, "Multiscale carbon nanotube–carbon fiber reinforcement for advanced epoxy composites", Langmuir, 2007, 23: 3970–3974.

[22] M.S. Kim, Y.B. Park, O.I. Okoli, C.H. Zhang, "Processing, characterization, and modeling of carbon nanotube-reinforced multiscale composites", Composites Science and Technology, 2009, 69: 335–342.

[23] E.T. Thostenson, W.Z. Li, D.Z. Wang, Z.F. Ren, T.W. Chou, "Carbon nanotube/carbon fiber hybrid multiscale composites", Journal of Applied Physics, 2002, 91: 6034–6037.

[24] A. Godara, L. Mezzo, F. Luizi, F. Warrier, S.V. Lomov, A.W.V. Vuure, L. Gorbatikh, P. Moldenaers, I. Verpoest, "Influence of carbon nanotube reinforcement on the processing and the mechanical behaviour of carbon fiber/epoxy composites", Carbon, 2009, 47: 2914–2923.

[25] K.J. Green, R.D. Derrick, K.V. Uday, N. Elijah, "Multiscale fiber reinforced composites based on a carbon nanofiber/epoxy nanophased polymer matrix synthesis, mechanical, and hermomechanical behavior" Composites Part A: Applied Science and Manufacturing, 2009, 40, 1470–1475.

[26] M. Rafiee, X.F. Liu, X.Q. He, S. Kitipornchai, "Geometrically nonlinear free vibration of shear deformable piezoelectric carbon nanotube/fiber/polymer multi-scale laminated composite plates", Journal of Sound and Vibration, 2014, 333 (14): 3236–3251.

[27] S. Kamarian, M. Shakeri, M. H. Yas, "Natural frequency analysis and optimal design of CNT/fiber/polymer hybrid composites plates using Mori-Tanaka approach, GDQ technique and Firefly algorithm", Polymer Composites, 2016, Available online.

[28] S. Kamarian, M. Bodaghi, A. Pourasghar, S. Talebi, "Vibrational behavior of non-uniform piezoelectric sandwich beams made of CNT-reinforced polymer nanocomposite by considering the agglomeration effect of CNTs", Polymer composites, 2016, Available online.

[29] D. Garijo, "Free vibration analysis of non-uniform Euler-Bernoulli beams by means of Bernstein pseudospectral collocation", Engineering with Computers, 2015, 31(4): 813-823.

[30] S. Khani, N. Tabandeh, M.M. Ghomshei, "Natural frequency analysis of non-uniform smart beams with piezoelectric layers, using differential quadrature method", Composites Part B, 2014, 58: 303-311.

[31] M. Ahmadi, A. Nikkho, "Utilization of characteristic polynomials in vibration analysis of non-uniform beams under a moving mass excitation". Applied Mathematical Modelling, 2014, 38(7-8): 2130-2140.

[32] K. Torabi, H. Afshari, F. Haji Aboutalebi, "A DQEM for transverse vibration analysis of multiple cracked non-uniform Timoshenko beams with general boundary conditions", Computers & Mathematics with Applications, 2014, 67(3): 527-541.

[33] Z. Zhang, F. Chen, Z. Zhang, H. Hua, "Vibration analysis of non-uniform Timoshenko beams coupled with flexible attachments and multiple discontinuities", International Journal of Mechanical Sciences, 2014, 80: 131-143.

[34] A.A. Mahmoud, R. Awadalla, M.M. Nassar, "Free vibration of non-uniform column using DQM", Mechanics Research Communications, 2011, 38(6): 443-448.

[35] S.W. Tsai, C.V. Hoa, D. Gay, "Composite Materials, Design and Applications", CRC Press, 2003.

[36] D.L. Shi, X.Q. Feng, Y.Y. Huang, K.C. Hwang, H. Gao, "The effect of nanotube waviness and agglomeration on the elasticproperty of carbon nanotube reinforced composites", Journal of Engineering Materials and Technology, 2004, 126: 250-257.

[37] C. Shu, B.E. Richards, "Application of generalized differential quadrature to solve two-dimensional incompressible Navier-Stockes equations", International Journal for Numerical Methods in Fluids, 1992, 15: 791-798.

[38] Bert CW., Malik M., Differential quadrature method in computational mechanics, a review. Appl. Mech. Rev. 1996, 49:1-28.

[39] F. Tornabene, A. Liverani, G. Caligiana, "FGM and laminated doubly curved shells and panels of revolution with a free-form meridian: A 2-D GDQ solution for free vibrations", International Journal of Mechanical Sciences, 2011, 53(6): 446–470.

[40] F. Tornabene, N. Fantuzzi, F. Ubertini, E. Viola, "Strong Formulation Finite Element Method Based on Differential Quadrature: A Survey", Applied Mechanics Reviews, 2015, 67..

[41] F. Tornabene, N. Fantuzzi, M. Bacciocchi, "The strong formulation finite element method: stability and accuracy" Frattura ed Integrità Strutturale, 2014, 29: 251-265.

[42] G.M. Odegard, T.S. Gates, K.E. Wise, C. Park, E.J. Siochi, "Constitutive modeling of nanotube–reinforced polymer composites", Composites Science and Technology, 2003, 63: 1671-1687.

[43] L. Shen, J. Li, "Transversely isotropic elastic properties of single-walled carbon nanotubes". Physical Review B, 2004, 69: 045414.

[44] V. Tita, J. de Carvalho, J. Lirani, 'Theoretical and Experimental Dynamic Analysis of Fiber Reinforced Composite Beams', Journal of the Brazilian Society of Mechanical Sciences and Engineerin, 2003, 25(3): 306-310.