

Freezing of gauge symmetries in the heterotic string on T^4

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ABSTRACT: We derive a map relating the gauge symmetry groups of heterotic strings on T^4 to other components of the moduli space with rank reduction. This generalizes the results for T^2 and T^3 which mirror the singularity freezing mechanism of K3 surfaces in F and M-theory, respectively. The novel feature in six dimensions is that the map explicitly involves the topology of the gauge groups, in particular acting only on non-simply-connected ones. This relation is equivalent to that of connected components of the moduli space of flat G -bundles over T^2 with G non-simply-connected. These results are verified with a reasonably exhaustive list of gauge groups obtained with a moduli space exploration algorithm.

KEYWORDS: Gauge Symmetry, Superstring Vacua, Superstrings and Heterotic Strings

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1 Introduction

The moduli space of heterotic string compactifications with maximal supersymmetry consists of different connected components which are labeled by the number of compact dimensions $d \geq 1$ and the rank $r \leq 16 + d$ of the gauge group. In rare cases there are more than one component with the same values for d and r , e.g. for $d = 7$ and $r = 3$ there are two [1]. For each value of d there is one component with $r = 16 + d$, the so-called Narain moduli space, which is realized by compactifying on T^d [2, 3]. The components with reduced gauge group rank can be realized by asymmetric orbifolds [4] of the T^d compactification involving outer automorphisms of the Narain lattice (most notably the CHL string [5, 6]), and for larger d one generically finds more components with various values of r .

A natural question is what gauge groups can be realized in each component of this moduli space. It has been known for a long time that in the case of T^d compactifications the allowed gauge algebras correspond to root lattices which embed into the Narain lattice (see e.g. [7]). For each algebra, the global data that specifies the full gauge group (i.e. its fundamental group) can similarly be obtained from the Narain lattice [8, 9]. These statements generalize to the reduced rank components for which there are analogues of the Narain lattice [1, 10], but in both cases some subtleties have to be taken into account, ultimately due to the fact that these lattices are generically not self-dual [9, 11].

For $d = 1$ the Narain lattice $\Gamma_{1,17} \simeq \Gamma_{1,1} \oplus E_8 \oplus E_8$ is Lorentzian, and its automorphism group is Coxeter, allowing to obtain all the possible gauge algebras from a Generalized Dynkin Diagram (GDD) [12]. The situation for the CHL string is yet simpler, since the momentum lattice is $\Gamma_{1,9} \simeq \Gamma_{1,1} \oplus E_8$ [10], and one finds in particular that all gauge groups are simply-connected (so that the algebras completely specify them) [9]. For $d \geq 2$ however the momentum lattices are not Lorentzian and such overarching GDDs do not exist. Instead one finds that there are many different GDDs for each moduli space component, each giving partial information [8, 9]. Still, however, these diagrams do not yield the full list of gauge groups in a controlled way.

In order to obtain the full list of gauge algebras, an algorithm was developed in [8, 9, 13], which basically works by taking a point in the moduli space with maximally enhanced gauge symmetry, i.e. with no $U(1)$ factors, and looking for others in its vicinity. By iterating this algorithm one is able to obtain all such maximal enhancements in $d = 1, 2, 3$ for the Narain and CHL components [8, 9, 13]. In [13] we obtained the full gauge groups for all the moduli space components in $d = 3$, basing our methods on the results of [11] for those with rank reduction.

It turns out that for $d = 1, 2, 3$ all the gauge groups of the reduced rank components can be obtained from those of the Narain component by means of a suitable map. At the level of the algebras this has been known for a long time, for $d = 2$, in the dual frame of F-theory on K3, where reduced rank algebras are obtained by partially “freezing” the singular fibers [14, 15]. This situation extends to $d = 3$ using M-theory on K3 [1, 16, 17]. In the heterotic string these results can be reproduced by using lattice embedding techniques, and in fact one can also see how the full gauge group is “frozen”. For $d = 2$ this extension was done in [11], and we have found similar results for $d = 3$ in [13]. As explained in the text, for $d = 1$ it suffices to delete an E_8 factor in any gauge group containing it, and both groups related by the map are simply connected.

The purpose of this work is to extend these results to $d = 4$, focusing on the map from the Narain component to the rank reduced components of the moduli space. The main novelty in $d = 4$ is that in order to apply the map one must know the fundamental group of the gauge group explicitly. It cannot be naively applied at the level of the algebras as in $d = 1, 2, 3$. This is due to the fact that the lattice which corresponds to the frozen singularity in the heterotic frame is a root lattice for $d = 1, 2, 3$ but the weight lattice of a non-simply-connected group for $d = 4$. Most remarkably, however, is the fact that the maps seem to be exactly those which relate the “topologically nontrivial” components of the moduli space of flat connections of a non-simply-connected gauge group over T^2 (and

not T^4) to the “topologically trivial” component [18, 19], although to our awareness an explanation for this is lacking.

For this work we have carried out an exploration of the possible maximally enhanced gauge groups realized in six components of the moduli space of heterotic strings with 16 supercharges, which can be accessed at [23]. Although we cannot guarantee its exhaustivity, it is extensive enough to check the freezing map we have derived.

This paper is structured as follows. In section 2 we explain how the freezing map is constructed for generic d , reviewing the cases $d = 2, 3$ and deriving the case $d = 4$. We also give the map for $d = 1$. In section 3 we extend the freezing map to the 6d heterotic vacua obtained by compactifying the 7d holonomy triples of [1] to 6d. Our results are summarized and discussed in section 4.

2 Mapping gauge groups from Narain to CHL

In this section we explain the general method for determining the map which connects the Narain component with the CHL component and explicitly derive it for $d = 1, 2, 3, 4$. The case $d = 2$ was first obtained in [11] and the case $d = 3$ in [13]. Extensions to other rank reduced components are considered in section 3.

2.1 Setup and basic facts

In order to determine the map which applies to the gauge groups of the Narain component of the moduli space to give those of the rank reduced components we have to relate the way in which these are obtained in each case from the corresponding momentum lattices. We will illustrate this procedure using the CHL string, and so the focus is on the Narain lattice Γ_N and the Mikhailov lattice Γ_M , which can be written as

$$\begin{aligned}\Gamma_N &\simeq \Gamma_{d,d} \oplus E_8 \oplus E_8, \\ \Gamma_M &\simeq \Gamma_{d-1,d-1}(2) \oplus \Gamma_{1,1} \oplus E_8.\end{aligned}\tag{2.1}$$

Here $\Gamma_{d,d} \simeq \bigoplus_{i=1}^d \Gamma_{1,1}$ is the unique even self-dual lattice with signature $(+^d, -^d)$, where $\Gamma_{1,1}$ is the hyperbolic lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The symbol (2) denotes a rescaling of the lattice by $\sqrt{2}$, hence a rescaling of the Gram matrix by 2. The lattice E_8 is just the lattice generated by the roots of the algebra \mathfrak{e}_8 , but for the later, as well as for the groups, we will use the symbol E_8 when there is no risk of ambiguity. The same applies for any other root lattice of A-to-G type. We convene in taking the momentum lattices to have signature with mostly pluses, unless stated otherwise.

For the Narain component of the moduli space one obtains all the possible gauge algebras by finding embeddings of root lattices L into Γ_N such that the intersection of $L \otimes \mathbb{R}$ with Γ_N is an overlattice $M \leftrightarrow L$ whose maximal root sublattice is L itself. Here we mean by overlattice any lattice of the same rank containing the lattice in question. Intersections of real slices such as $L \otimes \mathbb{R}$ with Γ_N give lattices which are said to be primitively embedded, in this case in Γ_N , hence the embedding $M \hookrightarrow \Gamma_N$ is primitive but $L \hookrightarrow \Gamma_N$ is not unless $M = L$. By roots we mean vectors $v \in \Gamma_N$ with norm $v \cdot v = 2$, since these are the ones associated to root states in the adjoint representation of the gauge algebra.

This discussion extends to the CHL component of the moduli space, with the only difference being that roots are not only vectors with norm 2 but also vectors v with norm 4 satisfying the condition $v \cdot u = 0 \pmod 2$ for all vectors $u \in \Gamma_M$ [10]. This last condition is equivalent to the statement that the coroot $v^\vee = \frac{1}{2}v$ is in the dual lattice Γ_M^* , which is the language used in [11]. Note that $v^\vee \cdot v^\vee = 1$, hence this condition cannot be satisfied by any vector in the Narain lattice which is even and self-dual. The same applies to Γ_M when $d = 1$. For $d \geq 2$, however, Γ_M is not self dual and Γ_M^* indeed contains vectors with norm 1. The appearance of non-simply-laced algebras seems therefore to be intimately connected with the non-self-duality of the momentum lattice for the moduli space component in question.

These facts allow to obtain the possible gauge algebras \mathfrak{g} in these moduli space components, but we are also interested in the full gauge groups G . For this we need to compute the fundamental group $\pi_1(G)$, which we denote by H . If \tilde{G} is the universal cover of G , then $G = \tilde{G}/H$. In the Narain component it suffices to compute the lattice quotient M/L , which gives a finite Abelian group isomorphic to H due to the self-duality of Γ_N [9]. For example, if $M = L$, then G is simply connected. For the CHL component one must do a more precise analysis [11], but the upshot is that H is given by the quotient M^\vee/L^\vee , where L^\vee is the coroot lattice of \mathfrak{g} embedded in the dual lattice Γ_M^* , and M^\vee its overlattice which embeds primitively into Γ_M^* . Clearly, this is a generalization of the computation for Γ_N . In both cases H is a subgroup of the center $Z(G)$, specified by a set of elements $k_i \in Z(G)$.

2.2 Construction of the map in $d = 1, 2, 3$

To relate the Narain and the CHL constructions just outlined we require some additional facts. For $d = 1, 2, 3, 4$, Γ_M can be written respectively as [10]

$$\Gamma_M \simeq \Gamma_{d,d} \oplus \Lambda, \quad \Lambda = \begin{cases} \text{E}_8 & d = 1 \\ \text{D}_8 & d = 2 \\ \text{D}_4 \oplus \text{D}_4 & d = 3 \\ \text{D}_8^*(2) & d = 4 \end{cases}, \quad (2.2)$$

to which we restrict our attention in the following. In each case there is an embedding

$$\Gamma_M \oplus \Lambda \hookrightarrow \Gamma_N, \quad (2.3)$$

where $\Gamma_M \hookrightarrow \Gamma_N$ and $\Lambda \hookrightarrow \Gamma_N$ are primitive. Furthermore, the primitive embedding of Λ into Γ_N is unique (up to automorphisms of Γ_N), so that by constructing any such embedding one may take its orthogonal complement which by necessity is just Γ_M . As we will review, Λ can be interpreted as the K3 frozen singularity (or singularities) in the dual geometric frame both for 8d and 7d, and so we will refer to it as the *frozen sublattice* in the heterotic string context. We also use the terms mapping (from Narain to reduced rank) and freezing interchangeably.

Consider now a lattice¹ M' primitively embedded into Γ_M , with root sublattice L' . It follows from (2.3) that there is an embedding

$$M' \oplus \Lambda \hookrightarrow \Gamma_N \quad (2.4)$$

¹Here we prime the lattice M in the Mikhailov lattice since we will later focus on the map *to* and not *from* the CHL component.

with M' (but not necessarily $M' \oplus \Lambda$) primitively embedded into Γ_N . The intersection $(M' \oplus \Lambda) \otimes \mathbb{R} \cap \Gamma_N$ gives a lattice M primitively embedded into Γ_N , with root sublattice L . This gives a priori a map φ from a gauge algebra $\mathfrak{g}_{\text{CHL}}$ in CHL moduli space to another $\mathfrak{g}_{\text{Narain}}$ in Narain moduli space, but since we are dealing with the full embedding data for each lattice, we can also obtain the fundamental group of the gauge group and promote this map to one at the level of groups,

$$\varphi : G_{\text{CHL}} \mapsto G_{\text{Narain}} . \tag{2.5}$$

Consider conversely a lattice M primitively embedded into Γ_N , with root sublattice L , such that Λ is in turn primitively embedded into M (note that primitivity in this case is guaranteed by the fact that $\Lambda \hookrightarrow \Gamma_N$ is primitive). It follows that M has a sublattice of the form $M' \oplus \Lambda$, where both M' and Λ are primitively embedded into M . Since the orthogonal complement of Λ in Γ_N is just Γ_M , it follows that M' is primitively embedded into Γ_M , and defines a gauge group G_{CHL} . This gives a map

$$\varphi^{-1} : G_{\text{Narain}} \mapsto G_{\text{CHL}} . \tag{2.6}$$

We note however that the embedding $\Lambda \hookrightarrow M$ is not necessarily unique so that this map is generically one-to-many. As we will see, the form of this map has markedly different qualitative features depending on the value of d . In the following we study explicitly the cases $d = 1, 2, 3, 4$.

2.2.1 $d = 1$

For $d = 1$, we have that $\Gamma_M \simeq \Gamma_{1,1} \oplus E_8$ and $\Lambda = E_8$ are even self-dual. Therefore, eq. (2.3) can be replaced by a stronger statement (cf. eq. (2.1)),

$$\Gamma_N \simeq \Gamma_M \oplus E_8 , \quad d = 1 . \tag{2.7}$$

In this case, the lattice M' that we consider is a root lattice L' , since in nine dimensions all gauge groups are simply-connected [9]. Therefore we have an embedding $L' \oplus E_8 \hookrightarrow \Gamma_N$. This embedding is primitive, since $L' \hookrightarrow \Gamma_M$ is primitive and E_8 is unimodular, so there does not exist an even overlattice of $L' \oplus E_8$ in Γ_N . Moreover, $L' \oplus E_8$ is a root lattice corresponding to a simply connected gauge group in Narain moduli space. We see therefore that to every gauge group G_{CHL} in the CHL component we can associate another group G_{Narain} in the Narain component by some map

$$\varphi : G_{\text{CHL}} \mapsto G_{\text{Narain}} = G_{\text{CHL}} \times E_8 , \quad d = 1 . \tag{2.8}$$

Conversely, consider some root lattice of the form $L' \oplus E_8$ in Γ_N . Similarly to the CHL component, all of the associated groups are simply connected. Since the primitive embedding of E_8 into Γ_N is unique, it follows that L' is primitively embedded into $E_8^\perp \simeq \Gamma_M$. This means that any gauge group of the form $G \times E_8$ in the Narain component necessarily has $G = G_{\text{CHL}}$ some group in the CHL component. At the end of the day, the result is that by taking all gauge groups in the Narain component which contain an E_8 factor and

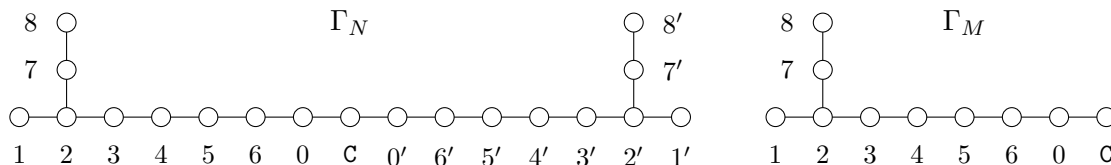


Figure 1. Generalized Dynkin diagrams for the Narain lattice $\Gamma_N \simeq \Gamma_{1,17}$ and the Mikhailov lattice $\Gamma_M \simeq \Gamma_{9,1}$ in nine dimension.

deleting it one obtains all of the gauge groups in the CHL component. If there are two E_8 factors, they are equivalent by an automorphism of Γ_N , so that there is no ambiguity in deleting one or the other.

This same result can be obtained in a more concrete way by considering the GDDs for the lattices Γ_N and Γ_M , shown in figure 1. Gauge algebras in the Narain moduli space can be obtained by deleting two or more nodes of the diagram such that the result is the Dynkin diagram for an ADE root lattice. The same applies to the CHL component, but the minimum number of nodes one can delete is one. As we can see, deleting the node $0'$ in the GDD for Γ_N gives the GDD for Γ_M accompanied by an E_8 Dynkin diagram, from which it follows that the gauge algebras that can be obtained in each moduli space component are related as deduced above. As commented, all of the relevant groups are simply connected.

2.2.2 $d = 2$

The map from the Narain to the CHL components of the moduli space for $d = 2$ was obtained at the level of the full gauge groups in [11] using more group-theoretical language, and proven explicitly by projecting the cocharacter lattice, which determines the topology, from Γ_N to Γ_M . Here we briefly explain how it can be obtained in the framework of this paper.

In eight dimensions we have $\Gamma_M \simeq \Gamma_{2,2} \oplus D_8$ and $\Lambda = D_8$. We will consider a lattice M primitively embedded into Γ_N , which is the overlattice of a root lattice L , containing in turn a primitively embedded D_8 lattice. This condition restricts L to be of the form

$$L \simeq D_{8+n} \oplus N, \tag{2.9}$$

where n is some non-negative integer and N is some other ADE lattice. The orthogonal complement of D_8 in L is of the form $D_n \oplus N$, and has an overlattice M' primitively embedded into Γ_M .

The question is if $D_n \oplus N$ is the maximal root sublattice of M' , according to the definition of roots in the CHL moduli space. This can indeed be verified for all points of symmetry enhancement. The subtlety here is that as lattices, D_n and C_n are equivalent. The actual contribution to the gauge algebra depends on which vectors correspond to massless states, and we find that in this case it is actually \mathfrak{sp}_n and not \mathfrak{so}_{2n} . We therefore write $L = C_n \oplus N$. We have then a simple rule for mapping gauge algebras from the Narain component to the CHL component of the moduli space. Just take any gauge algebra with a D_{8+n} factor and replace it with C_n . Since it is possible to have gauge algebras with terms $D_{8+n} \oplus D_{8+m}$, with $m \neq n$, this map is generically one-to-many.

To promote this map to one at the level of groups we compute the fundamental group of the gauge group associated to the embeddings $L \hookrightarrow \Gamma_N$ and $L \hookrightarrow \Gamma_M$ using the lattice methods outlined above, and then see how they are related. Each fundamental group is specified by a set of elements $\{k_i\}$ of the center of the universal cover, $Z(\tilde{G})$, each one of the form $k_i = (k_i^1, \dots, k_i^s)$, where k_i^j denotes the contribution of each of the s simple factors in \tilde{G} to k_i . For D_{2n} factors we write the corresponding contribution as a pair $(p, q) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we can separate each k_i into the contribution of the D_{8+n} factor to be replaced and that of the remaining factor given by the lattice N . Let us write $k_i = (k_i^N, k_i^S)$ for the gauge group in the Narain component and $k'_i = (k_i'^N, k_i'^S)$ for the associated one in the CHL component. We find that for n even,

$$k_i^S = (p, q) \rightarrow k_i'^S = p + q \pmod{2}, \tag{2.10}$$

while for n odd

$$k_i^S = p \rightarrow k_i'^S = p \pmod{2}, \tag{2.11}$$

while $k_i'^N = k_i^N$ in both cases. If $n = 0$, one just deletes k_i^S .

As an example, consider the gauge group $\frac{\text{Spin}(32)}{\mathbb{Z}_2} \times \text{SU}(2)^2$, whose fundamental group is generated by only one element $k = ((1, 0), 0, 0)$. Using the rules above, the associated gauge group in the CHL component is $\frac{\text{Sp}(8)}{\mathbb{Z}_2} \times \text{SU}(2)^2$ with $k = (1, 0, 0)$. If we had the gauge group $\text{SO}(32) \times \text{SU}(2)^2$ with $k = ((1, 1), 0, 0)$, it would map to the simply-connected $\text{Sp}(8) \times \text{SU}(2)^2$. However, $\text{SO}(2n)$ factors are not present in the theory so that this last example does not arise. It's interesting to note that $\text{SO}(2n)$ would map to the same gauge group as $\text{Spin}(2n)$, making the mapping generically many-to-many and not one-to-many.

Note also that the fundamental group of any two groups connected by this mapping are isomorphic. This is in accordance with the fact that the topology of the gauge groups in the dual frame of F-theory on elliptically fibered K3 is given by the torsional part of the Mordell-Weil group [20, 21] which can be obtained from the Weierstrass model for the fibration (see e.g. [22]), as the mechanism of singularity freezing does not alter the Weierstrass model itself [15].

2.2.3 $d = 3$

Let us now review the generalization of the above results to $d = 3$ [13]. In seven dimensions we have $\Lambda = D_4 \oplus D_4$. Each D_4 factor can be contained in algebras of D_{4+n} type, in which case the analysis for $d = 2$ goes through, including the way in which the contribution of these factors to the fundamental group transform. The difference now is that we have two such factors transforming simultaneously, e.g. $D_{n+4} \oplus D_{m+4} \rightarrow C_n \oplus C_m$. This is not the only possibility, however.

It is also possible for D_4 to be primitively embedded into E_6 , E_7 and E_8 . Taking the orthogonal complement of D_4 in each case we obtain the lattices $A_2(2)$, $3A_1 \simeq B_3$ and $D_4 \simeq F_4$, respectively. Similarly to the previous case, we can look at the points of symmetry enhancement in the CHL component and determine that the contributions to the algebra are respectively \mathfrak{su}_3 , \mathfrak{so}_7 and \mathfrak{f}_4 , hence the use of these lattice isomorphisms. With respect to the gauge group's topology, we have that $Z(\text{SU}(3)) \simeq Z(E_6) \simeq \mathbb{Z}_3$, $Z(\text{Spin}(7) \simeq E_7) \simeq \mathbb{Z}_2$ and $Z(F_4) \simeq E_8 \simeq \{0\}$, and that the contributions of these factors to the $\{k_i\}$ remain

invariant. This means that as for $d = 2$, the fundamental group of two gauge groups related by the mapping are isomorphic. As in the previous case, this coincides at the algebra level with results on the dual geometrical frame's mechanism of singularity freezing [1, 16, 17], in this case M-theory on K3 with two D_4 frozen singularities. We are not aware of how the fundamental group of the gauge group is encoded in the M-theory compactification, but it should in any case be invariant under singularity freezing.

2.3 Algebra projection

In the previous constructions we have seen that the root system of the CHL gauge algebra corresponds to a subset of the orthogonal complement lattice of Λ in the root lattice L' . This algebra is determined precisely by checking each case algorithmically and the result is seen to correspond to a simple general rule. Now we give a procedure whose result predicts this algebra directly, mapping the simple roots of $\mathfrak{g}_{\text{Narain}}$ to those of $\mathfrak{g}_{\text{CHL}}$. This procedure gives the correct results for $d = 1, 2, 3, 4$. We will illustrate it case by case starting with $d = 2$, which exhibits the non-trivial features that generalize to larger d .

2.3.1 $d = 2$

We start by considering a primitive embedding of $\Lambda = D_8$ into $\Gamma_N \simeq \Gamma_{2,2} \oplus \Gamma_{16}$, where Γ_{16} is the weight lattice of $\frac{\text{Spin}(32)}{\mathbb{Z}_2}$. This description makes calculations easier because D_8 embeds primitively into Γ_{16} but not into $E_8 \oplus E_8$. A particularly simple embedding is

$$\begin{aligned} \alpha_i &= |0, 0, 0, 0; 0^{i-1}, 1, -1, 0^{15-i}\rangle, \quad i = 1, \dots, 7, \\ \alpha_8 &= |0, 0, 0, 0; -1, -1, 0^{14}\rangle, \end{aligned} \tag{2.12}$$

where the first four entries correspond to the $\Gamma_{2,2}$ part and the other 16 to Γ_{16} . Suppose the associated gauge algebra is enhanced to D_{8+n} by adding n simple β_1, \dots, β_n roots forming an A_n chain, with $\beta_1 \cdot \alpha_7 = -1$. For example, take

$$\beta_i = |0, 0, 0, 0; 0^{i+6}, 1, -1, 0^{8-i}\rangle, \quad i = 1, \dots, n \leq 8. \tag{2.13}$$

We will take the projection of the roots β_i along the space orthogonal to D_8 . The roots β_2, \dots, β_8 are obviously invariant under this projection, but β_1 gets projected as

$$\beta_1 \rightarrow |0, 0, 0, 0; 0^8, -1, 0^7\rangle. \tag{2.14}$$

However, this projection is not in Γ_N , and so we multiply it by 2 to get a simple root $\beta'_1 = |0, 0, 0, 0; 0^8, -2, 0^7\rangle$. We see then that the simple roots of the A_n chain get projected into the simple roots of C_n . This construction is represented in figure 2, and applies to any other primitive embedding of D_8 since it is unique up to automorphisms of Γ_N .

2.3.2 $d = 3$

For $d = 3$ we have $\Lambda = D_4 \oplus D_4$, which has an easily describable primitive embedding into $E_8 \oplus E_8$, so we use the basis $\Gamma_N \simeq \Gamma_{3,3} \oplus E_8 \oplus E_8$. This embedding reads

$$\begin{aligned} \alpha_1 &= |0, 0, 0, 0; 1, -1, 0^6; 0^8\rangle, & \alpha_2 &= |0, 0, 0, 0; 0, 1, -1, 0^5; 0^8\rangle, \\ \alpha_3 &= |0, 0, 0, 0; 0^2, 1, -1, 0^4; 0^8\rangle, & \alpha_4 &= |0, 0, 0, 0; -1, -1, 0^6; 0^8\rangle, \\ \alpha'_1 &= |0, 0, 0, 0; 0^8; 1, -1, 0^6\rangle, & \alpha'_2 &= |0, 0, 0, 0; 0^8; 0, 1, -1, 0^5\rangle, \\ \alpha'_3 &= |0, 0, 0, 0; 0^8; 0^2, 1, -1, 0^4\rangle, & \alpha'_4 &= |0, 0, 0, 0; 0^8; -1, -1, 0^6\rangle. \end{aligned} \tag{2.15}$$

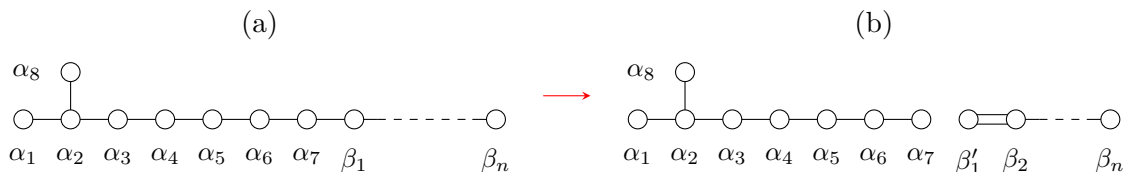


Figure 2. (a) Primitive embedding of D_8 in $\Gamma_{2,18}$ with simple roots α_i extended to D_{8+n} by β_j (see eqs. (2.12) and (2.13)). (b) Projection of the roots β_j to the orthogonal complement of D_8 gives a C_n lattice and associated \mathfrak{sp}_n algebra in the CHL component.

As in the previous case, we can extend each D_4 to D_{4+n} with an A_n chain, which gets projected to the orthogonal complement of Λ as a C_n . However, D_4 can also be extended to E_8 passing through D_5 , E_6 and E_7 . This D_5 coincides with that of the generic extension D_{4+n} with $n = 1$, and so it gives rise to an $A_1(2)$ lattice with simple root, say,

$$\beta'_1 = |0, 0, 0, 0; 0, 0, 0, 0, -2, 0^3; 0^8\rangle, \quad (2.16)$$

which arises from projecting $|0, 0, 0, 0; 0, 0, 0, 1, -1, 0^3; 0^8\rangle$. Extending D_5 to E_6 can be done by adding the root $|0, 0, 0, 0; \frac{1}{2}^8, 0^8\rangle$. Its projection multiplied by 2 is

$$\beta'_2 = |0, 0, 0, 0; 0^4, 1^4; 0^8\rangle, \quad (2.17)$$

and so we see that β'_1 and β'_2 give rise to an $A_2(2)$ lattice, as expected. We can further add the roots $|0, 0, 0, 0; 0^4, 1, -1, 0, 0\rangle$ and $|0, 0, 0, 0; 0^5, 1, -1, 0\rangle$, extending E_6 to E_7 and then E_8 . Since these roots are orthogonal to Λ , they are invariant under the projection and we see that they extend $A_2(2)$ to B_3 and then F_4 as predicted.

2.3.3 $d = 4$

Here we have $\Lambda = D_8^*(2)$. This lattice has a root sublattice $L_\Lambda = 8A_1$ and can be in fact interpreted as the weight lattice of $\frac{SU(2)^8}{\mathbb{Z}_2}$ with \mathbb{Z}_2 diagonal, i.e. $k = (1, \dots, 1)$. A suitable primitive embedding of this lattice into $\Gamma_N \simeq \Gamma_{4,4} \oplus E_8 \oplus E_8$ has simple roots

$$\begin{aligned} \alpha_1 &= |0, 0, 0, 0; 1, -1, 0^6; 0^8\rangle, & \alpha_2 &= |0, 0, 0, 0; 0, 0, 1, -1, 0^4; 0^8\rangle, \\ \alpha_3 &= |0, 0, 0, 0; 0^4, 1, -1, 0^2; 0^8\rangle, & \alpha_4 &= |0, 0, 0, 0; 0^6, 1, -1; 0^8\rangle, \\ \alpha_5 &= |0, 0, 0, 0; 0^8; 1, -1, 0^6\rangle, & \alpha_6 &= |0, 0, 0, 0; 0^8; 0, 0, 1, -1, 0^4\rangle, \\ \alpha_7 &= |0, 0, 0, 0; 0^8; 0^4, 1, -1, 0^2\rangle, & \alpha_8 &= |0, 0, 0, 0; 0^8; 0^6, 1, -1\rangle. \end{aligned} \quad (2.18)$$

The weight vector extending this root lattice to Λ is just

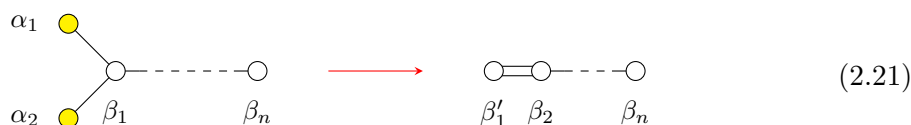
$$w = \frac{1}{2} \sum_{i=1}^8 \alpha_i = |0, 0, 0, 0; 1, 0, 1, 0, 1, 0, 1, 0; 1, 0, 1, 0, 1, 0, 1, 0\rangle. \quad (2.19)$$

Requiring orthogonality with the roots is enough to get orthogonality with Λ , so we will not worry about w . However we note that there exists also a primitive embedding of $8A_1$ into Γ_N , which should not be confused with Λ .

The first thing to note is that the lattice $\Lambda_L = 8A_1$ can be naively extended in many different ways but not all of them are allowed extensions of Λ itself. For example, no A_1 factor can be individually extended to A_2 with a root orthogonal to the other A_1 factors. Any attempt to do this is easily seen to fail. The next logical step is to attach a root to two A_1 factors at the same time, e.g. with $|0, 0, 0, 0; 0, 1, -1, 0^5; 0^8\rangle$, in this case giving an A_3 . This vector gets projected to

$$\beta = |0, 0, 0, 0; 1, 1, -1, -1, 0^4; 0^8\rangle, \tag{2.20}$$

and so we have that A_3 freezes to² $A_1(2)$. This is equivalent to $D_3 \rightarrow C_1$, and forms part of the more general rule $D_{2+n} \rightarrow C_n$, or $\mathfrak{so}_{2n+4} \rightarrow \mathfrak{sp}_n$, in analogy with those we have for $d = 2, 3$. This is depicted as



The next possibility is to attach $n - 1$ roots to n A_1 factors in pairs such that one gets an A_{2n-1} chain. The case $A_3 \rightarrow A_1(2)$ above can be generalized e.g. to $A_5 \rightarrow A_2(2)$ with roots

$$\beta_1 = |0, 0, 0, 0; 1, 1, -1, -1, 0^4; 0^8\rangle, \quad \beta_2 = |0, 0, 0, 0; 0^2, 1, 1, -1, -1, 0^2; 0^8\rangle, \tag{2.22}$$

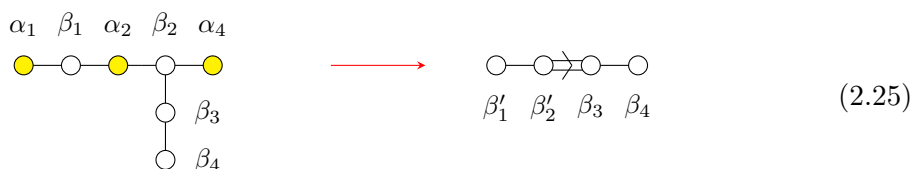
and more generally we find the rule $A_{2n-1} \rightarrow A_{n-1}(2)$, or $\mathfrak{su}_{2n} \rightarrow \mathfrak{su}_n$, depicted as



From this rule we can actually get another by simply attaching a root β_n to β_{n-1} , namely $D_{2n} \rightarrow B_n$, or $\mathfrak{so}_{4n} \rightarrow \mathfrak{so}_{2n+1}$,



Finally, we can take the particular case $n = 4$ and attach a root to β_3 to get the rule $E_7 \rightarrow F_4$, or $\mathfrak{e}_7 \rightarrow \mathfrak{f}_4$,



²It is more precise to say that $A_3 \oplus 6A_1$ freezes to $A_1(2)$, but we are now focusing on the behaviour under projection of sublattices corresponding to simple algebras and not the whole lattice containing Λ .

In summary we have found the following freezing rules at the level of the algebras:

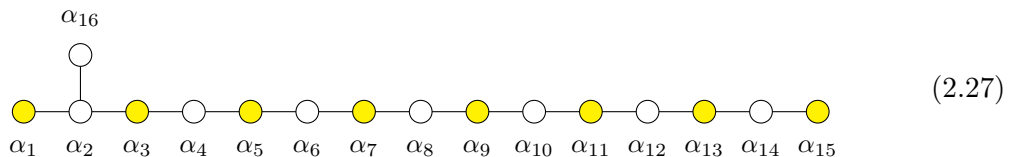
$$\begin{aligned}
 \mathfrak{so}_{2n+4} &\rightarrow \mathfrak{sp}_n \\
 \mathfrak{su}_{2n} &\rightarrow \mathfrak{su}_n \\
 \mathfrak{so}_{4n} &\rightarrow \mathfrak{so}_{2n+1} \\
 \mathfrak{e}_7 &\rightarrow \mathfrak{f}_4
 \end{aligned}
 \tag{2.26}$$

where both the l.h.s. and r.h.s. algebras are at level 1 (the algebras unaffected by the freezing become level 2). These rules cannot be applied arbitrarily, however. In order for the l.h.s. algebras to be reduced to those in the r.h.s., their roots must be connected with those of Λ as specified in each case above. Any root of Λ left by itself is simply projected out, $\mathfrak{su}_2 \rightarrow \emptyset$.

2.4 Applying the map in $d = 4$

Having seen the possible ways in which subalgebras of a gauge algebra in the Narain component in six dimensions can be transformed when mapping to the CHL component, we now treat the problem of when these rules are applicable for a given gauge group G . In the cases $d = 1, 2, 3$ this problem is trivial because the root lattices associated to Λ are uniquely embedded, so one always knows for any gauge group if its weight lattice contains Λ by a simple reading of the algebra. For $d = 4$, however, the relevant root lattice is $8A_1$, which may or may not be associated to Λ . It is necessary therefore to check explicitly, for each $8A_1$ sublattice, if it corresponds to Λ or not.

As a simple example let us consider the gauge group $\frac{\text{Spin}(32)}{\mathbb{Z}_2}$, ignoring the extra four $U(1)$ factors for now. It turns out that the weight lattice of this group contains Λ as a sublattice, whose $8A_1$ sublattice correspond to the yellow nodes in the diagram



This can be shown explicitly by deleting the white nodes and checking that the weight vector of the $\frac{\text{SU}(2)^8}{\mathbb{Z}_2}$ weight lattice is in the Narain lattice (cf. eq. (2.19)). At the level of the algebras, then, we have that \mathfrak{so}_{32} goes to \mathfrak{so}_{17} . This is to be contrasted with the gauge group $\text{Spin}(32)$, which is simply connected and therefore does not contain Λ in its weight lattice (which is a root lattice in this case). From this we learn that the topology of the group dictates what are the allowed freezings. Furthermore, we can explicitly compute the fundamental group of the gauge groups using the methods of [13], which extend to $d = 4$, and find that $\frac{\text{Spin}(32)}{\mathbb{Z}_2}$ gets mapped to $\text{Spin}(17)$. In other words, the element $k' = (1, 0)$ which generates $\pi_1(\frac{\text{Spin}(32)}{\mathbb{Z}_2})$ gets mapped to $k = 0$ in $\pi_1(\text{Spin}(17))$.

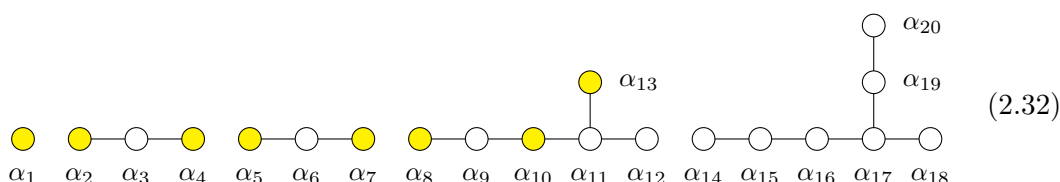
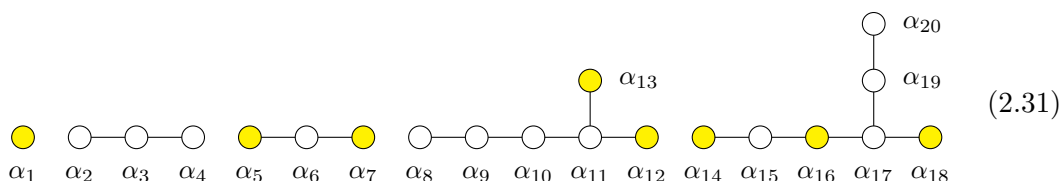
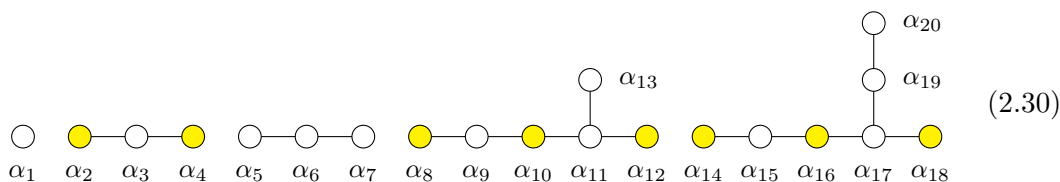
In general, the gauge group to be mapped has more than one nontrivial element in its fundamental group, which makes things more complicated. Consider for example

$$G = \frac{\text{SU}(2) \times \text{SU}(4) \times \text{SU}(4) \times \text{Spin}(12) \times \text{E}_7}{\mathbb{Z}_2 \times \mathbb{Z}_2},
 \tag{2.28}$$

where the $\mathbb{Z}_2 \times \mathbb{Z}_2$ consists of

$$k_1 = (0, 2, 0, (1, 0), 1), \quad k_2 = (1, 0, 2, (1, 1), 1), \quad k_3 = (1, 2, 2, (0, 1), 0). \quad (2.29)$$

Any pair of these elements, which generate $\pi_1(G)$, corresponds to two vectors which extend the root lattice L of G to its weight lattice M . They are inequivalent under translations in L . One can then delete nodes in the Dynkin diagram of L such that the reduced root lattice still has a nontrivial weight overlattice which might correspond to Λ , at which point any other reduction will not contain weight vectors. In this special case, all such reductions lead to inequivalent embeddings of Λ in M , represented by the yellow nodes in the diagrams



To each of these embeddings corresponds a different way of mapping G to the CHL component. Using the rules in (2.26) and computing the fundamental group in each case we get, respectively,

$$G \rightarrow \frac{\text{SU}(2) \times \text{SU}(2) \times \text{SU}(4) \times \text{Spin}(7) \times \text{F}_4}{\mathbb{Z}_2}, \quad k = (1, 0, 2, 1, 0), \quad (2.33)$$

$$G \rightarrow \frac{\text{SU}(4) \times \text{SU}(2) \times \text{Sp}(4) \times \text{F}_4}{\mathbb{Z}_2}, \quad k = (2, 0, 1, 0), \quad (2.34)$$

$$G \rightarrow \frac{\text{SU}(2) \times \text{SU}(2) \times \text{Spin}(7) \times \text{E}_7}{\mathbb{Z}_2}, \quad k = (0, 0, 1, 1). \quad (2.35)$$

The first thing to note is that in the resulting gauge group the fundamental group always reduces by a factor of \mathbb{Z}_2 (as already happened in the $\frac{\text{Spin}(32)}{\mathbb{Z}_2} \rightarrow \text{Spin}(17)$ case above). This can be understood by noting that one is taking the orthogonal complement of Λ , which contains weight vectors. These are also weight vectors in M , equivalent under translations in $8A_1$, so they can be related to one of the elements in $\pi_1(G)$. For any such weight vector w , we have that $2w \in 8A_1$ and so the associated $k \in \pi_1(G)$ generates a \mathbb{Z}_2 . This is precisely the factor which is eliminated in mapping G , corresponding respectively to k_1 , k_2 and k_3 above.

Now we need to know how the remaining k 's get transformed in each case. What we find is that it suffices to mod every k by the one that is eliminated, call it k_Λ ,

$$k \rightarrow k \pmod{k_\Lambda} \tag{2.36}$$

and then project it into the center of the resulting gauge group. In the case of a $\text{Spin}(4n)$ factor, we project the modded k contribution to $1 \in \pi_1(\text{Spin}(2n+1)) = \mathbb{Z}_2$ if it is not $(0,0)$. Of course, we also have that $k_\Lambda \rightarrow 0$ so that this rule applies equally well to all the k 's of $\pi_1(G)$.

We see then that the only information we require to know how to map a group G to the CHL component is the embedding of the roots of Λ into the root lattice L of G and its associated $k_\Lambda \in \pi_1(G)$. In fact, however, these two pieces of data are the same. One can take any $k \in \pi_1(G)$ of order 2 in $Z(G)$ and check if it corresponds to Λ in the following way. For each simple factor in G , if the corresponding entry in k is nonzero, its Dynkin diagram should be labeled according to one of the diagrams of section 2.3.3. The only simple factor which contains more than one order 2 element is D_{2n} , in which case $k_{D_{2n}} = (1, 1)$ corresponds to the diagram (2.21) and $k_{D_{2n}} = (0, 1), (1, 0)$ correspond to (2.24). Coloring the nodes appropriately lead to those shown in the example above, as one can easily check. If there are in total eight yellow nodes, this labeling will correspond to an embedding of Λ into M . With this we can apply the mapping rules to the algebra and to the fundamental group of G .

We verified all of these statements by applying the procedures outlined above to a reasonably exhaustive list of gauge groups in the Narain component, and checking the results against a list for the CHL string. In the next section we look at other rank reduced components, where the results are similarly verified against lists of symmetry enhancements. These lists can be obtained in the same way as those of the 7d case in [13], and can be accessed at [23]. We provide various examples in appendix A.

3 Other rank reduced components

In this section we extend the freezing procedure explained above to other rank reduced components in the moduli space of heterotic strings which appear in seven dimensions and below. These correspond to the holonomy triples constructed in [1] and their torus compactifications. We will focus our attention on the six dimensional case. Before this, however, let us review that of seven dimensions [13].

3.1 Review of the map in 7d

In seven dimensions there are six connected components in the moduli space of supersymmetric heterotic strings, including the Narain and the CHL component. They can be obtained as asymmetric orbifolds of the T^3 compactifications at points in the moduli space where the Narain lattice exhibits appropriate symmetries. These orbifolds are of order 2 to 6, and they correspond to non-trivial holonomy triples in the target space, hence they are called \mathbb{Z}_n -triples with $n = 2, \dots, 6$. The \mathbb{Z}_2 -triple is equivalent to the CHL string treated in section 2. Let us then treat the cases $n = 3, 4, 5, 6$.

n	Momentum lattice Γ	Frozen sublattice Λ
1	$\text{II}_{3,3} \oplus \text{E}_8 \oplus \text{E}_8$	\emptyset
2	$\text{II}_{3,3} \oplus \text{F}_4 \oplus \text{F}_4$	$\text{D}_4 \oplus \text{D}_4$
3	$\text{II}_{3,3} \oplus \text{G}_2 \oplus \text{G}_2$	$\text{E}_6 \oplus \text{E}_6$
4	$\text{II}_{3,3} \oplus \text{A}_1 \oplus \text{A}_1$	$\text{E}_7 \oplus \text{E}_7$
5	$\text{II}_{3,3}$	$\text{E}_8 \oplus \text{E}_8$
6	$\text{II}_{3,3}$	$\text{E}_8 \oplus \text{E}_8$

Table 1. Momentum lattices for the moduli spaces of heterotic \mathbb{Z}_n -triples and their orthogonal complements Λ in $\Gamma_{3,19}$.

For each \mathbb{Z}_n -triple, the momentum lattice can be obtained as the orthogonal complement of some other lattice Λ . This data is shown in table 1. For the \mathbb{Z}_3 -triple, we have $\Lambda = \text{E}_6 \oplus \text{E}_6$, which can only be embedded into $\text{E}_p \oplus \text{E}_q$ with $p, q = 6, 7, 8$. For each E_p factor, we have the algebra mapping

$$\mathfrak{e}_6 \rightarrow \emptyset, \quad \mathfrak{e}_7 \rightarrow \mathfrak{su}_2, \quad \mathfrak{e}_8 \rightarrow \mathfrak{g}_2, \tag{3.1}$$

while the corresponding contribution to any element k of the fundamental group is preserved. As with the $n = 2$ component, we have that the gauge groups related by the mapping have isomorphic fundamental groups.

For the \mathbb{Z}_4 -triple, we have $\Lambda = \text{E}_7 \oplus \text{E}_7$, which embeds only into $\text{E}_p \oplus \text{E}_q$ with $p, q = 7, 8$. For each E_p factor we have the algebra mapping

$$\mathfrak{e}_7 \rightarrow \emptyset, \quad \mathfrak{e}_8 \rightarrow \mathfrak{su}_2. \tag{3.2}$$

Here however, one finds that the resulting group is not simply-connected. If we take, for example, $G = \text{E}_8 \times \text{E}_8$, it maps to $\frac{\text{SU}(2) \times \text{SU}(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2} \simeq \text{PSO}(4)$, which in particular can be broken down to $\text{SO}(3)$. This is the only instance in which the fundamental groups of gauge groups connected by the moduli space mapping are not isomorphic.

The \mathbb{Z}_5 and \mathbb{Z}_6 -triples both have $\Lambda = \text{E}_8 \oplus \text{E}_8$ and so the only mapping allowed is $\mathfrak{e}_8 \rightarrow \emptyset$. All the possible gauge groups involved in this mapping are simply connected so here again they have isomorphic fundamental groups, namely trivial ones.

3.2 Extension of the freezing map in 6d

Let us now consider the compactifications of the 7d \mathbb{Z}_n -triples to 6d with $n = 3, 4, 5, 6$. Not surprisingly, the mappings that we find here generalize naturally those of the $n = 2$ case.

3.2.1 6d \mathbb{Z}_3 -triple

For $n = 3$, the momentum lattice is

$$\Gamma_{3,3} \oplus \Gamma_{1,1}(3) \oplus \text{A}_2 \oplus \text{A}_2, \tag{3.3}$$

which can be shown to be the orthogonal complement of a lattice Λ in $\Gamma_{4,20}$ isomorphic to the weight lattice of $\frac{\text{SU}(3)^6}{\mathbb{Z}_3}$, with \mathbb{Z}_3 diagonal. There are two types of root lattices which

can be obtained by attaching nodes to the Dynkin diagram of this $SU(3)^6$. First, we have those of the type A_{3n-1} , obtained by adding roots between each pair of A_2 's consecutively. These map to A_n . For example, we have that $A_8 \rightarrow A_2$,

$$\begin{array}{cccccccc}
 \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \bullet \\
 \alpha_1 & \alpha_2 & \beta_1 & \alpha_3 & \alpha_4 & \beta_2 & \alpha_5 & \alpha_6
 \end{array}
 \longrightarrow
 \begin{array}{cc}
 \circ & \circ \\
 \beta'_1 & \beta'_2
 \end{array}
 \tag{3.4}$$

The other possibility is to map E_6 to G_2 ,

$$\begin{array}{ccccccc}
 & & \circ & & & & \\
 & & \beta_2 & & & & \\
 \bullet & \bullet & \circ & \bullet & \bullet & & \\
 \alpha_1 & \alpha_2 & \beta_1 & \alpha_3 & \alpha_4 & &
 \end{array}
 \longrightarrow
 \begin{array}{cc}
 \circ & \circ \\
 \beta'_1 & \beta'_2
 \end{array}
 \tag{3.5}$$

A gauge group G in the Narain component can be mapped to this moduli space if $\pi_1(G)$ contains an order 3 element k_Λ such that its entries label exactly 12 nodes in the associated Dynkin diagram, in a manner completely analogous to the case for the CHL string (see section 2.4). The procedure for mapping all the elements of $\pi_1(G)$ is the same. For example, the gauge group $\frac{E_6^3}{\mathbb{Z}_3}$ with π_1 generator $k = (1, 1, 1)$ maps to G_2^3 , and $\frac{SU(3)^2 \times SU(6)^2 \times Spin(10)}{\mathbb{Z}_6}$ with π_1 generator $k = (1, 1, 1, 1, 2)$ maps to $\frac{SU(2)^2 \times Spin(10)}{\mathbb{Z}_2}$ with $k = (1, 1, 2)$. Similarly to the CHL string, the unaltered simple factors correspond to level 3 algebras and the altered to level 1 ones, so that e.g. the later has algebra $(\mathfrak{su}_2 \oplus \mathfrak{su}_2)_1 \oplus (\mathfrak{spin}_{10})_3$.

3.2.2 6d \mathbb{Z}_4 -triple

For $n = 4$, the momentum lattice is $\Gamma_{3,3} \oplus \Gamma_{1,1}(4) \oplus A_1 \oplus A_1$, whose associated Λ is the weight lattice of $\frac{SU(2)^2 \times SU(4)^4}{\mathbb{Z}_4}$ with \mathbb{Z}_4 generated by $k = (1, 1, 1, 1, 1, 1)$. The roots of this lattice can be extended in particular to A_{4n-1} and D_{2n+3} , the later with $n = 1, 2$. The algebras are respectively mapped to \mathfrak{su}_n and \mathfrak{b}_n . At the level of groups, however, we have that $Spin(4n + 6)$ maps to $\frac{Spin(2n+1)}{\mathbb{Z}_2}$. This situation is analogous to the one in 7d where E_8 maps to $SO(3)$, and is required to reproduce the results from compactifying the 7d \mathbb{Z}_4 -triple to 6d. For example, we have

$$\begin{array}{ccccccccccc}
 \bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet \\
 \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \alpha_4 & \alpha_5 & \alpha_6 & \beta_2 & \alpha_7 & \alpha_8 & \alpha_9
 \end{array}
 \longrightarrow
 \begin{array}{cc}
 \circ & \circ \\
 \beta'_1 & \beta'_2
 \end{array}
 \tag{3.6}$$

$$\begin{array}{ccccccc}
 \alpha_1 & \bullet & & & & & \\
 & \diagdown & & & & & \\
 & \bullet & & & & & \\
 \alpha_2 & \bullet & & & & & \\
 & \diagup & & & & & \\
 & \bullet & \circ & \bullet & \circ & \bullet & \\
 & \alpha_3 & \beta_1 & \alpha_4 & \beta_2 & \alpha_5 &
 \end{array}
 \longrightarrow
 \begin{array}{cc}
 \circ & \circ \\
 \beta'_1 & \beta'_2
 \end{array}
 \tag{3.7}$$

In the later case we see that the two A_1 's of Λ are used up, so that one cannot extend to D_9 and beyond.

The element k_Λ associated to this mapping is of order 4. In particular this means that $2k_\Lambda$ is an order 2 element, which turns out to be associated to the freezing to the CHL component of the moduli space. This is reflected in the fact that the frozen sublattice of this moduli space component contains the one for the CHL component. Indeed, the $2A_1$ part of L_Λ can be extended to D_n and frozen to C_{n-2} , as for the CHL freezing rule. This will be the case if k_Λ has an order 2 contribution to a D_n factor.

Algebra	Order of k_Λ	Transforms to	Contribution to Λ
A_{qn-1}	$q = 2, 3, 4, 5, 6$	A_{n-1}	nA_{q-1}
D_{n+2}	2	C_n	$2A_1$
D_{2n}	2	B_n	nA_1
E_7	2	F_4	$3A_1$
E_6	3	G_2	$2A_2$
D_5	4	B_1	$A_1 + A_3$
D_7	4	B_2	$2A_1 + A_3$

Table 2. Freezing rules for the simple factors in the gauge groups according to the order of the corresponding entry in the element k_Λ of the fundamental group associated to the freezing. In the first case, the resulting A_{n-1} has roots of length $2q$.

3.2.3 6d $\mathbb{Z}_{5,6}$ -triples

For $n = 5$ the momentum lattice is $\Gamma_{3,3} \oplus \Gamma_{1,1}(5)$, whose associated Λ is the weight lattice of $SU(5)^4/\mathbb{Z}_5$ with \mathbb{Z}_5 generated by $k = (1, 1, 1, 1)$. The only extension allowed here is A_{5n-1} , which maps to A_{n-1} , generalizing the similar freezings in the previous cases.

For $n = 6$, we have momentum lattice $\Gamma_{3,3} \oplus \Gamma_{1,1}(6)$, whose Λ is the weight lattice of $\frac{SU(2)^2 \times SU(3)^2 \times SU(6)^2}{\mathbb{Z}_6}$ with \mathbb{Z}_6 generated by $(1, 1, 1, 1, 1, 1)$. Again, the only allowed freezing here will be from A_{6n-1} to A_{n-1} , associated to an order 6 element in $\pi_1(G)$. However, this Λ includes the frozen sublattices of $n = 2$ and $n = 3$. Similarly to the $n = 4$ case including $n = 2$ freezing rules, here we also have the $n = 2$ and $n = 3$ rules which can be realized by two A_1 factors and two A_2 factors, respectively.

3.3 Relation with G -bundles over T^2

So far we have shown that depending on the topology of a gauge group G in the Narain component of the 6d moduli space one can map it to another gauge group G' in a different component using a simple set of rules. Associated to every freezing there is an element of the fundamental group $k_\Lambda \in \pi_1(G)$, and depending on the order of its entries with respect to the center of each simple factor, the associated algebra will transform in a specific way. This rules are summarized in table 2, where we've also indicated the contribution of the freezing rule to the overall root sublattice L_Λ of Λ .

These transformations also appear in a seemingly unrelated problem, namely in the relation between the moduli space components of flat bundles over T^2 with non-simply-connected structure group G [19] when G is simply-laced. The transformed group is simply-connected and describes the so-called topologically non-trivial components of the moduli space for a certain G . In this sense, what we find in the moduli space of 6d heterotic strings is a generalization to semisimple lie groups with many factors and more complicated fundamental groups.³ There is one important difference, however. In the case of $D_{5,7}$, the result is not simply-connected but rather has a \mathbb{Z}_2 quotient, although there is agreement at the level of algebras since $B_{1,2} \simeq C_{1,2}$. The reason we write B_n instead of C_n is that for

³We are not aware of a treatment of this general problem in the literature.

n	Momentum Lattice Γ	Frozen root lattice L_Λ	$\pi_1(G_\Lambda)$
1	$\Gamma_{4,20}$	\emptyset	
2	$\Gamma_{3,3} \oplus \Gamma_{1,1}(2) \oplus D_4 \oplus D_4$	$8A_1$	\mathbb{Z}_2
3	$\Gamma_{3,3} \oplus \Gamma_{1,1}(3) \oplus A_2 \oplus A_2$	$6A_2$	\mathbb{Z}_3
4	$\Gamma_{3,3} \oplus \Gamma_{1,1}(4) \oplus A_1 \oplus A_1$	$2A_1 \oplus 4A_3$	\mathbb{Z}_4
5	$\Gamma_{3,3} \oplus \Gamma_{1,1}(5)$	$4A_4$	\mathbb{Z}_5
6	$\Gamma_{3,3} \oplus \Gamma_{1,1}(6)$	$2A_1 \oplus 2A_2 \oplus 2A_5$	\mathbb{Z}_6

Table 3. Momentum lattices and corresponding orthogonal complements in $\Gamma_{4,20}$, given in terms of their root sublattices and fundamental group of the associated gauge group.

$n = 1$ the root is of length squared 2 in the momentum lattice, while its extension to B_2 is due to a root of length squared 4. This enhancement pattern corresponds to a B_n chain.

4 Summary of results and outlook

Let us summarize our results. The connected components of moduli space of the heterotic string studied in this paper have momentum lattices and corresponding orthogonal complements in $\Gamma_{4,20}$ (frozen sublattices) as shown in table 3. Here we have given Λ in terms of its root sublattice L_Λ and the fundamental group of the gauge group associated to Λ .⁴ The gauge symmetry groups that can be realized in the $n = 2, \dots, 6$ components can be obtained by applying a set of “freezing rules” to those of the $n = 1$ one. To check if one of these freezings can be done with a certain G , one looks for order n elements k_Λ in $\pi_1(G)$ such that they define an embedding of L_Λ into the root lattice L of G . If this is the case, one applies the rules shown in table 2 according to this embedding, and obtains the fundamental group of the resulting gauge group G' by modding the elements of $\pi_1(G)$ by k_Λ and projecting them onto the center of G' . Lists of gauge groups in these components can be found at [23], and we give some examples of these freezings in appendix A.

The moduli space components that we have studied are not all. In [1] it was shown that there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -quadruple in 6d, but, in any case, an exhaustive list of the components of the moduli space of heterotic strings in 6d with maximal supersymmetry is not known. However, the map we have obtained is defined in terms of the fundamental group elements of the gauge groups and seems to naturally extend to many other cases that may correspond to other moduli space components, some of which require an M-theory description. This extension is the subject of an upcoming work [24].

On the other hand, the relation between these freezing rules and the problem of non-simply-connected flat G -bundles over T^2 is not clear, as in the heterotic string we are considering bundles over T^4 . It may be better understood, perhaps, in a dual frame such as F-theory on $K3 \times T^2$ where one can more naturally isolate tori such as the fibers of the K3. As the former problem is rather high-level, it is tantalizing to think that it may play a role in constraining the possible theories with 16 supercharges that can be coupled to gravity (see e.g. [25] for recent results in this direction).

⁴The L_Λ 's correspond to the singularities of $K3 \times S^1$ orbifolds of order n in the dual M-theory [1].

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A Examples of gauge group freezings

Here we give some examples of freezings of gauge groups in 6d heterotic strings. For simplicity we use the A-to-G notation for gauge *groups*. Whenever the length of an A factor is not 2, superscript indicates half its length.

- a) $2A_1 + A_{11} + D_7$ with $H = \mathbb{Z}_4$ generated by $k = (0, 0, 3, 3)$, with center $(2, 2, 12, 4)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(0,0,9,1)	$8A_1/\mathbb{Z}_2$	$2A_1 + C_2 + A_2^4$	\mathbb{Z}_2	(0,0,1,0)	(2,2,2,3)
(0,0,6,2)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$2A_1 + C_5 + A_5^2$	\mathbb{Z}_2	(0,0,1,3)	(2,2,2,6)

- b) $A_3 + A_{11} + E_6$ with $H = \mathbb{Z}_6$ generated by $k = (2, 2, 2)$, with center $(4, 12, 3)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(2,6,0)	$8A_1/\mathbb{Z}_2$	$E_6 + A_1^2 + A_5^2$	\mathbb{Z}_3	(1,0,2)	(3,2,6)
(0,4,1)	$6A_2/\mathbb{Z}_3$	$A_3 + G_2 + A_3^3$	\mathbb{Z}_2	(2,0,2)	(4,1,4)
(2,2,2)	$(2A_1 + 2A_2 + 2A_5)/\mathbb{Z}_6$	$G_2 + A_1^2 + A_1^6$	1		(1,2,2)

- c) $3A_1 + 2A_4 + A_9$ with $H = \mathbb{Z}_{10}$ generated by $k = (1, 1, 1, 4, 4, 1)$, with center $(2, 2, 2, 5, 5, 10)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(1,1,1,0,0,5)	$8A_1/\mathbb{Z}_2$	$2A_4 + A_4^2$	\mathbb{Z}_5	(4,4,1)	(5,5,5)
(0,0,0,3,3,2)	$4A_4/\mathbb{Z}_5$	$3A_1 + A_1^5$	\mathbb{Z}_2	(1,1,1,1)	(2,2,2,2)

- d) $2A_2 + 2A_5 + E_6$ with $H = \mathbb{Z}_3^2$ generated by $k_1 = (0, 0, 2, 4, 1)$ and $k_2 = (1, 2, 0, 2, 1)$, with center $(3, 3, 6, 6, 3)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(0,0,2,4,1)	$6A_2/\mathbb{Z}_3$	$2A_2 + G_2 + 2A_1^3$	\mathbb{Z}_3	(1,1,0,0,0)	(3,3,1,2,2)
(2,1,2,2,0)	$6A_2/\mathbb{Z}_3$	$E_6 + 2A_1^3$	\mathbb{Z}_3	(1,0,0)	(3,2,2)
(1,2,0,2,1)	$6A_2/\mathbb{Z}_3$	$A_5 + G_2 + A_1^3$	\mathbb{Z}_3	(2,0,0)	(3,3,1,2,2)

- e) $3A_1 + D_{10} + E_7$ with $H = \mathbb{Z}_2^2$ generated by $k_1 = (0, 0, 0, 1, 0, 1)$ and $k_2 = (1, 1, 1, 0, 1, 0)$, with center $(2, 2, 2, (2, 2), 2)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(1,1,1,1,1,1)	$8A_1/\mathbb{Z}_2$	$C_8 + F_4$	\mathbb{Z}_2	(1,0)	(2,1)
(1,1,1,0,1,0)	$8A_1/\mathbb{Z}_2$	$B_5 + E_7$	\mathbb{Z}_2	(1,1)	(2,2)
(0,0,0,1,0,1)	$8A_1/\mathbb{Z}_2$	$3A_1 + B_5 + F_4$	\mathbb{Z}_2	(1,1,1,0)	(2,2,2,1)

- f) $2A_1 + 2A_2 + A_3 + A_{11}$ with $H = \mathbb{Z}_{12}$ generated by $k = (1, 1, 2, 2, 1, 1)$, with center $(2, 2, 3, 3, 4, 12)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(0,0,0,0,2,6)	$8A_1/\mathbb{Z}_2$	$2A_1 + 2A_2 + A_1^2 + A_5^2$	\mathbb{Z}_6	(1,1,2,2,1,1)	(2,2,3,3,2,6)
(0,0,2,2,0,4)	$6A_2/\mathbb{Z}_3$	$2A_1 + A_3 + A_3^3$	\mathbb{Z}_4	(1,1,1,1)	(2,2,4,4)
(1,1,0,0,3,3)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$2A_2 + A_2^4$	\mathbb{Z}_3	(2,2,1)	(3,3,3)
(0,0,1,1,2,2)	$(2A_1 + 2A_2 + 2A_5)/\mathbb{Z}_6$	$2A_1 + A_1^2 + A_1^6$	\mathbb{Z}_2	(1,1,1,1)	(2,2,2,2)

- g) $A_1 + 3A_5 + D_4$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_6$ generated by $k_1 = (0, 0, 3, 3, (1, 1))$ and $k_2 = (0, 1, 1, 2, (0, 1))$, with center $(2, 6, 6, 6, (2, 2))$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(0,0,3,3,(1,1))	$8A_1/\mathbb{Z}_2$	$A_1 + A_5 + C_2 + 2A_2^2$	\mathbb{Z}_6	(0,5,1,1,2)	(2,6,2,3,3)
(0,2,2,4,(0,0))	$6A_2/\mathbb{Z}_3$	$A_1 + D_4 + 3A_1^3$	\mathbb{Z}_2^2	(0, (1, 0), 0, 1, 1) (0, (0, 1), 1, 0, 1)	(2, (2, 2), 2, 2, 2)
(0,1,1,2,(0,1))	$(2A_1 + 2A_2 + 2A_5)/\mathbb{Z}_6$	$A_1 + C_2 + A_1^3$	\mathbb{Z}_2	(0,1,1)	(2,2,2)

- h) $4A_1 + A_2 + 2A_7$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_4$ generated by $k_1 = (0, 0, 1, 1, 0, 2, 2)$ and $k_2 = (1, 1, 1, 1, 0, 0, 4)$, with center $(2, 2, 2, 2, 3, 8, 8)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(1,1,1,1,0,0,4)	$8A_1/\mathbb{Z}_2$	$A_2 + A_7 + A_3^2$	\mathbb{Z}_4	(0,2,2)	(3,8,4)
(0,0,0,0,0,4,4)	$8A_1/\mathbb{Z}_2$	$4A_1 + A_2 + 2A_3^2$	\mathbb{Z}_2^2	(0, 1, 0, 1, 0, 2, 2) (1, 0, 1, 0, 0, 2, 2)	(2,2,2,2,3,4,4)
(0,0,1,1,0,2,2)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$2A_1 + A_2 + 2A_1^4$	\mathbb{Z}_2	(0,0,0,1,1)	(2,2,3,2,2)

- i) $4A_1 + 2A_3 + 2D_5$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_4$ generated by $k_1 = (0, 0, 0, 0, 1, 3, 1, 3)$ and $k_2 = (1, 1, 1, 1, 0, 0, 2, 2)$, with center $(2, 2, 2, 2, 4, 4, 4, 4)$ can be frozen to

k_Λ	Singularity	L	H	k	Center
(1,1,1,1,2,2,0,0)	$8A_1/\mathbb{Z}_2$	$2D_5 + 2A_1^2$	\mathbb{Z}_4	(1,3,1,1)	(4,4,2,2)
(1,1,1,1,0,0,2,2)	$8A_1/\mathbb{Z}_2$	$2A_3 + 2C_3$	\mathbb{Z}_4	(1,3,1,1)	(4,4,2,2)
(0,0,0,0,2,2,2,2)	$8A_1/\mathbb{Z}_2$	$4A_1 + 2C_3 + 2A_1^2$	\mathbb{Z}_2^2	(1, 1, 1, 1, 0, 0, 0, 0) (0, 0, 0, 0, 1, 1, 1, 1)	(2,2,2,2,2,2,2,2)
(0,0,0,0,1,3,1,3)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$6A_1$	\mathbb{Z}_2^3	(1, 1, 1, 1, 0, 0) (0, 0, 0, 0, 1, 0) (0, 0, 0, 0, 0, 1)	(2,2,2,2,2,2,2)

j) $5A_1 + D_4 + D_5 + D_6$ with $H = \mathbb{Z}_2^3$ generated by $k_1 = (0, 0, 0, 0, 1, (0, 1), 2, (0, 1))$, $k_2 = (0, 1, 1, 1, 0, (0, 0), 2, (1, 0))$ and $k_3 = (1, 0, 0, 1, 0, (1, 1), 2, (1, 1))$, with center $(2, 2, 2, 2, 2, (2, 2), 4, (2, 2))$ can be frozen to

k_Λ	Singularity	L	H	k	Center
$(0,0,0,0,1,(0,1),2,(0,1))$	$8A_1/\mathbb{Z}_2$	$4A_1 + B_3 + C_2 + C_3$	\mathbb{Z}_2^2	$(0, 1, 1, 1, 0, 1, 0)$ $(1, 0, 0, 1, 1, 1, 0)$	$(2,2,2,2,2,2)$
$(0,1,1,1,0,(0,0),2,(1,0))$	$8A_1/\mathbb{Z}_2$	$2A_1 + B_3 + C_3 + D_4$	\mathbb{Z}_2^2	$(0, 1, 1, 0, (0, 1))$ $(1, 0, 1, 0, (1, 1))$	$(2,2,2,2,(2,2))$
$(0,1,1,1,1,(0,1),0,(1,1))$	$8A_1/\mathbb{Z}_2$	$A_1 + C_2 + C_4 + D_5$	\mathbb{Z}_2^2	$(0, 0, 1, 2)$ $(1, 1, 0, 2)$	$(2,2,2,4)$
$(1,0,0,1,0,(1,1),2,(1,1))$	$8A_1/\mathbb{Z}_2$	$3A_1 + C_2 + C_3 + C_4$	\mathbb{Z}_2^2	$(0, 1, 1, 0, 0, 1)$ $(1, 0, 0, 1, 0, 1)$	$(2,2,2,2,2,2)$
$(1,0,0,1,1,(1,0),0,(1,0))$	$8A_1/\mathbb{Z}_2$	$2A_1 + B_3 + C_2 + D_5$	\mathbb{Z}_2^2	$(0, 0, 1, 1, 2)$ $(1, 1, 0, 0, 2)$	$(2,2,2,2,4)$
$(1,1,1,0,0,(1,1),0,(0,1))$	$8A_1/\mathbb{Z}_2$	$2A_1 + B_3 + C_2 + D_5$	\mathbb{Z}_2^2	$(1, 0, 1, 0, 2)$ $(0, 1, 0, 1, 2)$	$(2,2,2,2,4)$
$(1,1,1,0,1,(1,0),2,(0,0))$	$8A_1/\mathbb{Z}_2$	$A_1 + C_2 + C_3 + D_6$	\mathbb{Z}_2^2	$(0, 1, 0, (1, 0))$ $(1, 0, 0, (0, 1))$	$(2,2,2,(2,2))$

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