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Frequency conversion by exploiting time in transformation optics

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Abstract
The fundamental framework of transformation optics describes how coordinate transformations on electromagnetic fields can be physically realized through complex material parameters. This approach includes the possibility of transforming time as well as space, but the applications of such transformations remain relatively unexplored. Here we analyze the material properties and wave effects of a one-dimensional spatially varying time transformation. This transformation results in relatively simple electromagnetic material parameters defined by isotropic dielectric constants with a time-varying magnetoelectric coupling constant. We show that the resulting wave and field behavior in this medium is what is expected from the transformation, namely that an input frequency is scaled to a new and arbitrary output frequency defined by the overall magnitude of the time transformation. While the parameters required to realize such a medium are complex, we describe how the needed time-varying magnetoelectric coupling is feasible using an externally tunable electromagnetic metamaterials approach.

Keywords: transformation optics, time-varying materials, magnetoelectric coupling

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Transformation optics (Pendry *et al* 2006) has created a fundamentally new approach for the design of electromagnetic and optical devices. This coordinate transformation technique was previously shown to be applicable to electric current flow and impedance tomography (Greenleaf *et al* 2003) and has since been shown to be applicable to acoustics (Cummer and Schurig 2007, Chen and Chan 2007, Norris 2008), to quantum mechanical matter waves (Zhang *et al* 2008) and to surface water waves (Farhat *et al* 2008). To date, the majority of research on the applications of transformation optics has focused on the novel devices one can create using purely spatial transformations, such as scatter-reducing shells (Pendry *et al* 2006) and other beam-and field-manipulating devices (Rahm *et al* 2008, Jiang *et al* 2008, Kwon and Werner 2008). The coordinate transformation invariance of the Maxwell equations extends to time as well, however, and in principle combined space–time transformations can yield an even wider range of possible devices.

Following earlier work by Plebanski (1960), Leonhardt and Philbin (2006) described the mechanics of transformations involving time and showed that, in general, time transformations lead to magnetoelectric coupling terms and time variation in the constitutive parameters of the resulting electromagnetic materials. In special cases, time transformations do not lead to time-varying parameters or even magnetoelectric coupling, and examples such as the Aharonov–Bohm effect and electromagnetic analog black holes were analyzed by Leonhardt and Philbin (2006).

Possible applications of materials with time-varying parameters should not be immediately dismissed, however. With a metamaterials approach to engineered electromagnetic material design, significant complexity and outside control can be embedded inside a material. Examples of this that have been demonstrated experimentally include frequency-

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tunable metamaterials (Reynet and Acher 2004, Gil *et al* 2004, Chen *et al* 2006, Hand and Cummer 2007), nonlinear metamaterials (Shadrivov *et al* 2008, Wang *et al* 2008) and powered active metamaterials (Syms *et al* 2008, Yuan *et al* 2009). Creating time-varying parameters is an extension of this past work. In fact, both nonlinear and tunable metamaterials are essentially time-varying metamaterials, the former where the time variation is controlled by the fields themselves and the latter where the time variation is controlled externally.

Here we consider employing time transformations in transformation optics to create linear devices that alter the frequency of electromagnetic waves. We show that a relatively simple space-dependent time transformation results in a simple dielectric medium with time-varying magnetoelectric coupling and we demonstrate how this general transformation can be used to create a linear, time-varying medium that changes an input frequency to a desired output frequency. Provided that the material properties can be realized in broadband form, the same time compression or expansion applies to the envelope of a broadband pulse. The conditions and constraints required to realize such a medium in practice are also discussed.

2. Analysis of time transformations

Leonhardt and Philbin (2006) built on earlier work by Plebanski (1960) to describe the basic approach for deriving the material parameters for a general spacetime coordinate transformation. That work analyzed some transformations involving time but only examples that resulted in time-independent material parameters that do not influence frequency. Here our goal is specifically to manipulate frequency. We begin by considering a time stretch that varies linearly in space, namely,

$$ct = (ax' + b)ct', \qquad x = x', \qquad y = y', \qquad z = z', \tag{1}$$

where a and b are constants. Such a transformation could be used to connect two regions created by different uniform time stretches and forms the building block for the frequency-converting linear materials described below. The important elements of the Jacobian matrix, written in terms of the unprimed coordinates, are

$$\frac{\partial ct}{\partial ct'} = ax' + b = ax + b, \qquad \frac{\partial (ct)}{\partial x'} = act' = \frac{act}{ax+b}. \tag{2}$$

The transformed metric tensor is given by Leonhardt and Philbin (2006)

$$g^{\alpha\beta} = \sum_{\alpha'\beta'} \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} g^{\alpha'\beta'} \tag{3}$$

and with an original metric tensor of $g^{\alpha'\beta'} = \mathrm{diag}[-1, 1, 1, 1]$, we obtain $g^{\alpha\beta}$ that is unity on the diagonal and zero off the diagonal except for the terms

$$g^{tt} = -(ax+b)^2 + \frac{(act)^2}{(ax+b)^2} \tag{4}$$

$$g^{tx} = g^{xt} = \frac{act}{ax+b}. \tag{5}$$

Needed quantities that follow from this are

$$\sqrt{-g} = \frac{1}{\sqrt{-\det(g^{\alpha\beta})}} = \frac{1}{ax+b} \tag{6}$$

and

$$g_{\alpha\beta} = -\frac{1}{(ax+b)^2}\begin{bmatrix} 1 & -\frac{act}{ax+b} & 0 & 0 \\ -\frac{act}{ax+b} & \frac{(act)^2}{(ax+b)^2} - (ax+b)^2 & 0 & 0 \\ 0 & 0 & -(ax+b)^2 & 0 \\ 0 & 0 & 0 & -(ax+b)^2 \end{bmatrix}. \tag{7}$$

Continuing to follow Leonhardt and Philbin (2006), the constitutive relations of the transformed medium are

$$D = \epsilon_0 \bar{\bar{\epsilon}} E + \frac{\bar{w}}{c} \times H, \qquad B = \mu_0 \bar{\bar{\mu}} E - \frac{\bar{w}}{c} \times E \tag{8}$$

where $\bar{w}$ is a vector containing three dimensionless scalars that describe the magnetoelectric coupling in the medium. From Leonhardt and Philbin (2006) we have

$$\bar{\bar{\epsilon}} = \bar{\bar{\mu}} = -\frac{\sqrt{-g}}{g_{tt}} g^{ij}, \qquad w_i = \frac{g_{ti}}{g_{tt}}, \tag{9}$$

and for this time transformation, the medium parameters are thus

$$\bar{\bar{\epsilon}} = \bar{\bar{\mu}} = (ax+b)\bar{\bar{I}}, \qquad \bar{w} = \begin{bmatrix} -\frac{act}{ax+b} & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{10}$$

Despite the space-dependent time variation inherent in the transformation, the effective permittivity and permeability are isotropic, time-invariant scalars. The time dependence appears only in the magnetoelectric coupling term, which itself contains only one component that depends on the x position and on time. The resulting medium is thus a linear but time-variant material. The linear-in-time magnetoelectric coupling is linked to the effect of a uniformly accelerating dielectric, although care must be taken to evaluate the conditions under which magnetoelectric coupling is truly equivalent to a velocity (Thompson *et al* 2010).

From the transformation we can determine what this medium should do to an incident wave. Time and frequency are inversely related. According to the transformation, as a wave travels through this medium, the linear stretch in time creates a frequency that changes inverse-linearly with position. We thus expect a frequency that varies as

$$\omega(x) = \pm \frac{ck}{ax+b} \tag{11}$$

as it travels through the medium. We confirm this in appendix A by assuming a wave of the form $\exp[\mathrm{j}\omega(x)t - \mathrm{j}kx]$ propagates in the material, and we show this is the $\omega(x)$ that satisfies the Maxwell equations in the above medium. Note that, even though the wavenumber k is uniform throughout the medium, the spatial dependence of ω means that the local wavelength is not uniform, and is in fact time-varying. This is described in more detail below.

3. Application to frequency-converting linear materials

Time transformations can alter the frequency of electromagnetic fields. By stretching time ($ax + b > 1$ in (1)), frequency will be reduced because it will take longer for any event (such as a wave period) to occur. Compressing time ($ax + b < 1$ in (1)) will have the opposite effect. To create a material of thickness d that smoothly changes the frequency of a wave propagating in the $+x$ direction in free space from an initial frequency ω to a new frequency $m\omega$, the required time transformation is

$$t = t' + \frac{x'}{d}(m^{-1} - 1)t', \tag{12}$$

where the material slab resides between $x = 0$ and d. The frequency scaling parameter m can be any real value, not just an integer. At $x = 0$, $t = t'$, and at $x = d$, $t = m^{-1}t'$. Equating this transformation to the general one in (1), we find that the material parameters required to create this frequency conversion are

$$\epsilon = \mu = 1 + \frac{m^{-1} - 1}{d}x \tag{13}$$

and

$$w_1 = -\frac{(m^{-1} - 1)}{1 + (m^{-1} - 1)\frac{x}{d}} \frac{ct}{d} = w_1^0 \left(\frac{ct}{d}\right), \tag{14}$$

which apply for $x = [0, d]$ where the slab resides.

The required relative permittivity and permeability vary linearly with position and, as noted above, are otherwise simple with isotropic and constant values. The time dependence of the material, required for frequency conversion, resides entirely in the time-and position-dependent magnetoelectric coupling term. While the material properties depend on absolute time, the frequency-converting effect of the medium does not depend on absolute time. This indicates that the critical property is the time rate of change of the magnetoelectric coupling. In other words, we are free to arbitrarily choose the absolute time that a signal is incident on this medium.

Assume that, for $x > d$, we wish to keep the time transformation uniform in space and we thus employ a time transformation of

$$t = m^{-1}t'. \tag{15}$$

It is straightforward to show that this medium is a simple dielectric with

$$\epsilon = \mu = m^{-1}, \tag{16}$$

with no magnetoelectric coupling or time dependence. This makes sense physically, as the medium required to support the new frequency and original wavelength is a medium with phase velocity increased by m.

As is the case for all transformation optics designs, the resulting medium is theoretically frequency-independent. The slab described above, provided its properties are frequency-independent, will increase the frequency of any incident signal by a factor of m, independent of the frequency itself. This implies that the basic time compression or expansion performed by the medium will apply to the envelope of a modulated pulse as well, again provided that the material properties can be realized so that they are essentially constant over the bandwidth of the pulse. This approach thus provides a method for manipulating the temporal properties of broadband pulses as well as single frequencies.

The overall medium and its effects on a time harmonic signal are illustrated in figure 1. Suppose we wish to increase the frequency of a 1.5 μm wavelength (200 THz) signal by a factor of 2.6 (an arbitrary value) over a distance of 10 cm. The top panel shows the resulting material parameters (permittivity, permeability and magnetoelectric coupling constant w_1^0) as a function of position through the material sample from 0 to 10 cm, including the free-space region before the slab and the simple dielectric after the slab. The bottom panel shows the resulting theoretical spatial distribution of normalized frequency and wavelength throughout the domain. Inside the time-varying material, the signal frequency increases inverse-linearly and the signal exits this slab with a new frequency but the same wavelength it had when entering.

The wavelength in the time-varying material is itself time-varying. This is evident from the three spatial field distributions shown in figure 2, which also provide useful insight into the overall field behavior in this complex medium. The fields were computed from the analytical distribution $\exp[\mathrm{j}\omega(x)t - \mathrm{j}kx]$ using $\omega(x)$ as shown in figure 1 and for a constant k. For these figures we have assumed a signal frequency of 10 GHz so that the individual wavelengths can be conveniently visualized. In the uniform dielectrics at $x < 0$ and $x > 10$ cm, wavelength is uniform and, while not shown, frequency is 2.6 times greater for $x > 10$ cm than for $x < 0$.

At $t = -300$ ps (top panel), the wavelength in the time-varying medium is compressed significantly, and the magnitude of the magnetoelectric coupling constant is large. Note, however, that the fields are continuous everywhere, as required by the boundary conditions. At $t = 0$ (middle panel), the instantaneous spatial field distribution has a uniform wavelength, but the frequency is different at different positions. At $t = 100$ ps (bottom panel), the wavelength in the time-varying medium is longer and getting larger as time advances further. It is evident that the total phase change across the time-varying medium slab is changing significantly with time. This is exactly what must happen in order to maintain field continuity between two completely different frequencies present at the input and output planes of the slab.

At the output edge of this material slab at $x = d$, the frequency will have converted to its new value mf and we would likely wish to bring this signal back into the original dielectric (vacuum in the above assumption). For a normally incident signal, the wave impedance E/H maintains a constant value equal to that of the original dielectric throughout the entire slab. Consequently we could simply place a half-space of the original dielectric at $x = d$, and the signal at the new frequency will emerge into the original dielectric with no reflection anywhere in the system.

4. Comments on physical realizability

There are some obvious challenges in physically realizing the kind of material needed to implement time transformations

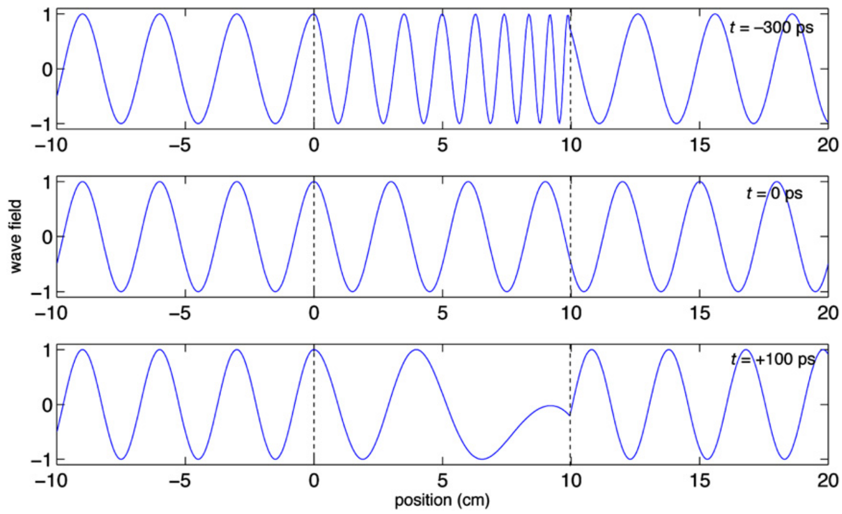

Figure 1. Electromagnetic material properties versus position that result from the transformation in (12).

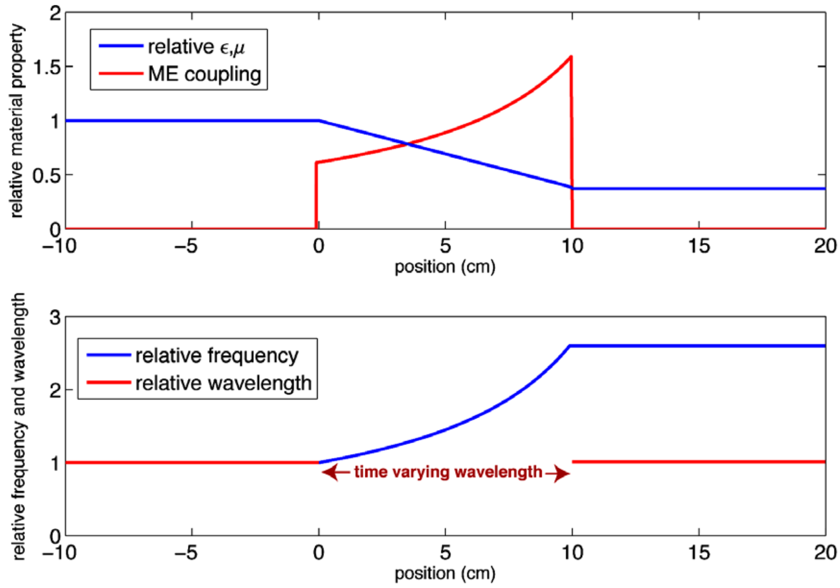

Figure 2. The spatial field distribution at three separate times that results from the time-varying material described by (13) and (14). The frequency that exits to the right is 2.6 times higher than that which enters to the left, and the time-varying total phase accumulation in the slab is required to maintain phase continuity with the input and output waves.

in transformation optics. Gradient material properties are relatively straightforward to realize with an electromagnetic metamaterials approach, either at microwave (Smith *et al* 2005) or optical (Gabrielli *et al* 2009, Valentine *et al* 2009) frequencies. The specific example considered here requires relative permittivity and permeability values less than unity, but these values are themselves relative to the background medium. If the input material is nonvacuum and $\epsilon > 1$, the permittivity and permeability in the time-varying slab will be larger. There are many more degrees of freedom in the transformation itself that could be exploited to control the resulting material parameters. One could, for instance, design a joint spacetime transformation that maintains $\epsilon = \mu = 1$ throughout the entire system in which the frequency conversion only required magnetoelectric coupling. Magnetoelectric coupling is a parameter of increasing interest in metamaterials designs, particularly in its importance in chiral materials (Lindell *et al* 1994). Inclusions can be designed to exhibit strong magnetoelectric coupling, for example omega particles (Saadoun and Engheta 1992) and helixes (Pendry 2004, Gansel *et al* 2009).

The more significant challenge is creating time-varying material properties. Voltage tunable metamaterials, which have been physically realized in many forms (Reynet and Acher 2004, Gil *et al* 2004, Chen *et al* 2006, Hand and Cummer 2007), are, in essence, time-varying materials. Such tunable metamaterials would simply need to be supplied with a tuning signal to change their parameters quickly enough for this application. How fast is fast enough?

In equation (14) the overall dimensionless magnetoelectric coupling constant w_1 has a leading term w_1^0 that is a dimensionless and order-unity constant. As noted above, for frequency conversion applications it is the time derivative of w_1 that is important, which is given by

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = w_1^0 \frac{c}{d}. \tag{17}$$

Therefore, to achieve the needed time derivative of w_1, the magnetoelectric coupling coefficient must vary of the order of unity in a time d/c, or approximately the travel time of the wave through the thickness of the medium. This requirement is consistent with physical expectations, as the frequency conversion must occur during the time window when a given packet of wave energy is inside the time-varying medium, and the medium parameters must change significantly in this window in order to significantly change the wave frequency.

This constraint creates a trade-off between the required rate of time variation of the medium and the length of the material slab performing the frequency conversion. For a slab 30 cm thick, the magnetoelectric coupling constant of the medium must change of the order of unity in $d/c \approx 1$ ns. This rate of parameter variation is feasible for voltage-tunable RF metamaterials; for example, nonlinear self-modulation experiments have shown that medium properties can be changed significantly on this timescale (Shadrivov *et al* 2008, Wang *et al* 2008). The faster modulation required to change frequency with thinner slabs will be more challenging to realize but is not infeasible.

Obtaining material parameters that vary monotonically with time for all time is also clearly impossible in practice. The duration over which one can monotonically change material properties with a sufficiently large time rate of change (as discussed above) limits the duration over which one can change the frequency of a signal. Physically larger material slabs will enable longer duration frequency shifts because of the smaller time rate of change of the material parameters required. Frequency conversion can also be done repeatedly with the application of a sawtooth tuning signal, i.e. one with a repeating slope of the desired duration and magnitude.

5. Conclusions

While time transformations are an acknowledged part of the transformation optics framework, the possible functionality and devices they enable remain relatively unexplored. In this work we have analyzed the material properties and wave effects of a one-dimensional spatially varying time transformation. This transformation results in relatively simple electromagnetic material parameters defined by an isotropic, time-invariant spatially varying dielectric with a time-varying magnetoelectric coupling constant. We show analytically that the resulting wave and field behavior in this medium is what is expected from the transformation, namely that an input frequency is scaled to a new and arbitrary output frequency defined by the overall magnitude of the time transformation. Provided that the material properties can be realized in broadband form, this approach provides a method for manipulating the temporal properties of broadband pulses as well as single frequencies. While the parameters required to realize such a medium are complex, we argue that the needed time-varying magnetoelectric coupling is feasible using an externally tunable electromagnetic metamaterials approach. Applying other, more complex time and joint spacetime transformations in the transformation optics framework will likely yield the electromagnetic material parameters for even more sophisticated devices.

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Appendix

Here we confirm that the electromagnetic material described in (13) and (14) exhibits the expected solution, i.e. a solution with spatially varying frequency. For fields that vary only in the x direction and polarized such that only E_y, D_y, B_z and H_z are nonzero, the Maxwell equations and constitutive relations reduce to

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \tag{18}$$

$$\frac{\partial H_z}{\partial x} = -\frac{\partial D_y}{\partial t} \tag{19}$$

$$D_y = (ax+b)\epsilon_0 E_y + \frac{at}{ax+b} H_z \tag{20}$$

$$Bz = (ax+b)\mu_0 H_z + \frac{at}{ax+b} E_y \tag{21}$$

into which the material properties of (13) and (14) have been inserted.

We now wish to determine whether these field equations admit a solution of the form $E, H \propto \exp[\mathrm{j}\omega(x)t - kx]$ in which the wave frequency varies with position and, if so, to confirm that $\omega(x)$ varies inverse-linearly as expected from the transformation theory. With this assumed field variation, time derivatives are simply multiplication by $\mathrm{j}\omega(x)$. Space derivatives are more complicated because of the x-dependent frequency and are a multiplication by $\mathrm{j}(t\frac{\mathrm{d}\omega}{\mathrm{d}x} - k) = \mathrm{j}\Delta_x$. After eliminating D_y and B_z the field equations are

$$\mathrm{j}\Delta_x E_y = \mathrm{j}\omega(ax+b)\mu_0 H_z - \mathrm{j}\omega\frac{at}{ax+b} E_y \tag{22}$$

$$\mathrm{j}\Delta_x H_z = -\mathrm{j}\omega(ax+b)\epsilon_0 E_y - \mathrm{j}\omega\frac{at}{ax+b} H_z. \tag{23}$$

After further combining we find the overall dispersion relation that defines $\omega(x)$ is given by

$$\left(\frac{\omega(ax+b)}{c}\right)^2 = \left(\Delta_x + \omega\frac{at}{ax+b}\right)^2. \tag{24}$$

Expanding Δ_x gives an ordinary differential equation that defines $\omega(x)$, namely,

$$t\frac{\mathrm{d}\omega}{\mathrm{d}x} + \left(\frac{at}{ax+b} \pm \frac{ax+b}{c}\right)\omega = k. \tag{25}$$

It is straightforward to verify that this equation has the solution

$$\omega(x) = \pm\frac{ck}{ax+b}. \quad (26)$$

Thus, the material properties derived from transformation optics create a wave solution in which the frequency varies with the inverse of the spatially varying time stretch. This is precisely the effect that should be created by the transformation.

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