Frequency Domain Analysis of a Dimensionless Cubic Nonlinear Damping System Subject to Harmonic Input

X. J. Jing and Z. Q. Lang

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield
S1 3JD, UK

Research Report No. 971
Feb 2008

Frequency Domain Analysis of a Dimensionless Cubic Nonlinear Damping System Subject to Harmonic Input

Xing Jian Jing* and Zi Qiang Lang
Department of Automatic Control and Systems Engineering, University of Sheffield
Mappin Street, Sheffield, S1 3JD, U.K.
X.J.Jing@sheffield.ac.uk

Abstract: The effects of cubic nonlinear damping on the system output spectrum are theoretically studied through a dimensionless mass-spring-damping system model subject to a harmonic input, based on the Volterra series approximation. It is for the first time shown theoretically that the cubic nonlinear damping has little effect on the system output spectrum at high or low frequencies but drives the system output spectrum to be an alternative series at the natural frequency 1 such that the system output spectrum can be suppressed by the cubic damping.
Keywords: Cubic nonlinear damping, Output frequency response, Alternative series

1 Introduction

Suppression of system output vibration covers a wide range of applications such as active control or isolation of the foundation vibration in many engineering systems [5]. Traditionally, an increase in damping can reduce the response at the resonance. However, this is often at the expense of degradation of isolation at high frequencies [4]. Some optimization methods have been proposed to deal with this problem, such as H-infinity control, “skyhook” damper, repetitive learning control, and optimization etc [4, 6, 9, 19]. On the other hand, it shall be noted that, exploitation of nonlinearities for improving the performance of vibration systems has also drawn the attention of researches [10-12, 14, 20]. Although nonlinearity complicates the analysis of system output response, it may provide superior performance for specific applications as demonstrated in Jing et al [14]. Frequency domain analysis for damping systems is also reported in some publications [19], which however focused mainly on linear damping. Recently, some progress has been achieved in the analysis of nonlinear systems in the frequency domain based on Volterra series approximation theory [1, 7, 8, 15-17]. For these reasons, frequency domain analysis of a cubic nonlinear damping is theoretically performed as a case study in this paper for a dimensionless mass-spring-damping system model under harmonic excitation. Some important properties of the nonlinear damping are revealed and demonstrated theoretically. These may provide a useful insight into the understanding of nonlinear damping and the design of nonlinear vibration systems to suppress output vibration.

2 A dimensionless vibration system and its frequency response functions

Consider a SDOF nonlinear mass-spring-damping system as show in Figure 1. The dynamics of the system can be described by

$$m\ddot{z}(t') + (c_1 + c_2\dot{z}(t')^2)\dot{z}(t') + k_c z(t') = u(t')$$

In this study, suppose the input u(t) to be a harmonic excitation given by

$$u(t') = A\sin(\Omega t')$$

which can be transformed into a dimensionless format as

$$\ddot{x}(t) + \xi_1\dot{x}(t) + \xi_2\dot{x}(t)^3 + x(t) = u(t) \qquad (1a)$$

$$u(t) = \frac{1}{k_c}\sin(\omega t) \qquad (1b)$$

where $x = \frac{z}{A}$, $\Omega_0 = \sqrt{\frac{k_c}{m}}$, $t = \Omega_0 t'$, $\omega = \frac{\Omega}{\Omega_0}$, $\xi_1 = \frac{c_1}{\sqrt{k_c m}}$, $\xi_2 = \frac{c_2}{\sqrt{k_c m}} F_d^2$. For convenience, let k_c=1 in what follows. Consider the output of interest for model (1ab) as

$$y(t) = \xi_1\dot{x}(t) + \xi_2\dot{x}^3(t) + x(t) \qquad (1c)$$

which is the transmitted force from u(t) to the base. The dimensionless model (1a) can be found in many engineering systems, usually acting or known as a vibration isolator [3, 12] as shown in Figure 1 with a nonlinear damping, or be found in circuit systems as shown in Figure 2 with a nonlinear resistor.

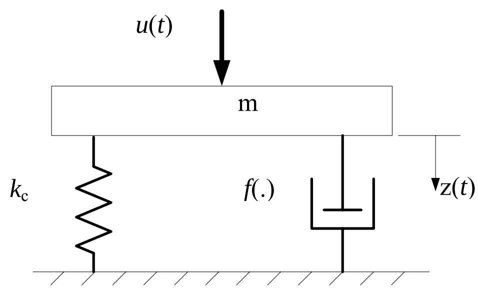

Fig. 1 A mass-spring-damping system with nonlinear damping

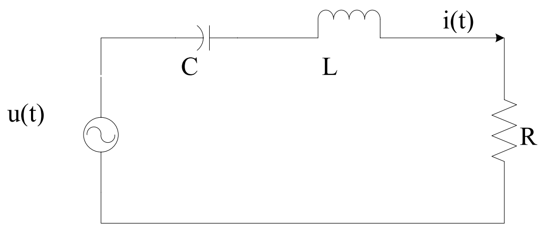

Figure 2. A similar circuit system with nonlinear resistor

Note that there is a cubic nonlinear damping terms in (1a). The objective of this study is to analyze the effect of the nonlinear damping on the system output response in the frequency domain, and therefore to demonstrate some notable advantages of the nonlinear damping compared with the linear damping known in the literature.

The frequency domain analysis of nonlinear systems can be carried out based on the Volterra series expansion theory. It is known that, nonlinear systems can be approximated by a Volterra series up to maximum order *N* in the neighbourhood of the zero equilibrium [2] as

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{2}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is called the *n*th-order Volterra kernel. The generalized frequency response function (GFRF) is defined as [13]

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{3}$$

which provides a basis for the frequency domain analysis of nonlinear systems. In order to conduct the frequency domain analysis of the cubic nonlinear damping, the GFRFs and output spectrum of system (1) can be obtained by using some recently developed results[15-17]. These are summarized in the following Lemmas.

Lemma 1. The GFRFs for the relationship between u(t) and y(t) of model (1) can be determined as: for n=0,1,2,3,…

$$H_{2n}^{y}(j\omega_1,\cdots,j\omega_{2n})=0 \tag{4}$$

and

$$\begin{aligned}&H_{2n+1}^{y}(j\omega_1,\cdots,j\omega_{2n+1})\\&=\left(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_{2n+1})\right)\cdot H_{2n+1}^{x}(j\omega_1,\cdots,j\omega_{2n+1})+\delta(\delta(n))\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})\end{aligned} \tag{5a}$$

(5a) can also be written as for n>0,

$$\begin{aligned}&H_{2n+1}^{y}(j\omega_1,\cdots,j\omega_{2n+1})\\&=-\xi_2\frac{(j\omega_1+\cdots+j\omega_{2n+1})^2}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})}\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=2n+1}}^{2n-1}\prod_{i=1}^{3}H_{r_i}^{x}(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})\end{aligned} \tag{5b}$$

where for *n*=0, 1,2,3,…

$$H_{2n}^{x}(j\omega_1,\cdots,j\omega_{2n})=0 \tag{6a}$$

$$H_{2n+1}^{x}(j\omega_1,\cdots,j\omega_{2n+1})=\frac{\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})} \tag{6b}$$

$$H_{2n+1,3}(\cdot)=\sum_{i=1}^{2n-1}H_i^{x}(j\omega_1,\cdots,j\omega_i)H_{2n+1-i,2}(j\omega_{i+1},\cdots,j\omega_{2n+1})(j\omega_1+\cdots+j\omega_i) \tag{6c}$$

$$H_{2n+1,1}(j\omega_1,\cdots,j\omega_{2n+1})=H_{2n+1}^{x}(j\omega_1,\cdots,j\omega_{2n+1})(j\omega_1+\cdots+j\omega_{2n+1})^{k_1} \tag{6d}$$

$$H_1^{x}(j\omega_1)=\frac{-1}{L_1(j\omega_1)} \tag{6e}$$

$$L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})=-\left(1+\xi_1(j\omega_1+\cdots+j\omega_{2n+1})+(j\omega_1+\cdots+j\omega_{2n+1})^2\right) \tag{6f}$$

Proof. See Appendix A. □

Therefore, based on (4-6), the GFRFs for the relationship between y(t) and u(t), can be determined up to any high orders. According to the results in [15, 17], and noting that there is only one nonlinear term with coefficient ξ_2, it can be obtained that the nth-order GFRF for the relationship between y(t) and u(t) can be expressed as for n=0,1,2,3,…

$$H_{2n+1}^{y}(j\omega_1,\cdots,j\omega_{2n+1})=\xi_2^{n}\cdot\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1}) \tag{7a}$$

$$H_{2n}^{y}(j\omega_1,\cdots,j\omega_{2n})=0 \tag{7b}$$

From the results in [16], it can be seen that $\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})$ is a function of $H_1^x(j\omega_i)$ with degree 2n+1 and also a function of $L_j(.)$ with degree n. Also from the recursive computation above, it can be verified that the complex valued function $\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})$ in (7a) can be written as

$$\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})$$
$$=-\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\frac{(j\omega_1+\cdots+j\omega_{2n+1})^2\cdot\prod_{i=1}^{2n+1}(j\omega_1)H_1^x(j\omega_i)\cdot\prod_{i=1}^{n-1}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})\cdot\prod_{i=1}^{n-1}L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})} \tag{8}$$

Assume that $\bar{f}_1(j\omega_1)=H_1^x(j\omega_1)$ for n=0 in (7a). This can also be proved rigorously by mathematical induction which is straightforward. Equation (8) can be determined explicitly from equations (4-6).

Lemma 2. The system output frequency response is

$$Y(j\omega)=\bar{Y}_0(j\omega)+\sum_{n=1}^{\lfloor N-1/2\rfloor}\bar{Y}_n(j\omega) \tag{9a}$$

$$\bar{Y}_0(j\omega)=-\frac{jF_d(1+j\xi_1\omega)H_1^x(j\omega)}{2} \tag{9b}$$

$$\bar{Y}_n(j\omega)=\xi_2^n\cdot\frac{j(F_d)^{2n+1}}{2^{2n+1}}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^{2n}\cdot\left(j\omega H_1^x(j\omega)\right)\cdot(j\omega)^2}{L_1(j\omega)}$$
$$\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})} \tag{9c}$$

for n>0 with assumption that $\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}=1$ when n=1.

Proof. See Appendix B. □

Equations (9a-c) are a qualitative description of the output spectrum of model (1), which facilitates the analysis in the following section. $\bar{Y}_0(j\omega)$ denotes the effect of the linear part of the system, and $\bar{Y}_n(j\omega)$ for n=1,2,3,… denotes the effect of the nonlinear damping. The detailed output spectrum can be determined by following the proof of Lemma 2 in Appendix B.

3 Analysis based on the frequency response functions

From Equations (9a-c), it can be seen that the nonlinear damping drives the system output spectrum to be an infinite series. When the nonlinear damping is zero, i.e., $\xi_2=0$, then the system is recovered to a linear system whose output spectrum is $\bar{Y}_0(j\omega)$. In order to demonstrate the effect of the nonlinear damping on the system output spectrum, the following two cases are discussed. Assume $0\le\xi_1$ and $0\le\xi_2$.

(1) Magnitude of $Y(j\omega)$ when $|\omega$ -1| is much larger than 0

It follows from (6f) that

$$\left|L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})\right|_{\omega_1+\cdots+\omega_{2n+1}=\Omega}=\left|1+\xi_1(j\Omega)+(j\Omega)^2\right|=\sqrt{(1-\Omega^2)^2+(\xi_1\Omega)^2}$$

then

$$\left|L_{2n+1}(j\Omega)\right|\approx\begin{cases}\sqrt{(\Omega^2)^2+(\xi_1\Omega)^2}\ge\Omega^2 & \text{for } \Omega>>1\\ \sqrt{1+(\xi_1\Omega)^2}\ge 1 & \text{for } \Omega<<1\end{cases}\tag{10a}$$

and it follows from (6e) that

$$\left|H_1^x(j\omega_1)\right|=\frac{1}{\left|L_1(j\omega_1)\right|}\le\begin{cases}\dfrac{1}{\omega_1^2} & \text{for } \omega_1>>1\\ 1 & \text{for } \omega_1<<1\end{cases}\tag{10b}$$

Consider two cases in the following: $\omega>>1$ and $\omega<<1$. For comparison, study the magnitude of the linear part in the output spectrum at first. It follows from (9b) that

$$\left|\overline{Y}_0(j\omega)\right|=\left|-\frac{jF_d(1+j\xi_1\omega)H_1^x(j\omega)}{2}\right|=\frac{\sqrt{1+(\xi_1\omega)^2}}{2\left|1+\xi_1(j\omega)+(j\omega)^2\right|}=\frac{\sqrt{1+(\xi_1\omega)^2}}{2\sqrt{(1-\omega^2)^2+(\xi_1\omega)^2}}\tag{11}$$

When ξ_2=0, the magnitude of the system output spectrum will be guided by (11). It can be seen that, an increase of the linear damping parameter ξ_1 will result in a decrease of the magnitude at $\omega=1$ but an increase of the magnitude at higher frequency, and $\left|\overline{Y}_0(j\omega)\right|\to 0.5$ when $\omega\to 0$. However, the nonlinear damping will not bring noticeable increase for the magnitude of the system output spectrum at higher or lower frequency than 1 as discussed below.

Case 1: high frequency response, *i.e.*, $\omega>>1$

Using (10) and (9c), it can be derived that for *n*>0 and $\omega>>1$

$$\left|\overline{Y}_n(j\omega)\right|\le\xi_2^n\cdot\frac{(F_d)^{2n+1}}{2^{2n+1}}\cdot\frac{\left|\omega H_1^x(j\omega)\right|^{2n+1}\cdot(\omega)^2}{\left|L_1(j\omega)\right|}\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\ \sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\ \prod_{i=1}^{n-1}\frac{\left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right|}{\left|L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})\right|}$$

$$\approx\xi_2^n\cdot\frac{(F_d)^{2n+1}}{2^{2n+1}}\cdot\frac{\left|\omega(\omega^{-2})\right|^{2n+1}\cdot(\omega)^2}{\omega^2}\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\ \sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\ \prod_{i=1}^{n-1}\frac{\left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right|}{(\omega_{l(1)}+\cdots+\omega_{l(j_i)})^2}$$

$$\approx\xi_2^n\cdot\frac{1}{2^{2n+1}}\cdot\frac{1}{\omega^{2n+1}}\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\ \sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\ \prod_{i=1}^{n-1}\frac{1}{\left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right|}$$

Fr

$$<\xi_2^n\cdot\frac{1}{2^{2n+1}}\cdot\frac{1}{\omega^{2n+1}}\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\ \sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\frac{1}{\omega^{n-1}}\tag{12a}$$

om (12a), it can be seen that, when $\omega>>1$ and for a bounded ξ_2, $\left|\overline{Y}_n(j\omega)\right|\to 0$, thus $\left|Y(j\omega)\right|\to\overline{Y}_0(j\omega)$. Therefore, it can be concluded that, after the nonlinear damping is introduced, the magnitude of system output spectrum may not be obviously increased at higher frequencies compared with the case without the nonlinear damping being introduced. Thus the nonlinear damping has little effect on the output spectrum of system (1) at high frequency. This may be held even for a larger but finite ξ_2.

Case 2: low frequency response, *i.e.*, $\omega << 1$

Similarly, using (10) and (9c), it can be derived that for *n*>0 and $\omega << 1$

$$\left|\bar{Y}_n(j\omega)\right| \le \xi_2^n \cdot \frac{(F_d)^{2n+1}}{2^{2n+1}} \cdot \frac{\left|\omega H_1^x(j\omega)\right|^{2n+1} \cdot (\omega)^2}{\left|L_1(j\omega)\right|} \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2k+1|1\le k\le n-1\}}} \prod_{i=1}^{n-1} \frac{\left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right|}{\left|L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})\right|}$$

$$\le \xi_2^n \cdot \frac{(F_d)^{2n+1}}{2^{2n+1}} \cdot \frac{\left|\omega\right|^{2n+1} \cdot (\omega)^2}{1} \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2k+1|1\le k\le n-1\}}} \prod_{i=1}^{n-1} \left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right|$$

$$= \xi_2^n \cdot \frac{1}{2^{2n+1}} \cdot \omega^{2n+3} \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2k+1|1\le k\le n-1\}}} \prod_{i=1}^{n-1} \left|\omega_{l(1)}+\cdots+\omega_{l(j_i)}\right| \qquad (12b)$$

From (12b), it can be seen that, when $\omega << 1$ and for a bounded ξ_2, $\left|\bar{Y}_n(j\omega)\right| \to 0$, thus $\left|Y(j\omega)\right| \to \bar{Y}_0(j\omega)$. Therefore, after the nonlinear damping is introduced, the magnitude of system output spectrum is not obviously increased at lower frequencies compared with the case without the nonlinear damping being introduced. Thus the nonlinear damping has also little effect on the output spectrum of system (1) at lower frequency.

The following discussion will show that a proper nonlinear damping will reduce the magnitude of system output spectrum at frequency $\omega \approx 1$.

(2) The frequency characteristic of $Y(j\omega)$ at frequency $\omega \approx 1$

For any $\upsilon \in \mathcal{C}$ (the set of all the complex numbers), define

$$signc(\upsilon) = \begin{bmatrix} sign(REAL(\upsilon)) & 0 \\ 0 & sign(IMAG(\upsilon)) \end{bmatrix} \qquad (13)$$

where $sign(x) = \begin{cases} +1 & x \ge 0 \\ -1 & x < 0 \end{cases}$ for $x \in \Re$, $REAL(\upsilon)$ is the real part of υ and $IMAG(\upsilon)$ is the imaginary part of υ. Obviously, $signc(.) \neq 0$, $signc(.)^{-1}$ exists and $signc(.)^{-1}=signc(.)$.

Definition 1. Considering a series *S* in $\mathcal{C}$, *i.e.*, $S=s_0+s_1+s_2+s_3+\ldots\ldots$, if the following conditions hold,
(1) $REAL(s_n)IMAG(s_n)REAL(s_{n+1})IMAG(s_{n+1}) \neq 0$, and additionally $signc(s_n)signc(s_{n+1})=-\mathbf{1}$ (2) $REAL(s_n)IMAG(s_n)REAL(s_{n+1})IMAG(s_{n+1}) = 0$, and additionally

$sign(REAL(s_n)REAL(s_{n+1})) = -1$ or $sign(IMAG(s_n)IMAG(s_{n+1})) = -1$

then it is said to be an alternating series, where $\mathbf{1}$ is two-dimensional unitary matrix.

The traditional definition for the alternating series can refer to [18]. The following result can be established.

Theorem 1. Consider the dimensionless system (1). At around natural frequency 1, the cubic nonlinear damping drives the system output spectrum to be an alternating series if the linear damping parameter ξ_1 is sufficiently small, i.e., $\xi_1 << 3$.

Proof. By using this function, it can be derived that

$$signc(\overline{Y}_n(j\omega)) = signc\left[\xi_2^n \cdot \frac{j(F_d)^{2n+1}}{2^{2n+1}} \cdot \frac{\left|(j\omega)H_1^x(j\omega)\right|^{2n} \cdot \left(j\omega H_1^x(j\omega)\right) \cdot (j\omega)^2}{L_1(j\omega)} \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}\right]$$

which further yields

$$\begin{aligned} & signc(\overline{Y}_{n+1}(j\omega))signc(\overline{Y}_n(j\omega))^{-1} = signc(\overline{Y}_{n+1}(j\omega))signc(\overline{Y}_n(j\omega)) \\ & = signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+3}}=\omega} \sum_{\substack{(j_1,\cdots,j_n) \\ j_i \in \{2n+3|n=0,2,\ldots\}}} \prod_{i=1}^{n} \frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}\right) \\ & \cdot signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}\right) \end{aligned} \tag{14}$$

Note that $\omega_{k_i} \in \{\omega, -\omega\}$, and the condition $\omega_{k_1}+\cdots+\omega_{k_{2n+1}} = \omega$, thus there are n+1 frequencies equal to ω and n frequencies equal to $-\omega$. $j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}$ is the summation of j_i frequencies chosen from the 2n+1 frequencies, where j_i is an odd number. Note also that $\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}(.)$ includes all the possible cases for different values of $\omega_{k_i} \in \{\omega, -\omega\}$ such that $\omega_{k_1}+\cdots+\omega_{k_{2n+1}} = \omega$. It can be verified that $j\omega_{l(1)}+\cdots+j\omega_{l(j_i)} = pj\omega$ for some odd integer p satisfying $|p|<n+2$. Thus

$$\begin{aligned} & signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}\right) \\ & = (-1)^{n-1} signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{1-(p_{ij_i}\omega)^2+\xi_1(jp_{ij_i}\omega)}\right) \\ & = (-1)^{n-1} signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)}\right) \end{aligned} \tag{15}$$

Consider $\frac{jp_{ij}\omega}{-L_1(jp_{ij}\omega)}$ at $\omega \approx 1$ where p_{ij} is an odd integer satisfying $|p_{ij}|<n+2$.

$$\frac{jp_{ij}\omega}{-L_1(jp_{ij}\omega)} = \frac{jp_{ij}}{1-p_{ij}^2+j\xi_1 p_{ij}} = \frac{|p_{ij}|e^{j\frac{\pi}{2}sign(p_{ij})}}{\sqrt{(1-p_{ij}^2)^2+(\xi_1 p_{ij})^2}\,e^{j\theta_{ij}}}$$

where $\theta_{ij} = \begin{cases} sign(p_{ij})\pi/2 & \text{if } |p_{ij}| = 1 \\ sign(p_{ij})(\pi - \delta_{ij}) & \text{if } |p_{ij}| \geq 3 \end{cases}$, $\delta_{ij} \geq 0$ and $\delta_{ij} \to 0$ when $|p_{ij}| >> 1$ and $\xi_1 << 1$. Thus

$$\frac{jp_{ij}\omega}{-L_1(jp_{ij}\omega)} = 1/\xi_1 \text{ when } |p_{ij}| = 1 \tag{16a}$$

and

$$\frac{jp_{ij}\omega}{-L_1(jp_{ij}\omega)} \approx \frac{e^{-j(\frac{\pi}{2}-\delta_{ij})sign(p_{ij})}}{\sqrt{p_{ij}^2 + \xi_1^2}} \leq \frac{e^{-j(\frac{\pi}{2}-\delta_{ij})sign(p_{ij})}}{|p_{ij}|} \text{ when } |p_{ij}| > 1 \tag{16b}$$

$p_{ij} = \pm 1$ corresponds to $j\omega_{l(1)} + \cdots + j\omega_{l(j)} = \pm j\omega$, and $\frac{\pm j\omega}{-L_1(\pm j\omega)}$ is a positive real number in this case. However, $|p_{ij}| \geq 3$ corresponds to $j\omega_{l(1)} + \cdots + j\omega_{l(j)} = jp_{ij}\omega$, and $\frac{jp_{ij}\omega}{-L_1(jp_{ij}\omega)}$ is approximately a pure negative or positive imaginary number. Note that there are more cases for $j\omega_{l(1)} + \cdots + j\omega_{l(j_i)} = \pm j\omega$ than those for $j\omega_{l(1)} + \cdots + j\omega_{l(j_i)} = jp_{ij_i}\omega$ ($|p_{ij_i}| > 1$). For example, let j_1=5, then there are C_5^3 cases for $j\omega_{l(1)} + \cdots + j\omega_{l(5)} = j\omega$, but only one case for $j\omega_{l(1)} + \cdots + j\omega_{l(5)} = j5\omega$ and C_5^1 cases for $j\omega_{l(1)} + \cdots + j\omega_{l(5)} = j3\omega$. Similarly, there are more cases for $|j\omega_{l(1)} + \cdots + j\omega_{l(j_i)}|$ to be a smaller number. And considering the multiplying terms $\prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)}$ in (15), it can be seen from (16) that $\prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)}$ is a negative real number only if there are only 2+4m (for m=0,1,2,…) terms with $p_{ij}>1$ in this multiplication. Obviously, there are more other cases which have positive real parts. Therefore, from the discussion above, there are more cases in which $-\pi \leq \arg(\prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)}) \leq 0$. Especially, it can be seen that for the value of $\prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)}$

(1) the largest magnitude value when the angular is $\pi/2$ or $-3\pi/2$ (+2kπ) is $\frac{1}{p_{ij}} \cdot \frac{1}{\xi_1^{n-2}}$; however, the largest magnitude value when the angular is -$\pi/2$ is also $\frac{1}{p_{ij}} \cdot \frac{1}{\xi_1^{n-2}}$;

(2) the largest magnitude value when the angular is $-\pi$ (+2k π) is $\frac{1}{p_{ij}^2} \cdot \frac{1}{\xi_1^{n-3}} \leq \frac{1}{3^2} \cdot \frac{1}{\xi_1^{n-3}}$, however the largest magnitude value when the angular is 0 is $\frac{1}{\xi_1^{n-1}}$.

Therefore, if ξ_1 is sufficiently small, i.e., $\xi_1 << 3$, then the following equality can hold

$$signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)} \right) = signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+3}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+3|n=1,2,\ldots\}}} \prod_{i=1}^{n} \frac{jp_{ij_i}\omega}{-L_1(jp_{ij_i}\omega)} \right)$$

which enables (15) to be

$$signc\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} \sum_{\substack{(j_1,\cdots,j_{n-1}) \\ j_i \in \{2n+1|n=1,2,\ldots\}}} \prod_{i=1}^{n-1} \frac{j\omega_{l(1)} + \cdots + j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)} + \cdots + j\omega_{l(j_i)})} \right) = (-1)^{n-1} \cdot const$$

Thus (14) gives for n>0, $signc(\overline{Y}_{n+1}(j\omega))signc(\overline{Y}_n(j\omega)) = -1$. Furthermore, if $\omega \approx 1$, it can be derived that

$$signc(\overline{Y}_0(j\omega)) = signc(-\frac{jF_d(1+j\xi_1\omega)H_1^x(j\omega)}{2}) = sign(-\frac{F_d}{2}(\frac{1}{\xi_1}+j)) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1$$

$$signc(\overline{Y}_1(j\omega)) = signc(\xi_2^n \cdot \frac{j(F_d)^{2n+1}}{2^{2n+1}} \cdot \frac{\left|(j\omega)H_1^x(j\omega)\right|^{2n} \cdot \left(j\omega H_1^x(j\omega)\right) \cdot (j\omega)^2}{L_1(j\omega)})$$

$$= signc(\xi_2^n \cdot \frac{(F_d)^3\left|(j\omega)H_1^x(j\omega)\right|^2}{2^3} \cdot \frac{\omega^3}{\xi_1^2}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

which yields $signc(\overline{Y}_1(j\omega))signc(\overline{Y}_0(j\omega)) = -1$. This shows that the output spectrum (9a), *i.e.*, $Y(j\omega) = \overline{Y}_0(j\omega) + \overline{Y}_1(j\omega) + \overline{Y}_2(j\omega) + \cdots$, is an alternating series at frequency $\omega \approx 1$ for a sufficiently small ξ_1. This completes the proof. □

Therefore, the nonlinear damping drives the system output spectrum to be an alternating series at the natural frequency. There are contradictions between different terms in the series. Thus this may result in suppression of the system output spectrum for a proper nonlinear damping parameter. The following result can be obtained.

Corollary 1. Consider the dimensionless system (1). At around natural frequency 1, there exist $\overline{\xi}_2$>0, if the linear damping is sufficiently small, *i.e.*, $\xi_1 << 3$, and $0 < \xi_2 < \overline{\xi}_2$ then the system output spectrum is convergent and the magnitude of the output spectrum can also be expressed into a convergent alternative series and can be suppressed for $0 < \xi_2 < \overline{\xi}_2$.

Proof. From Lemmas 1-2, the system output spectrum at around natural frequency 1 can be written as

$$Y(j\omega) = \overline{Y}_0(j\omega) + \xi_2\widetilde{Y}_1(j\omega) + \xi_2^2\widetilde{Y}_2(j\omega) + \cdots \quad (17)$$

From Theorem 1, it is an alternative series. From [2] and the continuity of $Y(j\omega)$ in ξ_2, there must exist $\overline{\overline{\xi}}_2$>0, such that it is a convergent series for $0 < \xi_2 < \overline{\overline{\xi}}_2$. The magnitude of $Y(j\omega)$ can be evaluated by

$$\begin{aligned} \left|Y(j\omega)\right|^2 &= Y(j\omega)Y(-j\omega) \\ &= (\overline{Y}_0(j\omega) + \xi_2\widetilde{Y}_1(j\omega) + \xi_2^2\widetilde{Y}_2(j\omega) + \cdots)(\overline{Y}_0(-j\omega) + \xi_2\widetilde{Y}_1(-j\omega) + \xi_2^2\widetilde{Y}_2(-j\omega) + \cdots) \quad (18) \\ &= \sum_{n=0,1,2,\ldots} \xi_2^n \sum_{i=0}^{n} \widetilde{Y}_i(j\omega)\widetilde{Y}_{n-i}(-j\omega) \end{aligned}$$

For n=2k, it can be verified that

$$\sum_{i=0}^{2k} \widetilde{Y}_i(j\omega)\widetilde{Y}_{2k-i}(-j\omega) = \widetilde{Y}_0(j\omega)\widetilde{Y}_{2k}(-j\omega) + \cdots + \widetilde{Y}_k(j\omega)\widetilde{Y}_k(-j\omega) + \cdots + \widetilde{Y}_{2k}(j\omega)\widetilde{Y}_0(-j\omega) > 0$$

For n=2k+1, it can be derived that

$$\begin{aligned} &\sum_{i=0}^{2k+1} \widetilde{Y}_i(j\omega)\widetilde{Y}_{2k+1-i}(-j\omega) = \widetilde{Y}_0(j\omega)\widetilde{Y}_{2k+1}(-j\omega) + \cdots + \widetilde{Y}_k(j\omega)\widetilde{Y}_{k+1}(-j\omega) + \cdots + \widetilde{Y}_{2k+1}(j\omega)\widetilde{Y}_0(-j\omega) \\ &= REAL(\widetilde{Y}_0(j\omega)\widetilde{Y}_{2k+1}(-j\omega) + \cdots + \widetilde{Y}_k(j\omega)\widetilde{Y}_{k+1}(-j\omega)) \end{aligned}$$

Note that (17) is an alternating series. That is, $REAL(\widetilde{Y}_i(j\omega)\widetilde{Y}_j(-j\omega)) \le 0$ for any i+j=odd integer. Therefore, for n=2k+1, $\sum_{i=0}^{2k+1} \widetilde{Y}_i(j\omega)\widetilde{Y}_{2k+1-i}(-j\omega) < 0$. This proves that (18) is an alternating series. Since (17) is convergent, (18) is also convergent. (18) further gives

$$\frac{\partial|Y(j\omega)|}{\partial\xi_2} = \frac{1}{2|Y(j\omega)|}\frac{\partial|Y(j\omega)|^2}{\partial\xi_2}$$

$$= \frac{1}{2|Y(j\omega)|}\left\{REAL(\widetilde{Y}_0(j\omega)\widetilde{Y}_1(-j\omega)) + \xi_2 \sum_{n=1,2,\ldots} n\xi_2^{n-1}\sum_{i=0}^{n}\widetilde{Y}_i(j\omega)\widetilde{Y}_{n-i}(-j\omega)\right\}$$

Because $REAL(\widetilde{Y}_0(j\omega)\widetilde{Y}_1(-j\omega))$ <0, it is obvious that there must exist 0< $\hat{\xi}_2 < \overline{\overline{\xi}}_2$ such that $\frac{\partial|Y(j\omega)|}{\partial\xi_2} < 0$ for $0 < \xi_2 < \hat{\xi}_2$. Note that $\sum_{n=1,2,\ldots} n\xi_2^{n-1}\sum_{i=0}^{n}\widetilde{Y}_i(j\omega)\widetilde{Y}_{n-i}(-j\omega)$ is also an alternative series, thus in practice, it may also be negative under certain conditions. Hence, there must exist $\hat{\xi}_2 < \overline{\xi}_2 < \overline{\overline{\xi}}_2$, such that $\frac{\partial|Y(j\omega)|}{\partial\xi_2} < 0$ for $0 < \xi_2 < \overline{\xi}_2$. □

Based on the discussions above, it can be concluded that the nonlinear damping for the dimensionless system (1) has the following properties:

(1) It has trivial effect on the system output spectrum at high or low frequencies.

(2) Given a proper small linear damping, the nonlinear damping drive the system output spectrum to be a convergent alternative series, and the magnitude of the output spectrum can be decreased by the increase of the nonlinear damping at the natural frequency.

4 Simulation study

To verify the theoretical analysis results above, computation of the output spectrum of system (1) can be conducted up to the 5th order according to Lemma 1 and Lemma 2. The GFRFs up to the 5th order are

$$H_1^y(j\omega_1) = (1+\xi_1\cdot(j\omega_1))\cdot H_1^x(j\omega_1) = -\frac{1+\xi_1\cdot(j\omega_1)}{L_1(j\omega_1)}$$

$$H_3^x(j\omega_1,\cdots,j\omega_{2n+1}) = \frac{\xi_2 H_{3,3}(j\omega_1,\cdots,j\omega_3)}{L_3(j\omega_1+\cdots+j\omega_3)} = \frac{\xi_2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_3)\cdot(j\omega_1)\cdots(j\omega_3)}{L_3(j\omega_1+\cdots+j\omega_3)}$$

$$\begin{aligned}
H_3^y(j\omega_1,\cdots,j\omega_3) &= (1+\xi_1\cdot(j\omega_1+\cdots+j\omega_3))\cdot H_3^x(j\omega_1,\cdots,j\omega_3) \\
&\quad + \xi_2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_3)\cdot(j\omega_1)\cdots(j\omega_3) \\
&= \frac{\xi_2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_3)\cdot(j\omega_1)\cdots(j\omega_3)\cdot(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_3))}{L_3(j\omega_1+\cdots+j\omega_3)} \\
&\quad + \xi_2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_3)\cdot(j\omega_1)\cdots(j\omega_3) \\
&= -\xi_2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_3)\cdot(j\omega_1)\cdots(j\omega_3)\cdot\frac{(j\omega_1+\cdots+j\omega_3)^2}{L_3(j\omega_1+\cdots+j\omega_3)}
\end{aligned}$$

$$H_5^x(j\omega_1,\cdots,j\omega_5)=\frac{\xi_2 H_{5,3}(j\omega_1,\cdots,j\omega_5)}{L_5(j\omega_1+\cdots+j\omega_5)}$$

$$=\frac{\xi_2}{L_5(j\omega_1+\cdots+j\omega_5)}\begin{pmatrix} H_1^x(j\omega_1)H_1^x(j\omega_2)H_3^x(j\omega_3,\cdots,j\omega_5)(j\omega_1)(j\omega_2)(j\omega_3+\cdots+j\omega_5) \\ +H_1^x(j\omega_1)H_3^x(j\omega_2,\cdots,j\omega_4)H_1^x(j\omega_5)(j\omega_1)(j\omega_2+\cdots+j\omega_4)(j\omega_5) \\ +H_3^x(j\omega_1,\cdots,j\omega_3)H_1^x(j\omega_4)H_1^x(j\omega_5)(j\omega_1+\cdots+j\omega_3)(j\omega_4)(j\omega_5) \end{pmatrix}$$

$$=\frac{\xi_2^2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_5)\cdot(j\omega_1)\cdots(j\omega_5)}{L_5(j\omega_1+\cdots+j\omega_5)}\begin{pmatrix} \dfrac{j\omega_3+\cdots+j\omega_5}{L_3(j\omega_3+\cdots+j\omega_5)} \\ +\dfrac{j\omega_2+\cdots+j\omega_4}{L_3(j\omega_2+\cdots+j\omega_4)} \\ +\dfrac{j\omega_1+\cdots+j\omega_3}{L_3(j\omega_1+\cdots+j\omega_3)} \end{pmatrix}$$

$$H_5^y(j\omega_1,\cdots,j\omega_5)$$
$$=\left(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_5)\right)\cdot H_5^x(j\omega_1,\cdots,j\omega_5)+\xi_2 H_{5,3}(j\omega_1,\cdots,j\omega_5)$$
$$=-\frac{\xi_2^2 H_1^x(j\omega_1)\cdots H_1^x(j\omega_5)\cdot(j\omega_1)\cdots(j\omega_5)\cdot(j\omega_1+\cdots+j\omega_5)^2}{L_5(j\omega_1+\cdots+j\omega_5)}\begin{pmatrix} \dfrac{j\omega_3+\cdots+j\omega_5}{L_3(j\omega_3+\cdots+j\omega_5)} \\ +\dfrac{j\omega_2+\cdots+j\omega_4}{L_3(j\omega_2+\cdots+j\omega_4)} \\ +\dfrac{j\omega_1+\cdots+j\omega_3}{L_3(j\omega_1+\cdots+j\omega_3)} \end{pmatrix}$$

Therefore, the output spectrum is

$$Y(j\omega)=\overline{Y}_0(j\omega)+\sum_{n=1}^{\lfloor 5-1/2 \rfloor}\overline{Y}_n(j\omega)=\overline{Y}_0(j\omega)+\overline{Y}_1(j\omega)+\overline{Y}_2(j\omega)+\cdots \tag{17a}$$

where

$$\overline{Y}_0(j\omega)=-\frac{j(1+j\xi_1\omega)H_1^x(j\omega)}{2} \tag{17b}$$

$$\overline{Y}_1(j\omega)=\xi_2\cdot\frac{j}{2^3}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^2\cdot\left(j\omega H_1^x(j\omega)\right)\cdot(j\omega)^2}{L_1(j\omega)}\cdot 3=\xi_2\cdot\frac{3\cdot\left|(\omega)H_1^x(j\omega)\right|^2\cdot H_1^x(j\omega)\cdot\omega^3}{8L_1(j\omega)} \tag{17c}$$

$$\begin{aligned}\overline{Y}_2(j\omega)&=\frac{\xi_2^2}{2^5}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^4\cdot H_1^x(j\omega)\cdot\omega^3}{L_1(j\omega)}\\&\quad\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_5}=\omega}\left(\frac{j\omega_3+\cdots+j\omega_5}{L_3(j\omega_3+\cdots+j\omega_5)}+\frac{j\omega_2+\cdots+j\omega_4}{L_3(j\omega_2+\cdots+j\omega_4)}+\frac{j\omega_1+\cdots+j\omega_3}{L_3(j\omega_1+\cdots+j\omega_3)}\right)\\&=\frac{\xi_2^2}{2^5}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^4\cdot H_1^x(j\omega)\cdot\omega^3}{L_1(j\omega)}\cdot 3\sum_{\omega_{k_1}+\cdots+\omega_{k_5}=\omega}\left(\frac{j\omega_1+\cdots+j\omega_3}{L_3(j\omega_1+\cdots+j\omega_3)}\right)\\&=\frac{3\xi_2^2}{2^5}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^4\cdot H_1^x(j\omega)\cdot\omega^3}{L_1(j\omega)}\cdot\left(\frac{6j\omega}{L_3(j\omega)}+\frac{-3j\omega}{L_3(-j\omega)}+\frac{j3\omega}{L_3(j3\omega)}\right)\end{aligned} \tag{17c}$$

Let $\xi_1=0.5$, then (6f) gives $L_1(j\omega')=-\left(1+\xi_1(j\omega')+(j\omega')^2\right)=-(1-(\omega')^2+0.5\cdot j\omega')$, thus

$$L_1(j\omega)=-\left(1+\xi_1(j\omega)+(j\omega)^2\right)=-(1-\omega^2+0.5\cdot j\omega) \tag{18a}$$

$$L_1(-j\omega)=-(1-\omega^2-0.5\cdot j\omega) \tag{18b}$$

$$L_1(j3\omega) = -(1 - 9\omega^2 + 1.5 \cdot j\omega) \tag{18c}$$

and it follows from (6e) that

$$H_1^x(j\omega) = \frac{-1}{L_1(j\omega)} = \frac{1}{1-(\omega)^2 + \xi_1(j\omega)} = \frac{1}{1-\omega^2 + j0.5\omega} \tag{18d}$$

Then at the natural frequency $\omega = 1$, it can be obtained that

$$\bar{Y}_0(j\omega)\Big|_{\omega=1} = -\frac{j(1+j\xi_1\omega)H_1^x(j\omega)}{2} = -(1+0.5j)$$

$$\bar{Y}_1(j\omega)\Big|_{\omega=1} = \xi_2 \cdot \frac{3\cdot\left|(\omega)H_1^x(j\omega)\right|^2 \cdot H_1^x(j\omega)\cdot\omega^3}{8L_1(j\omega)} = 6\xi_2$$

$$\bar{Y}_2(j\omega)\Big|_{\omega=1} = \frac{3\xi_2^2}{2^5}\cdot\frac{\left|(j\omega)H_1^x(j\omega)\right|^4 \cdot H_1^x(j\omega)\cdot\omega^3}{L_1(j\omega)}\cdot\left(\frac{6j\omega}{L_1(j\omega)} + \frac{-3j\omega}{L_1(-j\omega)} + \frac{j3\omega}{L_1(j3\omega)}\right)$$

$$= -6\left(18 + \frac{3j}{-8+1.5j}\right)\xi_2^2 = -\xi_2^2(108.4075 - 2.1736j)$$

Therefore,

$$Y(j1) = -(1+0.5j) + 6\xi_2 - (108.4075 - 2.1736j)\xi_2^2 + \cdots\cdots \tag{19}$$

Obviously, it is an alternating series. It can be verified by simulation that the magnitude of the output spectrum at the natural frequency $\omega = 1$ can be reduced when ξ_2 is increased (See Figure 3 and Figure 4). Figure 3 is obtained by using the first three terms of the series (19). Since only the first three terms of (19) are used, it is only valid for a small ξ_2 as shown in Figure 3. Figure 4 is a simulation result. For comparison, the magnitude characteristics of the system output spectrum with respect to the linear damping parameter ξ_1 under the condition ξ_2=0 and those with respect to the nonlinear damping parameter ξ_2 under the condition ξ_1=0.5 are studied, respectively (See Figures 5-7). In Figures 7 - 8, the results for ξ_1=0.5 or 0.54, 0.75 and ξ_2=0.028, 0.25 are studied. Compared with the linear damping, the nonlinear damping holds the magnitude of system output spectrum similar to the corresponding linear damping case for much higher or lower frequencies than 1 but reduce the magnitude to a lower level at around the natural frequency. However, although an increase of the linear damping will reduce the magnitude at around frequency 1, yet it also increases largely the magnitude of output at high or low frequencies.

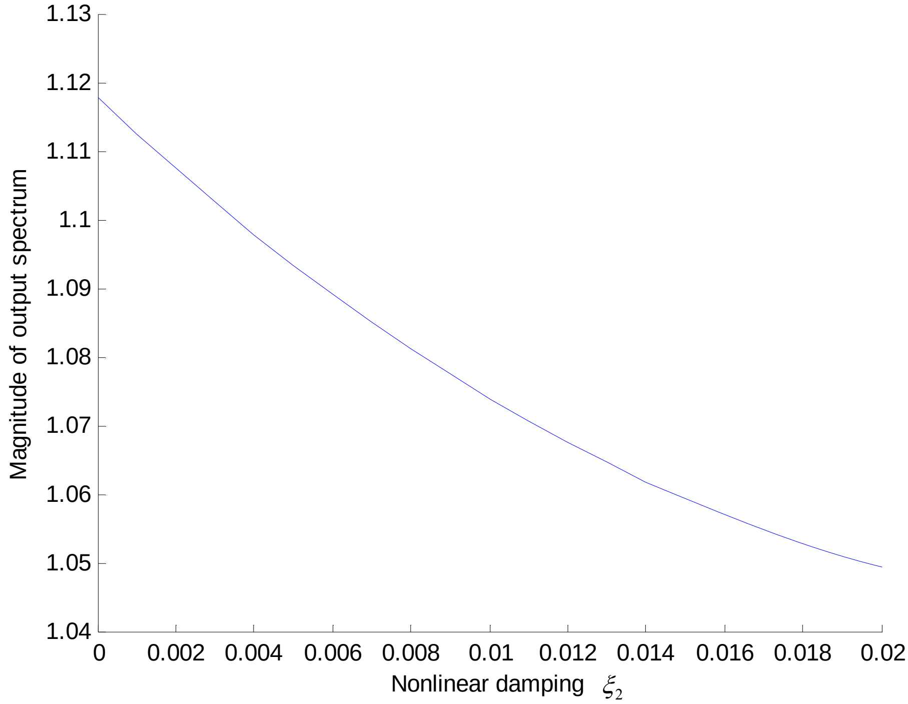

Figure 3. Theoretical computation of the magnitude of system output spectrum for different nonlinear damping parameter ξ_2 with ξ_1=0.5 at natural frequency $\omega = 1$

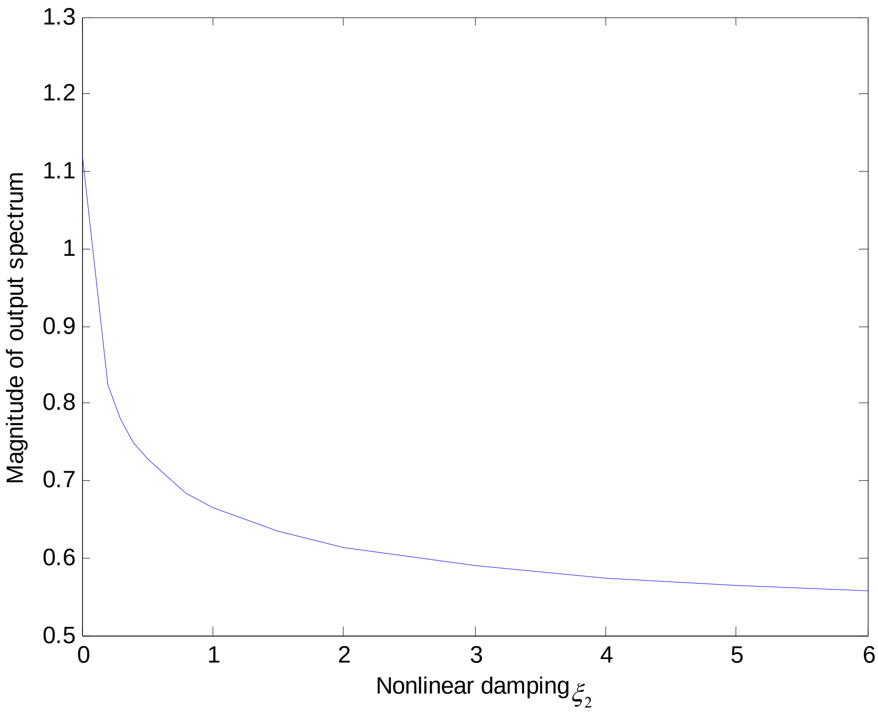

Figure 4. Simulation test for the magnitude of system output spectrum for different nonlinear damping parameter ξ_2 with ξ_1=0.5 at natural frequency $\omega = 1$

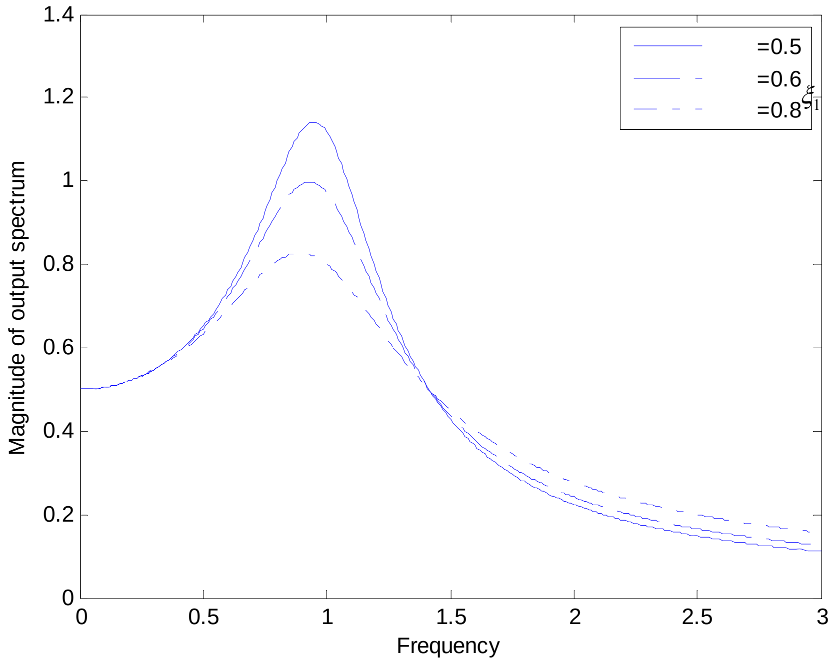

Figure 5. Magnitude characteristics of system output spectrum with respect to frequency ω and different linear damping ξ_1

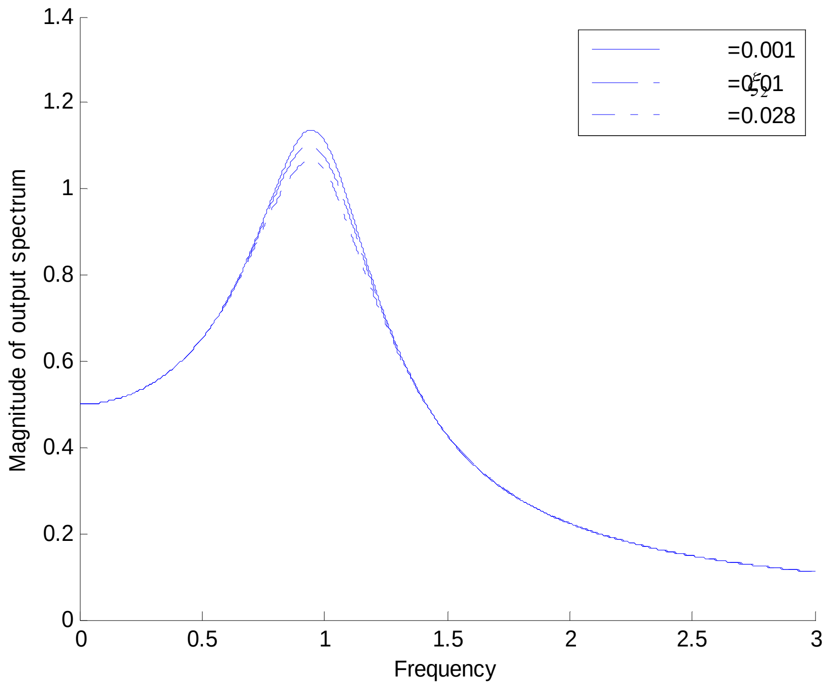

Figure 6. Magnitude characteristics of system output spectrum with respect to frequency ω and different nonlinear damping ξ_2 with ξ_1=0.5

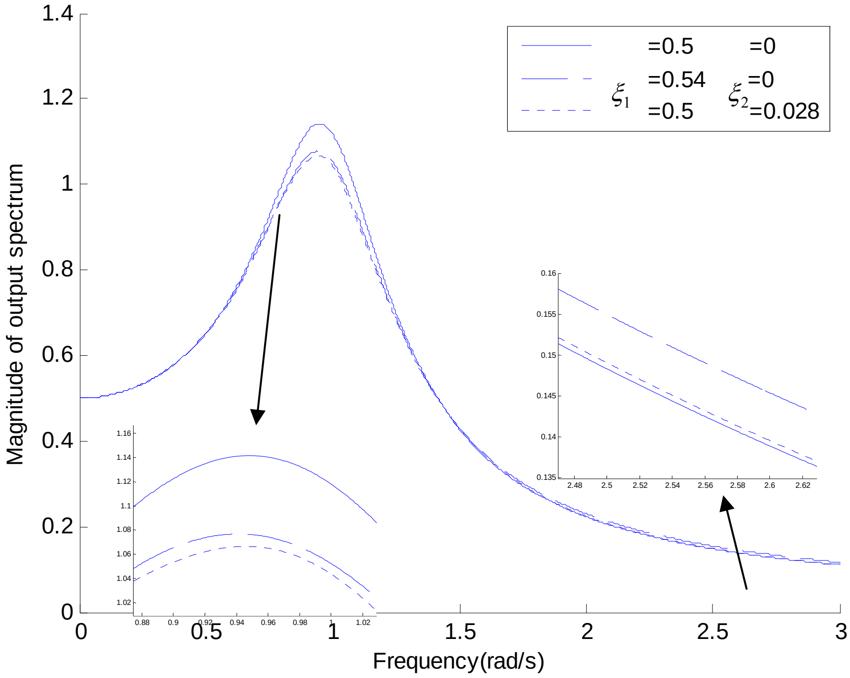

Figure 7. Magnitude characteristics of system output spectrum with respect to frequency ω and different nonlinear damping ξ_2 and ξ_1

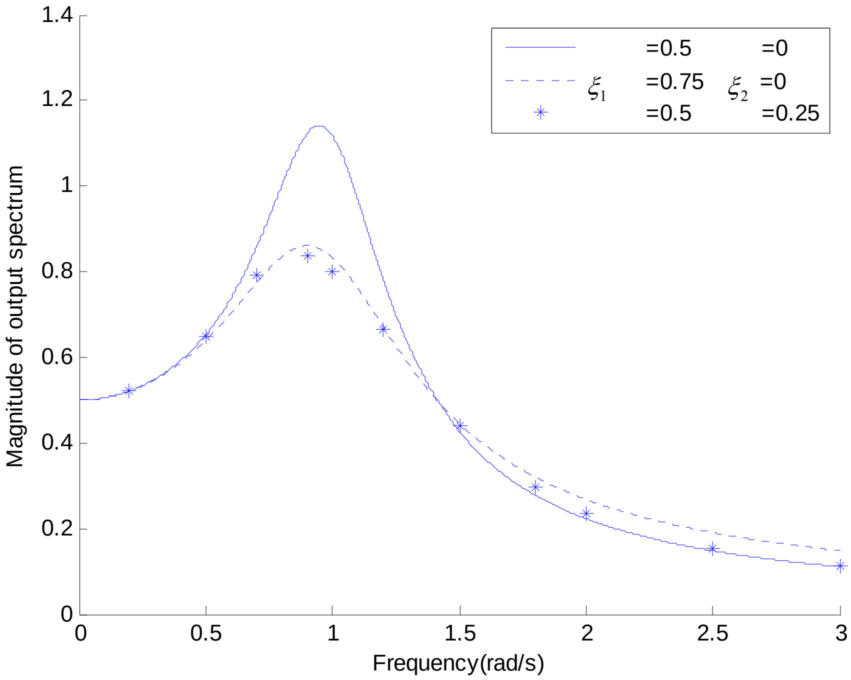

Figure 8. Magnitude characteristics of system output spectrum with respect to frequency ω and different nonlinear damping ξ_2 and ξ_1 ("*"from simulations)

5 Conclusions

The cubic nonlinear damping is studied in the frequency domain through a dimensionless vibration system model actuated by a harmonic input. Theoretical analysis and simulation show that the cubic nonlinearity drives the system output spectrum to be an alternative series at the natural frequency 1, which can be used to suppress the magnitude level of the output spectrum, and the magnitude frequency characteristics of the system output spectrum at higher or lower frequencies than the natural frequency 1 after the cubic nonlinear damping is introduced is very similar to those before the cubic nonlinear damping is introduced. These results can provide a significant insight into the design of active and passive nonlinear vibration systems in practice. Further study will extend these results to a more general case.

Appendix A: Proof of Lemma 1

To compute the GFRFs for system (1), consider a more general model described by the following nonlinear differential equation (NDE)

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \qquad \text{(A1)}$$

where $\left.\frac{d^k x(t)}{dt^k}\right|_{k=0}=x(t)$, $p+q=m$, $\sum_{k_1,k_{p+q}=0}^{K}(\cdot)=\sum_{k_1=0}^{K}(\cdot)\cdots\sum_{k_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of y(t) and u(t), and K is the maximum order of the derivative. It is obvious that model (1a) is just a very special case of model (A1) with $c_{10}(0)=1$, $c_{10}(2)=1$, $c_{10}(1)=\xi_1$, $c_{30}(111)=\xi_2$, $c_{01}(0)=-1$, K=2, M=3 and all other parameters zero.

For model (A1), the GFRFs can be computed through an algorithm provided in [1]

$$\begin{aligned} L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n) &= \sum_{k_1,k_n=0}^{K}c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \\ &+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \\ &+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \end{aligned} \qquad \text{(A2)}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \qquad \text{(A3)}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \qquad \text{(A4)}$$

where $L_n(j\omega_1+\cdots+j\omega_n)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1}$. Moreover, equation (A3) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{r_X+1},\cdots,j\omega_{r_X+r_i})(j\omega_{r_X+1}+\cdots+j\omega_{r_X+r_i})^{k_i} \qquad \text{(A5)}$$

where $X=\sum_{x=1}^{i-1}r_x$.

To obtain the frequency response functions for the SDOF vibration system (1), the system can be regarded that it has one input u(t) and two outputs x(t) and y(t). The GFRFs for the relationship between y(t) and u(t) are dependent on the GFRFs for the relationship between x(t) and u(t). Therefore, the later are derived first in the following.

By the parametric characteristic analysis in [15], the nth-order GFRF for the relationship between x(t) and u(t) can be expressed as

$$H_n^x(j\omega_1,\cdots,j\omega_n)=CE\left(H_n^x(j\omega_1,\cdots,j\omega_n)\right)\cdot f_n(j\omega_1,\cdots,j\omega_n)$$

where $f_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued vector, "$\oplus$" and "$\otimes$" are two operators defined in [15], $CE\left(H_n^x(j\omega_1,\cdots,j\omega_n)\right)$ is referred to as the parametric characteristic of the *n*th-order GFRF

$$CE(H_n^x(j\omega_1,\cdots,j\omega_n))=C_{0,n}\oplus\left(\underset{q=1}{\overset{n-1}{\oplus}}\underset{p=1}{\overset{n-q}{\oplus}}C_{p,q}\otimes CE(H_{n-q-p+1}^x(\cdot))\right)\oplus\left(C_{n,0}\oplus\underset{p=2}{\overset{\lfloor n+1/2\rfloor}{\oplus}}C_{p,0}\otimes CE(H_{n-p+1}^x(\cdot))\right)$$

According to the results in [17], for model (1)

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))=CE(H_n^x(j\omega_1,\cdots,j\omega_n))$$

Note also that there is only one nonlinear term with coefficient $c_{3,0}(1,1,1)$, thus it can be obtained that the nth-order GFRF for the relationship between y(t) and u(t) can be expressed as for n=0,1,2,3,…

$$H_{2n+1}^y(j\omega_1,\cdots,j\omega_{2n+1})=\xi_2^n\cdot\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1}) \quad \text{(A6)}$$

$$H_{2n}^y(j\omega_1,\cdots,j\omega_{2n})=0 \quad \text{(A7)}$$

Hence, only odd order GFRFs need to be computed. From (A2) it can be derived that for the first order GFRF,

$$H_1^x(j\omega_1)=\frac{-1}{L_1(j\omega_1)} \quad \text{(A8)}$$

and for *n*=1,2,3,…

$$H_{2n}(j\omega_1,\cdots,j\omega_{2n})=0 \quad \text{(A9)}$$

$$H_{2n+1}^x(j\omega_1,\cdots,j\omega_{2n+1})=\frac{\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})} \quad \text{(A10)}$$

$$H_{2n+1,3}(\cdot)=\sum_{i=1}^{2n-1}H_i^x(j\omega_1,\cdots,j\omega_i)H_{2n+1-i,2}(j\omega_{i+1},\cdots,j\omega_{2n+1})(j\omega_1+\cdots+j\omega_i) \quad \text{(A11)}$$

$$H_{2n+1,1}(j\omega_1,\cdots,j\omega_{2n+1})=H_{2n+1}^x(j\omega_1,\cdots,j\omega_{2n+1})(j\omega_1+\cdots+j\omega_{2n+1})^{k_1} \quad \text{(A12)}$$

$$L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})=-\left(1+\xi_1(j\omega_1+\cdots+j\omega_{2n+1})+(j\omega_1+\cdots+j\omega_{2n+1})^2\right) \quad \text{(A13)}$$

From (A5), (A11) can also be written as

$$H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=2n+1}}^{2n-1}\prod_{i=1}^{3}H_{r_i}^x(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i}) \quad \text{(A14)}$$

Based on equations (A8-14), the GFRFs for the relationship between y(t) and u(t) can be obtained by applying the results in [17]. For n=0,1,2,3,…

$$H_{2n}^y(j\omega_1,\cdots,j\omega_{2n})=0 \text{ and } H_{2n+1}^y(j\omega_1,\cdots,j\omega_{2n+1})=\sum_{p=1}^{2n+1}\sum_{k_1,k_p=0}^{1}\tilde{c}_{p,0}(k_1,\cdots,k_p)H_{2n+1,p}(j\omega_1,\cdots,j\omega_{2n+1})$$

Noting that $\tilde{c}_{1,0}(0)=1$, $\tilde{c}_{1,0}(1)=\xi_1$, $\tilde{c}_{3,0}(1,1,1)=\xi_2$ and other parameters of form $\tilde{c}_{p,0}(k_1,\cdots,k_p)$ zero and using (A11), it can be further obtained that for n=0,1,2,3,…

$$\begin{aligned}
&H^y_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})=\tilde{c}_{1,0}(0)H_{2n+1,1}(j\omega_1,\cdots,j\omega_{2n+1})+\tilde{c}_{1,0}(1)H_{2n+1,1}(j\omega_1,\cdots,j\omega_{2n+1})\\
&\quad+\delta(\delta(n))\tilde{c}_{3,0}(1,1,1)H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=H^x_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})+\xi_1 H^x_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})\cdot(j\omega_1+\cdots+j\omega_{2n+1})\\
&\quad+\delta(\delta(n))\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=\left(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_{2n+1})\right)\cdot H^x_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})+\delta(\delta(n))\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1}) \qquad (A15)
\end{aligned}$$

where $\delta(n)=\begin{cases}1 & n=0\\ 0 & \text{else}\end{cases}$. From (A10) and (A15), it can be obtained that for n>0,

$$\begin{aligned}
&H^y_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=\left(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_{2n+1})\right)\cdot H^x_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})+\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=\left(1+\xi_1\cdot(j\omega_1+\cdots+j\omega_{2n+1})\right)\frac{\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})}+\xi_2 H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=-\xi_2\frac{(j\omega_1+\cdots+j\omega_{2n+1})^2}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})}H_{2n+1,3}(j\omega_1,\cdots,j\omega_{2n+1})
\end{aligned}$$

Using (A14) yields for n>0

$$\begin{aligned}
&H^y_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})\\
&=-\xi_2\frac{(j\omega_1+\cdots+j\omega_{2n+1})^2}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})}\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=2n+1}}^{2n-1}\prod_{i=1}^{3}H^x_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i}) \qquad \text{(A16)}
\end{aligned}$$

This completes the proof. □

Appendix B: Proof of Lemma 2

When the system input is a multi-tone function described by

$$u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i) \qquad \text{(B1)}$$

the system output frequency response function can be obtained according to [7],

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \qquad \text{(B2)}$$

where $F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases}$ and $\omega_k=sign(k)\omega_{|k|}$.

Using the GFRFs determined by equations (4-8) for the relationship from u(t) to y(t), the system output spectrum can then be determined by substituting (8) in (B2) gives

$$\begin{aligned}
Y(j\omega)&=\sum_{n=0}^{\lfloor N-1/2\rfloor}\frac{1}{2^{2n+1}}\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}H^y_{2n+1}(j\omega_{k_1},\cdots,j\omega_{k_{2n+1}})F(\omega_{k_1})\cdots F(\omega_{k_{2n+1}})\\
&=\sum_{n=0}^{\lfloor N-1/2\rfloor}\frac{\xi_2^n}{2^{2n+1}}\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}\bar{f}_{2n+1}(j\omega_{k_1},\cdots,j\omega_{k_{2n+1}})F(\omega_{k_1})\cdots F(\omega_{k_{2n+1}})
\end{aligned} \qquad \text{(B3)}$$

For the specific input function (1b), it can be obtained from (B2) that

$F(\omega_{k_l})=-jk_l F_d$, for $k_l=\pm1, \omega_{k_l}=k_l\omega$, and $l=1,\cdots,n$

where F_d is the magnitude of the input signal, *i.e.*, F_d =1. Note that the condition $\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega$ means that there are n frequencies $\omega_{k_l}=-\omega$ and n+1 frequencies $\omega_{k_l}=\omega$. Hence,

$$F(\omega_{k_1})\cdots F(\omega_{k_{2n+1}})=(-jF_d)^{2n+1}k_1\cdot\cdots\cdot k_{2n+1}=-j^{2n}j\cdot(F_d)^{2n+1}\cdot(-1)^n=-j(F_d)^{2n+1}$$

Also consider $\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})$ in the sum under the condition $\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega$,

$$\bar{f}_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})\Big|_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega}$$

$$=-\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\frac{\left((-j\omega)H_1^x(-j\omega)\right)^n\cdot\left(j\omega H_1^x(j\omega)\right)^{n+1}\cdot(j\omega)^2\cdot\prod_{i=1}^{n-1}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}{L_{2n+1}(j\omega)\cdot L_{j_1}(j\omega_{l(1)}+\cdots+j\omega_{l(j_1)})\cdots L_{j_{n-1}}(j\omega_{l(1)}+\cdots+j\omega_{l(j_{n-1})})}$$

$$=-\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\frac{\left((-j\omega)H_1^x(-j\omega)\right)^n\cdot\left(j\omega H_1^x(j\omega)\right)^{n+1}\cdot(j\omega)^2}{L_{2n+1}(j\omega)}\cdot\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}$$

$$=-\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\frac{\left|(j\omega)H_1^x(j\omega)\right|^{2n}\cdot\left(j\omega H_1^x(j\omega)\right)\cdot(j\omega)^2}{L_{2n+1}(j\omega)}\cdot\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}$$

$$=-\frac{\left|(j\omega)H_1^x(j\omega)\right|^{2n}\cdot\left(j\omega H_1^x(j\omega)\right)\cdot(j\omega)^2}{L_1(j\omega)}\cdot\sum_{\substack{(j_1,\cdots,j_{n-1})\\ j_i\in\{2k+1|1\le k\le n-1\}}}\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(j_i)}}{L_{j_i}(j\omega_{l(1)}+\cdots+j\omega_{l(j_i)})}$$

Substituting these equations into (B3), (9a-c) can be obtained. This completes the proof. □

References

[1] S.A. Billings and J.C. Peyton-Jones, "Mapping nonlinear integro-differential equation into the frequency domain", International Journal of Control, Vol 54, 863-879, 1990

[2] S. Boyd and L. O. Chua, "Fading memory and the problem of approximating nonlinear operators with Volterra series", IEEE Trans. Circuits Syst., vol. CAS-32, 1150--1161, Nov. 1985

[3] S. Daley, J. Hatonen, D.H. Owens, "Active vibration isolation in a 'smart spring' mount using a repetitive control approach", Control Engineering Practice, 14, 991 – 997, 2006

[4] D. Graham and D. Mcruer. Analysis of nonlinear control systems. New York: John Wiley & Sons. Inc. 1961

[5] D. Hrovat. "Survey of advanced suspension developments and related optimal control applications". Automatica, Vol.33, No. 10, 1781-1817, 1997

[6] D. Karnopp, "Active and semi-active vibration isolation", ASME J. Mech. Des., Vol. 117, 177-185, 1995

[7] Z.Q. Lang, and S. A. Billings. "Output frequency characteristics of nonlinear systems". International Journal of Control, Vol. 64, 1049-1067, 1996

[8] Z.Q. Lang, S.A. Billings, R. Yue and J. Li, "Output frequency response functions of nonlinear Volterra systems", to appear in Automatica, 2007

[9] R. C.H. Lee, M. C. Smith. “Nonlinear control for robust rejection of periodic disturbances”, Systems & Control Letters 39, 97-107. 2000
[10] Mahony R.; Mareels I.; Bastin G.; Campion G. Output stabilization of square nonlinear systems, Automatica, Volume 33, Number 8, August 1997, pp. 1571-1577(7)
[11] S.A. Nayfeh and A.H. Nayfeh, “Energy transfer from high- to low frequency modes in flexible structure via modulation”, Journal of Vibration and Acoustics, 116, pp. 203 – 207, 1994.
[12] G. Nakhaie Jazar, R. Houim, A. Narimani and M. F. Golnaraghi. Frequency response and jump avoidance in a nonlinear passive engine mount. Journal of Vibration and Control, 12(11), 1205-1237, 2006
[13] W. J. Rugh. Nonlinear system theory --- the Volterra/Wiener approach. Baltimore and London: the Johns Hopkins University Press. 1981
[14] X.J. Jing, Z.Q. Lang and S.A. Billings. “Frequency Domain Analysis Based Nonlinear Feedback Control for Suppressing Periodic Disturbance”. The 6th World Congress on Intelligent Control and Automation, June 21-23, China, 2006a
[15] Jing X.J., Lang Z.Q., Billings S. A. and Tomlinson G. R. “The Parametric Characteristics of Frequency Response Functions for Nonlinear Systems”, International Journal of Control, Vol. 79, No. 12, December, 1552-1564, 2006b
[16] Jing X.J., Lang Z.Q., and Billings S. A., New Bound Characteristics of NARX Model in the Frequency Domain, International Journal of Control, 80(1), pp140-149, 2007a
[17] Jing X.J., Lang Z.Q., and Billings S. A., Frequency Domain Analysis for Nonlinear Volterra Systems with a General Nonlinear Output Function, accepted by International Journal of Control, 2007b
[18] Bromwich, T. J. I'A. and MacRobert, T. M. "Alternating Series." §19 in An Introduction to the Theory of Infinite Series, 3rd ed. New York: Chelsea, pp. 55-57, 1991.
[19] HARTLEY T. T., and LORENZO C. F. A Frequency-Domain Approach to Optimal Fractional-Order Damping. Nonlinear Dynamics 38: 69–84, 2004.
[20] Zhu W.Q., Yang Z.G. and Song T.T. “An optimal nonlinear feedback control strategy for randomly excited structural systems”, Nonlinear dynamics, 24:31-51, 2001