Frequency Domain Analysis of Nonlinear Systems Driven by Multiharmonic Signals

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***Abstract*—This paper examines the output properties of static power-series nonlinearities driven by periodic multiharmonic signals with emphasis given to their effect on linear frequency response function (FRF) measurements. The analysis is based on the classification of nonlinear distortions into harmonic and interharmonic contributions. The properties of harmonic contributions are examined in detail and explicit formulae are derived, by which the number of harmonic contributions generated at the test frequencies can be calculated for odd-order nonlinearities up to, and including, the ninth order. Although an analytic solution for any odd-order nonlinearity is still under investigation, a heuristic methodology is developed that solves this problem. It is shown that the derived formulae provide a useful tool in the examination of the behavior of FRF measurements in the presence of nonlinear distortions. Based on these formulae, different approaches in classifying nonlinear distortions are then compared with respect to their suitability in assessing the influence of system nonlinearities on linear FRF measurements.**

***Index Terms*—Frequency-domain analysis, linear systems, multisine signals, nonlinear distortions, signal processing, system identification.**

I. Introduction

LINEAR SYSTEM identification deals with the measurement and modeling of linear dynamic systems. However, in reality, the linearity assumption is only approximately valid, since all practical systems are nonlinear to some extent. This creates the need for an intuitive understanding of the behavior of nonlinear systems. This is the main topic of this paper, which examines the output properties of static power-series nonlinearities and their effect on linear frequency response function (FRF) measurements. As such, the study presented in this paper falls within the spectrum of work of other authors in this field [1], [2]. The paper is organized as follows.

The contributions generated by static power-series nonlinearities driven by periodic multiharmonic inputs are examined based on the frequency-domain approach introduced by Evans *et al.* [3], [4], by which nonlinear distortions are classified into harmonic and interharmonic contributions. This approach is closely related to extensive work conducted by Schoukens *et al.* [5] in this area. It is shown that interharmonic contributions depend entirely on the properties of the input harmonic vector; thus their behavior varies accordingly. Harmonic contributions, however, exhibit more systematic properties, since they only depend on the number of input frequencies and the order of the nonlinearity.

This leads to an investigation of the characteristics of harmonic contributions as the order of the nonlinearity increases and explicit formulae are derived, by which the number of harmonic contributions generated at the excitation frequencies can be calculated for odd-order nonlinearities up to, and including, the ninth order. Although an analytic solution for any odd-order nonlinearity is yet to be found, a heuristic methodology has been developed that solves this problem. It is shown that the derived formulae offer new insights into the output properties of power-series nonlinearities and also provide the tool by which their influence on different types of multiharmonic signals can be examined.

Based on these formulae, different approaches in classifying nonlinear distortions are then compared, with the aim of clarifying the suitability of each approach in examining the behavior of FRF measurements in the presence of nonlinear distortions, and the use of the Evans *et al.* approach is advocated.

II. Nonlinear Systems

Consider a periodic multiharmonic signal applied to a time invariant system. Any nonlinearities present will generate an output contribution that will be the same for each successive period of the signal. This will introduce a distortion into the estimated linear FRF, which in contrast to the error introduced by stochastic effects, will not reduce with averaging

$$\hat{H}(j\omega_k) = \frac{\overline{Y}(j\omega_k)}{\overline{U}(j\omega_k)} = \frac{\overline{Y}_l(j\omega_k) + \overline{Y}_n(j\omega_k)}{\overline{U}(j\omega_k)} \tag{1}$$

where $\bar{U}(j\omega_k)$ and $\bar{Y}(j\omega_k)$ are the input and output Fourier coefficients averaged across M periods, $\bar{Y}_l(j\omega_k)$ is the linear response, and $\bar{Y}_n(j\omega_k)$ the nonlinear distortion at the test frequencies ω_k.

The nature of this distortion will now be examined by employing the parallel nonlinear structure given in Fig. 1.

This is the most basic nonlinear model and it is composed of a linear system L in parallel with a static power-series nonlinear element N[6]. The nonlinear element is defined by

$$N\left(u(t)\right) = \sum_{n=1}^{\infty} a_n u(t)^n \tag{2}$$

where $u(t)$ is the input signal and α_n represents the coefficient of the nth-order term. The use of the parallel nonlinear structure of Fig. 1 does not affect in any way the generality of the analysis that follows.

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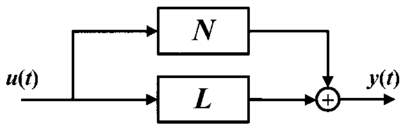

Fig. 1. Parallel nonlinear structure.

The input signal is a multisine of F cosines with dc excluded and a double-sided spectrum

$$U(j\omega) = \sum_{\substack{k=-F \\ k\neq 0}}^{F} a(k)e^{j\phi(k)}\delta\left(\omega - i(k)\omega_0\right) \tag{3}$$

with

$$\begin{aligned} a(-k) &= a(k) \\ i(-k) &= -\,i(k) \qquad\qquad k = 1 \cdot F \\ \phi(-k) &= -\,\phi(k) \end{aligned}$$

and

$$\begin{aligned} \mathbf{a} &= [a(-F)a(-F+1)\ldots a(F-1)a(F)] \\ \mathbf{i} &= [i(-F)i(-F+1)\ldots i(F-1)i(F)] \\ \Phi &= [\phi(-F)\phi(-F+1)\ldots\phi(F-1)\phi(F)] \end{aligned}$$

where $\mathbf{a}$ is a vector of harmonic amplitudes, $\mathbf{i}$ a vector of harmonic numbers, Φ a vector of harmonic phases, ω_0 is the signal fundamental frequency, and δ represents the Dirac delta function defined as

$$\delta\left(\omega - i_k\omega_0\right)\begin{cases} = 0 \;\; \omega - i_k\omega_0 \neq 0 \\ = 1 \;\; \omega - i_k\omega_0 = 0 \end{cases}. \tag{4}$$

The distortions introduced by static power-series nonlinearities are the product of time-domain multiplication of the input multisine and, hence, convolution in the frequency domain. The output from a nonlinear term of order n will consist of $(2F)^n$ contributions, made up of all possible combinations of n input harmonics. For example, in the specific case of a third-order nonlinearity, the output will consist of $(2F)^3$ contributions, made up of all possible combinations, with permutations, of three input harmonics

$$\begin{aligned} Y_n(j\omega) = a_3 &\sum_{\substack{p=-F \\ p\neq 0}}^{F}\sum_{\substack{m=-F \\ m\neq 0}}^{F}\sum_{\substack{k=-F \\ k\neq 0}}^{F}[a(p)a(m)a(k)] \\ &\times \exp(j[\phi(p)+\phi(m)+\phi(k)])\delta(\omega - [i(p)+i(m)+i(k)]\omega_0). \end{aligned} \tag{5}$$

It is clear from (5) that the third-order nonlinear contributions will be generated at frequencies which are sums of three input harmonics.

It is common practice to consider the contributions generated by an nth-order nonlinearity as two types: the nth *harmonic* contributions, generated by combinations of the same input frequencies (e.g., $+\omega_0+\omega_0+\omega_0$), and the *cross-talk* or *intermodulation* contributions, made up of combinations of different input frequencies (e.g., $+3\omega_0 + \omega_0 + \omega_0$). This distinction owes its origin to communications engineering [7] and will be termed as the *classic approach* for the remainder of this paper. This approach clearly has some value when considering inputs made up of only a few frequencies, but it can be misleading when studying the contributions generated by multiharmonic signals, since the nth *harmonic* contributions are only a very small fraction of the total number of contributions.

An alternative methodology that can be used to clarify the influence of power-series nonlinearities on such signals was proposed by Evans *et al.* [3], [4], by which the contributions are divided into two types.

1) *Harmonic Contributions:* These are generated by combinations of pairs of equal positive and negative frequencies, or another test frequency combined with pairs of equal positive and negative frequencies. For example, for a second-order nonlinearity, a combination such as $(+3\omega_0 - 3\omega_0)$ will result in a harmonic contribution at dc, and for a cubic nonlinearity, the combinations $(+3\omega_0 + 3\omega_0 - 3\omega_0)$ and $(+3\omega_0 + \omega_0 - \omega_0)$ will result in harmonic contributions at $+3\omega_0$. The number of harmonic contributions generated depends only on the order of the nonlinearity and the number of input frequencies. Altering the specific frequencies which are included in the signal will, in no way, affect the number of the harmonic contributions.
2) *Interharmonic Contributions:* These contributions are generated by frequency combinations that do not follow the pattern of the harmonic contributions. For example, for a second-order nonlinearity, the frequency combination $(+3\omega_0 - \omega_0)$ will result in an interharmonic contribution at $+2\omega_0$, and, for a cubic term, the frequency combination $(+3\omega_0 + \omega_0 + \omega_0)$ will generate an interharmonic contribution at $+5\omega_0$. The resulting frequencies at which interharmonic contributions will fall depend entirely on the input harmonic vector $\mathbf{i}$. The phases of these contributions will vary depending on the phases of the specific input harmonics that gave rise to them. Omitting certain harmonics from the input signal will influence the number of the interharmonic contributions that fall at a given test frequency. Since the number and relative phase of these contributions will vary from frequency to frequency, the resulting bias will also vary. These contributions will introduce a bias in the form of a *scatter*, which can easily be mistaken for a *stochastic* effect.

The specific influence of the two types of contributions will depend on the order of the nonlinearity.

1) *Even-Order Terms:* The harmonic contributions will all fall at dc in this case, since they will be generated by pairs of equal positive and negative frequencies. Therefore, they will have no influence on the estimated FRF. In the case of interharmonic contributions, if the input signal contains only harmonics which are odd multiples of the fundamental frequency, then all the contributions will fall at even frequencies. Thus, by using an *odd harmonic multisine*, the linear and even-order contributions will be orthogonal in the frequency domain. The even harmonics can then be omitted from the data set used for estimation.

2) *Odd-Order Terms:* The harmonic contributions will fall at the input frequencies. They will always have the same phase as the original input frequency, since they are generated by the combination of that frequency with pairs of equal positive and negative frequencies, for which the phases cancel out. However, it should be noted that although the phases cancel out, this is not true for the amplitudes, which are multiplied together and the resulting amplitudes depend on the specific frequency combinations. The number of harmonic contributions generated will be equal for each test frequency and depends only on the order of the nonlinearity and the number of harmonics included in the input signal. They will, therefore, tend to introduce a bias, in the form of a *systematic offset* at each frequency of the measured output signal, which will also depend on the power spectrum of the input signal.

Restricting the input signal to contain only odd harmonics will ensure that the interharmonic contributions will only fall at odd frequencies, which will include the input frequencies themselves. The number and phase of the interharmonic contributions that fall at a given input frequency will depend on the specific input harmonics selected and their relative phases. Employing an odd harmonic signal will also ensure that the interharmonic contributions generated by odd- and even-order terms will be orthogonal in the frequency domain.

Harmonic and interharmonic contributions have been previously described in the literature as Type I and Type II contributions respectively [3], [4]. However, it was considered more appropriate to introduce a broader terminology, which will aid a comparison with related work in Sections IV and V of this paper.

III. Analysis of Harmonic Contributions

A. Motivation

Consider a multisine signal containing F harmonics of equal amplitude applied to the parallel nonlinear model shown in Fig. 1, with the linear element of the model set to a gain of one and the nonlinear element set to the cubic order. The nonlinear distortion of (1) at the test frequencies ω_k can be expressed as

$$Y_n(j\omega_k) = T_1 \angle U(j\omega_k) + T_2(\omega_k)\angle\theta(\omega_k) \tag{6}$$

where $T_2(\omega_k)$ is the cumulative magnitude of the interharmonic contributions that fall at the test frequencies, phase $\theta(\omega_k)$ and T_1 are the cumulative magnitude of the harmonic contributions, which is equal for each test frequency (given a flat-amplitude spectrum), and can be calculated from

$$T_1 = |U(j\omega_k)|^3 \times T_1^3 \tag{7}$$

where $U(j\omega_k)$ are the input Fourier coefficients and T_1^3 is the number of harmonic contributions generated at each test frequency. Evans *et al.* [3] showed that for the case of a cubic nonlinearity T_1^3 is equal to

$$T_1^3 = 3 + 6(F-1) \tag{8}$$

where the superscript term denotes the order of the nonlinearity n. Equations (6)–(8) can be used to separate the harmonic and interharmonic components that fall at the test frequencies and thus examine their interaction at those frequencies and in extent their overall influence on the system response. In fact, this methodology has been previously employed by the authors in [8] to examine the mechanisms at work when low CF signals are subjected to a cubic nonlinearity and to design low CF multisines that minimize the nonlinear distortion. This provided the motivation to investigate the behavior of harmonic contributions for higher orders of nonlinearity, with the aim of deriving explicit formulae to calculate the number of harmonic contributions that fall at the test frequencies for any odd-order nonlinearity.

B. Formulae

Consider the case of a cubic nonlinear element driven by a multisine signal that contains F harmonics. The frequency-domain output will consist of $(2F)^3$ contributions, generated by all possible combinations and permutations of each combination of three input frequencies. The combinations of input frequencies that result in harmonic contributions can be considered as an arrangement that comprises one input frequency and one frequency pair position which accommodates F pairs of equal positive and negative input frequencies, taken a pair at a time.

This is illustrated in Fig. 2 where $\mathbf{i}$ is a vector of harmonic numbers. For example, in the case of a 5 odd-harmonic multisine signal, $F = 5$, $\mathbf{i}$ = 1, 3, 5, 7, and 9, and, therefore, $\mathbf{i}_2 = 3$, $\mathbf{i}_F = 9$. To simplify the analysis that follows, the fundamental harmonic frequency $+\omega_0$ will be considered as the test frequency under investigation. This does not have any effect on the generality of the results obtained since the number of harmonic contributions that fall at a test frequency is equal for all test frequencies.

From Fig. 2, it can be seen that the frequency combinations are initiated by the combination of $+\omega_0$ with its own pair of positive and negative frequencies. The number of contributions resulting from this combination is simply 3!/2!. The remaining number of harmonic contributions is generated by the combinations of the remaining $F-$ 1 frequency pairs with $+\omega_0$, which results in 3!$\times$($F-$1) contributions. Adding the two different terms gives (8).

The same methodology can be applied to calculate the number of harmonic contributions for the case of a fifth-order nonlinearity. The procedure is shown in Fig. 3, where $\mathbf{i}_K \neq \mathbf{i}_Z$ and $[\mathbf{i}_K, \mathbf{i}_Z] \in \mathbf{i}$. In this case, there are two frequency pair positions that need to be filled by pairs of equal positive and negative frequencies. The first combination of frequencies is that comprising $+\omega_0$ with its own frequency pairs, which generates 5!/(3! $\times$ 2!) contributions. Then one frequency pair position is left free by a frequency pair of $+\omega_0$ and can now be occupied by the remaining $F-$ 1 frequency pairs resulting in (5!/2!)$\times$($F-$1) contributions. The second frequency pair position can now be left free by the other frequency pair of $+\omega_0$. There are two scenarios in this situation. 1) Frequency pairs of the same input frequency occupy both frequency pair positions, which result in 5!/(2! $\times$ 2!) $\times$ ($F-$ 1) contributions. 2) Both frequency pair positions are occupied by different

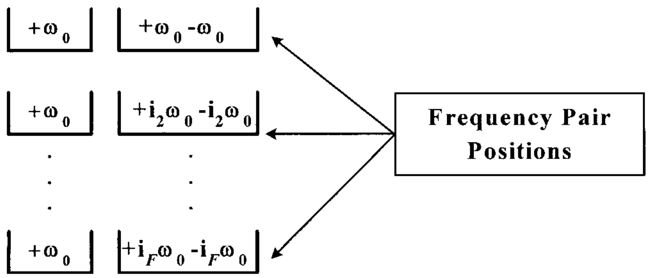

Fig. 2. Frequency combinations for harmonic contributions: cubic nonlinearity.

frequency pairs resulting in $5!/2\times(F-1\times F-2)$ contributions. The summation of the various terms leads to the Taylor series representation

$$T_1^5 = 10 + 90(F-1) + 60(F-1)(F-2). \tag{9}$$

Following the same methodology generated an expression for the case of a seventh-order nonlinearity

$$\begin{aligned} T_1^7 = 35 + 1190(F-1) + 2520(F-1)(F-2) \\ +840(F-1)(F-2)(F-3). \end{aligned} \tag{10}$$

From (8)–(10), it is obvious that the general expression for calculating the number of harmonic contributions T_1^n for any odd-order nonlinearity n will have the form

$$T_1^n = C_1^n + \sum_{k=2}^{(n+1)/2} \left[A_k^n \prod_{p=1}^{k-1} (F-p) \right] \quad n = 3, 5, 7 \ldots. \tag{11}$$

The first step in analyzing the general expression given in (11) begins with C_1^n, which is a constant term that depends only on the order of the nonlinearity n

$$C_1^n = \frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n+1}{2}-1\right)!}. \tag{12}$$

Equation (12) represents the number of harmonic contributions generated by the permutations of the test frequency under investigation (e.g., $+\omega_0$) with its own equal positive and negative frequency pairs. For example, in the case of a seventh-order n, C_1^7 can be calculated by taking all possible permutation terms in the combination $(+\omega_0 - \omega_0 + \omega_0 - \omega_0 + \omega_0 - \omega_0 + \omega_0)$, which simply results in 7!/(4!× 3!)= 35 contributions, as seen in (10).

The remaining unknown variable in (11) is A_k^n, which represents the coefficients of the $(F-p)$ terms. Although an explicit solution for the term coefficients A_k^n has not been found yet, explicit equations have been derived to calculate the first two and the last two term coefficients for any odd-order n, which are shown in the Appendix . Furthermore, a heuristic methodology has been developed by which any remaining term coefficient can be calculated. This methodology is based on the procedure followed to derive (9) (see Fig. 3), and it is also shown in the Appendix . Combining this methodology with the explicit formulae shown in the Appendix yielded the term coefficients for $n \leq 13$, which can be seen in Table I.

Recent work conducted by the authors in [9] demonstrated the use of the formulae derived in this section in examining the

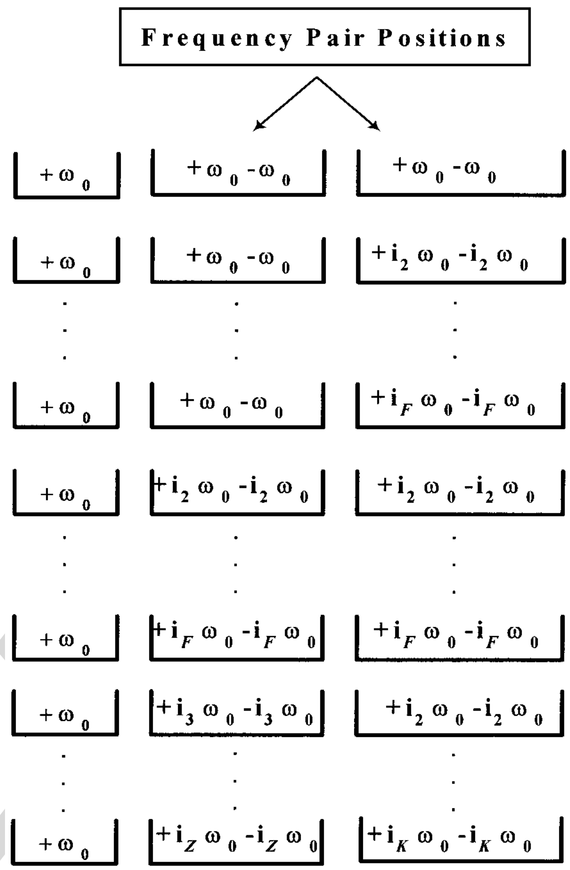

Fig. 3. Frequency combinations for harmonic contributions: fifth-order nonlinearity.

impact of odd-order nonlinearities ($n \leq 9$) on different types of multisine signals. This was achieved simply by modifying (7) to

$$T_1 = |U(j\omega_k)|^n \times T_1^n \tag{13}$$

where T_1^n is given by (11). Since the FRF estimate will depend on the type of multisine used, it is clear that these formulae contribute to a better understanding of the influence of static nonlinearities in the identification and modeling of linear systems. These formulae also offer new and important insights into the output properties of a wide class of nonlinearities that can be modeled using a power series expansion [6].

Based on the derived formulae, a comparison between different approaches in classifying nonlinear distortions will now be conducted, with the aim of clarifying the suitability of each approach in assessing the influence of static power-series nonlinearities on linear FRF measurements.

IV. Classification of Nonlinear Distortions

Two of the approaches under investigation, namely the *classic approach* and the Evans *et al.* approach, have already been presented in Section II. Two more approaches will now be considered in the analysis that follows.

A. Bussgang et al. Approach

This approach was first proposed by Bussgang *et al.* [2] and was further examined by Billings and Tsang [10], where it was

TABLE I
TERM COEFFICIENTS FOR ODD-ORDER NONLINEARITIES

Nonlinearity (n)	C_1^n	A_2^n	A_3^n	A_4^n	A_5^n	A_6^n	A_7^n
3	3	6					
5	10	90	60				
7	35	1190	2520	840			
9	126	15750	81900	75600	15120		
11	462	212982	2494800	4851000	2494800	332640	
13	1716	2942940	75087012	569368800	245044800	90810720	8648640

employed to investigate the properties of higher-order frequency response functions. According to the Bussgang *et al.* approach, the contributions generated by frequency combinations of the form $(+\omega_0+\omega_0+\omega_0)$, $(+3\omega_0+3\omega_0-\omega_0)$, $(+\omega_0+\omega_0-\omega_0)$, and $(+3\omega_0+\omega_0-3\omega_0)$ were classified as *harmonic*, *intermodulation, gain compression/expansion*, and *desensitization* contributions, respectively.

It is obvious that *harmonic* contributions possess exactly the same properties as the nth *harmonic* contributions defined under the classic approach and, therefore, they can be included in the wider class of interharmonic contributions according to the Evans *et al.* approach. The *intermodulation* contributions will fall at both the excited and the nonexcited frequencies at the output, depending on the specific frequency combinations that gave rise to them. For example, frequency combinations of the form $(+3\omega_0-\omega_0-\omega_0)$ will generate *intermodulation* contributions at $+\omega_0$, which is an excitation frequency, whereas frequency combinations of the form $(+3\omega_0+3\omega_0-\omega_0)$ will generate *intermodulation* contributions at $+5\omega_0$, which is not an excitation frequency. It can be seen that this class of contributions has the same properties as the interharmonic contributions.

Gain compression and *gain expansion* are terms that are used to describe the nonlinear variation in the gain of a system at a specific frequency, as the input amplitude at that frequency varies. For example, as the amplitude of $+\omega_0$ changes, frequency combinations of the form $(+\omega_0+\omega_0-\omega_0)$ will either compress or expand the gain of the system at $+\omega_0$ in a nonlinear manner. It is clear that *gain compression/expansion* contributions belong to the class of harmonic contributions according to the Evans *et al.* approach. *Desensitization* is a term used to describe the transfer of energy from one input frequency to another, a common phenomenon in nonlinear systems [11]. The *desensitization* effect can be illustrated by frequency combinations of the form $(+3\omega_0+\omega_0-3\omega_0)$. In this case, the system amplitude response at $+\omega_0$ will be affected by a change in the amplitude of $+3\omega_0$. It can be seen that *desensitization* contributions also belong to the class of harmonic contributions.

B. Schoukens et al. Approach

The Schoukens *et al.* [5] approach of classifying nonlinear distortions is very similar to the Evans *et al.* approach. Their work involved the exploitation of the properties of harmonic and interharmonic contributions to establish a framework to be used when conducting linear FRF measurements in the presence of nonlinear distortions. The kernel of their approach is based on the elimination of the influence of interharmonic contributions at the excitation frequencies by using random phase excitations. In this case the interharmonic contributions adopt Gaussian noise properties, thus termed by Schoukens *et al.* as *stochastic* contributions. However, the harmonic contributions are always present at the excitation frequencies and will introduce a bias in the form of an offset. Hence, Schoukens *et al.* termed harmonic contributions as *systematic* contributions.

Table II can be constructed as a brief summary of the approaches considered in this paper.

V. COMPARISON OF APPROACHES

From the analysis conducted in Section IV, it can be seen that the Bussgang *et al.* approach has many similarities to the Evans *et al.* approach. It is clear that the class of *gain expansion/compression* contributions combined with that of *desensitization* contributions form the class of harmonic contributions. This is an interesting observation, which illustrates that harmonic contributions can be further divided into *gain expansion/compression* contributions and *desensitization* contributions. However, the question arises of the benefits that such a division could provide. This can be investigated by considering the analysis conducted in Section III. It is obvious that the number of *gain compression/expansion* contributions can be calculated from (12). It follows that the remaining number of harmonic contributions can be considered as *desensitization* contributions.

By examining Table I and (11), it can be seen that the number of *desensitization* contributions will always be much larger than that of the *gain compression/expansion* contributions for any odd-order nonlinearity. It is obvious that, for input signals with many frequencies, the *gain compression/expansion* contributions will have negligible influence at the test frequencies and will be dominated by the *desensitization* contributions. Therefore, dividing the harmonic contributions into *compression/expansion* and *desensitization* contributions does not provide any additional benefits in the examination of the effect of harmonic contributions on FRF measurements.

Interharmonic contributions can also be divided into two different classes, namely the *harmonic* and the *intermodulation*

TABLE II
SUMMARY OF CLASSIFICATION APPROACHES

Approach	Frequency Combinations / Class of Contributions				
	$+\omega_0+\omega_0$ $+\omega_0$	$+\omega_0+\omega_0$ $-\omega_0$	$+3\omega_0-3\omega_0$ $-\omega_0$	$+3\omega_0+3\omega_0$ $-\omega_0$	$+3\omega_0-\omega_0$ $-\omega_0$
Classic	*n*th *harmonic*	inter-modulation	inter-modulation	inter-modulation	inter-modulation
Bussgang	*harmonic*	gain compression/ expansion	desensitisation	inter-modulation	inter-modulation
Schoukens	stochastic	systematic	systematic	stochastic	stochastic
Evans	inter-harmonic	harmonic	harmonic	inter-harmonic	inter-harmonic

contributions. However, the benefits of separating the interharmonic contributions when studying their influence on FRF measurements are negligible, since only the *intermodulation* contributions will fall at the test frequencies. Moreover, if the input signal contains many harmonics, the number of the *harmonic* contributions will only be a very small fraction of the total number of the *intermodulation* contributions and, therefore, the influence of *harmonic* contributions on the overall system response will be negligible.

The Schoukens *et al.* and the Evans *et al.* approaches only have terminology differences. It is clear that *systematic* and harmonic contributions are generated by the same type of frequency combinations and therefore they possess the same properties. A close agreement also exists for the *stochastic* and the interharmonic contributions. In this case, the *stochastic* contributions can be considered as interharmonic contributions with random phases since the term "stochastic" is used to describe the noise-like properties of the interharmonic contributions, which are imposed by the random phases of the input signal.

The overall analysis indicates that both the Evans *et al.* approach and the Schoukens *et al.* approach are suitable for examining the behavior of linear FRF measurements in the presence of nonlinear distortions and should be preferred over the classic approach and the Bussgang *et al.* approach. The Evans *et al.* approach, however, can be considered as broader since it also applies for signals that do not possess random phases.

VI. CONCLUSION

The output properties of static power-series nonlinearities driven by periodic multiharmonic inputs have been examined in this paper. The analysis was based on the classification of nonlinear distortions into harmonic and interharmonic contributions. It was shown that harmonic contributions possess properties that can be used to examine the effect of a cubic nonlinearity on linear FRF measurements. This provided the motivation to investigate the characteristics of harmonic contributions for higher orders of nonlinearity.

As a result, explicit formulae have been derived, by which the number of harmonic contributions that fall at the test frequencies can be calculated for odd-order nonlinearities up to, and including, the ninth order. Although an analytic solution for any odd-order nonlinearity is yet to be found, a heuristic methodology has been developed that solves this problem. It was shown that the derived formulae offer new insights into the output properties of static nonlinearities and can be used to examine their effect on linear system identification.

Based on these formulae, different approaches in classifying nonlinear distortions were then compared, with the aim of clarifying their suitability for assessing the effect of system nonlinearities on linear FRF measurements. These were termed the classic approach, the Bussgang *et al.* approach, the Evans *et al.* approach, and the Schoukens *et al.* approach. The analysis raised some interesting issues on the use of each approach when studying the contributions generated by periodic multiharmonic signals.

It was shown that the classic approach has some value for inputs made up of only a few harmonics but can prove misleading in the case of multiharmonic signals. The Bussgang *et al.* approach is also useful for signals containing a small number of harmonics but does not work as well for inputs with many harmonics. The analysis illustrated that, in this case, the Evans *et al.* approach works much better and should be preferred over the two approaches. It was finally shown that the Schoukens *et al.* approach is closely related to the Evans *et al.* approach, but it is most effective when studying the contributions generated by multiharmonic inputs with random phases.

APPENDIX
TERM COEFFICIENTS A_k^n

Given a set of frequency combinations **S** required to calculate a term coefficient A_k^n in (11), **S** will be initiated by the frequency combination where $(k-1)$ different frequency pairs occupy only one pair position each, with pairs of ω_0 completing all remaining positions. To calculate the permutations of this initial combination, the number of positive $\omega_0(Z_k^n)$ must be identified

$$Z_k^n = \frac{n-2k+3}{2}. \tag{14}$$

From (14), it follows that the number of negative ω_0 that are present in this initial combination is equal to $(Z_k^n - 1)$. All remaining frequency combinations are determined by considering that, for a given n, $(Z_k^n - 1)$ positions initially occupied by pairs of ω_0 will be replaced one at a time by $(k-1)$ available frequency pairs (see Fig. 3 for an example) to form $(Z_k^n - 1)$ combination subsets, with each subset having N_{i+1}^k frequency combinations, where

$$N_{i+1}^k = \frac{(k+i-2)!}{i!(k-2)!} \; i = 0, 1, 2, \ldots, (Z_k^n - 1). \tag{15}$$

From (14) and (15), a preliminary general expression for A_k^n has been derived

$$\begin{aligned} A_k^n = \frac{1}{(k-1)!} \times \Bigg[&\sum_{j=1}^{N_1^k} \frac{n!}{(Z_k^n)!(Z_k^n - 1)!} \\ &+ \sum_{j=1}^{N_2^k} \frac{n!}{(Z_k^n - 1)!(Z_k^n - 2)!\,[\mathrm{count}(P_j)]!} + \cdots \\ &+ \sum_{j=1}^{N_{Z_k^n - 1}^k} \frac{n!}{(Z_{n+1/2}^n)!(Z_{n+1/2}^n - 1)!\,[\mathrm{count}(P_j)]!} \Bigg]. \end{aligned} \tag{16}$$

The factor [count (P_j)] represents all the remaining factorials that need to be present in the denominators of all the terms in (16). This factor can be found by identifying all different groups of identical frequencies, other than ω_0, that are present in combination P_j. Then, counting the number of frequencies in each group will give the missing factorials. Although an analytical solution for A_k^n remains an open problem, explicit formulae have been derived to calculate the first two and last two coefficients for any n, as follows:

$$\begin{aligned} &A_2^n \\ &= \frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2-1})!} \\ &\quad + \sum_{i=1}^{n-1/2} \frac{n!}{(\frac{n-1}{2-i})!(\frac{n-1}{2-i-1})!\,[(1+i)!]^2} \end{aligned} \tag{17}$$

$$\begin{aligned} &A_3^n \\ &= \frac{1}{(k-1)!} \times \Bigg[\frac{n!}{\left(\frac{n-3}{2!}\right)} !(\frac{n-3}{2-1})! + + \sum_{i=2}^{n-3/2} \sum_{j=2}^{i+1} \\ &\quad \times \frac{n!}{(\frac{n-1}{2-i})!(\frac{n-1}{2-i-1})!\,[(i-(j-2))!]^2\,[(j-1)!]^2} \Bigg] \end{aligned} \tag{18}$$

$$\begin{aligned} &A_{n+1/2-1}^n \\ &= \frac{n!\,(n+1)}{(\frac{n-3}{2})! \times 8} \end{aligned} \tag{19}$$

$$\begin{aligned} &A_{n+1/2}^n \\ &= \frac{n!}{(\frac{n+1}{2-1})!} \end{aligned} \tag{20}$$

The results in Table I for $n \leq 9$ follow directly from (17)–(20) and by using (14)– (16), any remaining term coefficients for $n > 9$ were calculated.

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