Automatica 47 (2011) 363–367

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Frequency domain subspace-based identification of discrete-time power spectra from uniformly spaced measurements☆

Hüseyin Akçay*

Department of Electrical and Electronics Engineering, Anadolu University, Eskisehir 26470, Turkey

ARTICLE INFO

Article history:
Received 2 December 2009
Received in revised form
18 September 2010
Accepted 21 September 2010
Available online 26 November 2010

Keywords:
System identification
Power spectrum
Subspace methods
Uniformly spaced frequencies
Consistency

ABSTRACT

In this paper, we revisit the problem of identifying multi-input/multi-output linear-time-invariant discrete-time systems from measured power spectrum data on uniform grids of frequencies studied by Van Overschee, De Moor, Dehandschutter, and Swevers (1997). We show that the algorithm proposed by these authors is not consistent. Then, we propose an interpolatory identification algorithm which is strongly consistent when the corruptions in the spectrum measurements have a bounded covariance function. The performance of the proposed algorithm is demonstrated in a simulation example.

1. Introduction

Identification of multi-input/multi-output systems from measured power spectrum is a problem arising in many applications. For example, in vehicle dynamics *linear shape filters* are designed to model stochastic road disturbances experienced by a vehicle moving forward (Türkay & Akçay, 2005). In active vibration suppression, the *spectral factor* of the disturbance source plays an important role in designing optimal feedback controller (Fraanje, Verhaegen, Doelman, & Berkhoff, 2004).

Thus there is a need for algorithms to estimate minimum-phase spectral factors directly on the basis of power spectrum data in time or frequency domains. To tackle this problem, frequency-domain subspace identification algorithms were proposed in Akçay and Türkay (2004), Hinnen, Verhaegen, and Doelman (2005) and Van Overschee et al. (1997). When the measurements are noise-free, the algorithm in Van Overschee et al. (1997) yields the unknown minimum-phase spectral factor exactly from finite amounts of data. A key issue in Hinnen et al. (2005) was to ensure positivity of the estimated spectrum when recorded data are finite and corrupted by noise and unmodelled dynamics. In Akçay and Türkay (2004), a subspace algorithm which uses power spectrum measurements with non-uniformly spaced frequencies was developed. This algorithm is *strongly consistent* if the noise process has bounded moments of order four and its covariance function is known *a priori*.

A parametric or model-based approach to fitting a linear discrete-time power spectrum to given measured power spectrum samples uses a non-linear least-squares criterion, which is optimized by an iterative non-linear search in the parameter space. Discussion of parametric as well as non-parametric methods, which mostly use time-domain data, can be found in Kay (1988). Drawbacks of this approach are convergence problems and difficulty of parameterizing multi-input/multi-output systems.

This paper studies the consistency problem for the identification algorithm proposed in Van Overschee et al. (1997). Although power spectrum estimates delivered by this algorithm are exact on finite noise-free data sets, they are not consistent when the measurements are corrupted by random noise as demonstrated in this paper. However, we will show that a modified form of this algorithm is strongly consistent when the noise covariance function is bounded.

The paper is organized as follows. In Section 2, we formulate the identification problem. In Section 3, we present our subspace-based algorithm and show that this algorithm is not only strongly consistent, but also recovers finite-dimensional rational spectra given finite amounts of noise-free data. In Section 4, the properties of the proposed algorithm and the algorithm in Van Overschee et al. (1997) are studied by means of a numerical example. In particular, failure of the identification algorithm presented in Van Overschee et al. (1997) is demonstrated. Section 5 concludes the paper.

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Alessandro Chiuso under the direction of Editor Torsten Söderström.

* Tel.: +90 222 3350580x6459; fax: +90 222 323 9501.
E-mail address: huakcay@anadolu.edu.tr.

0005-1098/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2010.10.051

1.1. Notation

We end this introduction with some remarks on notation. The letters $\mathbf{R}$ and $\mathbf{C}$ denote the fields of real and complex numbers, respectively. Let I_n denote the n by n identity matrix. The k by l matrix of zeros is denoted by $0_{k\times l}$. Let $X^T, \overline{X}, X^H$, ReX, ImX denote respectively the transpose, the complex conjugate, the complex conjugate transpose, the real and the imaginary parts of a given complex matrix X. Let $X \otimes Y$ denote the Kronecker product of two given matrices $X \in \mathbf{C}^{m\times n}$ and $Y \in \mathbf{C}^{p\times q}$ defined as

$$X \otimes Y = \begin{bmatrix} X_{11}Y & \cdots & X_{1n}Y \\ \vdots & \ddots & \vdots \\ X_{m1}Y & \cdots & X_{mn}Y \end{bmatrix} \in \mathbf{C}^{mp\times nq}.$$

The Moore–Penrose pseudo inverse of a given full-column rank matrix X is denoted by $X^\dagger = (X^TX)^{-1}X^T$. Let $\|X\|_F = [\text{trace}\,(X^HX)]^{\frac{1}{2}}$ denote the Frobenius norm of a given matrix X. For a given complex measurable matrix $G(z)$ on the unit circle, its L_∞ norm is defined by

$$\|G\|_\infty = \text{ess sup}\{\sigma_{\max}(G(z)) : |z| = 1\}$$

where $\sigma_{\max}$ denotes the largest singular value. Let $\mathbf{E}(x)$ denote the expected value of a given random variable x. Let δ_{kl} denote the Kronecker delta, *i.e.,* $\delta_{kl} = 1$ if $k = l$ and 0 otherwise.

2. Problem formulation

Consider a multi-input/multi-output, linear-time-invariant, square, discrete-time system represented by the state–space equations:

$$\begin{aligned} x[t+1] &= Ax[t] + Bu[t], \\ y[t] &= Cx[t] + Du[t] \end{aligned} \tag{1}$$

where $x[t] \in \mathbf{R}^n$ is the state, $u[t] \in \mathbf{R}^m$, and $y[t] \in \mathbf{R}^m$ are respectively the input and the output of the system. The transfer function of the system in Eq. (1) denoted by $G(z)$ is calculated as

$$G(z) = D + C(zI_n - A)^{-1}B. \tag{2}$$

We summarize the requirements on $G(z)$ in the following:

Assumption 2.1. The system in Eq. (1) is stable and strictly minimum phase: all eigenvalues of A and $A - BD^{-1}C$ lie strictly inside the unit circle. The pairs $\{A, B\}$ and $\{C, A\}$ are controllable and observable, respectively. All eigenvalues of A are nonzero and distinct.

Thus, the system represented by Eq. (1) is a *minimal stochastic system*. The assumption that all eigenvalues of A are nonzero and distinct is technical and was also made in Akçay and Türkay (2004) and Van Overschee et al. (1997). The algorithms in this paper and Akçay and Türkay (2004), Van Overschee et al. (1997) retrieve A in Eq. (1) in the *Jordan canonical form*. See, Lemma 4 in Akçay and Türkay (2004). This choice imposes certain block structures on the estimated matrices. All such representations are internal and yield the same spectrum estimate when the data are noise-free. See, Lemma 5 in Akçay and Türkay (2004) and the preceding calculations. Lemma 5 and these calculations make use of the fact that matrices in Jordan canonical form with the same block sizes and types commute with respect to matrix multiplication and are also closed with respect to transposing and inversion operations. This fact was established in Akçay and Türkay (2004) for the distinct eigenvalues case, *i.e.,* Lemma 4 in Akçay and Türkay (2004).

It is possible to extend Lemma 4 to more general situations than the distinct eigenvalues case. But, this is unnecessary since the Jordan canonical form is not numerically stable and a slight perturbation of A will lead to distinct eigenvalues if A has repeated eigenvalues. Thus, small perturbations in A will lead to small perturbations in the estimated spectrum. When the data sets are finite and corrupted by unmodelled dynamics, the structural relations mentioned above are likely to be destroyed and it will be necessary to introduce a selection criterion. This problem is further discussed in Hinnen et al. (2005) where robust procedures are introduced. The consistency properties, however, do not change since they are asymptotic in nature. See, Theorem 7 in Akçay and Türkay (2004).

In this paper and Akçay and Türkay (2004), Hinnen et al. (2005) and Van Overschee et al. (1997) it is assumed that $G(z)$ is square. There are situations in which the number of noise inputs is smaller than the number of output channels. This subject warrants further work.

Assuming that $u[t]$ is zero-mean unity variance white-noise process, the power spectrum associated with Eq. (1) denoted by $S(z)$ is defined as

$$S(z) = G(z)G^T(z^{-1}). \tag{3}$$

The system in Eq. (1) is called the *innovation form*, unity variance, minimum phase spectral factor associated with the power spectrum $S(z)$.

From Eq. (3) and Assumption 2.1, note that

$$S(e^{i\theta}) > 0, \quad \text{forall}\,\theta. \tag{4}$$

This is the *positive realness* condition, and it imposes a constraint on the given spectrum samples S_k, *i.e.,* $S_k > 0$ for each k, as well as on the identified power spectrum denoted by $\widehat{S}_N(z)$. Several procedures to ensure the positivity of the estimated spectrum are outlined in the works Hinnen et al. (2005) and Van Overschee et al. (1997). Again, this problem is not pertinent to our consistency analysis. The measurement noise assumption is rephrased from McKelvey, Akçay, and Ljung (1996) and Van Overschee et al. (1997) as follows:

Assumption 2.2. The noise sequence $\tilde{s}_k$ corrupting the spectrum samples is a zero-mean complex white-noise process with a covariance function satisfying

$$\mathbf{E}\begin{bmatrix} \text{Re}\,\tilde{s}_k \\ \text{Im}\,\tilde{s}_k \end{bmatrix} [\text{Re}\,\tilde{s}_l^T \ \text{Im}\,\tilde{s}_l^T] = \begin{bmatrix} \frac{R_k}{2} & 0 \\ 0 & \frac{R_k}{2} \end{bmatrix} \delta_{kl} \tag{5}$$

and

$$\sup_k \|R_k\|_\infty < \infty. \tag{6}$$

The problem studied in this paper is stated as follows:

Given: $M + 1$ noisy samples $S_k \in \mathbf{C}^{m\times m}$ of the power spectrum $S(z)$ evaluated at $M + 1$ equidistantly spaced frequencies in the unit circle:

$$S_k = S(e^{i\frac{\pi(k-1)}{M}}) + \tilde{s}_k, \quad k = 1, \ldots, M+1, \tag{7}$$

where $\tilde{s}_k$ is a zero-mean complex white-noise process with a covariance function satisfying Eqs. (5) and (6) and the minimum phase spectral factor of $S(z)$ satisfies Assumption 2.1,

Find: a quadruplet $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ such that the estimated power spectrum

$$\widehat{S}_M(z) = \widehat{G}(z)\widehat{G}^T(z^{-1}) \tag{8}$$

is strongly consistent, *i.e.,*

$$\lim_{M\to\infty} \|\widehat{S}_M - S\|_\infty = 0, \quad \text{w.p.1} \tag{9}$$

where

$$\widehat{G}(z) = \widehat{C}(zI_n - \widehat{A})^{-1}\widehat{B} + \widehat{D}. \tag{10}$$

We also require the algorithm to produce the true model if the noise is zero given a finite amount of data M, *i.e.,* there exists an

$M_0 < \infty$ such that

$$\|\widehat{S}_M - S\|_\infty = 0, \quad \text{for all } M \geq M_0. \tag{11}$$

An identification algorithm which satisfies Eq. (11) when noise is zero is called *interpolatory*. In this paper, we present an algorithm with the properties in Eqs. (9) and (11).

3. Subspace identification algorithm

As in Akçay and Türkay (2004), Hinnen et al. (2005) and Van Overschee et al. (1997), we begin by splitting $S(z)$ into the so-called spectral summands as follows

$$S(z) = E + C(zI_n - A)^{-1}F + F^T(z^{-1}I_n - A^T)^{-1}C^T$$

with $E = CPC^T + DD^T$ and $F = APC^T + BD^T$ where P is the solution of the discrete-time Lyapunov equation $P = APA^T + BB^T$. Thus, the problem of identifying a spectral factor from spectrum samples is reduced to identifying the spectral summands.

Next, since $S(z)$ has a real-valued impulse response, its restriction to $[0, \pi]$ can be extended to $[\pi, 2\pi]$ from

$$S(z^{-1}) = S^H(z), \quad |z| = 1.$$

Hence, we let $N = 2M$ and $\omega_k = \frac{2\pi(k-1)}{N}$, $z_k = \mathrm{e}^{\mathrm{i}\omega_k}$ for $k = 1, \ldots, N$. Let p and r be two fixed positive integers satisfying the inequalities $p > 2n$, $r \leq p$, and $p + r \leq N$.

Let

$$\mathcal{F}_r = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_N & \cdots & z_N^r \end{bmatrix} \otimes I_m,$$

$$\Omega_p(\omega_k) = \begin{bmatrix} 1 \cdots z_k^{p-1} \end{bmatrix}^T,$$

$$\widehat{\mathcal{G}}_p = \frac{1}{\sqrt{N}}\left[\Omega_p(\omega_1) \otimes S_1 \cdots \Omega_p(\omega_N) \otimes S_N\right],$$

$$\widetilde{\mathcal{N}}_p = \frac{1}{\sqrt{N}}\left[\Omega_p(\omega_1) \otimes \tilde{s}_1 \cdots \Omega_p(\omega_N) \otimes \tilde{s}_N\right],$$

$$\mathcal{O}_p = \begin{bmatrix} C & F^T(A^T)^{p-1} \\ \vdots & \vdots \\ CA^{p-1} & F^T \end{bmatrix}.$$

The following formula is obtained from the derivations in Akçay (2010) by straightforward modifications

$$\widehat{\mathcal{G}}_p\mathcal{F}_r = \mathcal{O}_p\mathcal{D}_{N,p,r}\Delta_r + \widetilde{\mathcal{N}}_p\mathcal{F}_r \tag{12}$$

where

$$\Delta_r = \begin{bmatrix} F & \cdots & A^{r-1}F \\ (A^T)^{r-1}C^T & \cdots & C^T \end{bmatrix},$$

$$\mathcal{D}_{N,p,r} = \begin{bmatrix} \mathcal{I}_N & 0 \\ 0 & (A^T)^{N-p-r}\mathcal{I}_N^T \end{bmatrix},$$

$$\mathcal{I}_N = (I_n - A^N)^{-1}.$$

In the derivation of this formula, Lemma 5.1 in Akçay (2010) or equivalently Theorem 1 in Van Overschee et al. (1997) was used.

The following (incorrect) formula

$$\widehat{\mathcal{G}}_p\mathcal{F}_r = \mathcal{O}_p\mathcal{D}_{p,r}\Delta_r + \widetilde{\mathcal{N}}_p\mathcal{F}_r \tag{13}$$

where

$$\mathcal{D}_{p,r} = \begin{bmatrix} \mathcal{I}_N & 0 \\ 0 & \mathcal{I}_N^T \end{bmatrix}$$

was derived in Van Overschee et al. (1997). See, Eq. (5) in Van Overschee et al. (1997). Eq. (13) captures the so-called main theorem in Van Overschee et al. (1997), *i.e.*, Theorem 2. It differs from Eq. (12) by the factor $(A^T)^{N-p-r}$ in the right-bottom block of $\mathcal{D}_{N,p,r}$. The discrepancy between Eqs. (12) and (13) stems from an error in the derivation of Eq. (13) in Van Overschee et al. (1997). See, Eq. (B.1) in Van Overschee et al. (1997).

If $p \geq 2n$ and $r \geq 2n$, then $\mathcal{O}_p$ and Δ_r both have rank $2n$. Consequently, $\mathcal{O}_p\mathcal{D}_{N,p,r}\Delta_r$ has rank $2n$ if and only if $\mathcal{D}_{N,p,r}$ has rank $2n$. The latter matrix has rank $2n$ since it is block diagonal and all the eigenvalues of A lie inside the unit circle. This fact can be used to extract the observability range space of a realization of $S(z)$ from the singular-value decomposition (SVD) of $\widehat{\mathcal{G}}_p\mathcal{F}_r$ (Akçay & Türkay, 2004; Van Overschee et al., 1997).

For fixed p and r and as $N \to \infty$, $(A^T)^{N-p-r}$ tends to zero geometrically fast and Eq. (12) satisfies

$$(\widehat{\mathcal{G}}_p - \widetilde{\mathcal{N}}_p)\mathcal{F}_r \to \begin{bmatrix} C \\ \vdots \\ CA^{p-1} \end{bmatrix}[F \cdots A^{r-1}F]. \tag{14}$$

The right-hand side of Eq. (14) has rank n. This means that the subspace algorithm in Van Overschee et al. (1997) is not strongly consistent. This conclusion is reached by noting the facts that the convergence rate in Eq. (14) is geometric in N, Theorem 2 in Van Overschee et al. (1997) assesses a constant rank of $2n$ to the left-hand side of Eq. (14), and the consistency analysis in Van Overschee et al. (1997) relies upon the latter rank being $2n$. The error in Eq. (B.1) does not destroy the interpolation property of the algorithm in Van Overschee et al. (1997) since the error term is trapped into a similarity transformation. If $N - p - r$ remains bounded for all N, the algorithm in Van Overschee et al. (1997) is still interpolatory. But, consistency can be achieved only if the noise covariance information is available (Akçay & Türkay, 2004; McKelvey et al., 1996).

The convergence in Eq. (14) was also noted in Hinnen et al. (2005) where it was suggested that the left-hand side could be utilized to extract some initial estimates of A and C to be used in iterations for searching optimal values of A and C while ensuring positivity of the estimated spectrum.

A legitimate question is whether it is possible to derive a subspace algorithm which is both interpolatory and strongly consistent. The algorithms in Hinnen et al. (2005) and Van Overschee et al. (1997) have precisely either one of these properties. The answer is surprisingly simple as we will see next.

A dual factorization formula to Eq. (12) is obtained from the derivations in Akçay (2010) by straightforward modifications as

$$\Psi_p(\overline{\mathcal{G}}_p - \overline{\mathcal{N}}_p)\mathcal{F}_r\Psi_r = \mathcal{O}_p\begin{bmatrix} A^{N-p-r}\mathcal{I}_N & 0 \\ 0 & \mathcal{I}_N^T \end{bmatrix}\Delta_r \tag{15}$$

where

$$\overline{\mathcal{G}}_p = \frac{1}{\sqrt{N}}[\Omega_p(\omega_1) \otimes \overline{S}_1 \cdots \Omega_p(\omega_N) \otimes \overline{S}_N],$$

$$\overline{\mathcal{N}}_p = \frac{1}{\sqrt{N}}[\Omega_p(\omega_1) \otimes \overline{\tilde{s}}_1 \cdots \Omega_p(\omega_N) \otimes \overline{\tilde{s}}_N],$$

$$\Psi_k = \begin{bmatrix} 0 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & 0 \end{bmatrix} \in \mathbf{R}^{km \times km}.$$

Now, let

$$\mathcal{Y} = \widehat{\mathcal{G}}_p\mathcal{F}_r + \Psi_p\overline{\mathcal{G}}_p\mathcal{F}_r\Psi_r. \tag{16}$$

Then, from Eqs. (12), (15) and (16)

$$\mathcal{Y} = \mathcal{O}_p\mathcal{E}_{N,p,r}\Delta_r + \widetilde{\mathcal{N}}_p\mathcal{F}_r + \Psi_p\overline{\mathcal{N}}_p\mathcal{F}_r\Psi_r \tag{17}$$

where

$$\mathcal{E}_{N,p,r} = \left\{ I_{2n} + \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}^{N-p-r} \right\} \mathcal{D}_{p,r}. \tag{18}$$

Since the eigenvalues of A are inside the unit circle, $\mathcal{E}_{N,p,r}$ is always nonsingular. Hence, the range spaces of $\mathcal{O}_p$ and $\mathcal{Y}$ are equal when the data are noise-free. Moreover, in this case $\mathcal{Y} \to \mathcal{O}_p \Delta_r$ as $N \to \infty$.

It is easy to show (Akçay, 2010) that the entries of $\mathcal{Y}$ satisfy

$$\mathcal{Y}_{kl} = \hat{s}_{k+l-1} + \hat{s}_{N-p-r+k+l-1} \tag{19}$$

for $1 \le k \le p$ and $1 \le l \le r$ where $\hat{s}_k$ is the N-point inverse discrete Fourier transform of S_k:

$$\hat{s}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}jk} S_j. \tag{20}$$

Eq. (17) with the left-hand side computed from Eq. (19) is the basic relation which our subspace identification algorithm relies upon. In the sequel, we will present a strongly consistent identification algorithm. It should be noted that the parameters p and r can be chosen freely subject to the inequalities $p > 2n, r \ge 2n, p+r \le N$. In particular, $\mathcal{Y}$ does not have to be block square matrix as claimed in Hinnen et al. (2005). Combining Eq. (17) with the stages of the identification algorithm developed in Akçay and Türkay (2004) we propose:

Algorithm 1. (1) Given the spectrum samples S_k, expand the data according to $S_{M+k} = S_{M-k+2}$, $k = 2, \ldots, M$ to obtain signals of lengths $N = 2M$.

(2) Fix p and r as $p > 2n$, $r \ge 2n$, and $p + r \le N$ and for $1 \le k < p + r$ and $N - p - r < k < N$, compute the Fourier coefficients in Eq. (20).

(3) Calculate the SVD of $\mathcal{Y}$ in Eq. (19)

$$\mathcal{Y} = [U_{2n}\ U_n'] \begin{bmatrix} \Sigma_{2n} & 0 \\ 0 & \Sigma_n' \end{bmatrix} \begin{bmatrix} V_{2n} \\ V_n' \end{bmatrix} \tag{21}$$

where Σ_{2n} contains the $2n$ largest singular values.

(4) Determine the system order n by inspecting the singular values.

(5) With U_{2n} defined by Eq. (21) and J_u and J_d by

$$J_u = [0_{(p-1)m \times m}\ I_{(p-1)m}],$$
$$J_d = [I_{(p-1)m}\ 0_{(p-1)m \times m}],$$

calculate $\widetilde{A} = (J_d U_{2n})^{\dagger} J_u U_{2n}$.

(6) Put $\widetilde{A}$ into the following Jordan canonical form:

$$\widetilde{A} = [\Pi_c\ \Pi_{ac}] \begin{bmatrix} \Sigma_c & 0 \\ 0 & \Sigma_{ac} \end{bmatrix} [\Pi_c\ \Pi_{ac}]^{-1}$$

where the eigenvalues of Σ_c lie inside the unit circle.

(7) Let $\widehat{A} = \Sigma_c$, $\widehat{C} = J_f U_{2n} \Pi_c$ where $J_f = [I_m\ \ 0_{m \times (p-1)}]$.

(8) For $\widehat{E}$ and $\widehat{F}$, solve the least-squares problem

$$\min_{\widehat{E},\widehat{F}} \sum_{k=1}^{M+1} \| \widehat{\chi}(z_k)\widehat{F} + \widehat{F}^T \widehat{\chi}^T(z_k^{-1}) + \widehat{E} - S_k \|_F^2$$

where $\widehat{\chi}(z) = \widehat{C}(zI_n - \Sigma_c)^{-1}$.

(9) Solve the Riccati equation for $\widehat{P}$:

$$\widehat{P} = \widehat{A}\widehat{P}\widehat{A}^T + (\widehat{F} - \widehat{A}\widehat{P}\widehat{C}^T)(\widehat{E} - \widehat{C}\widehat{P}\widehat{C}^T)^{-1} \times (\widehat{F} - \widehat{A}\widehat{P}\widehat{C}^T)^T$$

and calculate $\widehat{B}$ and $\widehat{D}$ from

$$\widehat{B} = (\widehat{F} - \widehat{A}\widehat{P}\widehat{C}^T)(\widehat{E} - \widehat{C}\widehat{P}\widehat{C}^T)^{-\frac{1}{2}},$$
$$\widehat{D} = (\widehat{E} - \widehat{C}\widehat{P}\widehat{C}^T)^{\frac{1}{2}}.$$

(10) Calculate $\widehat{G}(z)$ and $\widehat{S}_M(z)$ from Eqs. (10) and (8).

Table 1
The 4th and the 5th largest singular values of $\widehat{\mathcal{G}}_9\mathcal{F}_8$ on noise-free data sets.

M	σ_4	σ_5
16	2.363	1.76×10^{-1}
32	3.445	2.17×10^{-2}
64	3.155	1.03×10^{-3}
128	3.146	3.13×10^{-6}
256	3.146	2.42×10^{-11}
512	3.146	6.53×10^{-16}

The main result of this paper is the following:

Theorem 1. *Suppose that Assumption 2.1 holds. Then, Algorithm 1 is interpolatory. If, in addition Assumption 2.2 holds, then Algorithm 1 is strongly consistent.*

Proof. We already established that the range space of $\mathcal{Y}$ equals the range space of $\mathcal{O}_p$ when the spectrum samples are noise-free. But, the range space of $\mathcal{Y}$ equals to the column space of U_{2n}. Moreover, Steps (4)–(10) of Algorithm 1 coincide with the same steps of the subspace algorithm in Akçay and Türkay (2004). When the spectrum samples are noise-free, the solution of the least-squares problem in Step (8) does not depend on the distribution of the frequencies as long as they are distinct. The first part then follows from the fact that the subspace algorithm in Akçay and Türkay (2004) is interpolatory. The proof of the second part is similar to the consistency proof in McKelvey et al. (1996), hence it is omitted. □

4. Illustrative example

In this section, we use a simulation example to illustrate the failure of the subspace algorithm in Van Overschee et al. (1997) and the properties of the identification algorithm proposed in this paper. Let the true system $G(z)$ be a fourth-order system described by the state–space model (Akçay & Türkay, 2004; McKelvey et al., 1996):

$$A = \begin{bmatrix} 0.8876 & 0.4494 & 0 & 0 \\ -0.4494 & 0.7978 & 0 & 0 \\ 0 & 0 & -0.6129 & 0.0645 \\ 0 & 0 & -6.4516 & -0.7419 \end{bmatrix},$$

$$B = [0.2247\ 0.8989\ 0.0323\ 0.1290]^T,$$

$$C = [0.4719\ 0.1124\ 9.6774\ 1.6129],$$

$$D = 0.9626.$$

Consider first the noise-free data case, *i.e.*, assume that $\tilde{s}_k = 0$ for all k in Eq. (7). With p and r fixed as $p = 9$ and $r = 8$, the fourth and the fifth largest singular values of $\widehat{\mathcal{G}}_p\mathcal{F}_r$ are displayed versus M in Table 1. From the table, we infer that the convergence in Eq. (14) takes place very rapidly as expected, and the algorithm in Van Overschee et al. (1997) is not consistent.

Now, assume that the noise in Eq. (7) are given as

$$\tilde{s}_k = \varepsilon \frac{0.2z_k^2 - 0.0904z_k + 0.1839}{z_k^2 - 1.1111z_k + 0.8520} \nu_k$$

where ν_k, $k = 1, \ldots, M+1$ are zero-mean, unit-variance, independent, and identically distributed complex normal random variables. The consistency properties of Algorithm 1 were examined by performing Monte Carlo simulations where the quality of the models was assessed by computing the (measured) L_2 and L_∞ norms of the estimation errors defined by

$$\|\widehat{S}_M - S\|_{m,2} = \left(\frac{1}{M+1} \sum_{k=1}^{M+1} |\widehat{S}_M(z_k) - S(z_k)|^2 \right)^{\frac{1}{2}}$$

$$\|\widehat{S}_M - S\|_{m,\infty} = \max_{1 \le k \le M+1} |\widehat{S}_M(z_k) - S(z_k)|,$$

Table 2
The average values of $\|\widehat{S}_M - S\|_{m,2}$ and $\|\widehat{S}_M - S\|_{m,\infty}$ from Monte Carlo simulations over 100 noise realizations with $p = r = 50$, $\varepsilon = 10^{-3}$ and using Algorithm 1.

M	$\|\widehat{S}_M - S\|_{m,2}$	$\|\widehat{S}_M - S\|_{m,\infty}$
1,000	1.55×10^{-4}	8.18×10^{-4}
2,000	4.79×10^{-5}	2.53×10^{-4}
4,000	3.11×10^{-5}	1.70×10^{-4}
8,000	1.87×10^{-5}	1.02×10^{-4}
16,000	1.37×10^{-5}	7.50×10^{-5}
32,000	9.47×10^{-6}	5.16×10^{-5}

and averaging them over 100 noise realizations. From Table 2, consistency of Algorithm 1 is evident.

Next, on the same data sets used by Algorithm 1 to form Table 2 we tried the identification algorithm proposed in Van Overschee et al. (1997) with $p = r = 50$ and $\varepsilon = 10^{-9}$. In all cases tried, the latter algorithm failed without returning spectral factors. The failures occurred in Step (6) of Algorithm 1. This step is also part of the algorithm in Van Overschee et al. (1997). Such failures have never been observed with Algorithm 1 when M is sufficiently large, say larger than or equal to 8000.

5. Conclusion

In this paper, we revisited the identification problem studied in Van Overschee et al. (1997) and demonstrated that the algorithm presented there lacks consistency. Then, we proposed an interpolatory identification algorithm which is strongly consistent under the same noise assumptions in Van Overschee et al. (1997). The performance of the proposed algorithm was demonstrated in a simulation example.

Acknowledgement

The work reported in this paper was done while the author was visiting the Center for Systems Engineering and Applied Mechanics (CESAME), Catholic University of Louvain, on leave from Anadolu University. This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its author. The author would like to thank the associate editor and the reviewers of the manuscript for helpful remarks.

References

Akçay, H. (2010). Subspace-based rational interpolation of analytic functions from phase data. *IEEE Transactions on Signal Processing*, *58*, 1069–1081.
Akçay, H., & Türkay, S. (2004). Frequency domain subspace-based identification of discrete-time power spectra from nonuniformly spaced measurements. *Automatica*, *40*, 1333–1347.
Fraanje, R., Verhaegen, M., Doelman, N., & Berkhoff, A. (2004). Optimal and robust feedback controller estimation for a vibrating plate. *Control Engineering Practice*, *12*, 1017–1027.
Hinnen, K., Verhaegen, M., & Doelman, N. (2005). Robust spectral factor approximation of discrete-time frequency domain power spectra. *Automatica*, *41*, 1791–1798.
Kay, S. M. (1988). *Modern spectral estimation: theory & application*. Englewood Cliffs, NJ: Prentice-Hall.
McKelvey, T., Akçay, H., & Ljung, L. (1996). Subspace-based multivariable system identification from frequency response data. *IEEE Transactions on Automatic Control*, *41*, 960–979.
Türkay, S., & Akçay, H. (2005). A study of random vibration characteristics of the quarter-car model. *Journal of Sound and Vibration*, *282*, 111–124.
Van Overschee, P., De Moor, B., Dehandschutter, W., & Swevers, J. (1997). A subspace algorithm for the identification of discrete-time frequency domain power spectra. *Automatica*, *33*, 2147–2157.

Hüseyin Akçay was born in Antalya, Turkey in 1958. He received his Engineering degree from the İstanbul Technical University in 1981, his M.Sc degree from the Massachusetts Institute of Technology in 1988, and his Ph.D degree from the University of Michigan, Ann Arbor in 1992, all in mechanical engineering, and his M.A. degree in mathematics from the University of Michigan, Ann Arbor in 1991. He has held visiting positions with Linköping, Newcastle, Bremen, and Catholic universities. He received a research fellowship award in 1998 from the Alexander Von Humboldt Foundation. He worked at the Tübitak, Marmara Research Center, Gebze, Turkey as a Research Scientist. He is currently professor of Electrical and Electronics Engineering in Anadolu University. His research interests include system identification, signal processing, applications to automotive and power systems.