Friction Damper Optimization: Simulation of Rainbow Tests

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Friction dampers have been used to reduce turbine blade vibration levels for a considerable period of time. However, optimal design of these dampers has been quite difficult due both to a lack of adequate theoretical predictions and to difficulties in conducting reliable experiments. One of the difficulties of damper weight optimization via the experimental route has been the inevitable effects of mistuning. Also, conducting separate experiments for different damper weights involves excessive cost. Therefore, current practice in the turbomachinery industry has been to conduct so-called "rainbow tests" where friction dampers with different weights are placed between blades with a predefined configuration. However, it has been observed that some rainbow test results have been difficult to interpret and have been inconclusive for determining the optimum damper weight for a given bladed-disk assembly. A new method of analysis—a combination of the harmonic balance method and structural modification approaches—is presented in this paper for the analysis of structures with friction interfaces and the method is applied to search for qualitative answers about the so-called "rainbow tests" in turbomachinery applications. A simple lumped-parameter model of a bladed-disk model was used and different damper weights were modeled using friction elements with different characteristics. Resonance response levels were obtained for bladed disks with various numbers of blades under various engine-order excitations. It was found that rainbow tests, where friction dampers with different weights are used on the same bladed-disk assembly, can be used to find the optimum damper weight if the mode of vibration concerned has weak blade-to-blade coupling (the case where the disk is almost rigid and blades vibrate almost independently from each other). Otherwise, it is very difficult to draw any reliable conclusion from such expensive experiments. [DOI: 10.1115/1.1391278]

1 Introduction

Friction dampers have been widely used in turbomachinery applications for a considerable period of time in order to provide mechanical damping to reduce resonance stresses. A typical application of the dry friction damping concept in gas turbines is the so-called ''friction damper,'' or underplatform damper, which is loaded by centrifugal force against the underside of the platforms of two adjacent blades. The main design criterion for such dampers is to determine the optimum damper mass for a given configuration in order to reduce the dynamic stresses by the maximum possible extent. If the damper mass is too small, the friction force will not be large enough to dissipate sufficient energy. On the other hand, if the damper mass is too large, it will ''stick,'' limiting the relative motion across the interface and thus the amount of energy dissipation. In both cases, the friction damper will be inefficient and between these two extremes there is an optimum mass.

Although a substantial effort has been devoted to understanding, modeling, and optimization of friction dampers for turbomachinery applications including: (i) modeling the basic contact characteristics, usually in the form of friction force-displacement hysteresis loops ([1–6]), (ii) modeling the friction damper element incorporating the basic contact characteristics ([7–9]) in (i), and (iii) developing analysis methods and application of these for friction damper optimization in practice ([10–15]). Although significant advances have been made in all these three categories it is still difficult to rely on computer-based predictions alone for assessing the response amplitude of turbomachinery blading and for optimizing the friction interfaces. This is due mainly to the marked nonlinearity of the contact mechanisms and the uncertainties of the actual dynamic forces acting on the blades. This situation has led the aero-engine manufacturers to rely mainly on previous experience and empirical data obtained from either simplified test rigs comprising a single or group of blades, or from more realistic, albeit more expensive, spin tests using a complete bladed-disk assembly which includes most of the important factors. Although most current research is focussed on validating theoretical models and methods so as to minimize these expensive experiments it is still common practice to carry out spin tests on complete bladed-disk assemblies in order to assess the effectiveness of the underplatform dampers. Conducting separate spin tests for different damper weights is, however, rarely employed in industry, due to excessive costs. Instead, so-called rainbow tests are adopted, where friction dampers with different weights are placed between adjacent blades with a predefined configuration as schematically illustrated in Fig. 1 for a special case of three different types of damper being installed. The experimental route, however, also has numerous difficulties in conducting reliable friction damping tests. More often than not, results obtained from such experiments have proven to be inconclusive for determining the optimum damper weight for a given bladed-disk assembly. One of the difficulties of damper weight optimization via the experimental route has been the inevitable effects of mistuning ([16–21]). There are mainly two problems associated with mistuning during damper optimization: (i) the difficulty of distinguishing the effect of mistuning from that of the dampers and (ii) in general, not being able to instrument every blade on an assembly, which makes it very likely that the maximum response levels experienced by a single blade will not be detected. Some recent measurement techniques using noncontact measurement systems allow (in principle) monitoring all the blades on a bladed-disk assembly although the interpretation of the results is also hampered by mistuning effects ([19]). Most of the research published in the literature is focussed on investigating the effects of stiffness and mass mistuning, resulting in variations of individual blade

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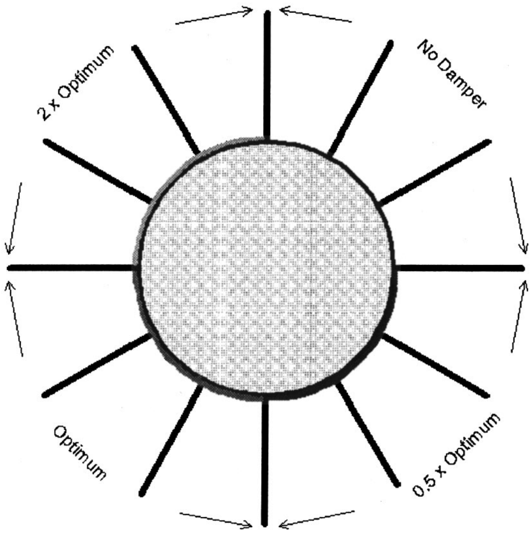


Fig. 1 Schematic illustration of "rainbow" tests

frequencies, some including the effect of uniform dry friction ([22]). One of the conclusions of [22] is that friction damped systems are more prone to localized vibrations. A number of researchers have also studied the effect of damping variations when all the blades within a bladed-disk are damped ([23,24]).

A recent study ([25]) investigated, using a linear analysis approach, whether accurate damping measurements for a single blade could be obtained by testing a bladed-disk with only a few blades damped. This paper is an extension of [25] and it seeks to identify those circumstances under which rainbow tests can be used for determining the optimum damping condition. Furthermore, it tries to establish how many blades need to be instrumented in such tests. The additional important and unique features of this paper are (i) the friction dampers are modeled as nonlinear friction elements, the analysis procedure being based on the harmonic balance method and (ii) a method based on the structural modification approach is used in conjunction with the harmonic balance method to analyze structures with friction interfaces efficiently.

2 Model Description

The lumped-parameter model used in this study is a variation of the bladed-disk model originally proposed by Dye and Henry [26], as shown in Fig. 2. A single mass (m) is used to model the blade while the other mass (M_d) represents the effective mass at the platform location and includes the sectorial mass of the disk as well as a proportion of the blade's mass. The dashpot attached between ground and the blade mass represents aerodynamic damping. The flexibility of the blade and the disk are also included and friction dampers are introduced between the lumped masses, M_d, indicated by a crossed box. The current friction dampers in practice are usually wedged shaped and their vibration characteristics are much more complicated than the simple one-dimensional damper model used here presented here. This simple model is, however, considered appropriate for the objective of this paper. A realistic model for wedge-shaped dampers, which is based on both measurements and theory, is addressed in [27]. The system shown in Fig. 2 can be described by the familiar equation as

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{f(t)\}+\{r(t)\} \tag{1}$$

where M, C, K are the mass, viscous damping, and stiffness matrices; and $f(t)$ and $r(t)$ are the external and friction forces due to damping, respectively. Assuming harmonic motion leads to the well-known counterpart of Eq. (1), as

$$([K]+i\omega[C]-\omega^2[M])\{Q\}=\{F\}+\{R\} \tag{2}$$

where ω is the frequency and $\{F\}$ and $\{R\}$ are the Fourier Transforms of $f(t)$ and $r(t)$, respectively. The external forcing considered here, $\{F\}$ is of the engine-order type, its magnitude being unity as in [25], applied to the blade coordinates only while $\{R\}$ represents the first harmonic components of the resulting friction forces which are applied to the platform coordinates.

3 Analysis Method

The new analysis method proposed in this paper is a combination of the harmonic balance method and a structural modification approach. These two approaches, and how to combine them for an efficient analysis tool to analyze structures, are described below. The solution algorithm adopted in the frequency domain is based on finding the response amplitudes iteratively, the starting point being the response levels of the underlying linear system. The behavior of the friction dampers is analyzed at a given relative response amplitude between the damper connection points and the individual dampers are represented as equivalent complex stiffnesses, representing both restoring and energy dissipation characters. The equivalent complex stiffnesses are then added to the

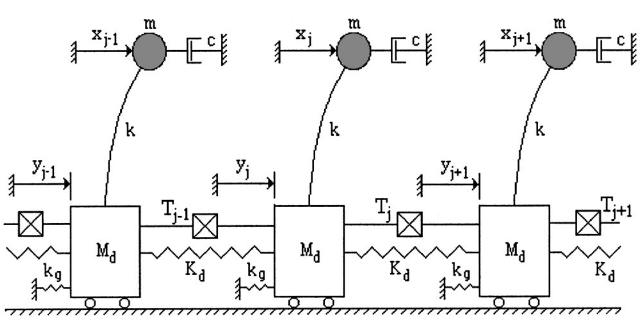


Fig. 2 Bladed-disk model with friction elements

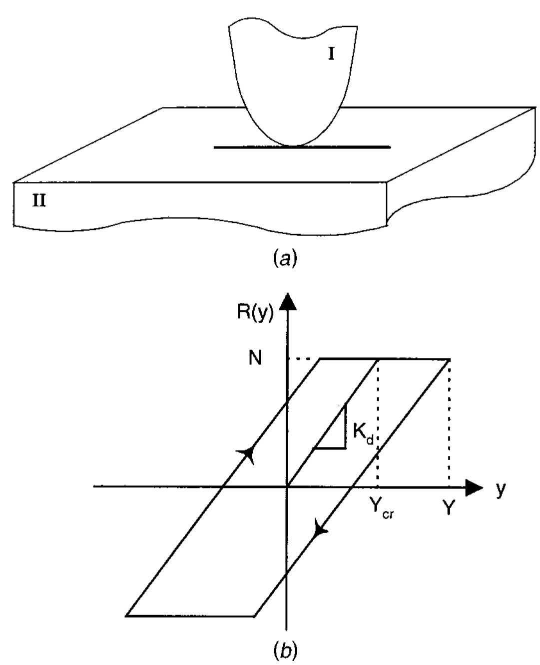


Fig. 3 (*a*) One-dimensional friction damper model, (*b*) force-displacement model of macroslip model

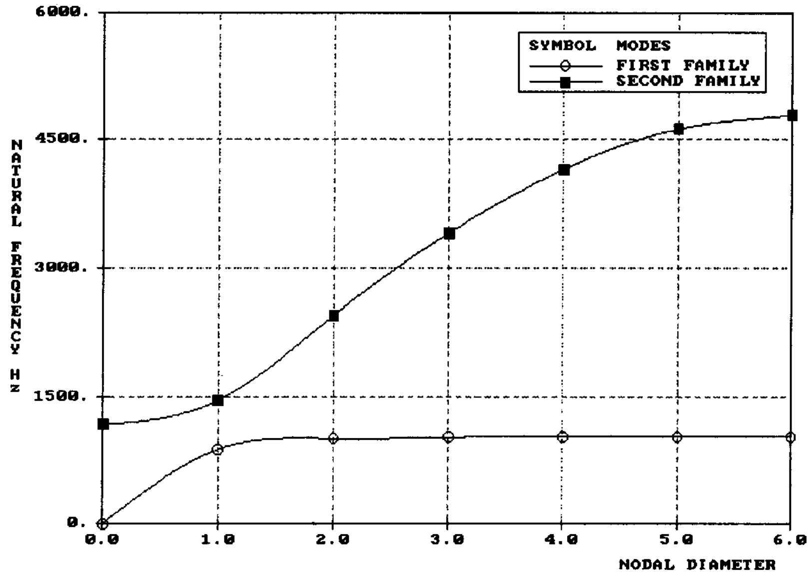


Fig. 4 Natural frequencies of the first and second family modes

otherwise linear system and the response levels of the modified system are calculated again, the procedure being repeated until convergence is achieved.

The following subsections are describing these two stages—representing a friction damper as a complex stiffness at a given relative response amplitude and—modifying the otherwise linear system to include frictional effects.

3.1 Harmonic Balance Approach. The friction damper element indicated by a crossed box in Fig. 2 is in fact a simplified representation of the situation depicted in Fig. 3(*a*) where two surfaces rub against each other along a line, parts I and II, representing the platforms of the neighboring blades. The macro-slip friction model of Fig. 3(*b*) has been used extensively in the analysis of various nonlinear systems and it will also be used in this paper although real contact characteristics can be quite different than that of macro-slip model ([6]). It should be noted, however, that the analysis method in this paper is equally applicable to any other type of hysteresis loop model.

In the traditional analysis of structures with friction joints, the nonlinear friction forces are calculated in an iterative fashion and are usually considered as external forces. As in [9,15] this paper converts these forces into amplitude (and phase)-dependent equivalent stiffness parameters for CPU reduction as well as numerical stability.

Now, let us consider the nonlinear force-displacement relationship of a friction damper element, denoted by T_j in Fig. 2. As mentioned earlier, the characterization of the damper is carried out at a given relative displacement as

$$Z_j = Y_{j+1} - Y_j \tag{3}$$

where Y_{j+1} and Y_j are complex quantities representing the platforms of neighboring blades. Z_j is also a complex quantity whose motion for a cycle can be described more explicitly as

$$z_j(t) = |Z_j|\cos(\omega t + \phi_j) = |Z_j|\cos(\theta_j) \tag{4}$$

where ϕ_j is the phase angle and $\theta_j = \omega t + \phi_j$. Now, let us consider the nonlinear force-displacement relationship of a friction damper, acting between coordinates y_{j+1} and y_j, given by (the subscripts will be omitted for clarity)

$$R = R(z) \tag{5}$$

where R is the friction force acting between the neighboring platforms. Then, the linearized stiffness coefficient, or the describing function of friction element, k_{eq}^*, can be written as ([9])

$$k_{eq}^*(Z) = k_{eq}^r(Z) + i k_{eq}^i(Z) \tag{6}$$

where k_{eq}^r and k_{eq}^i are the amplitude-dependent real and imaginary parts of the equivalent stiffness, respectively, as given by

$$k_{eq}^r = \frac{1}{\pi|Z|}\int_0^{2\pi} R(|Z|\cos(\theta))\cos(\theta)\,d\theta \tag{7a}$$

$$k_{eq}^i = \frac{-1}{\pi|Z|}\int_0^{2\pi} R(|Z|\cos(\theta))\sin(\theta)\,d\theta. \tag{7b}$$

If the system is tuned and the friction dampers are identical, all blades will experience the same level of vibration and the equivalent complex stiffness would be the same for all the friction dampers. However, friction dampers with different characteristics will result in blades having different vibration levels and these will, in turn, result in individual dampers having different equivalent stiffness parameters.

3.2 An Efficient Analysis Method for Structures With Friction Joints. The so-called Sherman-Morrison formula has already been proposed in the literature ([28]) to calculate the frequency response of a (linear) modified structure. It is shown in [28] that the Sherman-Morrison identity allows a direct inversion of the modified matrix efficiently using the data related to the initial matrix and to the modification. A brief summary of the Sherman-Morrison formula is appropriate here.

Let $[A]^{-1}$ be the inverse of a nonsingular square matrix, $[A]$. If the inverse of a modified matrix, $[A']^{-1}$, is needed where $[A']$ is of the form

Table 1 Linear structural parameters

$\frac{1}{2\pi}\sqrt{k/m} = 1033.7$ Hz	$\frac{\sqrt{K_d/M_d}}{\sqrt{k/m}} = 2.3$
$k = 2.109$ MN/m $k_g = 63.9$ N/m	$K_d = 40.17$ MN/m $c = 3.247$ Ns/m

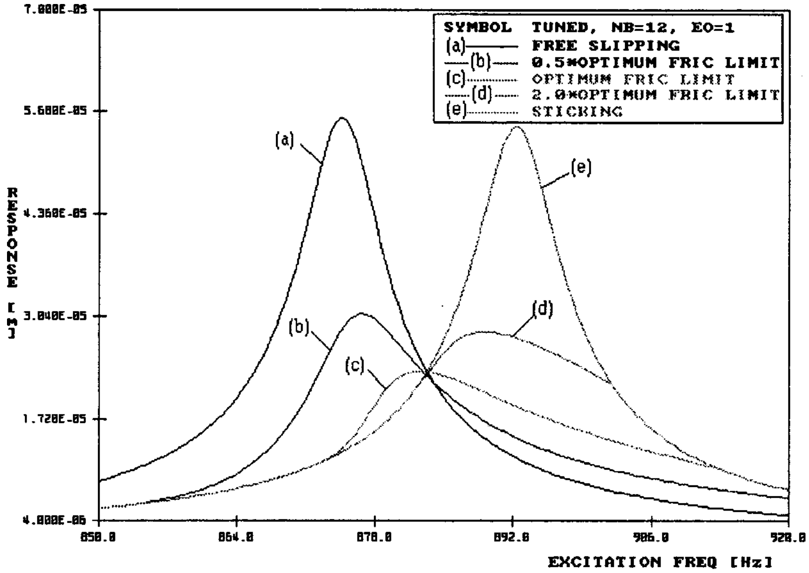


Fig. 5 Tuned response levels for various friction limits

$$[A']=[A]+\{u\}\{v\}^T, \tag{8}$$

it can be calculated using the Sherman-Morrison formula as

$$[A']^{-1}=[A]^{-1}-\frac{([A]^{-1}\{u\})(\{v\}^T[A]^{-1})}{1+\lambda} \tag{9}$$

where

$$\lambda=\{v\}^T[A]^{-1}\{u\}. \tag{10}$$

It should be noted that if $[A]^{-1}$ is known, Eq. (9) does not require any further matrix inversion to find the inverse of $[A']^{-1}$. The generalization of Eq. (9) is also available and is known as Sherman-Morrison-Woodbury formula which considers a modification as a product of two rectangular matrices such as $[U]\times[V]^T$. A more detailed coverage of this approach and the numerical aspects are discussed in [29].

The aim of this paper is quite different in the sense that the purpose of the analysis is to calculate the nonlinear response levels of structures with localized nonlinearities rather than the linear modification analysis as reported in the above literature. The analysis method presented here treats the linear and the nonlinear parts of a structure separately, the linear part being the structure excluding the nonlinear parts. The nonlinear part is considered as a linearized modification to the original system, the linearized parameters being obtained using the harmonic balance method as described in the previous section. Suppose that the linear structure is given by its dynamic stiffness matrix $[Z]$ and its frequency response function matrix $[\alpha]$, $[\alpha]=[Z]^{-1}$, and the modification matrix to be made to $[Z]$ is $[\Delta]$. The dynamic stiffness matrix of the modified system $[Z']$ can then be written as

$$[Z']=[Z]+[\Delta]. \tag{11}$$

If the modification matrix is written of the form

$$[\Delta]=\{u\}\{v\}^T, \tag{12}$$

the FRF matrix of the modified system $[\beta]$ can be computed from

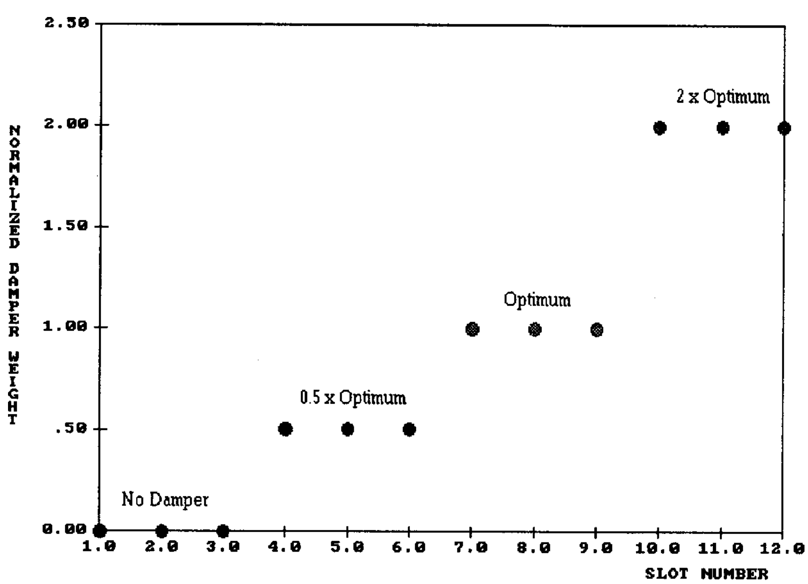


Fig. 6 Distribution of friction dampers

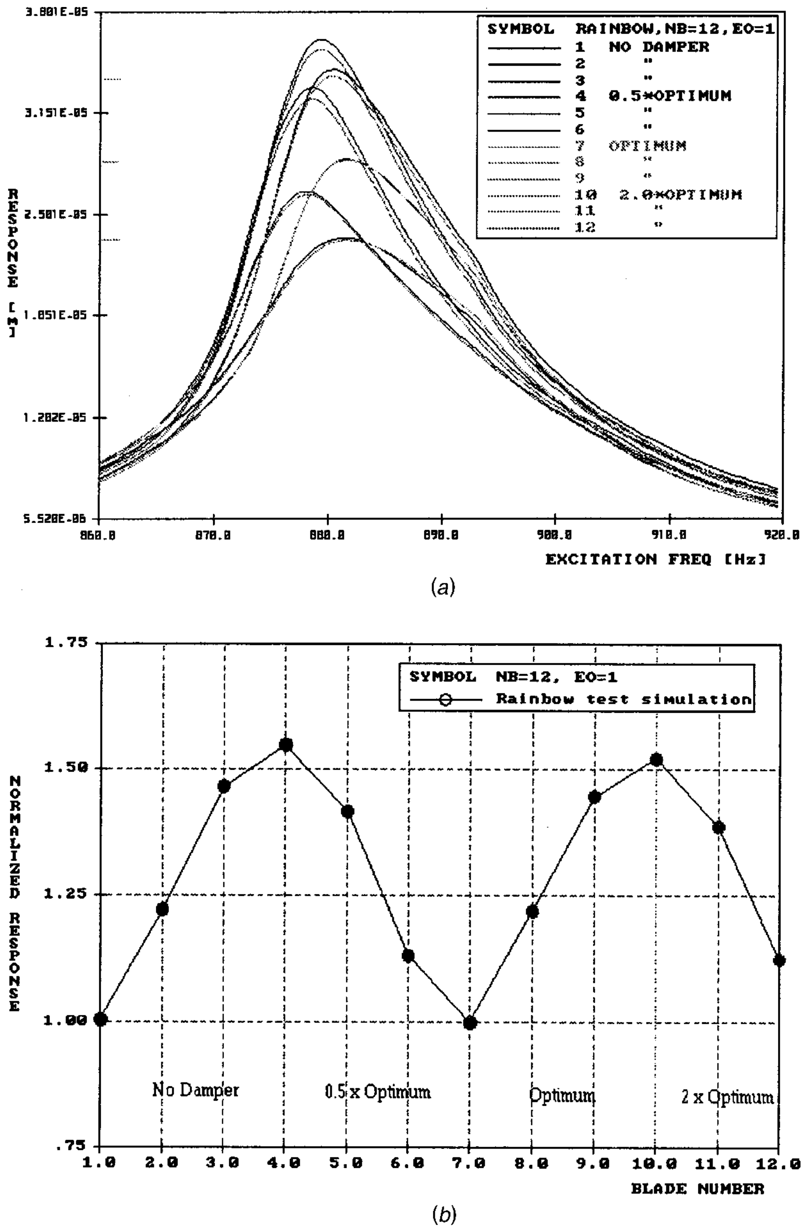


Fig. 7 (*a*) Blades' response levels versus excitation frequency (*N*=12, EO=1); (*b*) blades' maximum response levels with friction dampers (*N*=12, EO=1)

$$[\beta]=[Z']^{-1}=[\alpha]-\frac{([\alpha]\{u\})(\{v\}^T[\alpha])}{1+\{v\}^T[\alpha]\{u\}} \tag{13}$$

which allows the FRF matrix of the modified system to be calculated without any matrix inversion. It should be noted that if the total modification matrix $[\Delta]$ cannot be written as a multiplication of two vectors as in Eq. (12), it can be decomposed into several, say p, modification matrices, such as

$$[\Delta]=[\Delta_1]+[\Delta_2]+[\Delta_3]+\ldots+[\Delta_p] \tag{14}$$

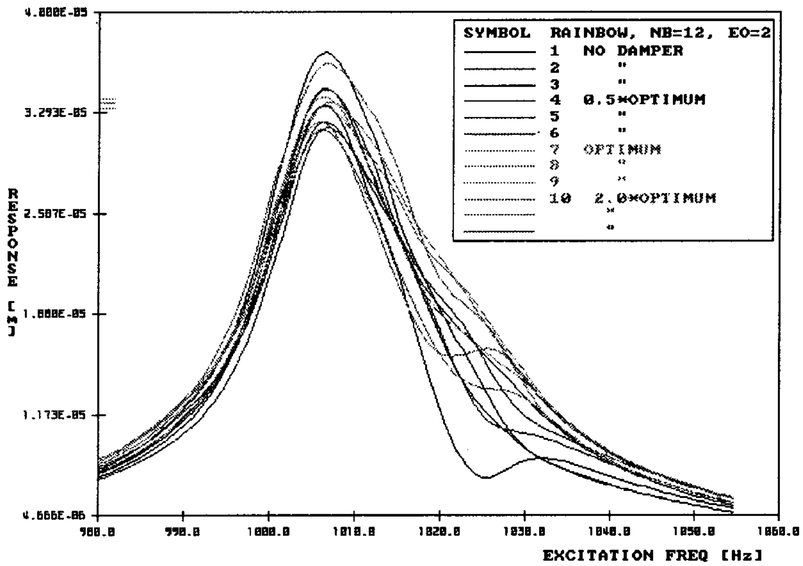
where $[\Delta_i]=\{u_i\}\{v_i\}^T$. This allows the FRF of the system to be calculated by considering each $[\Delta_i]$ individually.

It is also possible that the solutions can be obtained very efficiently at active coordinates only, active coordinates being the nonlinear coordinates, excitation coordinates, and the other coordinates where the response levels are needed. This approach allows local solutions to be obtained no matter how large the whole model is, as addressed in [27] including the application of this method to realistic turbine blades with friction dampers and validation of the predictions by experiments.

4 Analysis and Results

As the primary objective of this work is to investigate qualitative aspects of rainbow tests and to demonstrate the new analysis procedure, it was convenient to keep the number of blades reasonably small. Therefore, the bladed-disk system studied here had 12 blades, although a case with 24 blades is also studied in order to investigate the effect of varying the number of blades. However, the blade-to-blade coupling ratio was selected such that the bladed-disk studied here had a similar first family characteristic (see Fig. 4) to that of a realistic turbine stage. The corresponding linear structural parameters are listed in Table 1.

Having determined the linear structural model, the next step



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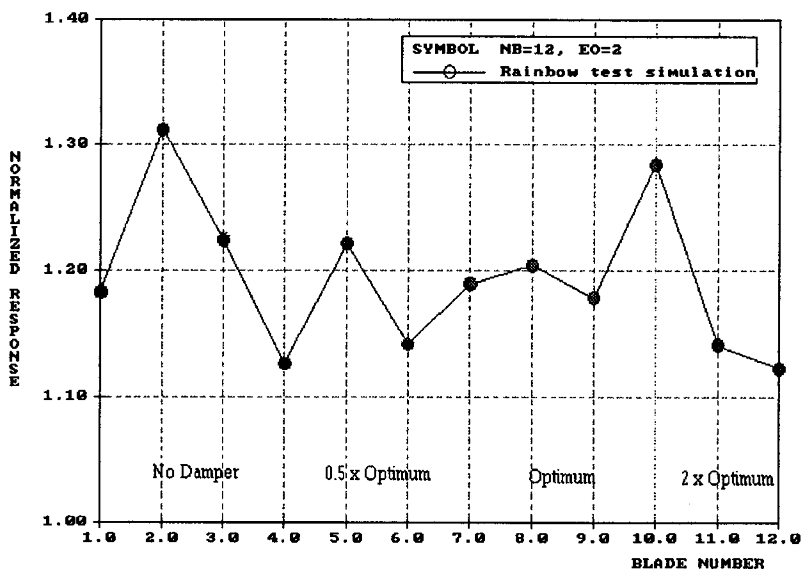


(*b*)

Fig. 8 (*a*) Blades' response levels versus excitation frequency (*N*=12, EO=2); (*b*) blades' maximum response levels with friction dampers (*N*=12, EO=2)

was to determine the parameters for the friction dampers. Macro-slip elements were selected to simulate friction dampers since it was easier to relate friction limits to damper weights. The contact stiffness of the macro-slip element was assumed to be the same for all dampers, and was chosen to give a clear natural frequency shift between slipping and sticking cases. A typical response plot is illustrated in Fig. 5 (free-slipping curve) for a tuned 12-bladed disk subjected to a first engine order (1EO) excitation (referring to Fig. 4, this particular mode of vibration can be classified as a mode with relatively strong blade-to-blade coupling). It is worth here to emphasize the definition of weak and strong blade-to-blade coupling used in this paper as some researchers use the same terminology for different meanings. The authors of this paper prefer to use the definition of "weak blade-to-blade coupling" for those cases where the disk is almost rigid and the assembly natural frequencies approach the cantilevered blade alone frequency (i.e., the stiffer the disk, the weaker the coupling hence zero coupling means rigid disk). Strong blade-to-blade coupling, on the other hand, refers to the other extreme where the disk is quite flexible. One way of quantifying the degree of coupling strength is to define a coupling ratio CR, $CR=(1-(\omega_{as}/\omega_n)^2)$, where ω_{as} is the tuned bladed-disk assembly frequency for a given nodal diameter mode of vibration and ω_n is the blade alone cantilever frequency.

The corresponding response levels of the tuned system were calculated for various friction limits (normal load times friction coefficient) in order to determine the optimum damper friction limit which is directly related to optimum damper weight. The analysis was done using a dedicated program based on the analysis procedure summarized in the previous section. It should be stressed that knowing the optimum friction damper beforehand is an essential part of this simulation since the rest of the study aims to determine whether the forced response levels of the blades with different friction dampers can be used to identify this known op-

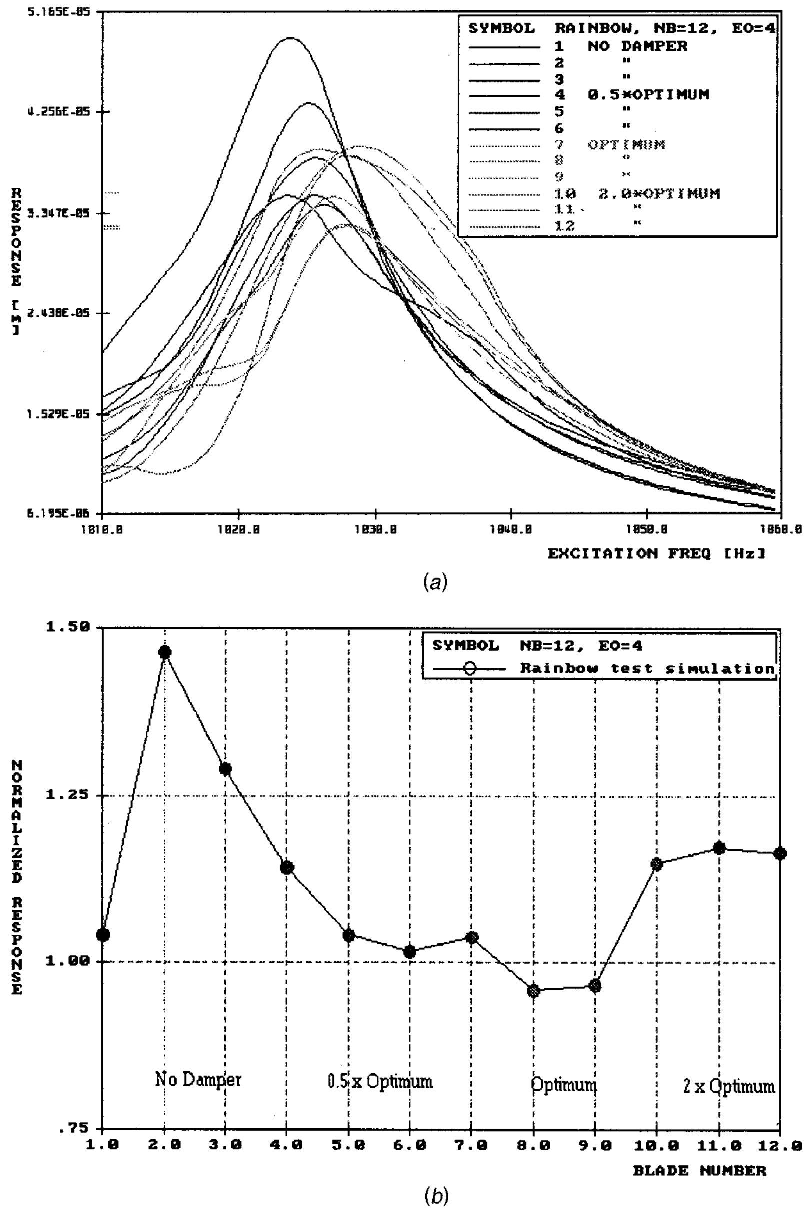


Fig. 9 (*a*) Blades' response levels versus excitation frequency (*N*=12, EO=4); (*b*) blades' maximum response levels with friction dampers (*N*=12, EO=4)

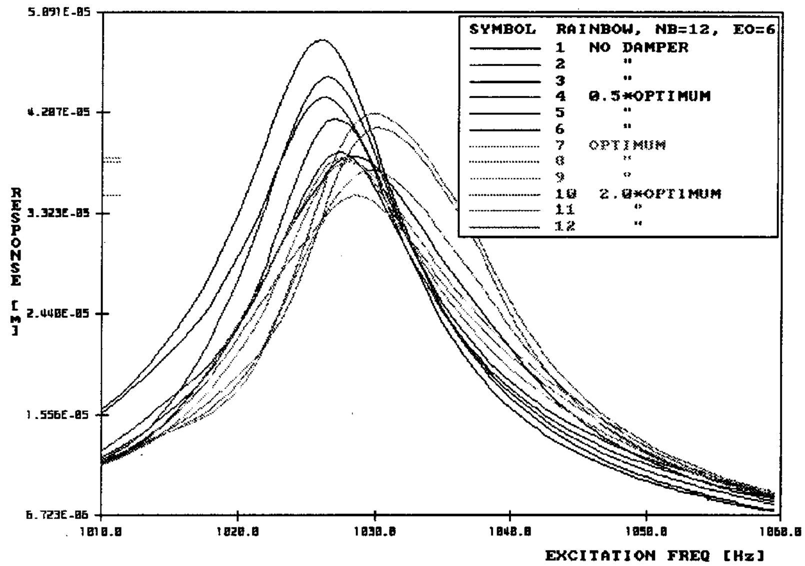
timum damper. The tuned system response amplitudes for the cases of the optimum damper weight, half of the optimum and twice the optimum, are illustrated in Fig. 5.

A particular rainbow test configuration studied here is illustrated in Fig. 6. It is seen that there are three different sizes of damper around the disk: optimum, half optimum, and twice the optimum in terms of damper weight. It should also be noted that some blades do not have any dampers at all. This type of simulation is quite representative of the actual tests in practice as it is common during this sort of test to install relatively heavy, normal, and light dampers so as to determine which one will produce the maximum damping. It should be stressed that the optimum damper in this simulation is known beforehand, the whole idea is to examine whether the measured results can identify this fact.

As can be seen in Fig. 6, the dampers are identified as "No Damper," "0.5×Optimum," "Optimum," and "2×Optimum" and this convention will be used throughout the rest of the plots so that results are presented in a consistent manner.

Most of the results were obtained for a 12-bladed disk, a typical set of results being illustrated in Fig. 7(*a*) for 1 engine-order (1EO) excitation. It is seen that the response levels of those blades with optimum dampers do not give any indication that those blades are the ones with optimum friction dampers (the maximum response levels of those blades with optimum dampers are identified by marks along the vertical axis in Fig. 7(*a*) and in other similar plots). The maximum response levels of individual blades in Fig. 7(*a*) were found and the results are presented in Fig. 7(*b*) (the maximum response levels were normalized to the tuned maximum response level with optimum friction damper). It is clear that there is no correlation between response amplitude and the damper weight but there is a strong suggestion that the maximum response amplitudes are determined by the mode shape rather than the distribution of dampers around the disk for this particular case.

A similar analysis was performed for the same bladed-disk assembly under different EO excitations. Each time, the optimum



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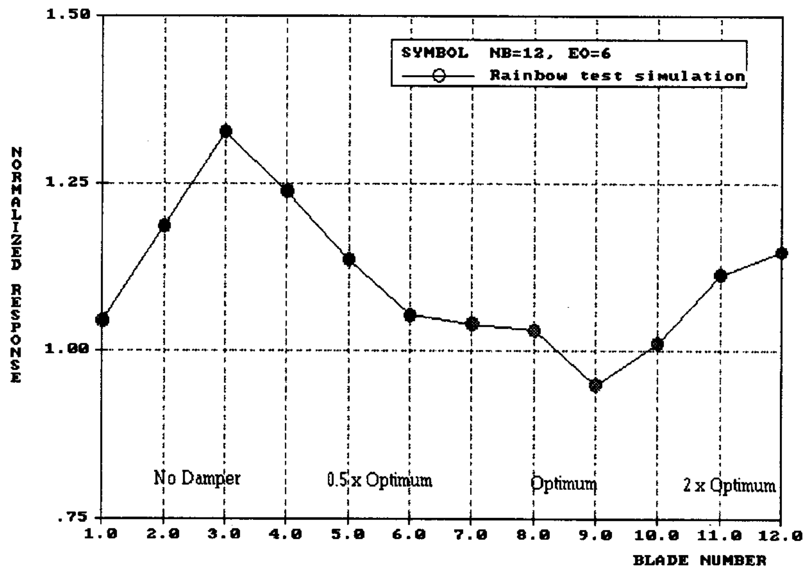


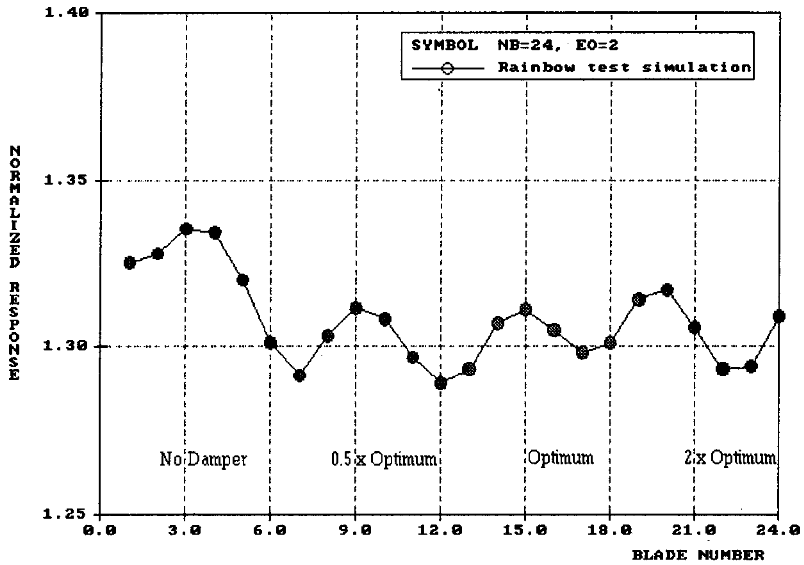
(*b*)

Fig. 10 (*a*) Blades' response levels versus excitation frequency (*N*=12, EO=6); (*b*) blades' maximum response levels with friction dampers (*N*=12, EO=6)

damper weight was determined by analysing the tuned system under various friction limit conditions (different damper weights), and the response levels of the blades corresponding to the friction damper distribution, as shown in Fig. 5, were determined. The results, similar to those in Fig. 7, are presented in Figs. 8–10, the difference being the order of the EO excitation. Inspection of these figures (Figs. 7 to 10) reveals that there is no relationship between blade response levels and the damper weights for low EO excitations. However, as the order of the excitation increases (see Figs. 9 and 10), a pattern starts to emerge, showing that those blades with optimum friction dampers tend to experience minimum response levels. It should be noted that a specified EO excitation predominantly excites the corresponding nodal diameter modes and blade-to-blade coupling decreases with increasing nodal diameter mode of vibration. Therefore, the argument above in terms of EO can be equally valid in terms of nodal diameter mode of vibration, or corresponding strength of blade-to-blade coupling.

So far, all the results presented here were for 12-bladed disk under various EO excitations. Similar calculations were also performed for a 24-bladed disk under 2EO and 12EO excitations so as to verify that the previous findings are not specific to the particular bladed disk studied here. Results presented in Fig. 11 fully support the findings from the 12-blade disk study. This is not surprising since there are very strong indications that the underlying parameter for forced vibration characteristics of a bladed disk is the coupling between the disk and the blade rather than number of blades or the order of the excitation alone ([21]).

All the findings in this study suggest that rainbow tests, where friction dampers with different weights are used on the same bladed-disk assembly, can be used to find the optimum damper weight if and only if the mode of vibration concerned has very weak blade-to-blade coupling (the case where the disk is almost rigid and blades vibrate almost independently from each other). Otherwise, it seems that it may be very difficult to draw any reliable conclusion form such expensive experiments. It is inter-



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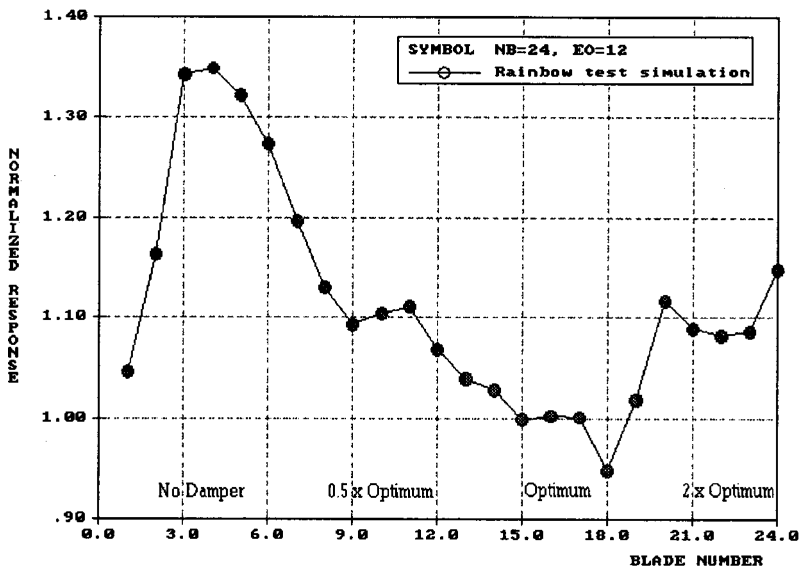

(*b*)

Fig. 11 (*a*) Blades' maximum response levels with friction dampers (*N*=24, EO=2); (*b*) blades' maximum response levels with friction dampers (*N*=24, EO=12)

esting to note that the nonlinear analysis method here yielded the same conclusion as that of [25] even though their simulation was based on a linear analysis.

Another important qualitative finding of this study is related to the number of blades that will require instrumenting in such experiments. The response levels of all the blades were assumed to be "measured" in this simulation in order to identify the conditions where rainbow tests can and cannot produce satisfactory results. In practice, however, it is hardly possible to monitor all the blades around the disk, especially for disks with large number of blades. The results presented in this paper also give some guide as to which and how many blades need instrumentation. Inspection of all the results suggest that at least one blade from each group needs to be instrumented although instrumenting two blades from each group is expected to yield a more reliable assessment of the results. Furthermore, it is better to instrument those blades which are close to the middle of a group of dampers around the circumference. It is expected, however, that this finding may not be applicable to other situations where the distribution of dampers are quite different than what is examined here, nor for those cases where the effect of other sources of mistuning is stronger than that provided by the friction dampers.

It is worth restating here that the mistuning due to the differences between blades' mechanical properties are deliberately excluded from the study reported in this paper. The main reason for this exclusion was to establish an upper limit of what to expect from rainbow test results under idealistic conditions. It is, however, necessary to include these effects in order to make more realistic assessment of such expensive tests as the additional mistuning effects are inevitable in practical tests. It is quite likely that the results of a rainbow test may not indicate the optimum damper weight, even for lightly coupled bladed disks, when the additional mistuning level exceeds a certain threshold. The simulation of this situation requires a better damper model as well as more realistic (empirical) contact parameters for the friction dampers, so that the relative importance of the nonlinear damping and blade-alone mistuning can be identified and their effects can be distinguished from each other.

5 Conclusions

A new method has been presented for the analysis of structures with friction interfaces. This method is a combination of the harmonic balance method and a structural modification approach and is based on modifying the otherwise-linear structure with the amplitude-dependent equivalent complex stiffness representing both restoring and energy dissipation characteristics of joints.

This method of analysis has been applied to investigate whether, and under which conditions, the so-called "rainbow" tests can be used for damper optimization purposes in turbomachinery applications. Results of the rainbow-test simulation presented in this paper suggest that these tests can be used to determine the optimum damper weight only if there is a weak blade-to-blade coupling for the mode of vibration concerned. It has also been found that instrumenting two blades from each group of dampers is expected to yield reliable assessment of the results when other sources of mistuning are negligible.

Although the results of this investigation have established the condition when rainbow tests can be used to identify optimum damper weight, a more detailed and representative analysis need to be carried out in order to establish the effect of additional mistuning present in these tests.

Acknowledgments

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References

[1] Goodman, L. E., and Brown, C. B., 1962, "Energy Dissipation in Contact Friction: Constant Normal and Cyclic Tangential Loading," ASME J. Appl. Mech., **29**, pp. 17–22.

[2] Iwan, W. D., 1967, "On a Class of Models for the Yielding Behaviour of Continuous and Composite Systems," ASME J. Appl. Mech., **34**, pp. 612–617.

[3] Rogers, P. F., and Boothroyd, G., 1975, "Damping at Metallic Interfaces Subjected to Oscillating Tangential Loads," J. Eng. Ind., **97**, pp. 1087–1093.

[4] Burtekin, M., Cowley, A., and Back, N., 1978, "An Elastic Mechanism for the Micro-sliding Characteristics Between Contacting Machined Surfaces," J. Mech. Eng. Sci., **20**, No. 3, pp. 121–127.

[5] Shoukry, S. N., 1985, "A Mathematical Model for the Stiffness of Fixed Joints Between Machine Parts," J. Middleton and G. N. Pande, eds., *Proc. Int. NUMETA Conf.*, Swansea, U.K., A. A. Balkama, Rotterdam, pp. 851–858.

[6] Sanliturk, K. Y., Stanbridge, A. B., and Ewins, D. J., 1995, "Friction Dampers: Measurement, Modelling and Application to Blade Vibration Control," *Proc. Des. Eng. Conf.*, Vol. 3, Part B, DE-Vol. 84-2, ASME, New York, pp. 1377–1382.

[7] Pfeiffer, F., and Hajek, M., 1992, "Stick-Slip Motion of Turbine Blade Dampers," Philos. Trans. R. Soc. London, Ser. A, **338**, No. 1651, pp. 503–517.

[8] Menq, C.-H., Bielak, J., and Griffin, J. H., 1986, "The Influence of Microslip on Vibratory Response, Part I: A New Microslip Model," J. Sound Vib., **107**, No. 2, pp. 279–293.

[9] Sanliturk, K. Y., and Ewins, D. J., 1996, "Modelling Two-Dimensional Friction Contact and Its Application Using Harmonic Balance Method," J. Sound Vib., **193**, No. 2, pp. 511–523.

[10] Menq, C.-H., and Griffin, J. H., 1985, "A Comparison of Transient and Steady State Finite Element Analysis of the Forced Response of a Frictionally Damped Beam," ASME J. Vibr. Acoust., **107**, pp. 19–25.

[11] Pierre, C., Ferri, A. A., and Dowel, E. H., 1985, "Multi-Harmonic Analysis of Dry Friction Damped Systems Using an Incremental Harmonic Balance Method," ASME J. Appl. Mech., **52**, pp. 958–964.

[12] Cameron, T. M., and Griffin, J. H., 1989, "An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Non-linear Dynamic Systems," ASME J. Appl. Mech., **56**, pp. 149–154.

[13] Ostachowicz, W., 1989, "The Harmonic Balance Method for Determining the Vibration Parameters in Damped Dynamic Systems," J. Sound Vib., **131**, No. 3, pp. 465–473.

[14] Cameron, T. M., Griffin, J. F., Kielb, R. E., and Hoosac, T. M., 1990, "An Integrated Friction Damper Design," ASME J. Vibr. Acoust., **112**, pp. 175–182.

[15] Sanliturk, K. Y., Imregun, M., and Ewins, D. J., 1997, "Harmonic Balance Vibration Analysis of Turbine Blades With Friction Dampers," ASME J. Vibr. Acoust., **119**, pp. 96–103.

[16] Muszynska, A., and Jones, D. I. G., 1983, "On Tuned Bladed Disk Dynamics: Some Aspects of Friction Related Mistuning," J. Sound Vib., **86**, No. 1, pp. 107–128.

[17] Ewins, D. J., and Han, Z. S., 1984, "Resonant Vibration Levels of a Mistuned Bladed Disks," ASME J. Vibr. Acoust., **106**, pp. 211–217.

[18] Sanliturk, K. Y., Imregun, M., and Ewins, D. J., 1992, "Statistical Analysis of Random Mistuning of Bladed Assemblies," *IMechE Proc. Int. Conf. on Vib. in Rotating Machinery*, Univ. of Bath, UK., Sept. 7–10, pp. 51–58.

[19] Schaber, U., 1997, "Non-Contact Vibration Measurements of Mistuned Coupled Blades," Paper No. 97-GT-190.

[20] Chen, S., and Sinha, A., 1990, "Probabilistic Method to Compute the Optimal Slip Load for a Mistuned Bladed Disk Assembly With Friction Dampers," ASME J. Vibr. Acoust., **112**, pp. 214–221.

[21] Sanliturk, K. Y., 1992, "Vibration Analysis of Mistuned Bladed Systems," Ph.D. thesis, Mechanical Eng. Dept., Imperial College, University of London.

[22] Wei, S.-T., and Pierre, C., 1989, "Effects of Dry Friction Damping on the Occurrence of Localised Forced Vibrations in Nearly Cyclic Structures," J. Sound Vib., **129**, No. 2, pp. 379–416.

[23] Lin, C.-C., and Mignolet, M. P., 1996, "Effects of Damping and Damping Mistuning on the Forced Vibration Response of Bladed Disks," J. Sound Vib., **193**, No. 2, pp. 525–543.

[24] Sextro, W., Popp, K., and Wolter, I., 1997, "Improved Reliability of Bladed Disc to Friction Dampers," ASME Paper No. 97-GT-189.

[25] Hollkamp, J. J., Hustedde, C. L., and Gordon, R. W., 1997, "Effects of Damping on a Blade-Disk Assembly," AIAA Paper No. AIAA-97-1395.

[26] Dye, R. C. F., and Henry, T. A., 1969, "Vibration Amplitudes of Compressor Blades Resulting From Scatter in Blade Natural Frequencies," ASME J. Eng. Gas Turbines Power, **91**, pp. 182–188.

[27] Sanliturk, K. Y., Ewins, D. J., and Stanbridge, A. B., 1999, "Underplatform Dampers for Turbine Blades: Theoretical Modeling, Analysis and Comparison with Experimental Data," ASME J. Eng. Gas Turbines Power, **123**, pp. 919–929.

[28] Level, P., Moraux, D., Drazetic, P., and Tison, T., 1996, "On a Direct Inversion of the Impedance Matrix in Response Reanalysis," Communications in Numerical Methods in Engineering, **12**, pp. 151–159.

[29] Hager, W. G., 1989, "Updating the Inverse of a Matrix," SIAM Rev., **31**, No. 2, pp. 221–239.