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Friedmann-like singularities in Szekeres' cosmological models

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Summary. We establish the existence of Friedmann-like singularities in a subclass of the Szekeres cosmological models which admits no Killing vector fields, and which depends on three arbitrary functions of a single variable. We describe the asymptotic properties of the solutions as the singularity is approached, for example the behaviour of the matter density, the rate of shear of the matter, and the Weyl conformal curvature tensor. The solutions are formulated in a way which permits comparison with linear perturbations of the Einstein–de Sitter solution.

1 Introduction

Attempts to study the nature of the singularity in general solutions of Einstein's field equations (with an irrotational, expanding perfect fluid as source) led to the discovery of a simple special case in which the leading time dependence of the metric near the singularity is that of an (isotropic) Friedmann–Robertson–Walker (FRW) solution [1, 2, 3]. In the Russian literature, a solution with this type of singularity is called 'quasi-isotropic', while Eardley, Liang & Sachs [2] use the terminology 'Friedmann-like singularity', which we shall adopt.

Friedmann-like singularities are quite different from those that occur generically, and which are dominated by anisotropy, and possibly spatial curvature. One might expect that the initial singularity in the Universe should be of the generic type, and this idea formed the basis of the chaotic cosmology program [4, 5]. More recently, however, it has been suggested on the basis of entropy considerations [6, 7, 8], and using the theory of quantum mechanical creation of particles by the gravitational field [9], that the initial singularity should be constrained in some way so as to be similar to the singularity in an exact FRW solution. For example, Penrose [7, 10], has proposed, as a possible constraint, that the Weyl conformal curvature tensor should tend to zero as the initial singularity is approached. These ideas suggest that the class of solutions with Friedmann-like singularity are possibly of significance as models of the initial big bang.

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Friedmann-like singularities have been studied primarily by power series expansion methods [1, 11] and by iterative methods [2, 3], but in neither case has convergence been established. However, certain anisotropic exact solutions do admit Friedmann-like singularities, which establishes that such singularities do exist in solutions other than exact FRW ones. These examples are either plane symmetric [2] or spatially homogeneous (e.g. [12] p. 76). In this paper we discuss a class of solutions with no Killing vector fields which admit a Friedmann-like singularity, and use them to illustrate certain aspects of this type of singularity. The solutions belong to the Szekeres [13] class of solutions with pressure-free dust as source.

We formulate the solutions in such a way as to facilitate comparison with solutions of the linearized Einstein equations which describe spatially inhomogeneous perturbations of FRW solutions. Linear perturbations of the FRW solutions with flat space sections and with dust as source, exhibit two modes of density fluctuations, called the growing and decaying modes. If μ_0 denotes the density of the unperturbed FRW solution, the density contrast $\delta\mu/\mu_0 \equiv (\mu - \mu_0)/\mu_0$, can be written in the form

$$\frac{\delta\mu}{\mu_0} = A_+ t^{2/3} + A_- t^{-1}, \tag{1.1}$$

where $A_\pm$ have only spatial dependence, and t denotes comoving proper time along the worldlines of the dust [14, 15]. The function A_+ determines the growing mode (increases in magnitude into the future) while A_- determines the decaying mode (decreases in magnitude into the future). In stating this we should point out that the density contrast is a gauge-dependent quantity [14]. Nevertheless, for dust, both modes of the density contrast are physically significant, although the particular form of $\delta\mu/\mu_0$ can depend on the choice of gauge [14, 16]. The exact solutions that we will discuss contain both density fluctuation modes, via arbitrary functions of a single variable, which we shall denote by $k_\pm(z)$. As one might expect, a Friedmann-like singularity occurs only when the *decaying* mode is absent, since this mode becomes unbounded as $t \to 0^+$.

The solutions and their required properties are presented in Section 2. Their relationship with linear perturbations of FRW solutions is given in Section 3. A detailed discussion of the Friedmann-like singularities which occur in these solutions is given in Section 4. There is a concluding Section 5 in which some general remarks and suggestions for further research concerning Friedmann-like singularities are made.

2 The solutions

The line-element is given by

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + Z^2 dz^2), \tag{2.1}$$

where

$$Z = A + k_+ t^{2/3} + k_- t^{-1} \tag{2.2}$$

and

$$A = ax + by + c + {}^5\!/_9\, k_+ (x^2 + y^2). \tag{2.3}$$

Here a, b, c and $k_\pm$ are arbitrary but sufficiently smooth functions of z. The coordinate freedom $\tilde{z} = f(z)$ leads to a rescaling of these functions by the expression $1/f'(z)$, so that there are in fact four essential arbitrary functions. This line-element satisfies the Einstein field equations with irrotational dust as source. The coordinates are comoving and synchronous, so that the four-velocity u of the dust is

$$u = \frac{\partial}{\partial t},$$

and the hypersurfaces orthogonal to u are t = const. The matter density is

$$\mu = {}^4\!/_3\, t^{-2}\,(1 - \chi_+ - \chi_-), \tag{2.4}$$

where

$$\chi_+ = k_+ t^{2/3}/Z, \qquad \chi_- = k_- t^{-1}/Z. \tag{2.5}$$

This solution is in fact the Szekeres [13] solution which was labelled PI by Bonnor & Tomimura [17], with some changes in notation. Bonnor *et al.* [18] have proved that if the functions a, b, c and $k_\pm$ are unrestricted, this solution admits no Killing vector fields. We have shown that even if $k_-(z) = 0$, which is the restriction that we will subsequently impose, there are no Killing vector fields, if the remaining functions are unrestricted.

The expressions $\chi_\pm$ in (2.5) in fact determine all the geometric quantities which are associated with these solutions, as follows. We use the natural orthonormal basis for the line element (2.1) i.e.

$$w_{(0)} = -dt,\ w_{(1)} = t^{2/3}\,dx,\ w_{(2)} = t^{2/3}\,dy,\ w_{(3)} = t^{2/3}\,Zdz. \tag{2.6}$$

The rate of expansion scalar* for the matter is

$$\theta = 2t^{-1}\,(1 + {}^1\!/_3\,\chi_+ - {}^1\!/_2\,\chi_-), \tag{2.7}$$

and the non-zero components (relative to equation 2.6) of the rate of shear tensor* are

$$\sigma_{11} = \sigma_{22} = -{}^1\!/_2\,\sigma_{33} = -{}^2\!/_3\,t^{-1}\,({}^1\!/_3\,\chi_+ - {}^1\!/_2\,\chi_-). \tag{2.8}$$

The acceleration and vorticity of the matter are zero.

The non-zero components of the electric part* of the Weyl conformal curvature tensor relative to (2.6) are

$$E_{11} = E_{22} = -{}^1\!/_2\,E_{33} = {}^4\!/_9\,t^{-2}\,(\chi_+ + \chi_-), \tag{2.9}$$

while the magnetic part $H_{\alpha\beta}$ is zero. Finally the Ricci scalar and tracefree Ricci tensor of the hypersurfaces orthogonal to the matter flow are

$$R^* = -{}^{40}\!/_9\,t^{-2}\,\chi_+, \tag{2.10}$$

$$S^*_{11} = S^*_{22} = -{}^1\!/_2\,S^*_{33} = -{}^{20}\!/_{27}\,t^{-2}\,\chi_+. \tag{2.11}$$

Certain restrictions must be imposed on a, b, c and $k_\pm$ in order that the density be physically reasonable. If for some z_0, $k_+(z_0) = 0$, then, unless $a(z_0) = 0 = b(z_0)$, there will exist values of x and y for which the metric function Z will be zero, for each $t = t_0 > 0$. This will lead to a matter singularity and the density becoming negative (if $k_-(z_0) \neq 0$), or a coordinate singularity (if $k_-(z_0) = 0$). To avoid this, we write a and b in the form

$$a = {}^{10}\!/_9\,\alpha k_+, \qquad b = {}^{10}\!/_9\,\beta k_+,$$

where α and β are arbitrary functions of z, and complete the square in x and y in (2.3). Then provided that $c(z)$ is sufficiently large, we can use the coordinate freedom in z to obtain

$$A = 1 + {}^5\!/_9\,k_+\,[(x + \alpha)^2 + (y + \beta)^2], \tag{2.12}$$

with $k_\pm$ being rescaled. We now assume that

$$k_\pm(z) \geqslant 0 \tag{2.13}$$

for all z, and that α, β are defined for all z. This implies $A \geqslant 1$, $Z \geqslant 1$ and $\mu > 0$ for all x, y, z and all $t > 0$. In summary, if the coordinates satisfy

$$x, y, z \in R, \qquad 0 < t < +\infty, \tag{2.14}$$

and α, β, $k_\pm$ are defined for all z, with $k_\pm$ non-negative, then equations (2.1), (2.2), (2.4) and (2.5), with A given by (2.12), define an exact solution with positive density. There is a

* See Ellis [19] for these terminologies.

matter singularity as $t \to 0^+$, which is simultaneous for all matter particles, but the density is bounded on each hypersurface $t = t_0 = \text{const.} > 0$.

3 Relation with linear perturbations of FRW solutions

In this section we show that the exact solution of Section 2, subject to certain restrictions, is arbitrarily close to an exact FRW solution. When $k_\pm(z) = 0$, the solution is an FRW solution, in fact the Einstein–de Sitter solution, with energy density

$$\mu_0 = 4/(3t^2), \tag{3.1}$$

as follows from (2.1), (2.2), (2.4) and (2.12).

Suppose now that t is restricted by

$$0 < t_1 \leqslant t \leqslant t_2, \tag{3.2}$$

and that $k_\pm(z)$ are restricted by

$$0 \leqslant k_+ t_2^{2/3} \leqslant \epsilon, \qquad 0 \leqslant k_- t_1^{-1} \leqslant \epsilon, \tag{3.3}$$

where t_1, t_2 and ϵ are positive constants. These restrictions, in conjunction with (2.5) and the fact that $Z \geqslant 1$, imply that

$$0 \leqslant \chi_\pm \leqslant \epsilon, \tag{3.4}$$

for all t satisfying (3.2) and all real x, y, z. Let X denote any rational scalar formed from the quantities (2.4) and (2.7)–(2.11), and which has a well-defined value X_0 when $k_\pm(z) = 0$. For example, the shear scalar $\sigma^2 = \tfrac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ is a suitable scalar, but the ratio σ^2/R^* is not. We measure the closeness of the solution to the Einstein–de Sitter solution by considering the quantity

$$\Delta(X) = \max |X - X_0|,$$

where X denotes *any* of the scalars mentioned above and X_0 its FRW counterpart, and the maximum is taken over all real x, y, z, and t subject to (3.3). It follows from (3.4), (2.4) and (2.7)–(2.11) that

$$\lim_{\epsilon \to 0} \Delta(X) = 0.$$

Thus we regard the exact solution, subject to (3.2) and (3.3) with $\epsilon \ll 1$, as being close to the Einstein–de Sitter solution, even though the respective metric components are not close for all x, y, z, due to the fact that A, as given by (2.12), becomes unbounded as $x^2 + y^2 \to +\infty$.

Using (2.4) and (3.1), the density contrast is

$$\frac{\delta\mu}{\mu_0} = -\chi_+ - \chi_-. \tag{3.5}$$

In the linear approximation, for $\epsilon \ll 1$, the function (2.2) becomes

$$Z \approx A,$$

where A is given by (2.12). We cannot write $A \approx 1$, since the term in k_+ in A can become arbitrarily large as $x^2 + y^2 \to +\infty$. Thus

$$\frac{\delta\mu}{\mu_0} \approx -(k_+ t^{2/3} + k_- t^{-1})/A,$$

which is of the form (1.1). This substantiates our claim that the exact solution contains both the growing (k_+) and decaying (k_-) modes of density fluctuations. The density fluctuations are negative, on account of (3.3) and the fact that $A \geqslant 1$. Positive fluctuations, with associated positive spatial curvature R^*, can be obtained only if we restrict x, y, z.

These solutions also illustrate another aspect of linear perturbations of spatially flat FRW solutions. It follows from (2.10) and (2.11) that only the increasing density mode, described by χ_+, appears in the spatial curvature. On the other hand both modes contribute to the rate of shear (and to the Weyl tensor), although near the initial singularity, the decreasing mode dominates. This supports the conclusion reached by Liang [15] (based on the results of [2, 3]) that 'the A-mode (i.e. decreasing mode) arises from primordial shear fluctuations while the B-mode (i.e. increasing mode) arises from primordial curvature fluctuations'.

In the next section we will show in detail that the absence of the decaying mode (k_-) gives rise to a Friedmann-like singularity as $t \to 0$.

4 The singularity

We now discuss the nature of the initial singularity as $t \to 0$. It follows from (2.7), (2.10) and (2.11) that

$$\lim_{t \to 0} \frac{R^*}{\theta^2} = 0, \qquad \lim_{t \to 0} \frac{S^*_{\alpha\beta}}{\theta^2} = 0. \tag{4.1}$$

This means that the spatial curvature is dynamically negligible relative to the expansion as $t \to 0$, and hence the singularity is 'velocity-dominated' in the sense of Eardley *et al.* [2]. The nature of the singularity depends significantly on whether or not the decaying mode of the density fluctuation is present. (i.e. on whether or not $k_-(z)=0$). If $k_-(z) \neq 0$, the singularity is of the Kasner type, with $P = (2/3, 2/3, -1/3)$ in the terminology of [2]. On the other hand, when $k_-(z) = 0$, the line-element assumes the following asymptotic form as $t \to 0$:

$$ds^2 = -dt^2 + t^{4/3}\,[g^{(0)}_{\alpha\beta}\,dx^\alpha\,dx^\beta + O(t^{2/3})], \tag{4.2}$$

where $g^{(0)}_{\alpha\beta} = \mathrm{diag}\,(1, 1, A)$, and $O(t^{2/3})$ denotes terms in $t^{2/3}$ or higher powers of t. Thus in this case the singularity is Friedmann-like, and in the terminology of [2], $g^{(0)}_{\alpha\beta}$ is the metric of the singularity, defined to be the three-dimensional manifold $t = 0$.

The name 'Friedmann-like singularity' suggests that the solution should become isotropic and spatially homogeneous in some sense as the singularity is approached. We now discuss to what extent this is the case for the solutions of Section 2. First, we note that the rate of shear scalar $\sigma^2 = \tfrac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$, which is related to the anisotropy of the matter flow, tends to infinity* as $t \to 0$:

$$\lim_{t \to 0} \sigma^2 = +\infty. \tag{4.3}$$

Despite this fact, the matter flow does approach isotropy in the following sense as $t \to 0$. The length scales l_α, $\alpha = 1, 2, 3$, in the eigendirections of the rate of expansion (or rate of shear) tensor all tend to zero at the same rate as $t \to 0$:

$$\lim_{t \to 0} \frac{l_\alpha}{l} = 1, \qquad \alpha = 1, 2, 3, \tag{4.4}$$

where $l = (l_1 l_2 l_3)^{1/3}$ is the overall length scale. The l_α are defined, up to a scale change which is constant along the flow lines, by

$$\frac{\dot{l}_\alpha}{l_\alpha} = \theta_{\alpha\alpha},$$

*This fact was noted by Bonnor & Tomimura [17].

where $\cdot$ denotes differentiation along the flow lines and the $\theta_{\alpha\alpha}$ are the components of the rate of expansion tensor $\theta_{\alpha\beta}$ in its eigenframe [12]. For the solutions of Section 2 subject to $k_-(z) = 0$, we have, with suitable rescaling

$$l_1 = l_2 = t^{2/3}, \qquad l_3 = t^{2/3}\,[1 + (k_+/A)t^{2/3}],$$

so that (4.4) is satisfied. The result (4.4) does restrict the rate of growth of σ^2 as $t \to 0$, and in fact we have

$$\lim_{t\to 0} \frac{\sigma^2}{\theta^2} = 0, \tag{4.5}$$

as follows from (2.7) and (2.8).

Secondly it follows from (4.2) that the leading term in the space-time metric is not exactly the Einstein–de Sitter metric, since the three-metric $g^{(0)}_{\alpha\beta}$ is not flat – its scalar curvature is in fact

$$R^{(0)} = -{}^{40}\!/_{9}\,(k_+/A). \tag{4.6}$$

In other words, the spatial metric is not homogeneous as $t \to 0^+$. Moreover this inhomogeneity is essential in order that the solution not be an exact FRW solution, since $g^{(0)}_{\alpha\beta}$ is flat if and only if $k_+(z) = 0$. Indeed it is the information in this three-metric, together with the leading t-dependence, that determines the future evolution of the space time away from the singularity (*cf.* [9], p. 331). On the other hand, the solution is spatially homogeneous as $t \to 0$ in the following sense. The leading term in the matter density as $t \to 0$ is precisely the Einstein–de Sitter density (3.1), as follows from (2.4) with $k_-(z) = 0$. Thus the density becomes spatially homogeneous* as $t \to 0$. Another manifestation of this is that the density contrast $\delta\mu/\mu_0$ tends to zero as $t \to 0$ [*cf.* (3.5) with $k_-(z) = 0$]. The leading term in the rate of expansion scalar θ is also spatially homogeneous, since

$$\lim_{t\to 0} \frac{3\mu}{\theta^2} = 1, \tag{4.7}$$

a result which holds in all exact FRW solutions with $p = (\gamma - 1)\mu$ ([19,], p. 41). This is a consequence of the general first integral for irrotational perfect fluids

$$\mu = {}^1\!/_3\,\theta^2 - \sigma^2 + {}^1\!/_2\,R^* \tag{4.8}$$

(see, e.g. [19] p. 34, with $\Lambda = 0$), together with (4.1) and (4.5). The remaining scalars (e.g. σ^2, R^* etc.), which are all zero in the Einstein–de Sitter solution are, however, spatially inhomogeneous in their leading term as $t \to 0^+$.

The behaviour of the Weyl tensor as $t \to 0^+$ is also of interest in connection with the proposal of Penrose mentioned in the introduction. It follows from (2.9) that the Weyl tensor (orthonormal frame components, or scalars), tends to infinity as $t \to 0$, in violation of Penrose's hypothesis. However the Weyl tensor is dominated by the Ricci tensor as $t \to 0$ in the sense that the ratio of the Weyl tensor and the Ricci tensor tends to zero as $t \to 0$. This is exemplified, for example, by

$$\lim_{t\to 0} \frac{E_{\alpha\beta}}{\mu} = 0 = \lim_{t\to 0} \frac{H_{\alpha\beta}}{\mu} = 0, \tag{4.9}$$

We finally comment on the relative dynamical significance of the matter density, the shear and the spatial curvature, near the Friedmann-like singularity. Despite the fact that

*This feature of Friedmann-like singularities arises in the discussion of this topic using a power series expansion, and has been emphasized by Zeldovich in the $p = {}^1\!/_3\,\mu$ case (see the discussion in Belinskii *et al.* [20]).

the spatial curvature is dynamically unimportant relative to θ^2 and hence μ near the singularity (*cf.* equations 4.1 and 4.7), it does play an important role. As mentioned earlier, it determines the asymptotic spatial metric $g^{(0)}_{\alpha\beta}$. In addition, in (4.8), R^* is more significant than σ^2, since

$$\lim_{t\to 0} \frac{\sigma^2}{R^*} = 0, \tag{4.10}$$

as follows from (2.8) and (2.10).

5 Conclusion

It is clearly of interest to know which of the results discussed in Section 4 hold for more general Friedmann-like singularities. A few simple calculations using the Eardley, Liang & Sachs [2] asymptotic but non-exact solutions (and formula in [19]) show that all the results of the preceding section i.e. (4.1), (4.3), (4.5), (4.7), (4.9) and (4.10) hold for their general Friedmann-like singularity with dust source. In particular, the Weyl tensor becomes infinite but is dominated by the Ricci tensor. Similar conclusions also hold for the analysis of Friedmann-like singularities with equations of state $p = (\gamma - 1)\mu$, given by Liang [3], with one difference. The behaviour of the shear scalar σ^2 as $t \to 0$ depends on the equation of state as follows:

$$\lim_{t\to 0} \sigma^2 = \begin{cases} +\infty & \text{if } 1 \leqslant \gamma < 4/3 \\ \text{finite non-zero value,} & \text{if } \gamma = 4/3 \\ 0 & \text{if } 4/3 < \gamma \leqslant 2. \end{cases}$$

These tentative conclusions are supported by analysis of other exact solutions, for example spatially homogeneous solutions with equation of state $p = (\gamma - 1)\mu$ [21, 22], and certain generalizations [23] of the Szekeres solutions which have an asymptotic equation of state $p = (\gamma - 1)\mu$ at the singularity. Details will be published elsewhere.

We should stress that the above general remarks are at best tentative, since they are based on an iterative procedure whose convergence has not been established. It would certainly be desirable to fill this gap. Nevertheless, these remarks do emphasize the need for further investigation as to whether there exist non-FRW cosmological solutions for which the Weyl tensor tends to zero at the initial singularity. In connection with this, we point out that it is not completely clear how one should define the concept of Friedmann-like singularity. One possibility would be to simply require that the matter flow become isotropic as the singularity is approached in the sense of (4.4) or (4.5). It is by no means obvious that this would lead to the currently accepted class of Friedmann-like singularities.

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