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Frobenius Push-Forwards on Quadrics

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
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Instytut Matematyki

Maj 2010

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Abstrakt

We generalize, explain and simplify Langer's results concerning Frobenius direct images of line bundles on quadrics, describing explicitly the decompositions of higher Frobenius push-forwards of arithmetically Cohen-Macaulay bundles into indecomposables, with an additional emphasis on the case of characteristic two. These results are applied to check which Frobenius push-forwards of the structure sheaf are tilting.

Słowa kluczowe

Frobenius morphism, Frobenius push-forward, Frobenius direct image, quadric, positive characteristic, Cohen-Macaulay module, spinor bundle

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic geometry
14F (Co)homology theory
14F05 Sheaves, derived categories of sheaves and related constructions

14 Algebraic geometry
14F (Co)homology theory
14F17 Vanishing theorems

Tytuł pracy w języku angielskim

Frobenius Push-Forwards on Quadrics

Introduction

In [12], A. Langer computed the Frobenius push-forwards of line bundles on quadrics. However, the computations worked only for odd characteristic and explicit formulas for the push-forward were given only for the first Frobenius direct image. In this paper, we determine the push-forwards of line and spinor bundles on smooth quadrics in arbitrary positive characteristic. But mostly, we explain and simplify the aforementioned paper, reproving nearly all of the statements.

To illustrate our method, we briefly show how it can be used to determine Frobenius push-forwards of line bundles on a projective space $\mathbb{P}^N$ (this method is used in [17], Lemma 2.1). If the absolute Frobenius morphism on $\mathbb{P}^N$ is denoted by F, its s-th composition by F^s, the push-forward in question can be written as

$$\mathsf{F}^s_*(\mathscr{O}(a)) = \bigoplus_{t\in\mathbb{Z}} \mathscr{O}(t)^{\alpha^s(t,a)}$$

for some integers $\alpha^s(t,a)$ (the existence of such a decomposition follows directly from Horrocks' splitting criterion and the projection formula).

To compute $\alpha^s(t,a)$, let us write the projection formula using the bundle $\Omega^1_{\mathbb{P}^N}(-b)$:

$$\mathsf{F}^s_*(\mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(a-bq)) = \mathsf{F}^s_*(\mathscr{O}(a)) \otimes \Omega^1_{\mathbb{P}^N}(-b).$$

Comparing dimensions of the first cohomology groups we get

$$h^1(\mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(a-bq)) = \sum_{t\in\mathbb{Z}} \alpha^s(t,a)\cdot h^1(\Omega^1_{\mathbb{P}^N}(t-b)).$$

But $h^1(\Omega^1_{\mathbb{P}^N}(t-b)) = \delta_{t,b}$, so the right hand side is just $\alpha^s(b,a)$.

On the other hand, the dimension of $\mathsf{H}^1(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t))$ can be computed as

$$\dim\Big(\underbrace{k[x_0,\dots,x_N]/(x_0^q,\dots,x_N^q)}_{D^{(s)}}\Big)_t = \sum_{j=0}^{N+1}(-1)^j\binom{N+1}{j}\binom{N+t-jq}{N}$$

(see Lemma 3.1). Hence we obtain

$$\alpha^s(t,a) = \dim D^{(s)}_{a-tq} = \sum_{j=0}^{N+1}(-1)^j\binom{N+1}{j}\binom{N+a-tq-jq}{N}.$$

On quadrics, the situation is quite similar. It is well known that any ACM (*arithmetically Cohen-Macaulay*, i.e., with vanishing $h^i(\mathcal{E}(t))$ for $0<i<n$ and all t) bundle on a smooth n-dimensional quadric decomposes into a direct sum of line bundles and twisted spinor bundles. We use the above method to compute the coefficients in this decomposition. The result (see Theorem 1) is that

$$\mathsf{F}^s_*(\mathscr{O}(a)) = \bigoplus_{t\in\mathbb{Z}} \mathscr{O}(t)^{\beta^s(t,a)} \oplus \bigoplus_{t\in\mathbb{Z}} \mathsf{S}(t)^{\gamma^s(t,a)},$$

where S is the spinor bundle or the sum of the two half-spin bundles on Q_n (see Section 1.3) and the coefficients β and γ are given by the formulas

$$\beta^s(t,a) = \dim C^{(s)}_{a-tq}\,,$$
$$\gamma^s(t,a) = \frac{1}{2^{\lfloor n/2\rfloor+1}}\dim M^{(s)}_{a-(t-1)q}\,,$$

where $C^{(s)}$, and $M^{(s)}$ are certain graded modules defined in Section 2. The decomposition of $\mathsf{F}_*^s(\mathsf{S}(a))$ is also given. This description allows us to give explicit vanishing criteria for these coefficients (Theorems 2 and 3), from which we easily derive corollaries concerning the push-forwards being tilting (Theorem 4). The last section of the paper contains a comment on possible extension of these results to singular quadrics.

In particular, for $p = 2$ the formulas become easier and we can be a little bit more explicit. We extend the main theorems of [12] to this case.

The paper [12] was inspired by Samokhin's paper [18]. Frobenius direct images of the structure sheaf are of particular interest because they can produce tilting bundles and allow us to study $\mathscr{D}$-affinity in positive characteristic ([18], [12], [19]).

Acknowledgements. I would like to thank Prof. Adrian Langer for giving me the idea for writing this paper and for many helpful clues.

Contents

1. Preliminaries

1.1. The Frobenius morphism and some projection formulas

Let X be a projective variety over an algebraically closed field k of characteristic $p > 0$. The absolute Frobenius morphism $\mathsf{F} : X \to X$ is the mapping of schemes acting as identity on the underlying topological space and as the p-th power map on the structure sheaf. It is not a map of k-schemes. Denote by F^s the s-th composition of the Frobenius morphism and set $q = p^s$ once and for all.

Let $\mathscr{F}$ be a locally free sheaf of rank r on X. If X is smooth then F is flat and the sheaf $\mathsf{F}^s_*\mathscr{F}$ is also locally free, of rank $rq^{\dim X}$. The sheaf $\mathsf{F}^{s*}\mathscr{F}$ is locally free of rank r, and it is glued as a bundle using the cocycle obtained by raising the coefficients of the transition matrices defining $\mathscr{F}$ to the q-th power. If $\mathscr{F}$ is a line bundle, we infer from the above description of its pull-back that $\mathsf{F}^{s*}\mathscr{F} \simeq \mathscr{F}^{\otimes q}$.

Let $\mathscr{G}$ be a locally free sheaf. Since the Frobenius is an affine morphism, so that $\mathsf{H}^i(X, \mathscr{F}) = \mathsf{H}^i(X, \mathsf{F}_*\mathscr{F})$, we immediately deduce from the projection formula $\mathsf{F}^s_*(\mathscr{F} \otimes \mathsf{F}^{s*}\mathscr{G}) \simeq \mathsf{F}^s_*(\mathscr{F}) \otimes \mathscr{G}$ the following formulas concerning cohomology:

$$\mathsf{H}^i(X, \mathscr{F} \otimes \mathsf{F}^{s*}\mathscr{G}) \simeq \mathsf{H}^i(X, (\mathsf{F}^s_*\mathscr{F}) \otimes \mathscr{G}), \tag{1.1}$$

$$\mathsf{H}^i(X, \mathscr{F}(tq)) \simeq \mathsf{H}^i(X, (\mathsf{F}^s_*\mathscr{F})(t)), \tag{1.2}$$

$$\mathsf{H}^i(X, (\mathsf{F}^{s*}\mathscr{G})(a + tq)) \simeq \mathsf{H}^i(X, \mathsf{F}_*(\mathscr{O}(a)) \otimes \mathscr{G}(t)). \tag{1.3}$$

Remark. These isomorphisms are not k-linear, but the dimensions over k on both sides agree.

Definition 1.1. A coherent sheaf $\mathscr{F}$ on a projective variety X with a very ample line bundle $\mathscr{L}$ is called *arithmetically Cohen-Macaulay* (*ACM*) if

$$\bigoplus_{t \in \mathbb{Z}} \mathsf{H}^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes t}) = 0 \quad \text{for} \quad 0 < i < \dim X.$$

Formula (1.2) shows that the Frobenius push-forward of any coherent ACM sheaf is ACM.

1.2. Quadrics

Let n be a positive integer. The *smooth n-dimensional quadric* Q_n (or simply Q) is the hypersurface in $\mathbb{P}^N$, $N = n + 1$ defined by the equation $Q_n = 0$ where

$$Q_n = x_0^2 + x_1x_2 + \ldots + x_nx_{n+1}$$

if n is odd and

$$Q_n = x_0x_1 + \ldots + x_nx_{n+1}$$

if n is even. If char $k \neq 2$ then we can take a linear change of coordinates on $\mathbb{P}^N$ such that the quadric Q_n is given by the simpler equation $x_0^2 + \ldots + x_N^2 = 0$.

For completeness, let us also state here that by the adjunction formula Q_n is a Fano variety with the canonical bundle $\omega_X = \mathscr{O}_Q(-n)$ and Hilbert polynomial $q_t := \chi(\mathscr{O}_Q(t))$ equal to

$$q_t = \binom{N+t}{N} - \binom{N+t-2}{N}.$$

Remark. To simplify the calculations, we will assume that $n > 2$. This is not a real restriction since $Q_1 \simeq \mathbb{P}^1$ (Q_1 being the image of the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$) and $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (Q_2 being the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$) and everything we would want to say in these cases could be easily derived from what has been said in the example in the Introduction.

1.3. Spinor bundles

Now we shall recall the basic facts about the so-called *spinor bundles* on smooth quadrics. On Q_n, we have a single spinor bundle Σ if n is odd and two spinor bundles Σ_+, Σ_- (sometimes called half-spin) if n is even. There are many equivalent ways of introducing them present in the literature:

- **Spin representations.** a smooth quadric is a homogeneous space for the group $SO(n+2)$, thus also for $Spin(n+2)$. If the corresponding parabolic subgroup is denoted by P, we thus have a principal P-bundle $\mathscr{P}$ over P. The the Lie algebra of the Levi quotient L of P is $\mathfrak{o}(n)$. Then the spinor bundles can be defined as associated bundles: $\Sigma = \mathscr{P} \times_P V$, $\Sigma_\pm = \mathscr{P} \times_P V_\pm$ where V is the *spin* and $V_\pm$ are the *half-spin* representations of $\mathfrak{o}(n)$.

  References: [20], [12], Section 2.2.

- **Pull-backs** of the tautological bundle by explicit maps from Q_n to the Grassmannian $G(2^{\lfloor n/2 \rfloor+1}, 2 \cdot 2^{\lfloor n/2 \rfloor+1})$.

  References: [16].

- **Matrix factorizations.** A *matrix factorization* of a polynomial f with $f(0,\dots,0) = 0$ is a pair (φ, ψ) of square matrices of the same size such that $\varphi \cdot \psi = f \cdot id = \psi \cdot \varphi$. It was first observed by Eisenbud in [5] that given an appropriate notion of a morphism, the matrix factorizations of f form a category that is equivalent to the stable category of maximal Cohen-Macaulay modules over the local ring $\mathscr{O}_{k^n,0}/(f)$ of the hypersurface defined by $f = 0$. The module corresponding to (φ, ψ) is Coker φ where φ is regarded as a map $\mathscr{O}^m \to \mathscr{O}^m$, m being the size of both matrices; it is an $\mathscr{O}/(f)$–module.

  Using this technique, Eisenbud, Buchweitz and Herzog in [4] then classified all indecomposable *graded* maximal Cohen-Macaulay modules over $k[x_0,\dots,x_N]/(Q_n)$. Their description remains valid over any field k. It turns out that apart from the free MCMs, there is (up to shift) only one indecomposable module M if n is odd and there are two of them, M_+ and M_- if n is even. The corresponding matrix factorizations can be defined inductively as follows (see [12], Section 2.2):

  $$\varphi_{-1} = (x_0) = \psi_{-1}, \quad \varphi_0 = (x_0), \quad \psi_0 = (x_1),$$

  $$\varphi_n = \begin{pmatrix} \varphi_{n-2} & x_n \cdot id \\ x_{n+1} \cdot id & -\psi_{n-2} \end{pmatrix}, \quad \psi_n = \begin{pmatrix} \psi_{n-2} & x_n \cdot id \\ x_{n+1} \cdot id & -\varphi_{n-2} \end{pmatrix}.$$

  To define the spinor bundles using these matrix factorizations, we consider φ_n and ψ_n as maps between locally free sheaves on $\mathbb{P}^N$, i.e., $\varphi_n, \psi_n : \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor (n+1)/2 \rfloor}} \to \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor (n+1)/2 \rfloor}}$. Then for odd n we can define Σ to be the cokernel of $\varphi_n = \psi_n$, which is supported on Q_n. For even n we define Σ_+ to be the cokernel of φ_n and Σ_- to be the cokernel of ψ_n.

  References: [21], [11] and [1].

As mentioned above, we have the following exact sequences of sheaves on $\mathbb{P}^N$:

$$0 \to \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor (n+1)/2 \rfloor}} \xrightarrow{\varphi_n = \psi_n} \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor (n+1)/2 \rfloor}} \to i_*\Sigma \to 0$$

if n is odd and

$$0 \to \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor (n+1)/2 \rfloor}} \xrightarrow{\varphi_n} \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor (n+1)/2 \rfloor}} \to i_* \Sigma_+ \to 0,$$

$$0 \to \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor (n+1)/2 \rfloor}} \xrightarrow{\psi_n} \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor (n+1)/2 \rfloor}} \to i_* \Sigma_- \to 0$$

if n is even. It follows that the spinor bundles are arithmetically Cohen-Macaulay. In fact, as implied by the Eisenbud-Buchweitz-Herzog theorem, they provide a full description of ACM bundles on Q_n:

Theorem. *Any coherent ACM sheaf $\mathscr{F}$ on a smooth quadric Q_n is a direct sum of line bundles and twisted spinor bundles.*

In what follows, we shall use the bundle S defined by $\mathsf{S} = \Sigma$ for n odd and $\mathsf{S} = \Sigma_+ \oplus \Sigma_-$ for n even. We thus have the exact sequence of sheaves on $\mathbb{P}^N$:

$$0 \to \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor n/2 \rfloor +1}} \xrightarrow{\Phi} \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor n/2 \rfloor +1}} \to i_* \mathsf{S} \to 0, \tag{1.4}$$

where (Φ_n, Ψ_n) is the matrix factorization defined by $\Phi_n = \varphi_n$, $\Psi_n = \psi_n$ if n is odd and $\Phi_n = \varphi_n \oplus \psi_n$, $\Psi_n = \psi_n \oplus \varphi_n$ if n is even. The exact sequence (1.4) allows us to compute the Hilbert polynomial $s_t := \chi(\mathsf{S}(t))$ of S:

$$s_t = 2^{\lfloor n/2 \rfloor +1} \binom{n+t-1}{n}.$$

1.4. Structure of the derived category

The structure of the derived category of a smooth quadric was first studied by Kapranov ([10], [9]) in the case of characteristic zero, but it can also be easily seen in arbitrary characteristic from the aforementioned theorem of Buchweitz-Eisenbud-Herzog [4] together with Orlov's theorem on Gorenstein varieties.

Indeed, for a smooth quadric Q_n, Theorem 2.12 from [15] provides a semi-orthogonal decomposition

$$D^b(Q_n) = \langle \mathscr{O}_{Q_n}(-n), \ldots, \mathscr{O}_{Q_n}, D^{gr}_{Sg}(R) \rangle$$

where $R = k[x_0, \ldots, x_N]/(Q_n)$ and $D^{gr}_{Sg}(R)$ is the „graded category of singularities", which coincides with the category $\underline{MCM}_{gr}(R)$ (the stable category of graded maximal Cohen-Macaulay modules) by [3]. The description of $\underline{MCM}_{gr}(R)$ follows from the Buchweitz-Eisenbud-Herzog theorem [4] – this category is generated by the direct summands of $\Gamma_*(\mathsf{S})$.

2. Some graded algebras and modules

As we shall see in Section 3, the Euler sequence allows us to translate dimensions of sheaf cohomology groups into dimensions of gradings of certain 0-dimensional graded modules. In this section we develop technical results which let us compute the decompositions in Section 4.

2.1. Definitions

Let Q be the equation of the n-dimensional quadric as in Section 1.2. Recall that $q = p^s$ and $N = n + 1$. We set

$$\begin{aligned} S &= k[x_0, \dots, x_N], \\ R &= S/(Q), \\ A^{(s)} &= R/(x_0^q + x_1^q, x_2^q, \dots, x_N^q), \\ B^{(s)} &= A/(x_0^q) = R/(x_0^q, x_1^q, \dots, x_N^q), \\ C^{(s)} &= A/(0 : x_0^q), \\ D^{(s)} &= S/(x_0^q, \dots, x_N^q), \\ M^{(s)} &= (0 :_{A^{(s)}} x_0^q)A^{(s)}/(x_0^q)A^{(s)}. \end{aligned}$$

Remark. The strange generator $x_0^q + x_1^q$ in the definition of $A^{(s)}$ is used to make $A^{(s)}$ zero-dimensional (or to ensure that $(x_0^q + x_1^q, x_2^q, \dots, x_N^q)$ is an R-regular sequence). It is easy to check that the ring $S/(Q, x_1^q, \dots, x_N^q)$ is one-dimensional when n is even, i.e., $Q = x_0x_1 + x_2x_3 + \dots$ and $p = 2$. This is due to the fact that x_0^2 does not appear in Q. In any other case, we can assume that $A = S/(Q, x_1^q, \dots, x_N^q)$ as in [12].

By Section 1.3, we can write the module $\Gamma_*(\mathsf{S})$ as the cokernel of a map $\Phi : S[-2]^{2^{\lfloor n/2 \rfloor+1}} \to S[-1]^{2^{\lfloor n/2 \rfloor+1}}$ (Φ is a $2^{\lfloor n/2 \rfloor+1} \times 2^{\lfloor n/2 \rfloor+1}$ matrix of linear forms). The following definitions pertain to spinor bundles and will be needed in Section 4:

$$\begin{aligned} Z &= \Gamma_*(\mathsf{S}) = \operatorname{Coker}(\Phi), \\ \widetilde{A}^{(s)} &= Z/(x_0^q + x_1^q, x_2^q, \dots, x_N^q)Z, \\ \widetilde{B}^{(s)} &= Z/(x_0^q, x_1^q, \dots, x_N^q)Z, \\ \widetilde{C}^{(s)} &= \widetilde{A}/(0 : x_0^q), \\ \widetilde{M}^{(s)} &= (0 :_{\widetilde{A}^{(s)}} x_0^q)\widetilde{A}^{(s)}/(x_0^q)\widetilde{A}^{(s)}. \end{aligned}$$

Recall that Z is a maximal Cohen-Macaulay R-module and that $x_1, \dots, x_N$ is a Z-regular sequence when Z is considered as an S-module. Moreover, $\dim Z_d = 2^{\lfloor n/2 \rfloor+1}\binom{n+d-1}{n} = s_d$.

2.2. Dividing MCMs by q-th powers

Recall that in the example in the Introduction, $\dim D_d^{(s)}$ is the number of monomials in $x_0, \dots, x_N$ of degree d with all exponents $< q$, so by the inclusion-exclusion principle we obtain the combinatorial formula (which we already used there):

$$\dim D_d^{(s)} = \sum_{i=0}^{N+1} (-1)^j \binom{N+1}{j} \binom{N+d-jq}{N}. \tag{2.1}$$

In our study of spinor bundles, we shall need a more general statement. The following lemma explains this combinatorial formula in more algebraic terms.

Lemma 2.1. *Let M be a graded module over a graded algebra R generated by R_1 over a field $k = R_0$. Let $(x_1, \ldots, x_k) \in R_q$ be a regular sequence on M and $I = (x_1, \ldots, x_k)$. Then*

$$\dim_k (M/IM)_d = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \dim_k M_{d-jq}.$$

Proof. We construct the Koszul complex $C_* = M \otimes \mathsf{K}(x_1, \ldots, x_k)$. By [14], Theorem 43 (or [6], Corollary 17.5) we have $H_i(C_*) = 0$ for $i > 0$ and $H_0(C_*) = M/IM$. Hence

$$\dim_k (M/IM)_d = \sum_{i \geqslant 0} (-1)^j \dim_k (C_j)_{d-jq},$$

since the maps in the Koszul complex have degree q. But $C_j = \Lambda^{N+1-j} R^{N+1} \otimes M \simeq M^{\binom{k}{j}}$, which finishes the proof. ☐

Note also that by [6], Corollary 17.8, if $(x_0, \ldots, x_N)$ is an M-regular sequence then so is $(x_0^q, \ldots, x_N^q)$. We deduce (2.1) once again, together with

$$\dim A_d^{(s)} = \sum_{j=0}^{N} (-1)^j \binom{N}{j} q_{d-jq}, \tag{2.2}$$

$$\dim \widetilde{A}_d^{(s)} = \sum_{j=0}^{N} (-1)^j \binom{N}{j} s_{d-jq}. \tag{2.3}$$

2.3. Dimensions of $B_d^{(s)}$ and $\widetilde{B}_d^{(s)}$

We have the following two short exact sequences of graded modules:

$$0 \to C^{(s)}[-q] \xrightarrow{x_0^q} A^{(s)} \to B^{(s)} \to 0, \tag{2.4}$$

$$0 \to \widetilde{C}^{(s)}[-q] \xrightarrow{x_0^q} \widetilde{A}^{(s)} \to \widetilde{B}^{(s)} \to 0, \tag{2.5}$$

Seeing that $\dim M_d^{(s)} = \dim B_d^{(s)} - \dim C_d^{(s)}$, we obtain $\dim B_d^{(s)} = \dim A_d^{(s)} + \dim M_{d-q}^{(s)} - \dim B_{d-q}^{(s)}$ (and the same with the tildes). This gives the formulas

$$\dim B_d^{(s)} = \sum_{j \geqslant 0} (-1)^j \dim A_{d-jq}^{(s)} + \sum_{j \geqslant 0} (-1)^j \dim M_{d-(j+1)q}^{(s)}, \tag{2.6}$$

$$\dim \widetilde{B}_d^{(s)} = \sum_{j \geqslant 0} (-1)^j \dim \widetilde{A}_{d-jq}^{(s)} + \sum_{j \geqslant 0} (-1)^j \dim \widetilde{M}_{d-(j+1)q}^{(s)}. \tag{2.7}$$

3. The Frobenius morphism and the sheaf of differentials

Now let us relate the commutative algebra from Section 2 to some cohomology groups which will be used in Section 4. The following standard result can be found, e.g., in [2] (see Theorem 3).

Lemma 3.1. *Let $H \subseteq \mathbb{P}^N$ $(N > 2)$ be the hypersurface given by $f = 0$. Then there is an isomorphism of graded $S/(f)$-modules:*

$$\bigoplus_{t\in\mathbb{Z}} \mathsf{H}^1(H, (\mathsf{F}^{s*}(\Omega^1_{\mathbb{P}^N}|_H))(t)) \simeq D^{(s)}/(f)$$

For $s = 0$ we obtain

$$h^1(\Omega^1_{\mathbb{P}^N}|_H(t)) = \delta_{t,0} \tag{3.1}$$

When Q is our quadric and S the spinor bundle (or the sum of the two half-spin bundles) as defined in Section 1.3, we obtain the following lemma:

Lemma 3.2. *We have the following isomorphism of $R = S/(Q)$-modules:*

$$\bigoplus_{t\in\mathbb{Z}} \mathsf{H}^1(\mathsf{S} \otimes \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}|_Q(t)) \simeq \widetilde{B}^{(s)}.$$

Proof. Let us recall the exact sequence (1.4):

$$0 \to \mathscr{O}_{\mathbb{P}^N}(-2)^{2^{\lfloor n/2\rfloor+1}} \xrightarrow{\Phi} \mathscr{O}_{\mathbb{P}^N}(-1)^{2^{\lfloor n/2\rfloor+1}} \to \mathsf{S} \to 0.$$

Tensoring it by $\mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t)$ we get the following long cohomology exact sequence:

$$\begin{aligned} \dots \to &\mathsf{H}^1(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t-2))^{2^{\lfloor n/2\rfloor+1}} \xrightarrow{\Phi} \mathsf{H}^1(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t-1))^{2^{\lfloor n/2\rfloor+1}} \\ \to &\mathsf{H}^1(Q, \mathsf{S} \otimes \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}|_Q(t)) \to \mathsf{H}^2(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t-2))^{2^{\lfloor n/2\rfloor+1}}. \end{aligned}$$

Applying $\mathsf{F}^{s*}(-)\otimes\mathscr{O}_{\mathbb{P}^N}(d)$ to the Euler sequence and writing the cohomology exact sequence, we see that the last group in the above sequence vanishes for $N > 2$. Hence $\mathsf{H}^1(Q, \mathsf{S} \otimes \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t))$ is the cokernel of the map

$$\mathsf{H}^1(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t-2))^{2^{\lfloor n/2\rfloor+1}} \xrightarrow{\Phi} \mathsf{H}^1(\mathbb{P}^N, \mathsf{F}^{s*}\Omega^1_{\mathbb{P}^N}(t-1))^{2^{\lfloor n/2\rfloor+1}}.$$

Using our description of these groups from the previous lemma we see that it is just the t-th graded piece of the graded module $\widetilde{B}^{(s)}$. □

Clearly, this lemma works (with the definitions slightly adjusted) for an arbitrary ACM sheaf over a hypersurface (since ACM sheaves are in this case given by matrix factorizations).

As a corollary, for $s = 0$ we obtain the following formula (see [12], Proposition 4.1):

$$h^1(\mathsf{S} \otimes \Omega^1_{\mathbb{P}^N}|_Q(t)) = 2^{\lfloor n/2\rfloor+1} \cdot \delta_{t,1}. \tag{3.2}$$

4. Decompositions of $\mathsf{F}^s_*(\mathscr{O}(a))$ and $\mathsf{F}^s_*(\mathsf{S}(a))$

Let $\beta^s(t,a)$, $\gamma^s(t,a)$, $\delta^s(t,a)$ and $\varepsilon^s(t,a)$ be defined by the decompositions

$$\mathsf{F}^s_*(\mathscr{O}(a)) = \bigoplus_{t\in\mathbb{Z}} \mathscr{O}(t)^{\beta^s(t,a)} \oplus \bigoplus_{t\in\mathbb{Z}} \mathsf{S}(t)^{\gamma^s(t,a)},$$

$$\mathsf{F}^s_*(\mathsf{S}(a)) = \bigoplus_{t\in\mathbb{Z}} \mathscr{O}(t)^{\delta^s(t,a)} \oplus \bigoplus_{t\in\mathbb{Z}} \mathsf{S}(t)^{\varepsilon^s(t,a)},$$

where S is the spinor bundle or the sum of the two half-spin bundles as defined in 1.3.

(1) By the projection formula ((1.2) for $\mathscr{F} = \mathscr{O}(a)$ or $\mathsf{S}(a)$ and $t = b$) we obtain the following equalities of Hilbert polynomials:

$$q_{a+bq} = \sum_{t\in\mathbb{Z}} \beta^s(t,a)\cdot q_{t+b} + \sum_{t\in\mathbb{Z}} \gamma^s(t,a)\cdot s_{t+b}, \tag{4.1}$$

$$s_{a+bq} = \sum_{t\in\mathbb{Z}} \delta^s(t,a)\cdot q_{t+b} + \sum_{t\in\mathbb{Z}} \varepsilon^s(t,a)\cdot s_{t+b}. \tag{4.2}$$

(2) Let $\psi = \Omega^1_{\mathbb{P}^N}|_Q$. By the projection formula ((1.3) for $\mathscr{G} = \psi$, $i = 1$ and $t = -b$) we get

$$\mathsf{H}^1(Q_n, (\mathsf{F}^{s*}\psi)(a-bq)) = \mathsf{H}^1(Q_n, \mathsf{F}^s_*(\mathscr{O}(a))\otimes\psi(-b)).$$

By Lemma 3.1 we then have

$$\dim B^{(s)}_{a-bq} = \dim\left(k[x_0,\dots,x_N]/(Q, x_0^q,\dots,x_N^q)\right)_{a-bq} = h^1(\mathsf{F}^s_*(\mathscr{O}(a))\otimes\psi(-b))$$

which can be rewritten as

$$\dim B^{(s)}_{a-bq} = \sum_{t\in\mathbb{Z}} \beta^s(t,a)\cdot h^1(\psi(t-b)) + \sum_{t\in\mathbb{Z}} \gamma^s(t,a)\cdot h^1(\psi\otimes\mathsf{S}(t-b)).$$

But $h^1(\psi(t-b)) = \delta_{t,b}$ by (3.1) and $h^1(\psi\otimes\mathsf{S}(t-b)) = 2^{\lfloor n/2\rfloor+1}\cdot\delta_{t,b+1}$ by (3.2), so this reduces to $\dim B^{(s)}_{a-bq} = \beta^s(b,a) + 2^{\lfloor n/2\rfloor+1}\cdot\gamma^s(b+1,a)$. Equivalently, we have

$$\beta^s(t,a) = \dim B^{(s)}_{a-tq} - 2^{\lfloor n/2\rfloor+1}\cdot\gamma^s(t+1,a). \tag{4.3}$$

Similarly, using Lemma 3.2 and (3.1) one obtains

$$\delta^s(t,a) = \dim \widetilde{B}^{(s)}_{a-tq} - 2^{\lfloor n/2\rfloor+1}\cdot\varepsilon^s(t+1,a). \tag{4.4}$$

(3) We put (4.3) into (4.1), thus obtaining

$$\begin{aligned}
q_{a+bq} &= \sum_{t\in\mathbb{Z}}(\dim B^{(s)}_{a-tq} - 2^{\lfloor n/2\rfloor+1}\cdot\gamma^s(t+1,a))q_{t+b} + \sum_{t\in\mathbb{Z}}\gamma^s(t,a)\cdot s_{t+b} \\
&= \sum_{t\in\mathbb{Z}}\dim(B^{(s)}_{a-tq})q_{t+b} + \sum_{t\in\mathbb{Z}}\gamma^s(t,a)(s_{t+b} - 2^{\lfloor n/2\rfloor+1}\cdot q_{t+b-1}) \\
&= \sum_{t\in\mathbb{Z}}\dim(B^{(s)}_{a-tq})q_{t+b} - 2^{\lfloor n/2\rfloor+1}\sum_{t\in\mathbb{Z}}\gamma^s(t,a)\binom{n+t-2+b}{n}.
\end{aligned}$$

We rewrite this as

$$\sum_{t\in\mathbb{Z}}\dim(B^{(s)}_{a-tq})q_{b+t} - q_{a+bq} = 2^{\lfloor n/2\rfloor+1}\sum_{t\in\mathbb{Z}}\gamma^s(t+2,a)\binom{n+t+b}{n}. \tag{4.5}$$

Similarly, we get

$$\sum_{t\in\mathbb{Z}} \dim(\widetilde{B}^{(s)}_{a-tq})q_{b+t} - s_{a+bq} = 2^{\lfloor n/2\rfloor+1}\sum_{t\in\mathbb{Z}} \varepsilon^s(t+2,a)\binom{n+t+b}{n}. \tag{4.6}$$

We treat both sides as polynomials in b. Our goal is to rewrite the left hand side as a combination of $\binom{n+t_i+b}{n}$ for some t_i $(i=0,\dots,n)$ and conclude that this determines the numbers $\gamma^s(t+2,a)$. This will follow from the fact that for any pairwise distinct numbers $t_0,\dots,t_n$ the polynomials $\binom{t_i+x}{n}$ are linearly independent, and $\gamma^s(t,a)$, $\varepsilon^s(t,a)$ do not vanish only when $t=t_i$ for some $i\in\{0,\dots,n\}$.

(4) Now we use the formulas (2.6) and (2.7) for $\dim B^{(s)}_d$ and $\dim \widetilde{B}^{(s)}_d$ to expand the left hand sides of (4.5) and (4.6), first calculating the sums

$$\sum_{t\in\mathbb{Z}} \dim(B^{(s)}_{a-tq})q_{b+t} = \underbrace{\sum_{t\in\mathbb{Z}}\sum_{j\geqslant 0}(-1)^j\dim(A^{(s)}_{a-tq-jq})q_{b+t}}_{S_1} + \underbrace{\sum_{t\in\mathbb{Z}}\sum_{j\geqslant 0}(-1)^j\dim(M^{(s)}_{a-q-tq-jq})q_{b+t}}_{S_2},$$

$$\sum_{t\in\mathbb{Z}} \dim(\widetilde{B}^{(s)}_{a-tq})q_{b+t} = \underbrace{\sum_{t\in\mathbb{Z}}\sum_{j\geqslant 0}(-1)^j\dim(\widetilde{A}^{(s)}_{a-tq-jq})q_{b+t}}_{S_1'} + \underbrace{\sum_{t\in\mathbb{Z}}\sum_{j\geqslant 0}(-1)^j\dim(\widetilde{M}^{(s)}_{a-q-tq-jq})q_{b+t}}_{S_2'}.$$

Lemma 4.1. *Let $\alpha(t)=\sum_{j\geqslant 0}(-1)^j\binom{n+1}{j}f(t-jq)$. Then*

$$f(a+bq) = \sum_{t\in\mathbb{Z}}\alpha(a-tq)\binom{n+t+b}{n}.$$

Proof. Expanding the right hand side gives $\sum_{u\in\mathbb{Z}} f(a+uq)\left(\sum_{i+j=b-u}(-1)^j\binom{n+1}{j}\binom{n+i}{n}\right)$ and the nested sum is equal to the coefficient of z^{b-u} in $(1-z)^{n+1}\cdot(1-z)^{-n-1}=1$. So it is just $\delta_{b,u}$. ☐

Lemma 4.2. *The following identities hold*

$$q_{a+bq} = \sum_{t\in\mathbb{Z}}\dim A^{(s)}_{a-tq}\binom{n+b+t}{n}, \quad s_{a+bq} = \sum_{t\in\mathbb{Z}}\dim \widetilde{A}^{(s)}_{a-tq}\binom{n+b+t}{n}.$$

Proof. This follows immediately from Lemma 4.1 for $f(t)=q_t$ and $f(t)=s_t$ and from the formulas (2.2), (2.3) for the dimensions of A_d and $\widetilde{A}_d$. ☐

Lemma 4.3. *Let $\alpha(t)=\sum_{j\geqslant 0}(-1)^j f(t-jq)$. Then*

$$\sum_{t\in\mathbb{Z}}\alpha(a-tq)q_{b+t} = \sum_{t\in\mathbb{Z}} f(a-tq)\binom{n+t+b}{n}.$$

Proof. We expand the left hand side

$$LHS = \sum_{t\in\mathbb{Z}}\sum_{j\geqslant 0}(-1)^j f(a-tq-jq)q_{b+t} = \sum_{u\in\mathbb{Z}} f(a+qu)\left(\sum_{i\leqslant b+u}(-1)^{b+u-i}q_i\right).$$

and observe that $\sum_{i\leqslant x}(-1)^{x-i}q_i=\binom{n+x}{n}$, which yields the result. ☐

Now by Lemma 4.2, S_1 and S_1' cancel out with q_{a+bq} and s_{a+bq} on the left hand sides of (4.5) and (4.6), respectively. Hence Lemma 4.3 shows that

$$S_2 = \sum_{t\in\mathbb{Z}} \dim M^{(s)}_{a-(t+1)q} \binom{n+t+b}{n}, \quad S_2' = \sum_{t\in\mathbb{Z}} \dim \widetilde{M}^{(s)}_{a-(t+1)q} \binom{n+t+b}{n}.$$

Putting these into (4.5) and (4.6) (and replacing t by $t-2$) yields

$$\sum_{t\in\mathbb{Z}} \left(\frac{1}{2^{\lfloor n/2\rfloor+1}} \dim M^{(s)}_{a-(t-1)q} - \gamma^s(t,a) \right) \binom{n+t-2+b}{n} = 0, \tag{4.7}$$

$$\sum_{t\in\mathbb{Z}} \left(\frac{1}{2^{\lfloor n/2\rfloor+1}} \dim \widetilde{M}^{(s)}_{a-(t-1)q} - \varepsilon^s(t,a) \right) \binom{n+t-2+b}{n} = 0. \tag{4.8}$$

(5) We want to conclude from (4.7) and (4.8) that

$$\gamma^s(t,a) = \frac{1}{2^{\lfloor n/2\rfloor+1}} \dim M^{(s)}_{a-(t-1)q} \quad \text{and} \quad \varepsilon^s(t,a) = \frac{1}{2^{\lfloor n/2\rfloor+1}} \dim \widetilde{M}^{(s)}_{a-(t-1)q},$$

which with the formulas (4.3) and (4.4) immediately gives

$$\beta^s(t,a) = \dim C^{(s)}_{a-tq} \quad \text{and} \quad \delta^s(t,a) = \dim \widetilde{C}^{(s)}_{a-tq}.$$

Observe that by the formula (4.3), $\gamma^s(t+1,a) \neq 0$ implies $B^{(s)}_{a-tq} \neq 0$. Note that $B^{(s)}_d \neq 0$ only for $0 \leqslant d \leqslant (q-1)(n+1)$ and $K^{(s)}_d \neq 0$ only for $1 \leqslant d \leqslant (q-1)(n+1)+1$ (since $D^{(s)}_d \neq 0$ if and only if $0 \leqslant d \leqslant (q-1)(n+1)$). Therefore if $\frac{1}{2^{\lfloor n/2\rfloor+1}} \dim M^{(s)}_{a-(t-1)q} - \gamma^s(t,a)$ is non-zero, then $0 \leqslant a-(t-1)q \leqslant (n+1)(q-1)$. This can happen for at most $n+1$ values of t, so (4.7) is an equation of linear dependence of the polynomials $\binom{t_i+x}{n}$ for $n+1$ distinct values t_i (similarly with (4.8)). As they are clearly linearly independent (by the Vandermonde determinant), we conclude that all the coefficients are zero. This yields

Theorem 1. *The coefficients $\beta^s(t,a)$ and $\gamma^s(t,a)$ (resp. $\delta^s(t,a)$ and $\varepsilon^s(t,a)$) of $\mathscr{O}(t)$ and $\mathsf{S}(t)$ in $\mathsf{F}^s_*(\mathscr{O}(a))$ (resp. $\mathsf{F}^s_*(\mathsf{S}(a))$) are given by the formulas*

$$\beta^s(t,a) = \dim C^{(s)}_{a-tq}, \qquad \gamma^s(t,a) = \frac{1}{2^{\lfloor n/2\rfloor+1}} \dim M^{(s)}_{a-(t-1)q}.$$

$$\delta^s(t,a) = \dim \widetilde{C}^{(s)}_{a-tq}, \qquad \varepsilon^s(t,a) = \frac{1}{2^{\lfloor n/2\rfloor+1}} \dim \widetilde{M}^{(s)}_{a-(t-1)q}.$$

Remark. Since $h^1(S(t)) = 0$ and $h^1(S \otimes S(t)) = \delta_{t,0}$ for n odd and $2 \cdot \delta_{t,0}$ for n even ([12], Lemma 2.3), by the projection formula ((1.1) with $\mathscr{F} = \mathscr{O}(d)$, $\mathscr{G} = \mathsf{S}$ and $i = 1$) we obtain

$$\dim M^{(s)}_d = 2^{\lceil n/2\rceil} h^1(\mathsf{F}^{s*}\mathsf{S}(d-q)) \quad \text{and} \quad \dim \widetilde{M}^{(s)}_d = 2^{\lceil n/2\rceil} h^1(\mathsf{S} \otimes \mathsf{F}^{s*}\mathsf{S}(d-q)).$$

5. Vanishing and non-vanishing

5.1. Symmetry

For smooth complete varieties X, Y and a proper morphism $f : X \to Y$, the relative Serre duality ([7]) can be expressed in the following form (e.g. [8], 3.4, formula 3.20):

$$Rf_* D(\mathcal{E}) = D(Rf_* \mathcal{E}),$$

where $D(\mathcal{E}) = \mathcal{E}^\vee \otimes \omega$. Now since the Frobenius morphism is an affine morphism, the higher direct images vanish, and we get

Proposition. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$ and let $\mathsf{F} : X \to X$ be the absolute Frobenius morphism. Then for any vector bundle $\mathcal{E}$ on X we have*

$$\mathsf{F}_*(\mathcal{E}^\vee \otimes \omega_X) = (\mathsf{F}_*\mathcal{E})^\vee \otimes \omega_X.$$

On a smooth n-dimensional quadric Q_n, we have $\omega_{Q_n} = \mathcal{O}_{Q_n}(-n)$ and $\mathsf{S}^\vee = \mathsf{S}(1)$. This shows that, in the notation of Section 4,

$$\beta^s(t,a) = \beta^s(-t-n, -a-n), \qquad \delta^s(t,a) = \delta^s(-t-n, -a+1-n),$$
$$\gamma^s(t,a) = \gamma^s(-t+1-n, -a-n), \qquad \varepsilon^s(t,a) = \varepsilon^s(-t+1-n, -a+1-n).$$

Setting $t = 0$ and using Theorem 1 we deduce

Proposition 5.1.

$$C_d^{(s)} = C_{n(q-1)-d}^{(s)}, \qquad \widetilde{C}_d^{(s)} = \widetilde{C}_{n(q-1)+1-d}^{(s)},$$
$$M_d^{(s)} = M_{n(q-1)+q-d}^{(s)}, \qquad \widetilde{M}_d^{(s)} = \widetilde{M}_{n(q-1)+q+1-d}^{(s)}.$$

We also need the symmetry of $A^{(s)}$ and $\widetilde{A}^{(s)}$:

Proposition 5.2.

$$A_d^{(s)} = A_{n(q-1)+q-d}^{(s)}, \qquad \widetilde{A}_d^{(s)} = \widetilde{A}_{n(q-1)+q+1-d}^{(s)}.$$

Proof. Use formulas (2.2) and (2.3). □

5.2. Which summands appear ($p > 2$)

In this section we assume that $p > 2$. We will be able to show precisely which summands do appear in higher Frobenius push-forwards of ACM bundles. In the view of Theorem 1, this is equivalent to determining which graded parts of the zero-dimensional graded modules $C^{(s)}$, $M^{(s)}$, $\widetilde{C}^{(s)}$ and $\widetilde{C}^{(s)}$ treated in Section 2 are non-zero.

For brevity, let $D = D^{(1)} = k[x_0, \ldots, x_N]/(x_0^p, \ldots, x_N^p)$.

Langer's Lemma (Proposition 3.1 in [12], see also [13])**.** *Let $0 \leqslant e \leqslant p$ and let $x \in D_d$ with $d \leqslant \frac{1}{2}(N+1)(p-1) - e$. Assume that $Q^e \cdot x = 0$. Then there exists a $y \in D_{d-2(p-e)}$ such that $x = Q^{p-e} \cdot y$.*

Lemma 5.3. *Let (Φ, Ψ), $\Phi, \Psi \in \mathrm{M}_{k \times k}(D_1)$ be an* arbitrary *matrix factorization of Q over the ring D. Let $0 < e \leqslant p$ and let $h \in D_d^k$ with $d \leqslant \frac{1}{2}(N+1)(p-1) - e$.*

1. *If $Q^e \cdot h = 0$ then there exists g such that $h = Q^{p-e} \cdot g$.*

2. *If $Q^{e-1} \cdot \Phi(h) = 0$ then there exists g such that $h = Q^{p-e} \cdot \Psi(g)$.*

Proof.
1. This is Langer's Lemma above.

2. Let us first show that there exists f such that $h = \Psi(f)$. If $e < p$ then since $Q^e \cdot h = \Psi(Q^{e-1} \cdot \Phi(h)) = 0$, by (1) there exists f' such that $h = Q^{p-e} \cdot f' = \Psi(Q^{p-e-1} \cdot \Phi(f'))$. So we take $f = Q^{p-e-1} \cdot \Phi(f')$. Assume that $e = p$. Applying (1) to $\Phi(h)$ and $e = p-1$ gives us u such that $\Phi(h) = Q \cdot u$. Therefore $\Phi(h - \Psi(u)) = 0$. Now, because of what we have just proven for $e = 1$, there exists v such that $h - \Psi(u) = \Psi(v)$. So we can put $f = u + v$.

To finish the proof, we observe that since $h = \Psi(f)$, we have $0 = Q^{e-1}\Psi(h) = Q^e \cdot f$. So again by (1) there exists g such that $f = Q^{p-e} \cdot g$ and hence $h = Q^{p-e} \cdot \Psi(g)$. □

Proposition 5.4.

1. *$M^{(1)}_d = 0$ for $d \leqslant \frac{1}{2}n(p-1)$ or $d \geqslant \frac{1}{2}n(p-1) + p$.*

2. *$\widetilde{M}^{(1)}_d = 0$ for $d \leqslant \frac{1}{2}n(p-1)$ or $d > \frac{1}{2}n(p-1) + p$.*

Proof. By Proposition 5.1 it is sufficient to show the vanishings for $d \leqslant \frac{1}{2}n(p-1)$.

1. See the proof of Proposition 3.4 from [12].

2. We mimic the proof of the aforementioned Proposition. We need to prove that if g_0 is a vector of homogeneous polynomials of degree $\leqslant \frac{1}{2}n(p-1) - 1$ such that

$$x_0^p \cdot g_0 = \Phi(h) + \sum_{i=1}^{N} x_i^p \cdot g_i \qquad (*)$$

then there exist h', h_i, $i = 0, \ldots, N$ such that $g_0 = x_0^p \cdot h_0 + \sum_{i=1}^N x_i^p \cdot h_i + \Psi(h')$.

By $(*)$ and the previous lemma, there exist $h', h_0, h'_1, \ldots, h'_n$ such that $h = Q^{p-1}\Phi(h') + x_0^p \cdot h_0 + \sum_{i=1}^N x_i^p \cdot h'_i$. Putting this back into $(*)$ yields

$$\begin{aligned} g_0 \cdot x_0^p &= Q^p \cdot h' + x_0^p \cdot \Psi(h_0) + \sum_{i=1}^{N} x_i^p \cdot (\Psi(h'_i) + g_i) \\ &= x_0^{2p} \cdot h' + (Q - x_0^2)^p \cdot h' + x_0^p \cdot \Psi(h_0) + \sum_{i=1}^{N} x_i^p \cdot (\Psi(h'_i) + g_i). \end{aligned}$$

Hence $x_0^p \cdot (g_0 - x_0^p \cdot h' - \Psi(h_0)) = \sum_{i=1}^N x_i^p \cdot h''_i$ for some h''_i. But x_0^p is not a zero divisor in $k[x_0, \ldots, x_N]/(x_1^p, \ldots, x_N^p)$, which shows that $g_0 - x_0^p \cdot h' - \Psi(h_0) = \sum_{i=1}^N x_i^p \cdot h_i$ for some h_i. □

Proposition 5.5.

1. *$C^{(1)}_d \neq 0$ if and only if $0 \leqslant d \leqslant n(p-1)$.*

2. *$\widetilde{C}^{(1)}_d \neq 0$ if and only if $1 \leqslant d \leqslant n(p-1)$.*

Proof. Since $\dim C^{(1)}_0 = 1$, $\dim C^{(1)}_{-1} = 0$ and $\dim C^{(1)}_d = \dim C^{(1)}_{n(p-1)-d}$, it suffices to check that $\dim C^{(1)}_d$ is increasing for $d \leqslant \frac{1}{2}n(p-1)$. But by the previous lemma and the exact sequence (2.4)

$$\dim C^{(1)}_d = \dim B^{(1)}_d = \sum_{i \geqslant 0} (-1)^i \dim A^{(1)}_{d-pi}.$$

Now the formula (2.2) yields the result. The proof for $\widetilde{C}^{(1)}$ is analogous. □

Proposition 5.6.

1. $M^{(1)}_d \neq 0$ *if and only if* $\frac{1}{2}n(p-1) < d < \frac{1}{2}n(p-1)+p$,

2. $\widetilde{M}^{(1)}_d \neq 0$ *if and only if* $\frac{1}{2}n(p-1) < d \leqslant \frac{1}{2}n(p-1)+p$.

Proof. The exact sequences (2.4) and (2.5) together with Proposition 5.4 yield

$$\dim M^{(1)}_d = \sum_{i\in\mathbb{Z}} (-1)^i \dim A^{(1)}_{d+pi}$$

for $d \in (\frac{1}{2}n(p-1), \frac{1}{2}n(p-1)+p]$ and $M^{(1)}_d = 0$ otherwise. The same is true for $\widetilde{M}^{(1)}$ and $\widetilde{A}^{(1)}$ in place of $M^{(1)}$ and $A^{(1)}$.

Let $D(d,N) = \sum_{j=0}^{N} (-1)^j \binom{N}{j}\binom{n+d-pj}{n}$ and $E(d,N) = \sum_{i\in\mathbb{Z}} (-1)^i D(d+ip,N)$. Then by formulas (2.2) and (2.3)

$$\dim A^{(1)}_d = D(d,N) + D(d-1,N) \text{ and } \dim \widetilde{A}^{(1)}_d = 2^{\lfloor n/2\rfloor+1} D(d-1,N).$$

So, in the view of the above formulas for $\dim M^{(1)}_d$ and $\dim \widetilde{M}^{(1)}_d$, we want to prove that for p odd, $E(d-1,N)$ is always non-zero and that $E(d,N) + E(d-1,N) = 0$ if and only if p divides $d - \frac{1}{2}n(p-1)$.

We proceed by induction on N, proving also that $E(d,N)$ is increasing with respect to d for $d \in (\frac{1}{2}n(p-1), \frac{1}{2}N(p-1)]$. For $N=1$ we have $D(d,1) = 1$ for $d = 0,\dots,p-1$ and 0 otherwise, so $E(d,1) \neq 0$ for all d and $E(d,1) = -E(d-1,1)$ if and only if p divides d.

For the induction step, we use the formula $E(d,N) = \sum_{j=0}^{p-1} E(d-j,N-1)$, the fact that $E(d,N-1) > 0$ for $d \in (\frac{1}{2}(n-1)(p-1), \frac{1}{2}(n-1)(p-1)+p]$ and $E(d,N)+E(d-1,N) > 0$ for $d \in (\frac{1}{2}(n-1)(p-1), \frac{1}{2}(n-1)(p-1)+p)$ (being the dimension of a vector space) and the symmetry for $M^{(1)}$ and $\widetilde{M}^{(1)}$. □

Theorem 2. *Let* $p > 2$, $s \geqslant 1$ *and* $n > 2$. *Then*

1. $\mathsf{F}^s_*(\mathscr{O}(a))$ *contains* $\mathscr{O}(t)$ *if and only if* $0 \leqslant a - tq \leqslant n(q-1)$,

2. $\mathsf{F}^s_*(\mathscr{O}(a))$ *contains* $\mathsf{S}(t)$ *if and only if*

$$\left(\frac{1}{2}n(p-1) - p + 1\right) q/p \leqslant a - tq \leqslant \left(\frac{1}{2}n(p-1) - 1\right) q/p + n(q/p-1),$$

3. $\mathsf{F}^s_*(\mathsf{S}(a))$ *contains* $\mathscr{O}(t)$ *if and only if* $1 \leqslant a - tq \leqslant n(q-1)$,

4. $\mathsf{F}^s_*(\mathsf{S}(a))$ *contains* $\mathsf{S}(t)$ *if and only if*

$$\left(\frac{1}{2}n(p-1) - p + 1\right) q/p + 1 - \delta_{s,1} \leqslant a - tq \leqslant \left(\frac{1}{2}n(p-1) - 1\right) q/p + n(q/p-1) + \delta_{s,1}.$$

Proof. Denote the upper and lower bounds in 1 – 4 by β^s_0, β^s_1, …, ε^s_0 and ε^s_1. By Propositions 5.5 and 5.6 together with Theorem 1 we obtain the required assertion for $s=1$. Observe that

$$\beta^s_0 \leqslant \delta^s_0 \leqslant \gamma^s_0 \leqslant \varepsilon^s_0 \leqslant \gamma^s_1 \leqslant \varepsilon^s_1 \leqslant \beta^s_1 = \delta^s_1.$$

1. $\mathsf{F}^s_*\mathscr{O}(a)$ contains $\mathscr{O}(t)$ if and only if either there exists an i such that $\mathsf{F}^{s-1}_*(\mathscr{O}(a))$ contains $\mathscr{O}(i)$ and $\mathsf{F}_*(\mathscr{O}(i))$ contains $\mathscr{O}(t)$, or there exists an i such that $\mathsf{F}^{s-1}_*(\mathscr{O}(a))$ contains $\mathsf{S}(i)$ and

$\mathsf{F}_*(\mathsf{S}(i))$ contains $\mathscr{O}(t)$. By the induction assumption, this holds if and only if there exists an integer i such that either

$$\beta_0^{s-1} \leqslant a - iq/p \leqslant \beta_1^{s-1} \quad \text{and} \quad \beta_0^1 \leqslant i - tp \leqslant \beta_1^1 \tag{$*$}$$

or

$$\gamma_0^{s-1} \leqslant a - iq/p \leqslant \gamma_1^{s-1} \quad \text{and} \quad \delta_0^1 \leqslant i - tp \leqslant \delta_1^1. \tag{$**$}$$

We have the following simple observation: *if A, B, C, D, a, t, p, q' are integers satisfying $B - A \geqslant q' > 0$, $D - C > 0$, then there exists an integer i such that*

$$A \leqslant a - iq' \leqslant B \quad \textit{and} \quad C \leqslant i - tp \leqslant D$$

if and only if $Cq' + A \leqslant a - tpq' \leqslant Dq' + B$ (and the „only if" part remains true if we omit the assumption that $B - A \geqslant q'$).

Using this observation with $(A, B, C, D) = (\beta_0^{s-1}, \beta_1^{s-1}, \beta_0^1, \beta_1^1)$ and $q' = q/p$, we see that $(*)$ is equivalent to $\beta_0^s \leqslant a - tq \leqslant \beta_1^s$. Again with $(A, B, C, D) = (\gamma_0^{s-1}, \gamma_1^{s-1}, \delta_0^1, \delta_1^1)$ this shows that $(**)$ *implies* $q/p\delta_0^1 + \gamma_0^{s-1} \leqslant a - tq \leqslant q/p\delta_1^1 + \gamma_1^{s-1}$. Now because the first interval contains the second one, we see that $\mathsf{F}_*^s\mathscr{O}(a)$ contains $\mathscr{O}(t)$ if and only if $\beta_0^s \leqslant a - tq \leqslant \beta_1^s$.

2. Analogously, $\mathsf{F}_*^s\mathscr{O}(a)$ contains $\mathsf{S}(t)$ if and only if there exists an i such that either $\gamma_0^{s-1} \leqslant a - iq/p \leqslant \gamma_1^{s-1}$ and $\varepsilon_0^1 \leqslant i - tp \leqslant \varepsilon_1^1$ or $\beta_0^{s-1} \leqslant a - iq/p \leqslant \beta_1^{s-1}$ and $\gamma_0^1 \leqslant i - tp \leqslant \gamma_1^1$. Using the observation from (1), we see that this happens if and only if $q'\varepsilon_0^1 + \beta_0^{s-1} \leqslant a - tq \leqslant q'\gamma_1^1 + \beta_1^{s-1}$ and these bounds are equal to γ_0^s and γ_1^s.

The proofs of (3) and (4) are similar. □

5.3. Which summands appear ($p = 2$)

In this section we investigate the case when $p = 2$. As before, we first deal with the case $s = 1$. Let us first establish the following version of Langer's lemma used in the preceding section.

Lemma 5.7. *Let $\operatorname{char}(k) = 2$, $N \geqslant 0$. Let M_N be the set of all monomials in $k[x_0, \dots, x_N]$ not in $I := (x_0^2, \dots, x_N^2)$ which contain at least one variable each monomial of Q (except for possibly x_0^2), but are not divisible by any monomial of Q. Then M_N forms a basis of $(I : (Q))/(I + (Q))$.*

Proof. The proof is by induction on N, starting with $N \leqslant 0$, for which $Q \in I$. Then the statement is obvious.

Induction step: Renaming the last two variables, we have $Q = xy + Q'$. Take $f \in (I : (Q))$ and write $f \equiv f_{00} + xf_{10} + yf_{01} + xyf_{11}$, $f_{\alpha\beta} \in k[x_0, \dots, x_{N-2}]$, so that

$$0 \equiv (xy + Q')(\sum_{\alpha,\beta} x^\alpha y^\beta f_{\alpha\beta});$$

comparing coefficients in x and y yields the equations $f_{00} + f_{11}Q' \equiv 0$ and $f_{\alpha\beta}Q' \equiv 0$ for $(\alpha, \beta) \neq (1, 1)$ (modulo $(x_0^2, \dots, x_{N-2}^2)$). By the induction assumption, $f_{10} = g_{10}Q' + r_{10}$ and $f_{01} = g_{01}Q' + r_{01}$, where $r_{\alpha\beta}$ is a unique linear combination of elements of M_{N-2} of appropriate degree. We then have

$$\begin{aligned} f &= f_{00} + x(g_{10}Q' + r_{10}) + y(g_{01}Q' + r_{01}) + xyf_{11} \\ &= Q(f_{11} + xg_{10} + yg_{01}) + xr_{10} + yr_{01}. \end{aligned}$$

But $M_N = xM_{N-2} \cup yM_{N-2}$, so we see that M_N spans the quotient in question.

For linear independence, let us write $Q \cdot g \equiv \sum_{m \in M_N} a_m m$ with $a_m \in k$ and $g \in k[x_0, \ldots, x_N]$. Then for any monomial $x_i x_{i+1}$ of Q, monomials divisible by $x_i x_{i+1}$ do not occur on the left-hand side, so g is in the ideal spanned by the variables x_i and x_{i+1}. In other words, every term of g has at least one variable from each term of Q (except possibly x_0^2). But that means that $Q \cdot g = 0$, forcing the combination to be trivial in $k[x_0, \ldots, x_N]/(x_0^2, \ldots, x_N^2)$. But M_N is clearly linearly independent in this ring, which finishes the proof. □

Corollary 5.8. *$\gamma^1(t,a) = 1$ if $a - 2(t-1) = \lfloor \frac{n}{2} \rfloor + 1$ or if $a - 2(t-1) = \lfloor \frac{n}{2} \rfloor + 2$ and n is odd, and $\gamma^1(t,a) = 0$ otherwise.*

Proof. Let us set $M' = (I : (Q))/(I + (Q))$. By the exact sequence

$$0 \to D^{(1)}[-2]/(0:Q) \xrightarrow{Q} D^{(1)} \to B^{(1)} \to 0$$

we have $\dim B_d = \dim D_d + \dim M'_{d-2} - \dim B_{d-2}$. So

$$\dim B_d = \sum_{j \geqslant 0} (-1)^j \dim D_{d-2j} + \sum_{j \geqslant 0} (-1)^j \dim M'_{d-2(j+1)}. \tag{5.1}$$

Proceeding exactly as in Section 4, but replacing the use of 2.6 by 5.1 gives

$$\gamma^1(t,a) = \frac{1}{2^{\lfloor n/2 \rfloor + 1}} \dim M'_{a-2(t-1)},$$

which, together with Lemma 5.7, yields the result. □

Now we shall prove an analogue of Lemma 5.3:

Lemma 5.9. *Let* char $k = 2$, *$n > 0$ and let φ_n and ψ_n be the matrices defined in Section 1.3. Let h be a vector with polynomial entries of length $2^{\lfloor (n+1)/2 \rfloor}$. Suppose that all entries of h are homogeneous polynomials of degree d.*

1. *If $Q_n \cdot h \in (x_0^2, \ldots, x_{n+1}^2)$ and $d \leqslant \lfloor n/2 \rfloor$, then there exists a vector g with polynomial entries for which $h \equiv Q_n \cdot g$ modulo $(x_0^2, \ldots, x_{n+1}^2)$.*
2. *If $Q_n \cdot \varphi_n(h) \in (x_0^2, \ldots, x_{n+1}^2)$ and $d \leqslant \lceil n/2 \rceil - 1$, then there exists a vector g with polynomial entries for which $h \equiv \psi_n(g)$ modulo $(x_0^2, \ldots, x_{n+1}^2)$ (the same is true with φ_n and ψ_n exchanged).*
3. *If $\varphi_n(h) \in (x_0^2, \ldots, x_{n+1}^2)$ and $d \leqslant \lceil n/2 \rceil$, then there exists a vector g with polynomial entries for which $h \equiv Q_n \cdot \psi_n(g)$ modulo $(x_0^2, \ldots, x_{n+1}^2)$ (the same is true with φ_n and ψ_n exchanged).*

Proof. We work in the ring $D_n = k[x_0, \ldots, x_{n+1}]/(x_0^2, \ldots, x_{n+1}^2)$ and proceed by induction on n. For brevity let $\varphi = \varphi_n$, $\psi = \psi_n$, $\varphi' = \varphi_{n-2}$, $\psi' = \varphi_{n-2}$, $x = x_n$, $y = x_{n+1}$, $Q = Q_n$ and $Q' = Q_{n-2}$.

1. This follows from Lemma 5.7 above.

2. Let us divide h in two pieces: $h = (h_0, h_1)$. We can write h_i, $i = 0, 1$ as $h_i = h_i^{00} + xh_i^{01} + yh_i^{10} + xyh_i^{11}$ where h_i^{jk} are polynomials in $x_0, \ldots, x_{n-1}$.

Using the recurrence relations

$$\varphi = \begin{pmatrix} \varphi' & x \cdot id \\ y \cdot id & \psi' \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi' & x \cdot id \\ y \cdot id & \varphi' \end{pmatrix}, \quad Q = xy + Q',$$

our assumption on h takes the form

$$(xy+Q')(\varphi'(h_0)+xh_1)=0, \quad (xy+Q')(\psi'(h_1)+yh_0)=0.$$

By comparing coefficients in x and y we see that

$$Q'\varphi'(h_0^{00})=0, \qquad Q'\psi'(h_1^{00})=0, \tag{5.2}$$

$$Q'(\varphi'(h_0^{01})+h_1^{00})=0 \qquad Q'(\psi'(h_1^{10})+h_0^{00})=0, \tag{5.3}$$

$$Q'\varphi'(h_0^{10})=0, \qquad Q'\psi'(h_1^{01})=0, \tag{5.4}$$

$$\varphi'(h_0^{00})+Q'\varphi'(h_0^{11})+Q'h_1^{10}=0, \qquad \psi'(h_1^{00})+Q'\psi'(h_1^{11})+Q'h_0^{01}=0. \tag{5.5}$$

By (5.4) and the induction assumption, there exist g_0^{10} and g_1^{01} such that $h_0^{10}=\psi'(g_0^{10})$ and $h_1^{01}=\varphi'(g_1^{01})$. Observe also that by (5.3) and (1), there exist g_0^{01} and g_1^{10} such that $\varphi'(h_0^{01})+h_1^{00}=Q'g_0^{01}$ and $\psi'(h_1^{10})+h_0^{00}=Q'g_1^{10}$. Putting this into (5.5) and using the induction assumption once again gives us g_0^{11} and g_1^{11} such that $g_1^{10}+h_0^{11}=\psi'(g_0^{11})$ and $g_0^{01}+h_1^{11}=\varphi'(g_1^{11})$. Finally define $g_0^{00}=\varphi'(g_1^{10})+h_1^{10}$ and $g_1^{00}=\psi'(g_0^{01})+h_0^{01}$ and observe that $g=(g_0,g_1)$ defined by $g_i=g_i^{00}+xg_i^{01}+yg_i^{10}+xyg_i^{11}$ satisfies $\varphi(g)=h$.

3. Let us first prove that there exists an f such that $h=\psi(f)$. Decomposing h as before, we have

$$\varphi'(h_0^{00})=0, \qquad \psi'(h_1^{00})=0, \tag{5.6}$$

$$\varphi'(h_0^{01})+h_1^{00}=0, \qquad \psi'(h_1^{10})+h_0^{00}=0, \tag{5.7}$$

$$\varphi'(h_0^{10})=0, \qquad \psi'(h_1^{01})=0, \tag{5.8}$$

$$\varphi'(h_0^{11})+h_1^{10}=0, \qquad \psi'(h_1^{11})+h_0^{01}=0. \tag{5.9}$$

By (5.8) and the induction assumption, there exist f_0^{10} and f_1^{01} such that $h_0^{10}=\psi'(f_0^{10})$ and $h_1^{01}=\varphi'(f_1^{01})$. Observe also that

$$Q'\cdot\varphi'(h_0^{11})=Q'\cdot h_1^{10}=\varphi'(\psi'(h_1^{10}))=\varphi'(h_0^{00})=0,$$

and similarly $Q'\cdot\psi'(h_1^{11})=0$, therefore by (1) there exist f_0^{11} and f_1^{11} such that $h_0^{11}=\psi'(f_0^{11})$ and $h_1^{11}=\varphi'(f_1^{11})$. Finally set $f_0^{00}=h_1^{10}$, $f_1^{00}=h_0^{01}$, $f_0^{01}=0$ and $f_1^{10}=0$ and observe that $f=(f_0,f_1)$, $f_i=f_i^{00}+xf_i^{01}+yf_i^{10}+xyf_i^{11}$ satisfies $h=\psi(f)$.

Now since $Q\cdot f=\varphi(\psi(f))=\varphi(h)=0$, by (2) there exists a g such that $f=Q\cdot g$. Therefore $h=\psi(f)=Q\cdot\psi(g)$. □

Proceeding exactly as in Propositions 5.5 and 5.6 and Theorem 2, one obtains

Theorem 3.

1. *$F_*^s(\mathscr{O}(a))$ contains $\mathscr{O}(t)$ if and only if* $0\leqslant a-tq\leqslant n(q-1)$,
2. *$F_*^s(\mathscr{O}(a))$ contains $S(t)$ if and only if*

$$\left(\lfloor\frac{n}{2}\rfloor-1\right)\frac{q}{2}\leqslant a-tq\leqslant n(q-1)-q-\left(\lfloor\frac{n}{2}\rfloor-1\right)\frac{q}{2},$$

3. *$F_*^s(\mathsf{S}(a))$ contains $\mathscr{O}(t)$ if and only if* $1\leqslant a-tq\leqslant n(q-1)$,
4. *$F_*^s(\mathsf{S}(a))$ contains $\mathsf{S}(t)$ if and only if*

$$\left(\lfloor\frac{n}{2}\rfloor-1\right)\frac{q}{2}+1+\delta_{s,1}\cdot\delta_{n,odd}\leqslant a-tq\leqslant n(q-1)-q-\left(\lfloor\frac{n}{2}\rfloor-1\right)\frac{q}{2}-\delta_{s,1}\cdot\delta_{n,odd},$$

where $\delta_{n,odd}=1$ *if* n *is odd and* 0 *otherwise.* □

6. Corollaries

The following simple fact follows from Theorems 2 and 3.

Corollary 6.1. *For any ACM bundle $\mathcal{E}$ on Q_n, there are only finitely many $t \in \mathbb{Z}$ for which there exists an s such that $\mathscr{O}(t)$ or $\mathsf{S}(t)$ appears in $\mathsf{F}_*^s\mathcal{E}$.*

Now we proceed to extend the main results from [12].

Definition 6.2. A coherent sheaf $\mathscr{F}$ on a variety X is called *quasi-exceptional* if $\operatorname{Ext}^i(\mathscr{F},\mathscr{F}) = 0$ for $i > 0$. $\mathscr{F}$ is *tilting* if it is quasi-exceptional, Karoubian generates the bounded derived category $D^b(X)$ and the algebra $\operatorname{Hom}_X(\mathscr{F},\mathscr{F})$ has finite global dimension.

Lemma 6.3. *We have* $\operatorname{Ext}^1(\mathsf{S}(a), \mathsf{S}(a+1)) \neq 0$ *and* $\mathsf{S}(a)$ *is quasi-exceptional.*

Proof. For the first statement, tensor the sequence (1.4) by $\mathsf{S}(a)$ and write the long cohomology exact sequence. The second statement follows even simpler from (1.4). □

The following theorem extends slightly the main Theorem 1.1 from [12].

Theorem 4. *Let $n > 2$. Then $\mathsf{F}_*^s\mathscr{O}_{Q_n}$ is tilting if and only if one of the following holds:*

1. *$s = 1$ and $p > n$,*
2. *$s = 2$, $n = 4$ and $p = 2, 3$,*
3. *$s \geqslant 2$, n is odd and $p \geqslant n$.*

Proof. If $p > 2$, this is Theorem 1.1 from [12] (and can also be easily deduced from Theorem 2). Thus the only new part here is to show that in the case $p = 2$, $\mathsf{F}_*^s\mathscr{O}$ is not tilting, except for the case $s = 2$ and $n = 4$.

By Theorem 2, we see that $\mathsf{F}_*^s\mathscr{O}$ contains as direct summands only the line bundles

$$\mathscr{O}, \mathscr{O}(-1), \ldots, \mathscr{O}(-\lfloor n - \frac{n}{q}\rfloor).$$

So if $n > q$ then $\mathsf{F}_*^s\mathscr{O}$ does not generate the derived category.

We also see that the $\mathsf{F}_*^s\mathscr{O}$ contains $\mathsf{S}(t)$ for $\gamma_0^s \leqslant -tq \leqslant \gamma_1^s$ with $\gamma_1^s - \gamma_0^s = \frac{nq}{2} - n \geqslant 2q$ for $q \geqslant n \geqslant 6$, so in this case $\mathsf{F}_*^s\mathscr{O}$ contains two consecutive twists of S. Therefore it is not quasi-exceptional by the above lemma.

Finally, we work out the cases $n = 3, 4, 5$ by hand: for $n = 3$, $\mathsf{F}_*^2\mathscr{O}$ contains S and $\mathsf{S}(-1)$. For $n = 4$, $\mathsf{F}_*^3\mathscr{O}$ contains $\mathsf{S}(-1)$ and $\mathsf{S}(-2)$. For $n = 5$, $\mathsf{F}_*^2\mathscr{O}$ contains $\mathsf{S}(-1)$ and $\mathsf{S}(-2)$. So in all these cases the bundles (and their Frobenius push-forwards) are not quasi-exceptional. For $n = 3, 4, 5$ and $s = 1$, we have $n > q$. It remains to check the case $n = 4$, $s = 2$. In this case, $\mathsf{F}_*^2\mathscr{O}$ contains $\mathsf{S}(-1)$ and $\mathscr{O}(-i)$ for $i = 0, 1, 2, 3$, so it is tilting. □

A note on singular quadrics

It would be interesting to extend the above results to singular quadrics. It should be noted first that the ring $S/(Q)$ with Q a quadratic form not of full rank is no longer of finite Cohen-Macaulay type. Recently, N. Addington in [1] constructed the so-called *spinor sheaves*, which are analogues of spinor bundles. Among them, there are always one or two (depending on the parity of the rank of Q) *maximal* spinor sheaves (i.e., coming from a maximal linear subspace on the quadric) and they have nearly the same cohomological properties as the spinor bundles. In particular, if we denote by S the maximal spinor sheaf of the sum of the two and assume that $\mathsf{F}_*(\mathscr{O}(a))$ and $\mathsf{F}_*(\mathsf{S}(a))$ decompose into direct sums of twists of $\mathscr{O}$ and S, it is easy to see that the results from Section 4 hold true almost without change (one has to replace the factors $2^{\lfloor n/2\rfloor+1}$ by $2^{\lfloor r/2\rfloor}$, r being the rank of Q).

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