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Frobenius Theory for Positive Maps of von Neumann Algebras

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Abstract. Frobenius theory about the cyclic structure of eigenvalues of irreducible non negative matrices is extended to the case of positive linear maps of von Neumann algebras. Semigroups of such maps and ergodic properties are also considered.

1. Introduction

The spectral theory of positive maps has its origin in the classical work of Perron [1] and Frobenius [2], who considered the case of matrices with positive entries on finite dimensional vector spaces. For a compact exposition of Perron-Frobenius results see [3]. Let us distinguish two types of results in this theory. The first, due to Perron [1], is concerned with the existence and uniqueness of the maximal eigenvalue, the second, due to Frobenius [2], is concerned with the cyclic structure of the spectrum. Frobenius showed more particularly that a non negative irreducible matrix has always a simple eigenvalue r such that all other eigenvalues are contained in a circle of radius r around the origin. If the matrix is normalized such that $r = 1$ then the eigenvalues on the unit circle form a finite subgroup of the circle group which maps the system of all eigenvalues into itself.

In this paper we extend Frobenius results to the case of 2-positive maps of von Neumann algebras. Let us first give some references to previous work. As the literature is quite extensive, especially concerning extensions of Perron's results, we shall mainly mention work related to Frobenius results (for additional references see [4]).

Frobenius type of results for compact operators on commutative C*-algebras and ordered vector spaces can be found in Krein and Rutman [5], who also extended Jentsch's work [6] on Perron type of results. For other extensions in the case of ordered vector spaces see e.g. [7]–[9].

Automorphisms of commutative C*-algebras have been studied particularly in connection with ergodic theory, originating from classical work by Koopman [10], Carleman [11] and von Neumann [12], see [13]. Results of Frobenius type for groups of automorphisms in the general case of non commutative C*-algebras have been obtained by Størmer [14].

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For some particular spectral results which appeared in different contexts see the references in [14] and for recent related results see [15]–[17].

The extension of the entire Perron-Frobenius theory to the case of positive maps on finite-dimensional C*-algebras has been obtained by Evans and Høegh-Krohn [4].

We shall now briefly discuss our results.

We consider a von Neumann algebra M and positive linear normalized maps Φ of M into itself, which are 2-positive and thus satisfy the Kadison-Schwarz inequality $\Phi(a^*a) \geqq \Phi(a)^*\Phi(a)$ for any $a \in M$ (see e.g. [18]–[21]).

We recall that a map Φ is called 2-positive if $\Phi \otimes 1$ is positive on $M \otimes M_2$, where 1 is the unit mapping of the space M_2 of 2×2 matrices, so that in particular completely positive maps are 2-positive. Such maps have found several applications recently, see e.g. [22]–[26].

Consider now a state invariant under Φ and extend Φ to the Hilbert space $\mathscr{H}$ generated by applying M to the cyclic separating vector given by the state. Let Φ be ergodic, then we show that the set of eigenvalues on the unit circle for Φ in M and for Φ in $\mathscr{H}$ is the same, it consists of simple eigenvalues ("roots") α which form a subgroup of the circle group acting by complex multiplication on the spectrum of Φ as an operator in $\mathscr{H}$. The corresponding eigenvectors give unitary operators u_α and the map $\alpha \to u_\alpha$ is a unitary multiplier representation of the group $\Gamma(\Phi)$ of roots. The restriction of Φ to the subalgebra M_Γ of M generated by the operators u_α is an ergodic automorphism and the restriction of the state to M_Γ is a trace. We give also more detailed results for the cases where $\Gamma(\Phi)$ is cyclic or finite.

We then extend (Theorem 2.8–2.10) the considerations to the case of semigroups Φ_t, $t \geqq 0$ obtaining Frobenius type of results for their infinitesimal generators.

2. Dynamical Systems

Let M be a von Neumann algebra and Φ a positive linear normalized map of M, i.e. $\Phi(M^+) \subseteq M^+$, where M^+ is the positive cone in M and $\Phi(1) = 1$. We say that Φ is 2-positive if $\Phi \otimes \mathbb{1}_2$ is positive on $M \otimes M_2$, where M_2 is the algebra of 2×2-matrices and $\mathbb{1}_2$ is the identity map of M_2. It is easily seen that any 2-positive map of M satisfies the Schwarz inequality

$$\Phi(a^*a) \geqq \Phi(a)^*\Phi(a) \tag{2.1}$$

for any $a \in M$.

We say that the triplet (M, Φ, ξ) is a *dynamical system* if Φ is a 2-positive map of the von Neumann algebra M and ξ is a faithful normal state on M which is invariant under Φ, i.e. $\xi \circ \Phi = \xi$. Any *-automorphism θ of M is 2-positive and a dynamical system (M, θ, ξ), where θ is a *-automorphism is called a *closed* dynamical system. By the GNS construction we may assume that ξ is a vector state

$$\xi(a) = (\Omega, a\Omega). \tag{2.2}$$

Let $\mathscr{H} = \overline{M\Omega}$ be the Hilbert space generated by M and the cyclic vector Ω. For $a \in M$ we define

$$\Phi a\Omega = \Phi(a)\Omega. \tag{2.3}$$

By (2.1) we then have

$$\|\Phi a\Omega\|^2 = \xi(\Phi(a)^*\Phi(a)) \leqq \xi(\Phi(a^*a)) = \xi(a^*a) = \|a\|^2. \tag{2.4}$$

Hence Φ with dense domain $M\Omega$ is a contraction on $\mathscr{H}$ and thus extends uniquely to a contraction on $\mathscr{H}$.

Let $a\in M$ and $b'\in M'$, where M' is the commutant of M. Let ξ on $B(\mathscr{H})$ be given by (2.2), then $b'\in(M')^+$ if and only if $\xi(ab')\geqq 0$ for all $a\in M^+$, since Ω is a cyclic and separating vector for M. Now if $b'\in(M')^+$ then for $a\in M^+$ we have

$$\xi(\Phi(a)b') \leqq \|b'\|\,\xi(\Phi(a)) = \|b'\|\,\xi(a). \tag{2.5}$$

Hence if $b'\in(M')^+$ is normalized such that $\xi(b')=1$, then $\xi(\Phi(a)b')$ is a state on M which is majorized by the state ξ. It is well known that such a state is of the form

$$\xi(\Phi(a)b') = \xi(a\Phi'(b')) \tag{2.6}$$

where $\Phi'(b')$ is some positive operator affiliated with M'. The correspondence $b'\to\Phi'(b')$ is obviously linear and monotone and $\xi\circ\Phi=\xi$ implies that $\Phi'(1)=1$. By the monotonicity we then get for $b'\in(M')^+$ that $\Phi'(b')\leqq\|b'\|$. Hence Φ' is a positive map of M'. Since $(M\otimes M_2)'\simeq M'\otimes M_2$ and if $\alpha\in M\otimes M_2$ and $\beta'\in(M\otimes M_2)'$ then

$$\xi\otimes\tau_2(\Phi\otimes \mathbb{1}_2(\alpha)\beta') = \xi\otimes\tau_2(\alpha(\Phi'\otimes \mathbb{1}_2)(\beta)), \tag{2.7}$$

where τ_2 is the normalized trace on M_2, we see that $(\Phi\otimes \mathbb{1}_2)' = \Phi'\otimes \mathbb{1}_2$. This together with the above argument gives us that Φ' is a 2-positive map of M'.

Moreover $\Phi(1)=1$ implies that $\xi\circ\Phi'=\xi$ and therefore (M',Φ',ξ) is again a dynamical system. We thus have the following theorem

Theorem 2.1. *Let (M,Φ,ξ) be a dynamical system. If $\mathscr{H}=\overline{M\Omega}$ is the Hilbert space generated by M and Ω, where Ω is the vector given by the state ξ, then for $a\in M$ and $b'\in M'$ the equation*

$$\xi(\Phi(a)b') = \xi(a\Phi'(b'))$$

defines uniquely a 2-positive map of the commutant M' such that (M',Φ',ξ) is a dynamical system. Moreover $\Phi a\Omega=\Phi(a)\Omega$ defines a contraction on $M\Omega$ which extends to a contraction on $\mathscr{H}$.

We shall call (M',Φ',ξ) the *dual dynamical system.* We say that a dynamical system (M,Φ,ξ) is *ergodic* if ξ is the only normal state on M' which is invariant under Φ'. The following lemma follows from the fact that if $a\in M^+$, $\xi(a)=1$, then $\xi(ab')$ is a normal state on M'.

Lemma 2.1. *If (M,Φ,ξ) is an ergodic dynamical system then the only elements in M invariant under Φ are those proportional to the identity.*

From (2.1) we get with $a\in M$ and $b'\in(M')^+$ that

$$\xi(b'\Phi(a^*a)) \geqq \xi(\Phi(a)^*b'\Phi(a)) \tag{2.8}$$

or

$$\xi(a^*\Phi'(b')a) \geqq \xi(\Phi(a)^*b'\Phi(a)) \tag{2.9}$$

which gives

$$(a\Omega, \Phi'(b')a\Omega) \geqq (\Phi a\Omega, b'\Phi a\Omega). \tag{2.10}$$

Since $M\Omega$ is dense in $\mathscr{H}$ we have for arbitrary $x \in \mathscr{H}$ that for $b' \in (M')^+$

$$(x, \Phi'(b')x) \geqq (\Phi x, b'\Phi x). \tag{2.11}$$

Since Φ is a contraction on $\mathscr{H}$, the spectrum of Φ is contained inside the unit circle. If x_α is an eigenvector of Φ corresponding to an eigenvalue α on the unit circle i.e.

$$\Phi x_\alpha = \alpha x_\alpha, \quad |\alpha| = 1 \tag{2.12}$$

we say that x_α is a *root vector* for Φ corresponding to the *root* α. The set of roots $\Gamma(\Phi)$ of Φ consists then of the eigenvalues for Φ on the unit circle. Let now x_α be a normalized root vector, by (2.11) we then have

$$(x_\alpha, \Phi'(b')x_\alpha) \geqq (x_\alpha, b'x_\alpha) \tag{2.13}$$

for any $b' \in (M')^+$. Both sides of the inequality (2.13) define normalized states on M' and since one dominates the other they must be equal, hence

$$(x_\alpha, \Phi'(b')x_\alpha) = (x_\alpha, b'x_\alpha) \tag{2.14}$$

for arbitrary $b' \in M'$. Thus the vector state $b' \to (x_\alpha, b'x_\alpha)$ corresponding to a root vector x_α is an invariant state for Φ'.

Let us now assume that the dynamical system (M, Φ, ξ) is ergodic, then we have that

$$(x_\alpha, b'x_\alpha) = \xi(b') \tag{2.15}$$

for any $b' \in M'$. Set now $b' = c'^* c'$, then (2.15) gives us

$$\|c'x_\alpha\| = \|c'\Omega\|. \tag{2.16}$$

Let us now define the densely defined operator $\hat{x}_\alpha$ by

$$\hat{x}_\alpha c'\Omega = c'x_\alpha \tag{2.17}$$

for $c' \in M'$ i.e. $D(\hat{x}_\alpha) = M'\Omega$. From (2.16) we then have

$$\|\hat{x}_\alpha c'\Omega\| = \|c'\Omega\|, \tag{2.18}$$

hence $\hat{x}_\alpha$ is an isometry

$$\hat{x}_\alpha^* \hat{x}_\alpha = 1. \tag{2.19}$$

From the definition (2.17) we have that

$$\hat{x}_\alpha b' = b'\hat{x}_\alpha \tag{2.20}$$

for arbitary $b' \in M'$. Thus $\hat{x}_\alpha \in M$ and by (2.12) we have

$$\Phi(\hat{x}_\alpha) = \alpha \hat{x}_\alpha. \tag{2.21}$$

Since Φ is a positive map of M we have that

$$\Phi(a^*) = \Phi(a)^*,$$

which gives us that

$$\Phi(\hat{x}_\alpha^*) = \bar{\alpha}\hat{x}_\alpha^*. \tag{2.22}$$

Thus if α is a root so is $\bar{\alpha}$ and the corresponding root vector is $\hat{x}_\alpha^*\Omega$. Now (2.19) with $\hat{x}_\alpha^*$ replacing $\hat{x}_\alpha$ gives

$$\hat{x}_\alpha \cdot \hat{x}_\alpha^* = 1, \tag{2.23}$$

so that $\hat{x}_\alpha$ is unitary.

Let us now consider the quadratic form defined on M by

$$\mu_{b'}(a,a) = \xi(b'(\Phi(a^*a) - \Phi(a)^*\Phi(a))). \tag{2.24}$$

Then by (2.1) we have for $b' \in (M')^+$ that $\mu_{b'}(a,a)$ is a positive semi-definite form. From (2.19) and (2.21) we have that

$$\mu_{b'}(\hat{x}_\alpha, \hat{x}_\alpha) = 0. \tag{2.25}$$

Thus by Schwarz inequality we have

$$\mu_{b'}(\hat{x}_\alpha, a) = 0 \tag{2.26}$$

for arbitrary $b' \in (M')^+$ and $a \in M$. Since Ω is separating and cyclic also for M' we therefore have

$$\Phi(\hat{x}_\alpha^* a) = \bar{\alpha}\hat{x}_\alpha^*\Phi(a)). \tag{2.27}$$

which again implies that

$$\Phi(\hat{x}_\alpha a) = \alpha\hat{x}_\alpha\Phi(a), \tag{2.28}$$

$\hat{x}_\alpha\Omega$ being again a root vector. Since $M\Omega$ is dense in $\mathscr{H}$ we get from (2.28) that

$$\hat{x}_\alpha^*\Phi\hat{x}_\alpha = \alpha\Phi. \tag{2.29}$$

Still under the assumption that (M, Φ, ξ) is an ergodic dynamical system we get from (2.29) that if $\alpha \in \Gamma(\Phi)$ then the operators Φ and $\alpha\Phi$ are unitary equivalent. Especially we get that if α and β are in $\Gamma(\Phi)$ then $\alpha \cdot \beta$ is in $\Gamma(\Phi)$, and we already had that if $\alpha \in \Gamma(\Phi)$ then $\bar{\alpha} \in \Gamma(\Phi)$. Hence $\Gamma(\Phi)$ is a subgroup of the circle group. Let $\alpha \in \Gamma(\Phi)$ and let u_α be the corresponding unitary root operator in M. From (2.28) we get that if α and β are in $\Gamma(\Phi)$ then

$$\Phi(u_\alpha u_\beta) = \alpha\beta u_\alpha u_\beta, \tag{2.30}$$

so that $u_\alpha u_\beta$ is a root operator corresponding to the root $\alpha\beta$. Let u_α and v_α be two unitary root operators corresponding to the same root α. By (2.27) we then have

$$\Phi(u_\alpha^* v_\alpha) = u_\alpha^* v_\alpha, \tag{2.31}$$

which by Lemma 2.1 gives us the following

$$u_\alpha^* v_\alpha = c1, \ |c| = 1, \tag{2.32}$$

so that $v_\alpha = cu_\alpha$. Hence we have proved that any root α is a simple eigenvalue for Φ. Especially we get that for α and β in $\Gamma(\Phi)$

$$u_\alpha u_\beta = \gamma(\alpha, \beta)u_{\alpha\beta} \tag{2.33}$$

with

$$|\gamma(\alpha, \beta)| = 1. \tag{2.34}$$

So that

$$\alpha \to u_\alpha \tag{2.35}$$

is a unitary multiplier representation of the group $\Gamma(\Phi)$, with multiplier $\gamma(\alpha, \beta)$. If $\Gamma(\Phi)$ is cyclic, i.e. has a simple generator, then the multiplier $\gamma(\alpha, \beta)$ is necessarily trivial, hence $\alpha \to u_\alpha$ is a unitary representation of the abelian group $\Gamma(\Phi)$. Therefore in this case the algebra generated by the root operators is abelian and

$$u_\alpha^* = u_\alpha. \tag{2.36}$$

Note that one may choose (2.36) also in the general case.
We have now proved the following theorem.

Theorem 2.2. *Let (M, Φ, ξ) be an ergodic dynamical system, where ξ is a cyclic separating vector state for M invariant under Φ. Let $\mathscr{H}$ be the corresponding Hilbert space. Then the discrete eigenvalues on the unit circle for Φ as an operator in $\mathscr{H}$ coincide with the discrete eigenvalues on the unit circle for Φ in M. Let $\Gamma(\Phi)$ be the set of all roots of Φ, i.e. the discrete eigenvalues on the unit circle. $\Gamma(\Phi)$ is a subgroup of the circle group which acts by complex multiplication on the spectrum* Sp(Φ) *of Φ in $\mathscr{H}$. If $\alpha \in \Gamma(\Phi)$ then α is a simple eigenvalue of Φ and the corresponding root operator u_α in M is proportional to a unitary operator in M and $x_\alpha = u_\alpha \Omega$ is the corresponding root vector in $\mathscr{H}$, where Ω is the vector corresponding to the vector state ξ. The invariance of* Sp(Φ) *under multiplication by the root α is given by the unitary equivalence*

$$u_\alpha^* \Phi u_\alpha = \alpha \Phi,$$

if the root operator u_α is normalized so that it is unitary. If α and β are in $\Gamma(\Phi)$ with root operators u_α and u_β, then $u_\alpha u_\beta$ is a root operator for the root $\alpha\beta$ and u_α^* is a root operator for $\bar{\alpha}$. Hence if we select for each $\alpha \in \Gamma(\Phi)$ a unitary operator u_α then $u_\alpha u_\beta = \gamma(\alpha, \beta) u_{\alpha\beta}$, where $\gamma(\alpha, \beta)$ is a multiplier for the group $\Gamma(\Phi)$ and $\alpha \to u_\alpha$ is a unitary multiplier representation of the group $\Gamma(\Phi)$ with multiplier $\gamma(\alpha, \beta)$. If $\Gamma(\Phi)$ is cyclic, i.e. has a single generator, then $\alpha \to u_\alpha$ is a unitary representation of the abelian group $\Gamma(\Phi)$ and therefore the algebra generated by the root operators is abelian. ■

Remark. Results of this type were proven by Frobenius [2] for commutative, finite-dimensional von Neumann algebras. For the commutative infinite dimensional case with Φ compact, results were given by Krein and Rutman [5] and for the commutative infinite dimensional case with Φ an automorphism by Koopman [10] and von Neumann [12]. In the infinite dimensional non-commutative case with Φ an automorphism results of this type were obtained by Størmer [14] and in the finite dimensional non-commutative case with general Φ by Evans and Høegh-Krohn [4].
If Φ is compact in $\mathscr{H}$, $\Gamma(\Phi)$ must be a finite subgroup of the unit circle and since any such group has the form

$$\Gamma_m = \{e^{2\pi i k/m}, k = 0, 1, \ldots, m-1\} \tag{2.37}$$

we have that $\Gamma(\Phi) = \Gamma_m$ where $m = |\Gamma(\Phi)|$ is the order of $\Gamma(\Phi)$. We shall say that Φ is *primitive* if $|\Gamma(\Phi)| = 1$ i.e. $\Gamma(\Phi) = \{1\}$ and *imprimitive* if not, and following Frobenius we call $|\Gamma(\Phi)|$ the *imprimitivity* of Φ. Especially we have that if Φ is compact in $\mathscr{H}$ then it has finite imprimitivity. If Φ is of trace class in $\mathscr{H}$ then the Fredholm determinant $|1 - z\Phi|$ of $1 - z\Phi$ exists and defines an entire function

$$f_\Phi(z) = |1 - z\Phi| \tag{2.38}$$

such that $f_\Phi(z_0) = 0$ if and only if z_0^{-1} is an eigenvalue for Φ. Especially we get that the set of zeros of f on the unit circle is $\Gamma(\Phi)$. Recalling now that for $\alpha \in \Gamma(\Phi)$

$$u_\alpha^* \Phi u_\alpha = \alpha\Phi \tag{2.39}$$

by the unitary equivalence of $\alpha\Phi$ and Φ, we get then

$$f_\Phi(\alpha z) = f_\Phi(z) \tag{2.40}$$

because the Fredholm determinant is a unitary invariant. Since α in (2.40) is any mth root of the unit and f is entire, we have that there exists an entire function $g(z)$ such that $f_\Phi(z) = g(z^m)$. Let us also remark that since $\Gamma(\Phi) = \Gamma_m$ is cyclic, we have that the algebra generated by the root operators is commutative. Let now $\gamma = e^{2\pi i/m}$ and u' be a root operator corresponding to γ then $u'^m = \bar{c}.1$ where $|\bar{c}| = 1$. Let now $u = c^{1/m}u'$ then $u^m = 1$. Since u is unitary and $u^m = 1$ we have the spectral decomposition

$$u = \sum_{k=0}^{m-1} \gamma^k P_k \tag{2.41}$$

where P_k are the spectral projections for u.
Since $\Phi(u) = \gamma u$ we see that

$$\Phi(P_k) = P_{k-1} \text{ and } \Phi(P_0) = P_{m-1},\ k = 1, \ldots, m-1. \tag{2.42}$$

Especially we have that

$$\Phi^m(P_k) = P_k \tag{2.43}$$

so that Φ^m is not ergodic. It is easy to see that the restriction of Φ^m to the algebra $M_k = P_k M P_k$ is ergodic and in fact primitive. These results depend obviously only on the fact that $\Gamma(\Phi)$ is of finite order. We have thus the following theorem

Theorem 2.3. *Let (M, Φ, ξ) be as in Theorem 2.2. Then if Φ has finite imprimitivity we have $\Gamma(\Phi) = \Gamma_m$, where Γ_m is the group of m-th roots of the unit. Let $\gamma = e^{2\pi i/m}$ then a root operator u corresponding to $\bar{\gamma}$ may be normalized so that $u^m = 1$. For this u we have that $u = \sum_{k=0}^{m-1} \gamma^k P_k$ is the spectral resolution of the unitary operator u. Hence $\{P_k\}$ is a resolution of the identity in M and the algebra generated by the root operators is the abelian algebra generated by $\{P_k\}$. Moreover $\Phi(P_k) = P_{k-1}$ and $\Phi(P_0) = P_{m-1}$. Especially $\Phi^m(P_k) = P_k$, so that Φ^m is not ergodic. However the restriction of Φ^m to the algebra $M_k = P_k M P_k$ is ergodic and primitive. In fact $|\Gamma(\Phi)| = m$ only if Φ^m is not ergodic. If Φ is compact, then Φ has finite imprimitivity. If in addition Φ is of* *trace class in $B(\mathscr{H})$, then there is an entire function $g(z)$ such that*

$$|1 - z\Phi| = g(z^m)$$

where $|1 - z\Phi|$ is the Fredholm determinant of Φ. ■

Let now $\Gamma(\Phi)$ be cyclic but not finite. Then for any root $\gamma \in \Gamma(\Phi)$ we have that $\gamma/2\pi$ is irrational. If γ generates $\Gamma(\Phi)$ then

$$\Gamma(\Phi) = \{\gamma^n; n = 0, \pm 1, \dots\}. \tag{2.44}$$

Let now u be a root operator corresponding to $\bar{\gamma}$, normalized so that u is unitary. A root operator corresponding to γ^n is then given by u^{-n}. Let v be the spectral measure on the unit circle for the unitary operator u. Since obviously Φ restricted to the subalgebra generated by u is an automorphism, we have that Φ induces a transformation of the spectrum of u, and since $\Phi(u^n) = \gamma^n u^n$ it follows that this transformation coincides with the restriction to the spectrum of u of the transformation $z \to \gamma z$. Hence if

$$u = \int_{|z|=1} z dE_z \tag{2.45}$$

is the spectral resolution of u, we must have that

$$\Phi(E_z) = E_{\gamma z} \tag{2.46}$$

for v-almost all z in the unit circle. Since there are no other root operators than the $u^n, n = 0, \pm 1, \dots$, it follows that v is ergodic with respect to the transformation $z \to \gamma z$ of the unit circle. That v is invariant under this transformation follows from $\xi = \xi \circ \Phi$ and $\Phi(u^n) = \gamma^n u^n$ for all $n \in \mathbb{Z}$. Hence we have proved the following theorem.

Theorem 2.4. *Let (M, Φ, ξ) be an ergodic dynamical system, such that $\Gamma(\Phi)$ is cyclic but not finite. Then for any $\gamma \in \Gamma(\Phi)$ we have that $\gamma/2\pi$ is irrational. Let γ generate $\Gamma(\Phi)$, i.e. $\Gamma(\Phi) = \{\gamma^n; n = 0, \pm 1, \pm 2, \dots\}$. Let u be the root operator corresponding to $\bar{\gamma}$ normalized so that u is unitary.*

Let v be the spectral measure on the unit circle for the unitary operator u corresponding to the state ξ, i.e. $\xi(u^n) = \int_{|z|=1} z^n dv(z)$, and let $u = \int_{|z|=1} z dE_z$ be the spectral resolution of u. Then the projection valued measure dE_z is absolutely continuous with respect to v. v is an invariant ergodic measure with respect to the transformation $z = \gamma z$ of the unit circle and

$$\Phi(E_z) = E_{\gamma z}$$

for v-almost all z on the unit circle. ■

Let now α and β be two roots of the ergodic dynamical system (M, Φ, ξ) with corresponding root operators u_α and u_β. Since $u_\alpha \cdot u_\beta$ then is a root operator for the root $\alpha \cdot \beta$ we have

$$\Phi(u_\alpha u_\beta) = \alpha\beta u_\alpha u_\beta = \Phi(u_\alpha) \cdot \Phi(u_\beta). \tag{2.47}$$

Hence if M_Γ is the strongly closed subalgebra of M generated by the root operators $u_\alpha, \alpha \in \Gamma \equiv \Gamma(\Phi)$, then Φ maps M_Γ into M_Γ and restriction of Φ to M_Γ is an automorphism. Let $\mathscr{H}_\Gamma = \overline{M_\Gamma \Omega}$ then $\mathscr{H}_\Gamma$ is a Φ invariant subspace of $\mathscr{H}$ and the restriction of Φ to $\mathscr{H}_\Gamma$ is obviously unitary with discrete spectrum equal to Γ, and $u_\alpha \Omega, \alpha \in \Gamma$ is a complete set of orthogonal eigenvectors for Φ in $\mathscr{H}_\Gamma$. Hence Ω is the only invariant eigenvector. Since Φ is ergodic on M the restriction of Φ to M_Γ is ergodic. From the orthogonality of Ω and $u_\alpha \Omega$ for $\alpha \neq 1$ we have that $\xi(u_\alpha) = 0$ for $\alpha \neq 1$. But then $\xi(u_\alpha u_\beta) = \xi(u_\beta u_\alpha) = 0$ for $\alpha \neq \bar{\beta}$ and if $\beta = \bar{\alpha}$ then $u_\beta = c u_\alpha^*$, where c is an element in the unit circle, and since u_α is unitary we have that if $\beta = \bar{\alpha}$ then $u_\alpha u_\beta = u_\beta u_\alpha$ so that $\xi(u_\alpha u_\beta) = \xi(u_\beta u_\alpha)$ in any case. This shows that for a and b in M_Γ then $\xi(ab) = \xi(ba)$ i.e. the restriction of ξ to M_Γ is a trace. (That the restriction of an ergodic state to the root algebra M_Γ is a trace was observed by Størmer [14] in the case where Φ is an automorphism.) We have thus proven the following theorem.

Theorem 2.5. *Let (M, Φ, ξ) be an ergodic dynamical system with root system Γ. Let M_Γ be the root algebra, i.e. the strongly closed subalgebra of M generated by the root operators and let Φ_Γ be the restriction of Φ to M_Γ. Then Φ_Γ is an automorphism of M_Γ and $(M_\Gamma, \Phi_\Gamma, \xi_\Gamma)$, where ξ_Γ is the restriction of ξ to M_Γ, is an ergodic dynamical system. Moreover ξ_Γ is a trace on M_Γ.*

One could now ask if it is so that M_Γ is always commutative for an ergodic dynamical system. The following example shows that this is not the case.

Example 2.6. Let $\mathscr{H} = L_2(\mathbb{R})$ and set $(V(x)f)y) = f(y-x)$ and $(U(x)f)y) = e^{ixy}f(y)$. Then V and U are both strongly continuous unitary representations of the abelian group $\mathbb{R}$ on $L_2(\mathbb{R})$. Moreover

$$U(x)V(y) = e^{ixy}V(y)U(x).$$

Let $\lambda \geqq 0$ and n_1, n_2 in $\mathbb{Z}$, then

$$U(\lambda n_1)V(\lambda n_2) = e^{i\lambda^2 n_1 n_2}V(\lambda n_1)U(\lambda n_2).$$

Let $u_n \equiv U(\lambda n_1)V(\lambda n_2)$ for $n = (n_1, n_2) \in \mathbb{Z}^2$. Then $n \to u_n$ is a unitary multiplier representation of $\mathbb{Z}^2$. Let C be the C^*-algebra spanned by u_n and set $\tau(u_n) = \delta_{0n}$. It is easy to see that τ defines a faithful trace on C. In fact let $a = \sum \alpha_n u_n$, then we have $\tau(a^*a) = \sum \bar{\alpha}_n \alpha_m \tau(u_n^* u_m) = \sum |\alpha_n|^2$, which shows that τ is faithful. Moreover we have $\tau(u_n^* u_m) = 0 = \tau(u_m u_n^*)$ for $n \neq m$ and $\tau(u_n^* u_n) = 1 = \tau(u_n u_n^*)$, which shows that τ is a trace on C. Let now M be the von Neumann algebra given by the representation of C induced by the trace τ. Then M is non commutative if and only if λ^2 is not an integral multiple of 2π. Let α and β be two real numbers and set $W = U(\alpha)V(\beta)$. Then

$$W^*U(\lambda n)W = V^*(\beta)U(\lambda n)V(\beta) = e^{i\lambda\beta n}U(\lambda n)$$

and

$$W^*V(\lambda n)W = V^*(\beta)U^*(\alpha)V(\lambda n)U(\alpha)V(\beta) = e^{-i\lambda\alpha n}V(\lambda n).$$

Set now for $a \in M, \theta(a) = W^*aW$, then θ is an automorphism of M and (M, θ, ξ) is a dynamical system. Moreover it follows from the above equations that if α, β and $2\pi/\lambda$ are independent over $\mathbb{Z}$ (the ring of integers) then (M, θ, ξ) is an ergodic dynamical system. We have from the equation above that the root system

$\Gamma = \Gamma(\theta)$ is given by

$$\Gamma = \{e^{i\lambda(\alpha m + \beta n)}; (m, n) \in \mathbb{Z} \times \mathbb{Z}\}$$

and a root operator corresponding to $e^{i\lambda(\alpha m+\beta n)}$ is given by $U(\lambda m)V(-\lambda n)$. M is noncommutative if λ^2 is not an integral multiple of 2π, and $M = M_\Gamma$.

We shall now consider the case of semigroups of positive maps, instead of the iterates of a single positive map Φ.

Let M be a von Neumann algebra and $\Phi_t, t \geqq 0$ a semigroup of positive normalized maps of M i.e. $\Phi_0 = 1, \Phi_t \circ \Phi_s = \Phi_{t+s}, \Phi_t(M^+) \subseteq M^+$ and $\Phi_t(1) = 1$ such that the Φ_t are 2-positive, hence satisfy the Schwarz inequality

$$\Phi_t(a^*a) \geqq \Phi_t(a)^*\Phi_t(a) \tag{2.48}$$

for any $a \in M$ and all t. Moreover if ξ is a cyclic and separating normal state on M such that $\xi(a\Phi_t(b))$ is measurable as a function of t, and ξ is invariant under Φ_t i.e. $\xi \circ \Phi_t = \xi$, we say that (M, Φ_t, ξ) is a *dynamical system with continuous time* or a *dynamical flow*. We say that the dynamical flow is *ergodic* iff ξ is the only invariant normal state for the dual flow (M', Φ'_t, ξ). As for the discrete dynamical systems (M, Φ, ξ) considered before, (2.48) implies that Φ_t extends to a measurable, hence strongly continuous, contraction semigroup on $\mathscr{H}$, where $\mathscr{H}$ is the Hilbert space obtained by the GNS construction from the state ξ. We denote the continuous extension to $\mathscr{H}$ also by Φ_t, and we let iA be the infinitesimal generator of Φ_t in $\mathscr{H}$ i.e.

$$\Phi_t = e^{itA} \quad t \geqq 0. \tag{2.49}$$

Since Φ_t is a contraction, we have that $i(A - A^*) \geqq 0$ so that the spectrum of A is confined to the closed upper half plane. Let Γ be the discrete part of the spectrum of A on the real line. Then of course for any $t \geqq 0$ we have that $e^{it\Gamma}$ is the discrete spectrum of Φ_t on the unit circle. Let now $\alpha \in \Gamma$ and x_α be a corresponding normalized eigenvector. As in the proof of Theorem 2.2 we get that $x_\alpha = u_\alpha \Omega$ with u_α unitary in M. In this way we prove the following theorem.

Theorem 2.7. *Let (M, Φ_t, ξ) be an ergodic dynamical flow. Then the discrete eigenvalues on the real line for the infinitesimal generator of Φ_t in $\mathscr{H}$ coincide with the discrete eigenvalues on the real line for the infinitesimal generator of Φ_t in M. Let the set of these discrete eigenvalues on the real line be denoted by Γ, the root system of the flow, then Γ is a subgroup of the additive group of the real line. Moreover the spectrum of the semigroup Φ_t in $\mathscr{H}$ is invariant under this additive group Γ. Moreover, for any $\alpha \in \Gamma$, α is a simple eigenvalue of the semigroup Φ_t and a corresponding root operator $u_\alpha \in M$ is proportional to a unitary operator in M. The invariance of the spectrum of the semigroup Φ_t is given by the unitary equivalence*

$$u_\alpha^* \Phi_t u_\alpha = e^{2\pi i \alpha t} \Phi_t$$

where u_α is a normalized root operator corresponding to $\alpha \in \Gamma$. If α and β are in Γ with root operators u_α and u_β then $u_\alpha u_\beta$ is a root operator for $\alpha + \beta$ and u_α^* is a root operator for $-\alpha$. Hence if we select for each $\alpha \in \Gamma$ a unitary root operator u_α then $u_\alpha u_\beta = \gamma(\alpha, \beta) u_{\alpha+\beta}$, where $\gamma(\alpha, \beta)$ is a multiplier for Γ, and $\alpha \to u_\alpha$ is a unitary multiplier *representation with multiplier γ. Γ is either a dense subgroup of $\mathbb{R}$ or discrete i.e. $\Gamma = \{n\alpha, n\in\mathbb{Z}\}$. If Γ is discrete, then the strongly closed subalgebra M_Γ generated by the root operators is abelian.* ■

The restriction of Φ_t to M_Γ is obviously an automorphism and as in the discrete case we get that the restriction of ξ to M_Γ is a trace. In the special case where Γ is discrete, so that $\Gamma = \{n\alpha, n\in\mathbb{Z}\}$, M_Γ is abelian and generated by the root operator u corresponding to α. Let u be normalized to be unitary, then M_Γ is simply the von Neumann algebra generated by u. Since Φ_t restricted to M_Γ is a one parameter group of automorphism, it is induced by a one parameter flow on the spectrum of u. Since $\Phi_t(u^n) = e^{it\alpha n}u^n$ this flow on the spectrum of u must coincide with the flow $e^{i\varphi} \to e^{i(\varphi+\alpha t)}$ on the spectrum of u. From the fact that 1 is an eigenvalue of multiplicity one for the semigroup Φ_t restricted to M_Γ it follows that (M_Γ, Φ_t, ξ) is an ergodic dynamical flow so that the flow $e^{i\varphi} \to e^{i(\varphi+\alpha t)}$ is ergodic with respect to the spectral measure μ for u in ξ, i.e. the measure μ such that

$$\xi(f(u)) = \int_{|z|=1} f(z)d\mu(z). \tag{2.50}$$

Hence μ is an invariant and ergodic measure with respect to the flow induced by the rotation of the unit circle. Hence since Φ_t is also strongly continuous, we have that $d\mu$ is the Haar measure on the unit circle, and that u has constant spectral multiplicity. We have thus the following theorem.

Theorem 2.8. *Let (M, Φ_t, ξ) be an ergodic dynamical flow, and let Γ be its root system. Then the restriction of ξ to the von Neumann algebra M_Γ generated by the root operators is a trace, Φ_t leaves M_Γ invariant and the restriction of Φ_t to M_Γ is a one parameter group of automorphisms. Moreover (M_Γ, Φ_t, ξ) is an ergodic dynamical flow. Γ consists either of one point, or is discrete or is dense. In the discrete case we have $\Gamma = \{n\alpha; n\in\mathbb{Z}\}$. Let in this case u be a normalized root operator corresponding to α. Then u has Lebesgue spectrum and in fact the spectral measure for u in the state ξ is the Haar measure on the unit circle and u has constant spectral multiplicity. Moreover the flow (M_Γ, Φ_t, ξ) is induced by rotating the spectrum of u at the constant speed α.* ■

Remark. $t \to \Phi_t$ is an ergodic action of $\mathbb{R}$ on the von Neumann algebra M_Γ with a discrete spectrum. In [27, Chapter 8] we have an exhaustive discussion of such actions and we refer the interested reader to this reference for further information.

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