# From a $(p,2)$-Theorem to a Tight $(p,q)$-Theorem


**Chaya Keller**[1]

Department of Mathematics, Ben Gurion University of the NEGEV
Be'er Sheva,Israel
kellerc@math.bgu.ac.il

**Shakhar Smorodinsky**

Department of Mathematics, Ben Gurion University of the NEGEV
Be'er Sheva, Israel
shakhar@math.bgu.ac.il



## Abstract

A family $\mathcal{F}$ of sets is said to satisfy the $(p,q)$-property if among any $p$ sets of $\mathcal{F}$ some $q$ have a non-empty intersection. The celebrated $(p,q)$-theorem of Alon and Kleitman asserts that any family of compact convex sets in $\mathbb{R}^d$ that satisfies the $(p,q)$-property for some $q \geq d+1$, can be pierced by a fixed number (independent on the size of the family) $f_d(p,q)$ of points. The minimum such piercing number is denoted by $\mathsf{HD}_d(p,q)$. Already in 1957, Hadwiger and Debrunner showed that whenever $q > \frac{d-1}{d}p+1$ the piercing number is $\mathsf{HD}_d(p,q) = p-q+1$; no exact values of $\mathsf{HD}_d(p,q)$ were found ever since.

While for an arbitrary family of compact convex sets in $\mathbb{R}^d$, $d \geq 2$, a $(p,2)$-property does not imply a bounded piercing number, such bounds were proved for numerous specific families. The best-studied among them is axis-parallel boxes in $\mathbb{R}^d$, and specifically, axis-parallel rectangles in the plane. Wegner (1965) and (independently) Dol'nikov (1972) used a $(p,2)$-theorem for axis-parallel rectangles to show that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $q > \sqrt{2p}$. These are the only values of $q$ for which $\mathsf{HD}_{\mathrm{rect}}(p,q)$ is known exactly.

In this paper we present a general method which allows using a $(p,2)$-theorem as a bootstrapping to obtain a tight $(p,q)$-theorem, for families with Helly number 2, even without assuming that the sets in the family are convex or compact. To demonstrate the strength of this method, we show that $\mathsf{HD}_{\mathrm{d-box}}(p,q) = p-q+1$ holds for all $q > c' \log^{d-1} p$, and in particular, $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $q \geq 7\log_2 p$ (compared to $q \geq \sqrt{2p}$, obtained by Wegner and Dol'nikov more than 40 years ago).

In addition, for several classes of families, we present improved $(p,2)$-theorems, some of which can be used as a bootstrapping to obtain tight $(p,q)$-theorems. In particular, we show that any family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ with Helly number 2 admits a $(p,2)$-theorem with piercing number $O(p^{2d-1})$, and thus, satisfies $\mathsf{HD}_{\mathcal{F}}(p,q) = p-q+1$ for all $q > cp^{1-\frac{1}{2d-1}}$, for a universal constant $c$.



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# 1 Introduction

## 1.1 Helly's theorem and (p,q)-theorems

The classical Helly's theorem says that if in a family of compact convex sets in $\mathbb{R}^d$ every $d+1$ members have a non-empty intersection then the whole family has a non-empty intersection.

For a pair of positive integers $p \geq q$, we say that a family $\mathcal{F}$ of sets satisfies the $(p,q)$-property if $|\mathcal{F}| \geq p$, none of the sets in $\mathcal{F}$ is empty, and among any $p$ sets of $\mathcal{F}$ there are some $q$ with a non-empty intersection. A set $P$ is called a *transversal* (or alternatively, a *piercing set*) for $\mathcal{F}$ if it has a non-empty intersection with every member of $\mathcal{F}$. In this language, Helly's theorem states that any family of compact convex sets in $\mathbb{R}^d$ satisfying the $(d+1,d+1)$-property has a singleton transversal (alternatively, can be pierced by a single point).

In general, $d+1$ is clearly optimal in Helly's theorem, as any family of $n$ hyperplanes in a general position in $\mathbb{R}^d$ satisfies the $(d,d)$-property but cannot be pierced by less than $n/d$ points. However, for numerous specific classes of families, a $(d',d')$-property for some $d' < d+1$ is already sufficient to imply piercing by a single point. The minimal number $d'$ for which this holds is called the *Helly number* of the family. For example, any family of *axis-parallel boxes* in $\mathbb{R}^d$ has Helly number 2.

In 1957, Hadwiger and Debrunner [13] proved the following generalization of Helly's theorem:

▶ **Theorem 1** (Hadwiger-Debrunner Theorem [13]). *For all $p \geq q \geq d+1$ such that $q > \frac{d-1}{d}p+1$, any family of compact convex sets in $\mathbb{R}^d$ that satisfies the $(p,q)$-property can be pierced by $p-q+1$ points.*

▶ Remark. The bound in Theorem 1 is tight. Indeed, any family of $n$ sets which consists of $p-q$ pairwise disjoint sets and $n-(p-q)$ copies of the same set satisfies the $(p,q)$-property but cannot be pierced by less than $p-q+1$ points.

Hadwiger and Debrunner conjectured that while for general $p \geq q \geq d+1$, a transversal of size $p-q+1$ is not guaranteed, a $(p,q)$-property does imply a bounded-size transversal. This conjecture was proved only 35 years later, in the celebrated $(p,q)$-theorem of Alon and Kleitman.

▶ **Theorem 2** (Alon-Kleitman $(p,q)$-Theorem [2]). *For any triple of positive integers $p \geq q \geq d+1$, there exists an integer $s = s(p,q,d)$ such that if $\mathcal{F}$ is a family of compact convex sets in $\mathbb{R}^d$ satisfying the $(p,q)$-property, then there exists a transversal for $\mathcal{F}$ of size at most $s$.*

The smallest value $s$ that works for $p \geq q > d$ is called 'the Hadwiger-Debrunner number' and is denoted by $\mathsf{HD}_d(p,q)$. For various specific classes of families, a stronger $(p,q)$-theorem can be obtained. In such cases, we denote the minimal $s$ that works for the family $\mathcal{F}$ by $\mathsf{HD}_{\mathcal{F}}(p,q)$.

The $(p,q)$-theorem has a rich history of variations and generalizations. To mention a few: In 1997, Alon and Kleitman [3] presented a simpler proof of the theorem (that leads to a somewhat weaker quantitative result). Alon et al. [1] proved in 2002 a 'topological' $(p,q)$-theorem for finite families of sets which are a *good cover* (i.e., the intersection of every subfamily is either empty or contractible), and Bárány et al. [4] obtained in 2014 colorful and fractional versions of the theorem.

The size of the transversal guaranteed by the $(p, q)$-theorem is huge, and a large effort was invested in proving better bounds on $\mathsf{HD}_d(p, q)$, both in general and in specific cases. The most recent general result, by the authors and Tardos [16], shows that for any $\varepsilon > 0$, $\mathsf{HD}_d(p, q) \leq p - q + 2$ holds for all $(p, q)$ such that $p > p_0(\varepsilon)$ and $q > p^{\frac{d-1}{d}+\varepsilon}$. Yet, no exact values of the Hadwiger-Debrunner number are known except for those given in the Hadwiger-Debrunner theorem. In fact, even the value $\mathsf{HD}_2(4, 3)$ is not known, the best bounds being $3 \leq \mathsf{HD}_2(4, 3) \leq 13$ (obtained by Kleitman et al. [18] in 2001).

## 1.2 $(p, 2)$-theorems and their applications

As mentioned above, while no general $(p, q)$-theorems exist for $q \leq d$, such theorems can be proved for various specific families. Especially desirable are $(p, 2)$-theorems, which relate the *packing number*, $\nu(\mathcal{F})$, of the family $\mathcal{F}$ (i.e., the maximum size of a subfamily all of whose members are pairwise disjoint) to its *piercing number*, $\tau(\mathcal{F})$ (i.e., the minimal size of a piercing set for the family $\mathcal{F}$).

In the last decades, $(p, 2)$-theorems were proved for numerous families. In particular, in 1991 Károlyi [15] proved a $(p, 2)$-theorem for axis-parallel boxes in $\mathbb{R}^d$, guaranteeing piercing by $O(p \log^{d-1} p)$ points. Kim et al. [17] proved in 2006 that any family of translates of a fixed convex set in $\mathbb{R}^d$ that satisfies the $(p, 2)$-property can be pierced by $2^{d-1}d^d(p-1)$ points; five years later, Dumitrescu and Jiang [8] obtained a similar result for homothets of a convex set in $\mathbb{R}^d$. In 2012, Chan and Har-Peled proved a $(p, 2)$-theorem for families of pseudo-discs in the plane ([5], Theorem 4.6), with a piercing number linear in $p$. Two years ago, Govindarajan and Nivasch [11] showed that any family of convex sets in the plane in which among any $p$ sets there is a pair that intersects on a given convex curve $\gamma$, can be pierced by $O(p^8)$ points.

In 2004, Matoušek [20] showed that families of sets with bounded dual VC-dimension have a bounded fractional Helly number. Recently, Pinchasi [21] has drawn a similar relation between the union complexity and the fractional Helly number. Each of these results implies a $(p, 2)$-theorem for the respective families, using the proof technique of the Alon-Kleitman $(p, q)$-theorem.

Besides their intrinsic interest, $(p, 2)$-theorems serve as a tool for obtaining other results. One such result is an *improved Ramsey Theorem*. Consider, for example, a family $\mathcal{F}$ of $n$ axis-parallel rectangles in the plane. The classical Ramsey theorem implies that $\mathcal{F}$ contains a subfamily of size $\Omega(\log n)$, all whose elements are either pairwise disjoint or pairwise intersecting. As was observed by Larman et al. [19], the aforementioned $(p, 2)$-theorem for axis-parallel rectangles [15] allows obtaining an improved bound of $\Omega(\sqrt{n/\log n})$. Indeed, either $\mathcal{F}$ contains a subfamily of size $\left\lceil \sqrt{n/\log n} \right\rceil$ all whose elements are pairwise disjoint, and we are done, or $\mathcal{F}$ satisfies the $(p, 2)$-property with $p = \left\lceil \sqrt{n/\log n} \right\rceil$. In the latter case, by the $(p, 2)$-theorem, $\mathcal{F}$ can be pierced by $O(p \log p) = O(\sqrt{n \log n})$ points. The largest among the subsets of $\mathcal{F}$ pierced by a single point contains at least $\Omega(\frac{n}{\sqrt{n \log n}}) = \Omega(\sqrt{n/\log n})$ rectangles, and all its elements are pairwise intersecting.

Another result that can be obtained from a $(p, 2)$-theorem is an improved $(p, q)$-theorem; this will be described in detail below.

### 1.3 $(p,2)$-theorems and $(p,q)$-theorems for axis-parallel rectangles and boxes

The $(p,q)$-problem for axis-parallel boxes is almost as old as the general $(p,q)$-problem, and was studied almost as thoroughly (see the survey of Eckhoff [9]). It was posed in 1960 by Hadwiger and Debrunner [14], who proved that any family of axis-parallel rectangles in the plane that satisfies the $(p,q)$-property, for $p \geq q \geq 2$, can be pierced by $\binom{p-q+2}{2}$ points. Unlike the $(p,q)$-problem for general families of convex sets, in this problem a finite bound on the piercing number was known from the very beginning, and the research goal has been to improve the bounds on this size, denoted $\mathsf{HD}_{\mathrm{rect}}(p,q)$ for rectangles and $\mathsf{HD}_{\mathrm{d-box}}(p,q)$ for boxes in $\mathbb{R}^d$.

For rectangles and $q=2$, the quadratic upper bound on $\mathsf{HD}_{\mathrm{rect}}(p,2)$ was improved to $O(p \log p)$ by Wegner (unpublished), and independently, by Károlyi [15]. The best currently known upper bound, which follows from a recursive formula presented by Fon Der Flaass and Kostochka [10], is

$$\mathsf{HD}_{\mathrm{rect}}(p,2) \leq p\lceil \log_2 p \rceil - 2^{\lceil \log_2 p \rceil} + 1, \tag{1}$$

for all $p \geq 2$. On the other hand, it is known that the 'optimal possible' answer $p-q+1 = p-1$ fails already for $p=4$. Indeed, Wegner [24] showed that $\mathsf{HD}_{\mathrm{rect}}(4,2)=5$, and by taking $\lceil p/3 \rceil - 1$ pairwise disjoint copies of his example, one obtains a family of axis-parallel rectangles that satisfies the $(p,2)$-property but cannot be pierced by less than $\approx 5p/3$ points.

Wegner [24] conjectured that $\mathsf{HD}_{\mathrm{rect}}(p,2)$ is linear in $p$, and is possibly even bounded by $2p-3$. While Wegner's conjecture is believed to hold (see [9, 12]), no improvement of the bound (1) was found so far.

For rectangles and $q > 2$, Hadwiger and Debrunner showed that the exact bound $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $q \geq p/2+1$. Wegner [24] and (independently) Dol'nikov [7] presented recursive formulas that allow leveraging a $(p,2)$-theorem for axis-parallel rectangles into a tight $(p,q)$-theorem. Applying these formulas along with the Hadwiger-Debrunner quadratic upper bound on $\mathsf{HD}_{\mathrm{rect}}(p,2)$, Dol'nikov showed that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $2 \leq q \leq p < \binom{q+1}{2}$. Applying the formulas along with the improved bound (1) on $\mathsf{HD}_{\mathrm{rect}}(p,2)$, Scheller ([22], see also [9]) obtained by a computer-aided computation upper bounds on the minimal $p$ such that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds, for all $q \leq 12$. These values suggest that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds already for $q = \Omega(\log p)$. However, it appears that the method in which Dol'nikov proved a tight bound in the range $p < \binom{q+1}{2}$ does not extend to show a tight bound for all $q = \Omega(\log p)$ (even if (1) is employed), and in fact, no concrete improvement of Dol'nikov's result was presented (see the survey [9]).

Dol'nikov [7] claimed that if $\mathsf{HD}_{\mathrm{rect}}(p,2)$ is linear in $p$ as conjectured by Wegner [24], then one can deduce that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $q \geq c$, for some constant $c$. Eckhoff [9] wrote that the proof of this claim presented in [7] is flawed, but it is plausible that the claim does hold. On the other direction, nothing is known about the minimal $q$ for which $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ may hold; in particular, it is not impossible that the optimal bound $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds already for $q=3$.

For axis-parallel boxes in $\mathbb{R}^d$, the aforementioned recursive formula of [10] implies the bound $\mathsf{HD}_{\mathrm{d-box}}(p,2) \leq O(p \log^{d-1} p)$. While it is believed that the correct upper bound is $O(p)$, the result of [10] was not improved ever since; the only advancement is a recent result of Chudnovsky et al. [6], who proved an upper bound of $O(p \log\log p)$ for any family of axis-parallel boxes in which for each two intersecting boxes, a corner of one is contained in the other.

## 1.4 Our results

### From $(p, 2)$-theorems to $(p, q)$-theorems

The main result of this paper is a general method for leveraging a $(p, 2)$-theorem into a tight $(p, q)$-theorem, applicable to families with Helly number 2. Interestingly, the method does not assume that the sets in $\mathcal{F}$ are convex or compact.

▶ **Theorem 3.** *For any $m \in \mathbb{N}$, there exists $c' = c'(m)$ such that the following holds. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ such that $\mathsf{HD}_{\mathcal{F}}(2,2) = 1$. Assume that for all $2 \leq p \in \mathbb{N}$ we have $\mathsf{HD}_{\mathcal{F}}(p,2) \leq pf(p)$, where $f : [2,\infty) \to [1,\infty)$ is a differentiable function of $p$ that satisfies $f'(p) \geq \frac{\log_2 e}{p}$ and $\frac{f'(p)}{f(p)} \leq \frac{m}{p}$ for all $p \geq 2$. Denote $T_c(p) = T_c(p,f) = \min\{q : q \geq 2c \cdot f(2p/q)\}$. Then for any $p \geq q \geq 2$ such that $q \geq T_{c'}(p)$, we have $\mathsf{HD}_{\mathcal{F}}(p,q) = p - q + 1$.*

While the condition on the function $f(p)$ looks a bit “scary”, it actually holds for any function $f$ whose growth rate (as expressed by its derivative $f'(p)$ and by the derivative of its logarithm $(\log f(p))' = \frac{f'(p)}{f(p)}$) is between the growth rates of $f(p) = \log_2 p$ and $f(p) = p^m$ (where $m$ can be any integer, and $c'$ in the assertion depends on it), including all cases needed in the current paper.

The first application of our general method is the following theorem for families of axis-parallel rectangles in the plane, obtained using (1) as the basic $(p, 2)$-theorem and some local refinements.

▶ **Theorem 4.** $\mathsf{HD}_{\mathrm{rect}}(p,q) = p - q + 1$ *holds for all $q \geq 7\log_2 p$.*

▶ Remark. Theorem 4 improves significantly on the best previous result of Wegner (1965) and Dol’nikov (1972), that obtained the exact value $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ only for $q > \sqrt{2p}$.

Another corollary is a tight $(p, q)$-theorem for axis-parallel boxes in $\mathbb{R}^d$:

▶ **Theorem 5.** $\mathsf{HD}_{\mathrm{d-box}}(p,q) = p - q + 1$ *holds for all $q > c\log^{d-1} p$, where $c$ is a universal constant.*

In the proof of Theorem 3 we deploy the following observation of Wegner and Dol’nikov, which holds for any family $\mathcal{F}$ with Helly number 2:

$$\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1, \tag{2}$$

where $\lambda = \nu(\mathcal{F})$ is the packing number of $\mathcal{F}$ (Observation 8 below). We use an inductive process in which (2) is applied as long as $\mathcal{F}$ contains a sufficiently large pairwise-disjoint set. To treat the case where $\mathcal{F}$ does not contain a ‘large’ pairwise-disjoint set (and thus, $\nu(\mathcal{F})$ is small), we make use of a combinatorial argument, based on a variant of a ‘combinatorial dichotomy’ presented by the authors and Tardos [16], which first leverages the $(p, 2)$-theorem into a ‘weak’ $(p, q)$-theorem, and then uses that $(p, q)$-theorem to show that if $\nu(\mathcal{F})$ is ‘small’ then $\tau(\mathcal{F}) < p - q + 1$.

### From $(2, 2)$-theorems to $(p, 2)$-theorems

It is natural to ask, under which conditions a $(2, 2)$-theorem implies a $(p, 2)$-theorem for all $p > 2$.

While in general, a $(2, 2)$-theorem does not imply a $(p, 2)$-theorem (see an example in the full version of the paper), we prove such an implication for several kinds of families. Our first result here concerns families with Helly number 2 (i.e., families $\mathcal{F}$ with $\mathsf{HD}_{\mathcal{F}}(2,2) = 1$).

▶ **Theorem 6.** *Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$ with Helly number 2. Then* $\mathsf{HD}_{\mathcal{F}}(p,2) \leq p^{2d-1}/2^{d-1}$*, and consequently,* $\mathsf{HD}_{\mathcal{F}}(p,q) = p-q+1$ *holds for all* $q > cp^{1-\frac{1}{2d-1}}$*, where $c = c(d)$ is a constant depending only on the dimension $d$.*

The second result only assumes the existence of a $(2,2)$-theorem (where the piercing set may contain more than one point).

▶ **Theorem 7.** *Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$ that admits a $(2,2)$-theorem. Then:*

1. *$\mathcal{F}$ admits a $(p,2)$-theorem for piercing with a bounded number $s = s(p,d)$ of points.*
2. *If $d = 2$, then* $\mathsf{HD}_{\mathcal{F}}(p,2) = O(p^8 \log^2 p)$*.*
3. *If $d = 2$ and $\mathcal{F}$ has a bounded VC-dimension (see [23]), then* $\mathsf{HD}_{\mathcal{F}}(p,2) = O(p^4 \log^2 p)$*.*

Since families with a sub-quadratic union complexity admit a $(2,2)$-theorem and have a bounded VC-dimension, Theorem 7(3) implies that any family $\mathcal{F}$ of regions in the plane with a sub-quadratic union complexity satisfies $\mathsf{HD}_{\mathcal{F}}(p,2) = O(p^4 \log^2 p)$. This significantly improves over the bound $\mathsf{HD}_{\mathcal{F}}(p,2) = O(p^{16})$ that was obtained for such families in [16].

## 1.5 Organization of the paper

In Section 2 we demonstrate our general method for leveraging a $(p,2)$-theorem into a tight $(p,q)$-theorem and prove Theorem 4. Our new $(p,2)$-theorem for convex sets with Helly number 2 (i.e., Theorem 6 above) is presented in Section 3. We conclude the paper with a discussion and open problems in Section 4. For space reasons, the proofs of Theorems 3, 5, and 7 are presented only in the full version of the paper.

# 2 From $(p,2)$-theorems to tight $(p,q)$-theorems

In this section we present our main theorem which allows leveraging a $(p,2)$-theorem into a tight $(p,q)$-theorem, for families $\mathcal{F}$ that satisfy $\mathsf{HD}_{\mathcal{F}}(2,2) = 1$. As the proof of the theorem in its full generality is somewhat complex, we present here the proof in the case of axis-parallel rectangles in the plane, and provide the full proof in the full version of the paper. Before presenting the proof of the theorem, we briefly present the Wegner-Dol'nikov argument (parts of which we use in our proof) in Section 2.1, provide an outline of our method in Section 2.2, and prove two preparatory lemmas in Section 2.3.

## 2.1 The Wegner-Dol'nikov method

As mentioned in the introduction, Wegner and (independently) Dol'nikov leveraged the Hadwiger-Debrunner $(p,2)$-theorem for axis-parallel rectangles in the plane, which asserts that $\mathsf{HD}_{\text{rect}}(p,2) \leq \binom{p}{2}$, into a tight $(p,q)$-theorem, asserting that $\mathsf{HD}_{\text{rect}}(p,q) \leq p-q+1$ holds for all $p \geq q \geq 2$ such that $p < \binom{q+1}{2}$. The heart of the Wegner-Dol'nikov argument is the following observation.

▶ **Observation 8.** *Let $\mathcal{F}$ be a family that satisfies* $\mathsf{HD}_{\mathcal{F}}(2,2) = 1$*, and put $\lambda = \nu(\mathcal{F})$. Then*

$$\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1.$$

**Proof.** The slightly weaker bound $\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda$ holds trivially, and does not even require the assumption $\mathsf{HD}_{\mathcal{F}}(2,2) = 1$. Indeed, if $\mathcal{S}$ is a pairwise-disjoint subset of $\mathcal{F}$ of size $\lambda$, then $\mathcal{F} \setminus \mathcal{S}$ satisfies the $(p-\lambda, q-1)$-property, and thus, can be pierced by $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1)$ points. As $\mathcal{S}$ clearly can be pierced by $\lambda$ points, we obtain $\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda$.

To get the improvement by 1, let $\mathcal{S}$ be a pairwise-disjoint subfamily of $\mathcal{F}$ of size $\lambda = \nu(\mathcal{F})$ and let $T$ be a transversal of $\mathcal{F} \setminus \mathcal{S}$ of size $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1)$. Take an arbitrary $x \in T$, and consider the subfamily $\mathcal{X} = \{A \in \mathcal{F} \setminus \mathcal{S} : x \in A\}$ (i.e., the sets in $\mathcal{F} \setminus \mathcal{S}$ pierced by $x$). By the maximality of $\mathcal{S}$, each $A \in \mathcal{X}$ intersects some $B \in \mathcal{S}$. Hence, we can write $\mathcal{X} = \cup_{B \in \mathcal{S}} \mathcal{X}_B$, where $\mathcal{X}_B = \{A \in \mathcal{X} : A \cap B \neq \emptyset\}$. Observe that for each $B$, the set $\mathcal{X}_B \cup \{B\}$ is pairwise-intersecting. Indeed, any $A, A' \in \mathcal{X}$ intersect in $x$, and all elements of $\mathcal{X}_B$ intersect $B$. Therefore, by the assumption on $\mathcal{F}$, each $\mathcal{X}_B \cup \{B\}$ can be pierced by a single point. Since $\mathcal{X} = \cup_{B \in \mathcal{S}} \mathcal{X}_B$, this implies that there exists a transversal $T'$ of $\mathcal{X} \cup \mathcal{S}$ of size $|\mathcal{S}| = \lambda$. Now, the set $(T \setminus \{x\}) \cup T'$ is the desired transversal of $\mathcal{F}$ with $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1$ points. ◀

▶ Remark. We note that a generally similar argument of dividing the family $\mathcal{F}$ into a 'good' subfamily that satisfies a 'stronger' property (like the $(p-\lambda, q-1)$-property in Observation 8) and a small 'bad' subfamily that does not admit a 'good' property (like the independent set in Observation 8 which clearly cannot be pierced by less than $\lambda$ points) appears also in the improved bound on the Hadwiger-Debrunner number for general convex sets presented in [16]. While in [16], the bound on the piercing number is $p-q+2$, Observation 8 leads to the optimal piercing number $p-q+1$, as shown below. The advantage of Observation 8 is the 'improvement by 1' step, which reduces the piercing number by 1; this step cannot be applied for general families of compact convex sets, since it relies on the fact that axis-parallel rectangles have Helly number 2.

Using Observation 8, Wegner and Dol'nikov proved the following theorem, which we will use in our proof below.

▶ **Theorem 9** ([7], Theorem 2; [24]). *Let $\mathcal{F}$ be a family of axis-parallel rectangles in the plane. Then for any $p \geq q \geq 2$ such that $p < \binom{q+1}{2}$, we have $\mathsf{HD}_{\mathcal{F}}(p,q) = p-q+1$.*

**Proof.** The proof is by induction. The induction basis is $q = 2$: for this value, the assertion is relevant only for $p = 2$, and we indeed have $\mathsf{HD}_{\mathrm{rect}}(2,2) = 1 = 2-2+1$ as asserted.

For the inductive step, we consider $\lambda = \nu(\mathcal{F})$. Note that $\mathcal{F}$ satisfies the $(\lambda+1, 2)$-property. Thus, if $\binom{\lambda+1}{2} \leq p-q+1$ then we have $\mathsf{HD}_{\mathcal{F}}(p,q) \leq p-q+1$ by the aforementioned Hadwiger-Debrunner $(p,2)$-theorem for axis-parallel rectangles. On the other hand, if $\binom{\lambda+1}{2} > p-q+1$ then it can be checked that $p-\lambda < \binom{q}{2}$, so by the induction hypothesis we have $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) = (p-\lambda)-(q-1)+1$. By Observation 8, this implies $\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1 = p-q+1$, as asserted. ◀

## 2.2 Outline of our method

Let $\mathcal{F}$ be a family of axis-parallel rectangles in the plane. Instead of leveraging the Hadwiger-Debrunner $(p,2)$-theorem for $\mathcal{F}$ into a $(p,q)$-theorem as was done by Wegner and Dol'nikov, we would like to leverage the stronger bound $\mathsf{HD}_{\mathrm{rect}}(p,2) \leq p \log_2 p$ which follows from (1). We want to deduce that $\mathsf{HD}_{\mathrm{rect}}(p,q) = p-q+1$ holds for all $q \geq 7 \log p$.

Basically, we would like to perform an inductive process similar to the process applied in the proof of Theorem 9. As above, put $\lambda = \nu(\mathcal{F})$. If $\lambda$ is 'sufficiently large' (namely, if $q-1 \geq 7 \log_2(p-\lambda)$), we apply the recursive formula $\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1$ and use the induction hypothesis to bound $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1)$. Otherwise, we would like to use the improved $(p,2)$-theorem to deduce that $\mathcal{F}$ can be pierced by at most $p-q+1$ points.

However, since we want to prove the theorem in the entire range $q \geq 7\log_2 p$, in order to apply the induction hypothesis to $\mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1)$, $\lambda$ must be at least *linear in p* (specifically, we need $\lambda \geq 0.1p$, as is shown below). Thus, in the 'otherwise' case we have to show that if $\lambda < 0.1p$, then $\mathcal{F}$ can be pierced by at most $p-q+1$ points. If we merely use the fact that $\mathcal{F}$ satisfies the $(\lambda+1, 2)$-property and apply the improved $(p, 2)$-theorem, we only obtain that $\mathcal{F}$ can be pierced by $O(p \log p)$ points – significantly weaker than the desired bound $p-q+1$.

Instead, we use a more complex procedure, partially based on the following observation, presented in [16] (and called there a 'combinatorial dichotomy'):

▶ **Observation 10.** *Let $\mathcal{F}$ be a family that satisfies the $(p, q)$-property. For any $p' \leq p, q' \leq q$ such that $q' \leq p'$, either $\mathcal{F}$ satisfies the $(p', q')$-property, or there exists $S \subset \mathcal{F}$ of size $p'$ that does not contain an intersecting $q'$-tuple. In the latter case, $\mathcal{F} \setminus S$ satisfies the $(p-p', q-q'+1)$-property.*

First, we use Observation 10 to leverage the $(p, 2)$-theorem by an inductive process into a 'weak' $(p, q)$-theorem that guarantees piercing with $p-q+1+O(p)$ points, for all $q = \Omega(\log p)$. We then show that if $\lambda < 0.1p$ then $\mathcal{F}$ can be pierced by at most $p-q+1$ points, by combining the weak $(p, q)$-theorem, another application of Observation 10, and a lemma which exploits the size of $\lambda$.

## 2.3 The two main lemmas used in the proof

Our first lemma leverages the $(p, 2)$-theorem $\mathsf{HD}_{\text{rect}}(p, 2) \leq p\log_2 p$ into a weak $(p, q)$-theorem, using Observation 10.

▶ **Lemma 11.** *Let $\mathcal{F}$ be a family of axis-parallel rectangles in the plane. Then for any $c > 0$ and for any $p \geq q \geq 2$ such that $q \geq c\log_2 p$, we have*

$$\mathsf{HD}_{\mathcal{F}}(p, q) \leq p - q + 1 + \frac{2p}{c}.$$

**Proof.** First, assume that both $p$ and $q$ are powers of 2. We perform an inductive process with $\ell = (\log_2 q) - 1$ steps, where we set $\mathcal{F}_0 = \mathcal{F}$ and $(p_0, q_0) = (p, q)$, and in each step $i$, we apply Observation 10 to a family $\mathcal{F}_{i-1}$ that satisfies the $(p_{i-1}, q_{i-1})$-property, with $(p', q') = (\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})$ which we denote by $(p_i, q_i)$.

Consider Step $i$. By Observation 10, either $\mathcal{F}_{i-1}$ satisfies the $(p_i, q_i) = (\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})$-property, or there exists a 'bad' set $S_i$ of size $\frac{p_{i-1}}{2}$ without an intersecting $\frac{q_{i-1}}{2}$-tuple, and the family $\mathcal{F}_{i-1} \setminus S_i$ satisfies the $(\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2}+1)$-property, and in particular, the $(\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})$-property. In either case, we are reduced to a family $\mathcal{F}_i$ (either $\mathcal{F}_{i-1}$ or $\mathcal{F}_{i-1} \setminus S_i$) that satisfies the $(p_i, q_i)$-property, to which we apply Step $i+1$.

At the end of Step $\ell$ we obtain a family $\mathcal{F}_\ell$ that satisfies the $(2p/q, 2)$-property. (Note that the ratio between the left term and the right term remains constant along the way.) By the $(p, 2)$-theorem, $\mathcal{F}_\ell$ can be pierced by $\frac{2p}{q}\log_2\left(\frac{2p}{q}\right)$ points. As $q \geq \max(c\log_2 p, 2)$, this implies that $\mathcal{F}_\ell$ can be pierced by

$$\frac{2p}{q}\log_2\left(\frac{2p}{q}\right) \leq \frac{2p}{q}\log_2 p \leq \frac{2p}{c}$$

points.

In order to pierce $\mathcal{F}$, we also have to pierce the 'bad' sets $S_i$. In the worst case, in each step we have a bad set, and so we have to pierce $S = \cup_{i=1}^{\ell} S_i$. The size of $S$ is

$|S| = \frac{p}{2} + \frac{p}{4} + \ldots + 2 + 1 = p - 1$. Since any family that satisfies the $(p,q)$-property also satisfies the $(p-k, q-k)$-property for any $k$, the family $S$ contains an intersecting $(q-1)$-tuple, which of course can be pierced by a single point. Hence, $S$ can be pierced by $(p-1)-(q-1)+1 = p-q+1$ points. Therefore, in total $\mathcal{F}$ can be pierced by $p-q+1+2p/c$ points, as asserted.

Now, we have to deal with the case where $p, q$ are not necessarily powers of 2, and thus, in some of the steps either $p_{i-1}$ or $q_{i-1}$ or both are not divisible by 2. It is clear from the proof presented above that if we can define $(p_i, q_i)$ in such a way that in both cases (i.e., whether there is a 'bad' set or not), we have $\frac{p_i}{q_i} \leq \frac{p_{i-1}}{q_{i-1}}$, and also the total size of the bad sets (i.e., $|S|$) is at most $p$, the assertion can be deduced as above (as the ratio between the left term and the right term only decreases). We show that this can be achieved by a proper choice of $(p_i, q_i)$ and a slight modification of the steps described above. Let

$$(p', q') = \left( \left\lfloor \frac{p_{i-1}}{2} \right\rfloor, \left\lceil \frac{q_{i-1}}{2} \right\rceil \right).$$

If $\mathcal{F}_{i-1}$ satisfies the $(p', q')$-property, we define $\mathcal{F}_i = \mathcal{F}_{i-1}$ and $(p_i, q_i) = (p', q')$. Otherwise, there exists a 'bad' set $S_i$ of size $p'$ that does not contain an intersecting $q'$-tuple, and the family $\mathcal{F}_{i-1} \setminus S_i$ satisfies the

$$(p_{i-1} - p', q_{i-1} - q' + 1) = \left( \left\lceil \frac{p_{i-1}}{2} \right\rceil, \left\lfloor \frac{q_{i-1}}{2} \right\rfloor + 1 \right)$$

property. In this case, we define $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus S_i$ and $(p_i, q_i) = (p_{i-1} - p', q_{i-1} - q' + 1)$.

It is easy to check that in both cases we have $\frac{p_i}{q_i} \leq \frac{p_{i-1}}{q_{i-1}}$, and that $|S| \leq p-1$ holds also with respect to the modified definition of the $S_i$'s. Hence, the proof indeed can be completed, as above. ◀

Our second lemma is a simple upper bound on the piercing number of a family that satisfies the $(p,2)$-property. We shall use it to show that if $\nu(\mathcal{F})$ is 'small', then we can save 'something' when piercing large subsets of $\mathcal{F}$.

▶ **Lemma 12.** *Any family $\mathcal{G}$ of $m$ sets that satisfies the $(p,2)$-property can be pierced by $\left\lfloor \frac{m+p-1}{2} \right\rfloor$ points.*

**Proof.** We perform the following simple recursive process. If $\mathcal{G}$ contains a pair of intersecting sets, pierce them by a single point and remove both of them from $\mathcal{G}$. Continue in this fashion until all remaining sets are pairwise disjoint. Then pierce each remaining set by a separate point.

As $\mathcal{G}$ satisfies the $(p,2)$-property, the number of sets that remain in the last step is at most $p-1$ if $m-(p-1)$ is even and at most $p-2$ otherwise. In the former case, the resulting piercing set is of size at most $\frac{m-(p-1)}{2} + (p-1) = \frac{m+p-1}{2}$. In the latter case, the piercing set is of size at most $\frac{m-(p-2)}{2} + (p-2) = \frac{m+p-2}{2}$. Hence, in both cases the piercing set is of size at most $\left\lfloor \frac{m+p-1}{2} \right\rfloor$, as asserted. ◀

▶ Remark. The assertion of Lemma 12 is tight, as for a family $\mathcal{G}$ composed of $m-p+2$ lines in a general position in the plane and $p-2$ pairwise-disjoint segments that do not intersect any of the lines, we have $|\mathcal{G}| = m$, $\mathcal{G}$ satisfies the $(p,2)$-property, and $\mathcal{G}$ clearly cannot be pierced by less than $\left\lfloor \frac{m+p-1}{2} \right\rfloor$ points.

▶ **Corollary 13.** *Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$, and put $\lambda = \nu(\mathcal{F})$. Then any subset $S \subset \mathcal{F}$ can be pierced by at most $\left\lfloor \frac{|S|+\lambda}{2} \right\rfloor$ points.*

The corollary follows from the lemma immediately, as any such family $\mathcal{F}$ satisfies the $(\lambda+1, 2)$-property.

## 2.4 Proof of Theorem 4

Now we are ready to present the proof of our main theorem, in the specific case of axis-parallel rectangles in the plane. Let us recall its statement.

▶ **Theorem 4.** *Let $\mathcal{F}$ be a family of axis-parallel rectangles in the plane. If $\mathcal{F}$ satisfies the $(p,q)$-property, for $p \geq q \geq 2$ such that $q \geq 7\log_2 p$, then $\mathcal{F}$ can be pierced by $p-q+1$ points.*

▶ Remark. We note that the parameters in the proof (e.g., the values of $(p',q')$ in the inductive step) were chosen in a sub-optimal way, that is however sufficient to yield the assertion with the constant 7. (The straightforward choice $(p',q') = (0.5p, 0.5q)$ is not sufficient for that). The constant can be further optimized by a more careful choice of the parameters; however, it seems that in order to reduce it below 6, a significant change in the proof is needed.

**Proof of Theorem 4.** The proof is by induction.

**Induction basis.** One can assume that $q \geq 37$, as for any smaller value of $q$, there are no $p$'s such that $7\log_2 p \leq q \leq p$. For $q = 37$, the theorem is only relevant for $(p,q) = (37,37)$, and in this case we clearly have $\mathsf{HD}_{\mathcal{F}}(p,q) = 1 = p-q+1$. Generally speaking, this is a sufficient basis, since in the inductive step, the value of $q$ is reduced by 1 every time. However, in the proof of the inductive step we would like to assume that $p,q$ are 'sufficiently large'; hence, we use Theorem 9 as the induction basis in order to cover a larger range of small $(p,q)$ values.

We observe that for $q \leq 70$, all relevant $(p,q)$ pairs (i.e., all pairs for which $7\log_2 p \leq q \leq p$) satisfy $p \leq \binom{q+1}{2}$. Hence, in this range we have $\mathsf{HD}_{\mathcal{F}}(p,q) = p-q+1$ by Theorem 9. Therefore, we may assume that $q > 70$; we also may assume $q < \sqrt{2p}$ (as otherwise, the assertion follows from Theorem 9), and thus, $p > 2450$ and so (using again the assumption $q < \sqrt{2p}$), also $p > 35q$.

**Inductive step.** Put $\lambda = \nu(\mathcal{F})$. By Observation 8, we have $\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1$. We want $\lambda$ to be sufficiently large, such that if $(p,q)$ lies in the range covered by the theorem (i.e., if $q \geq 7\log_2 p$), then $(p-\lambda, q-1)$ also lies in the range covered by the theorem (i.e., $q-1 \geq 7\log_2(p-\lambda)$). Note that the condition $q \geq 7\log_2 p$ is equivalent to $2^{q/7} \geq p$, which implies $2^{(q-1)/7} = \frac{2^{q/7}}{2^{1/7}} \geq 0.9p$. Hence, if $\lambda \geq 0.1p$ then $q-1 \geq 7\log_2(p-\lambda)$, and so we can deduce from the induction hypothesis that

$$\mathsf{HD}_{\mathcal{F}}(p,q) \leq \mathsf{HD}_{\mathcal{F}}(p-\lambda, q-1) + \lambda - 1 \leq (p-\lambda) - (q-1) + 1 + (\lambda - 1) = p-q+1,$$

as asserted. Therefore, it is sufficient to prove that $\mathsf{HD}_{\mathcal{F}}(p,q) \leq p-q+1$ holds when $\lambda < 0.1p$.

Under this assumption on $\lambda$, we apply Observation 10 to $\mathcal{F}$, with $(p',q') = (\lfloor 0.62p \rfloor, 0.5q)$. We have to consider two cases:

**Case 1: $\mathcal{F}$ satisfies the $(p',q')$-property.** By the assumption on $(p,q)$, we have $q \geq 7\log_2 p$, and thus, $0.5q \geq 3.5\log_2 p \geq 3.5\log_2\lfloor 0.62p \rfloor$. Hence, by Lemma 11,

$$\mathsf{HD}_{\mathcal{F}}\left(\lfloor 0.62p \rfloor, 0.5q\right) \leq 0.62p - 0.5q + 1 + \frac{2}{3.5}\cdot 0.62p < 0.975p - 0.5q + 1 \leq p-q+1,$$

where the last inequality holds because we may assume $q \leq 0.05p$, since $p > 35q$ as was written above. Thus, $\mathcal{F}$ can be pierced by at most $p-q+1$ points, as asserted.

**Case 2: $\mathcal{F}$ does not satisfy the $(p', q')$-property.** In this case, there exists a 'bad' subfamily $S$ of size $p' = \lfloor 0.62p \rfloor$ that does not contain an intersecting $0.5q$-tuple, and the family $\mathcal{F} \setminus S$ satisfies the $(\lceil 0.38p \rceil, 0.5q)$-property.

To pierce $\mathcal{F} \setminus S$, we use Lemma 11. Like above, we have $0.5q \geq 3.5 \log_2 \lceil 0.38p \rceil$, whence by Lemma 11,

$$\mathsf{HD}_{\mathcal{F}}\left(\lceil 0.38p \rceil, 0.5q\right) \leq 0.39p - 0.5q + 1 + \frac{2}{3.5} \cdot 0.39p < 0.613p - 0.5q + 1,$$

where the first inequality holds since we may assume $p \geq 100$ (as was written above), and thus, $\lceil 0.38p \rceil \leq 0.39p$.

To pierce the 'bad' subfamily $S$, we use Corollary 13, which implies that $S$ can be pierced by

$$\left\lfloor \frac{1}{2}(|S| + \lambda) \right\rfloor \leq \frac{1}{2}(0.62p + 0.1p) = 0.36p$$

points. Therefore, in total $\mathcal{F}$ can be pierced by $(0.613p - 0.5q + 1) + 0.36p < 0.975p - 0.5q + 1$ points. Since we may assume $q \leq 0.05p$ (like above), this implies that $\mathcal{F}$ can be pierced by $p - q + 1$ points. This completes the proof. ◀

## 3 From $(2, 2)$-theorems to $(p, 2)$-theorems

As was mentioned in the introduction, in general, the existence of a $(2, 2)$-theorem (and even Helly number 2) does not imply the existence of a $(p, 2)$-theorem. An example mentioned by Fon der Flaass and Kostochka [10] (in a slightly different context) is presented in the full version of the paper.

In this section we prove Theorem 6 which asserts that for families of convex sets with Helly number 2, a $(2, 2)$-theorem does imply a $(p, 2)$-theorem, and consequently, a tight $(p, q)$-theorem for a large range of $q$'s. Due to space constraints, the proof of our other new $(p, 2)$-theorem (i.e., Theorem 7) is presented in the full version of the paper.

Let us recall the assertion of the theorem:

▶ **Theorem 6.** *For any family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ that has Helly number 2, we have $\mathsf{HD}_{\mathcal{F}}(p, 2) \leq \frac{p^{2d-1}}{2^{d-1}}$. Consequently, we have $\mathsf{HD}_{\mathcal{F}}(p, q) = p - q + 1$ for all $q > cp^{1-\frac{1}{2d-1}}$, where $c = c(d)$.*

The 'consequently' part follows immediately from the $(p, 2)$-theorem via Theorem 3. Hence, we only have to prove the $(p, 2)$-theorem.

Let us present the proof idea first. The proof goes by induction on $d$. Given a family $\mathcal{F}$ of sets in $\mathbb{R}^d$ that satisfies the assumptions of the theorem and has the $(p, 2)$-property, we take $\mathcal{S}$ to be a maximum (with respect to size) pairwise-disjoint subfamily of $\mathcal{F}$, and consider the intersections of other sets of $\mathcal{F}$ with the elements of $\mathcal{S}$. We observe that by the maximality of $\mathcal{S}$, each set $A \in \mathcal{F} \setminus \mathcal{S}$ intersects at least one element of $\mathcal{S}$, and thus, we may partition $\mathcal{F}$ into three subfamilies: $\mathcal{S}$ itself, the family $\mathcal{U}$ of sets in $\mathcal{F} \setminus \mathcal{S}$ that intersect only one element of $\mathcal{S}$, and the family $\mathcal{M} \subset \mathcal{F} \setminus \mathcal{S}$ of sets that intersect at least two elements of $\mathcal{S}$.

We show (using the maximality of $\mathcal{S}$ and the $(2, 2)$-theorem on $\mathcal{F}$) that $\mathcal{U} \cup \mathcal{S}$ can be pierced by $p - 1$ points. As for $\mathcal{M}$, we represent it as a union of families: $\mathcal{M} = \cup_{C,C' \in \mathcal{S}} \mathcal{X}_{C,C'}$, where each $\mathcal{X}_{C,C'}$ consists of the elements of $F \setminus \mathcal{S}$ that intersect both $C$ and $C'$. We use a geometric argument to show that each $\mathcal{X}_{C,C'}$ corresponds to $\mathcal{Y}_{C,C'} \subset \mathbb{R}^{d-1}$ that has Helly number 2 and satisfies the $(p, 2)$-property. This allows us to bound the piercing number of

$\mathcal{Y}_{C,C'}$ by the induction hypothesis, and consequently, to bound the piercing number of $\mathcal{X}_{C,C'}$. Adding up the piercing numbers of all $\mathcal{X}_{C,C'}$'s and the piercing number of $\mathcal{U} \cup \mathcal{S}$ completes the inductive step.

**Proof of Theorem 6.** By induction on $d$.

**Induction basis.** For any family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^1$, by the Hadwiger-Debrunner theorem [13] we have $\mathsf{HD}_{\mathcal{F}}(p,2) = p - 2 + 1 < p = p^{2\cdot 1-1}/2^{1-1}$, and so the assertion holds.

**Inductive step.** Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ that satisfies the assumptions of the theorem and has the $(p,2)$-property. Let $\mathcal{S}$ be a maximum (with respect to size) pairwise-disjoint subfamily of $\mathcal{F}$. We may assume $|\mathcal{S}| = p - 1$.

By the maximality of $\mathcal{S}$, each set $A \in \mathcal{F} \setminus \mathcal{S}$ intersects at least one element of $\mathcal{S}$. Moreover, any two sets $A, B \in \mathcal{F}$ that intersect the same $C \in \mathcal{S}$ and do not intersect any other element of $\mathcal{S}$, are intersecting, as otherwise, the subfamily $\mathcal{S} \cup \{A, B\} \setminus \{C\}$ would be a pairwise-disjoint subfamily of $\mathcal{F}$ that is larger than $\mathcal{S}$, a contradiction. Hence, for each $C_0 \in \mathcal{S}$, the subfamily

$$\mathcal{X}_{C_0} = \{A \in \mathcal{F} : \{C \in \mathcal{S} : A \cap C \neq \emptyset\} = \{C_0\}\} \cup \{C_0\}$$

satisfies the $(2,2)$-property, and thus, can be pierced by a single point by the assumption on $\mathcal{F}$. Therefore, denoting $\mathcal{U} = \{A \in \mathcal{F} : |\{C \in \mathcal{S} : A \cap C \neq \emptyset\}| = 1\}$, all sets in $\mathcal{U} \cup \mathcal{S}$ can be pierced by at most $p - 1$ points.

Let $\mathcal{M} \subset \mathcal{F}$ be the family of all sets in $\mathcal{F}$ that intersect at least two elements of $\mathcal{S}$. For each $C, C' \in \mathcal{S}$, let

$$\mathcal{X}_{C,C'} = \{A \in \mathcal{F} \setminus \mathcal{S} : A \cap C \neq \emptyset \wedge A \cap C' \neq \emptyset\}.$$

(Note that the elements of $\mathcal{X}_{C,C'}$ may intersect other elements of $\mathcal{S}$.) Let $H \subset \mathbb{R}^d$ be a hyperplane that strictly separates $C$ from $C'$, and put $\mathcal{Y}_{C,C'} = \{A \cap H : A \in \mathcal{X}_{C,C'}\}$.

▶ **Claim 14.** *$\mathcal{Y}_{C,C'} \subset H \approx \mathbb{R}^{d-1}$ admits $\mathsf{HD}_{\mathcal{Y}_{C,C'}}(2,2) = 1$ and satisfies the $(p,2)$-property.*

**Proof.** To prove the claim, we observe that $A \cap H, A' \cap H \in \mathcal{Y}_{C,C'}$ intersect if and only if $A$ and $A'$ intersect. Indeed, assume $A \cap A' \neq \emptyset$. The family $\{A, A', C\}$ satisfies the $(2,2)$-property, and hence, can be pierced by a single point by the assumption on $\mathcal{F}$. Thus, $A \cap A'$ contains a point of $C$. For the same reason, $A \cap A'$ contains a point of $C'$. Therefore, $A \cap A'$ contains points on the two sides of the hyperplane $H$. However, $A \cap A'$ is convex, and so, $(A \cap A') \cap H \neq \emptyset$, which means that $(A \cap H)$ and $(A' \cap H)$ intersect. The other direction is obvious.

It is now clear that as $\mathcal{X}_{C,C'} \subset \mathcal{F}$ satisfies the $(p,2)$-property, $\mathcal{Y}_{C,C'}$ satisfies the $(p,2)$-property as well. Moreover, let $T = \{A_1 \cap H, A_2 \cap H, A_3 \cap H, \ldots\} \subset \mathcal{Y}_{C,C'}$ be pairwise-intersecting. The corresponding family $\tilde{T} = \{C, A_1, A_2, A_3, \ldots\}$ is pairwise-intersecting, and thus, can be pierced by a single point by the assumption on $\mathcal{F}$. Thus, $(A_1 \cap A_2 \cap A_3 \cap \ldots) \cap C \neq \emptyset$. For the same reason, $(A_1 \cap A_2 \cap A_3 \cap \ldots) \cap C' \neq \emptyset$. Since $A_1 \cap A_2 \cap A_3 \cap \ldots$ is convex, this implies that $(A_1 \cap A_2 \cap A_3 \cap \ldots) \cap H \neq \emptyset$, or equivalently, that the family $T$ can be pierced by a single point. Therefore, $\mathcal{Y}_{C,C'}$ satisfies $\mathsf{HD}_{\mathcal{Y}_{C,C'}}(2,2) = 1$, as asserted. ◀

Claim 14 allows us to apply the induction hypothesis to $\mathcal{Y}_{C,C'}$, to deduce that it can be pierced by less than $p^{2d-3}/2^{d-1}$ points. Since $\mathcal{S}$ contains only $\binom{p-1}{2}$ pairs $(C, C')$, and since any set in $\mathcal{M}$ belongs to at least one of the $\mathcal{X}_{C,C'}$, this implies that $\mathcal{M}$ can be pierced by less than $\binom{p-1}{2} \cdot p^{2d-3}/2^{d-2}$ points. As $\mathcal{U} \cup \mathcal{S}$ can be pierced by $p-1$ points as shown above, $\mathcal{F}$ can be pierced by less than

$$\binom{p-1}{2} \cdot \frac{p^{2d-3}}{2^{d-2}} + (p-1) < \frac{p^{2d-1}}{2^{d-1}}$$

points. This completes the proof. ◀

## 4 Discussion and open problems

A central problem left for further research is whether Theorem 3 which allows leveraging a $(p, 2)$-theorem into a $(p, q)$-theorem, can be extended to the cases $\mathsf{HD}_{\mathcal{F}}(p, 2) = pf(p)$ where $f(p) \ll \log p$ or $f(p)$ being super-polynomial in $p$. It seems that super-polynomial growth rates can be handled with a slight modification of the argument (at the expense of replacing $T_{c'}(p)$ with some worse dependence on $p$). For a sub-logarithmic growth rate, it seems that the current argument does not work, since the inductive step requires the packing number of $\mathcal{F}$ to be extremely large, and so, Lemma 12 allows reducing the piercing number of the 'bad' family $S$ only slightly, rendering Lemma 11 insufficient for piercing $\mathcal{F}$ with $p - q + 1$ points in total.

Extending the method for sub-logarithmic growth rates will have interesting applications. For instance, it will immediately yield a tight $(p, q)$-theorem for all $q = \Omega(\log \log p)$ for families of axis-parallel boxes in which for each two intersecting boxes, a corner of one is contained in the other, following the work of Chudnovsky et al. [6]. Furthermore, it will imply that if axis-parallel rectangles admit a $(p, 2)$-theorem with the size of the piercing set linear in $p$ (as conjectured by Wegner [24]), then $\mathsf{HD}_{\mathrm{rect}}(p, q) = p - q + 1$ holds for all $q \geq c$ for a constant $c$. As mentioned in Section 1.3, this was claimed by Dol'nikov [7], but with a flawed argument, as remarked by Eckhoff [9].

Another open problem is whether the method can be extended to families $\mathcal{F}$ that admit a $(2, 2)$-theorem, but satisfy $\mathsf{HD}_{\mathcal{F}}(2, 2) > 1$. Such an improvement would allow transformation into tight $(p, q)$-theorems of the $(p, 2)$-theorems presented in Section 1.3, such as the $(p, 2)$-theorem for pseudo-discs of Chan and Har-Peled [5].

A main obstacle here is that in this case, Observation 8 does not apply, and instead, we have the bound $\mathsf{HD}(p, q) \leq \mathsf{HD}(p - \lambda, q - 1) + \lambda$. While the bound is only slightly weaker, it precludes us from using the inductive process of Wegner and Dol'nikov, as in each application of the inductive step we have an 'extra' point.

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