

## From continuous time random walks to the fractional Fokker-Planck equation

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We generalize the continuous time random walk (CTRW) to include the effect of space dependent jump probabilities. When the mean waiting time diverges we derive a fractional Fokker-Planck equation (FFPE). This equation describes anomalous diffusion in an external force field and close to thermal equilibrium. We discuss the domain of validity of the fractional kinetic equation. For the force free case we compare between the CTRW solution and that of the FFPE.

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### I. INTRODUCTION

The linear Fokker-Planck equation

$$\dot{P}(x,t) = K_1 \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{F(x)}{k_B T} \right] P(x,t) \quad (1)$$

[1–6], also called the Smoluchowski equation, is usually applied to describe various types of normal Markovian diffusive phenomena. In the absence of an external force,  $F(x) = 0$ , the equation describes a Gaussian evolution as may be anticipated based on the central limit theorem. The equation describes an overdamped motion and hence has no explicit dependence on the velocity of the test particle. When the motion is bounded by an external potential field the stationary solution is the Boltzmann equilibrium defined by the temperature  $T$ .

Many works have focused on the domain of validity of Eq. (1). The derivation of this equation can be achieved using different approaches reviewed in the variety of text books on the subject [1–6]. In all derivations it has been assumed that a microscopic time scale exists, which is small compared to the observation time  $t$ . In a random walk picture this time is the characteristic time it takes a particle to perform a single microscopic jump. What happens when this characteristic time scale diverges? In this anomalous case we certainly do not expect the Markovian Fokker-Planck equation (1) to hold. However, as we show, there exists a natural generalization of the Fokker-Planck equation.

It is by now well established that the divergence of microscopic time scales in random walk schemes may lead to anomalous diffusion [6]. The continuous time random walk (CTRW) of Montroll and Weiss [7] has been used to describe such anomalous diffusion when  $F(x) = 0$  for over two decades [6,8–15]. Here we generalize the one-dimensional CTRW on a lattice for jump probabilities that depend on the site of jump. In this way we break the spatial invariance usually assumed within the context of CTRW. Shugard and Reiss [16] have already carried out a similar extension and used it to develop a theory to calculate nucleation rates. In this case the external potential field is the free energy barrier. Their Montroll-Weiss waiting time distribution has been as-

sumed to follow an exponential decay which is very different from the power law decay we assume here.

Using the CTRW in an external field we consider a continuum approximation where the lattice spacing  $a \rightarrow 0$ . Under certain conditions we show that the dynamics are described by a fractional kinetic equation

$$\dot{P}(x,t) = {}_0D_t^{1-\alpha} K_\alpha \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{F(x)}{k_B T} \right] P(x,t), \quad (2)$$

where  ${}_0D_t^{1-\alpha}$  is the the fractional Riemann-Liouville operator (see more details below). We call Eq. (2) the fractional Fokker-Planck equation (FFPE). This equation has been recently investigated [17,18] and derived from a generalized master equation [19]. Earlier, Balakrishnan [20] has derived the FFPE for the case  $F(x) = 0$  based upon a generalization of Brownian motion. Schneider and Wyss [21,22] have found a solution to the force free problem in terms of Fox functions. Here a detailed and different derivation of the FFPE in an external field is presented, the starting point being the extensively investigated CTRW model. We discuss the scaling regime in which the FFPE is valid and its limitations, and compare between the CTRW solution [6,23,24] and that of the FFPE in dimension  $D = 1, 2, 3$ .

It is worth mentioning that fractional kinetic equations have been suggested to model a quantum particle interacting with a chaotic bath [25], anomalous diffusion in random environments [26,27], and for chaotic Hamiltonian systems [28]. These fractional equations have been used to describe Lévy flights or diverging diffusion. In contrast, we describe subdiffusive systems where the mean square displacement [when  $F(x) = 0$ ] behaves as  $\langle x^2 \rangle \sim t^\alpha$  and  $\alpha < 1$ . For a review on anomalous diffusion see Refs. [12,14,29].

This paper is organized as follows. In Sec. II we introduce the CTRW model in an external field. In Sec. III we derive the FFPE from the CTRW model. A comparison between the CTRW solution and the FFPE solution for the case  $F(x) = 0$  is made in Sec. IV. Finally, in Sec. V, we discuss the domain of validity of the FFPE and some of its limitations.

### II. MODEL

We consider an unbounded random walk on a one dimensional lattice with a lattice spacing  $a$ . Lattice sites are de-

noted by  $\{\dots, -1, 0, 1, \dots, n, \dots\}$ . At time  $t=0$  the particle is located at site  $n=0$ .

Once the particle has arrived at site  $n$  it is trapped there for some random time. These waiting times are given by  $\{\tau_i\}$  and  $i=1, 2, \dots$ .  $\{\tau_i\}$  are independent random variables identically distributed according to a probability density function  $\psi(\tau)$ . It is assumed that  $\psi(\tau)$  is independent of the location of the particle  $n$  (i.e., it is independent of the external field). Different types of such probability densities  $\psi(\tau)$  have been used to describe a wide variety of physical phenomena [6, 8–15].

We assume that the particle can jump either to the left or the right and only nearest neighbor jumps are allowed. The probability of hopping from site  $n$  to  $n+1$  is  $R(n)$  and from site  $n$  to site  $n-1$  is  $L(n)$ , the normalization condition being  $L(n)+R(n)=1$ .  $R(n)$  and  $L(n)$  are time independent.

The random walk process is therefore described as follows. At time  $t=0$  a particle starts at site  $n=0$ . It stays there for a time  $\tau_1$  chosen randomly. Then with probability  $R(0)$  [or  $1-R(0)$ ] it jumps to site  $n=1$  [or  $n=-1$ ]. The process is then renewed. The jumping probabilities  $R(n)$  and  $L(n)$  are independent of the duration of trapping.

### III. FROM CTRW TO FFPE

The probability that a particle is trapped for a period  $t$  without executing a jump is

$$W(t) = \int_t^\infty \psi(\tau) d\tau. \quad (3)$$

In Laplace  $t \rightarrow u$  space

$$\hat{W}(u) = \frac{1 - \hat{\psi}(u)}{u}, \quad (4)$$

where  $\hat{\psi}(u)$  is the Laplace transform of  $\psi(\tau)$ . Since the waiting times are independent, identically distributed random variables, it is straightforward to show that  $Q_i(t)$ , the probability that the random walker has jumped  $i$  times in the interval  $(0, t)$ , is in Laplace space

$$\hat{Q}_i(u) = \frac{1 - \hat{\psi}(u)}{u} \hat{\psi}^i(u). \quad (5)$$

Let the probability of finding the particle at site  $n$  at time  $t$  be  $P(n, t)$ , and let  $p_i(n)$  be the probability to be on site  $n$  after step  $i$ . Then,

$$P(n, t) = \sum_{i=0}^{\infty} p_i(n) Q_i(t). \quad (6)$$

Using Eq. (5),

$$\hat{P}(n, u) = \frac{1 - \hat{\psi}(u)}{u} \sum_{i=0}^{\infty} p_i(n) \hat{\psi}^i(u). \quad (7)$$

The evolution of  $p_i(n)$  is determined by the discrete time and space equation

$$p_{i+1}(n) = R(n-1)p_i(n-1) + L(n+1)p_i(n+1). \quad (8)$$

In Eq. (8) we have used the assumption that the jumping probabilities  $R(n)$  and  $L(n)$  are independent of the waiting times. We now consider the continuum limit of this equation by using the replacement

$$p_i(n) \rightarrow p_i(x),$$

where  $p_i(x)dx$  is the probability of finding the particle after the  $i$ th jump in the interval  $(x, x+dx)$ . Similarly,  $R(n) \rightarrow R(x)$  and  $L(n) \rightarrow L(x)$  with the normalization  $L(x) + R(x) = 1$ . In addition we have  $L(n+1) \rightarrow L(x+a)$  and  $R(n+1) \rightarrow R(x+a)$  where  $a$  is the lattice spacing. We now expand Eq. (8) in a Taylor series in  $a$ , a typical term being

$$\begin{aligned} R(n-1)p_i(n-1) &\rightarrow R(x-a)p_i(x-a) \\ &= R(x)p_i(x) + \frac{\partial}{\partial x}[R(x)p_i(x)](-a) \\ &\quad + \frac{\partial^2}{\partial x^2}[R(x)p_i(x)]a^2/2 + \dots, \end{aligned} \quad (9)$$

where higher order terms proportional to  $a^3, a^4$  etc. are omitted. Similar expansions are used to derive Eq. (1).

We assume that our system is close to thermal equilibrium defined with a temperature  $T$ . For this case  $R(x) \approx L(x) \approx 1/2$  and according to detailed balance  $R(x) - L(x) \approx aF(x)/(2k_bT)$ , where  $F(x)$  is the external force field. We show below that such a requirement on  $R(x)$  and  $L(x)$  guarantees that the system relaxes to the thermal Boltzmann equilibrium. In this case we obtain from Eqs. (8)–(9) in the continuum limit

$$p_{i+1}(x) = p_i(x) + \frac{a^2}{2} \left[ \frac{\partial^2}{\partial x^2} p_i(x) - \frac{\partial}{\partial x} \frac{F(x)}{k_bT} p_i(x) \right] + \dots \quad (10)$$

We now rewrite Eq. (7) as

$$\hat{P}(x, u) = \frac{1 - \hat{\psi}(u)}{u} p_0(x) + \frac{1 - \hat{\psi}(u)}{u} \sum_{i=1}^{\infty} p_i(x) \hat{\psi}^i(u), \quad (11)$$

where the continuum approximation  $\hat{P}(n, u) \rightarrow \hat{P}(x, u)$  has been made. Inserting Eq. (10) into Eq. (11), and using  $p_0(x) = \delta(x)$ , we find

$$\begin{aligned} \hat{P}(x, u) &= \frac{1 - \hat{\psi}(u)}{u} \delta(x) + \frac{1 - \hat{\psi}(u)}{u} \\ &\quad \times \sum_{i=1}^{\infty} \left\{ p_{i-1}(x) + \frac{a^2}{2} \frac{\partial^2}{\partial x^2} p_{i-1}(x) \right. \\ &\quad \left. - \frac{a^2}{2} \frac{\partial}{\partial x} \left[ p_{i-1}(x) \frac{F(x)}{k_bT} \right] \dots \right\} \hat{\psi}^i(u). \end{aligned} \quad (12)$$

We notice that according to Eq. (7)

$$\frac{1 - \hat{\psi}(u)}{u} \sum_{i=1}^{\infty} p_{i-1}(x) \hat{\psi}^i(u) = \hat{P}(x, u) \hat{\psi}(u) \quad (13)$$

and hence from Eq. (12) we obtain

$$\hat{P}(x,u) = \frac{1-\hat{\psi}(u)}{u} \delta(x) + \hat{\psi}(u) \left\{ \hat{P}(x,u) + \frac{a^2}{2} \frac{\partial^2}{\partial x^2} \hat{P}(x,u) - \frac{a^2}{2} \frac{\partial}{\partial x} \left[ \hat{P}(x,u) \frac{F(x)}{k_b T} \right] + \dots \right\}. \quad (14)$$

We now introduce the waiting time probability density function, which for large  $\tau$  behaves as

$$\psi(\tau) \sim \frac{\alpha A_\alpha}{\Gamma(1-\alpha)\tau^{1+\alpha}}, \quad (15)$$

where  $\alpha \leq 1$ . In Laplace  $u$  space the waiting time probability density function behaves as

$$\hat{\psi}(u) = 1 - A_\alpha u^\alpha + c_1 (A_\alpha u^\alpha)^2 + \dots \quad (16)$$

when  $u$  is small. When  $\alpha < 1$  the first moment of the waiting times diverges. Inserting Eq. (16) into Eq. (14) we find

$$\begin{aligned} \hat{P}(x,u) = & \frac{A_\alpha u^\alpha - c_1 (A_\alpha u^\alpha)^2 + \dots}{u} \delta(x) + [1 - A_\alpha u^\alpha \\ & + c_1 (A_\alpha u^\alpha)^2 \dots] \left\{ \hat{P}(x,u) + \frac{a^2}{2} \frac{\partial^2}{\partial x^2} \hat{P}(x,u) \right. \\ & \left. - \frac{a^2}{2} \left[ \hat{P}(x,u) \frac{F(x)}{k_b T} \right] + \dots \right\}. \quad (17) \end{aligned}$$

We are now practically ready to derive the FFPE, but must first specify the limit in which this equation is derived.

Consider the limit  $a \rightarrow 0$ . In the standard diffusion approximation such a limit is meaningful only when both the mean waiting time and the lattice spacing  $a$  approach zero. For those cases when the mean waiting time diverges the standard limit of the diffusion approximation breaks down. We take  $a \rightarrow 0$  and  $A_\alpha \rightarrow 0$ , while the ratio

$$\lim_{a^2 \rightarrow 0, A_\alpha \rightarrow 0} \frac{a^2}{2A_\alpha} = K_\alpha \quad (18)$$

is kept finite.  $K_\alpha$  is the generalized diffusion coefficient whose units are  $[\text{m}^2]/[\text{sec}]^\alpha$ . When  $\alpha = 1$ ,  $K_1 = a^2/(2\langle\tau\rangle)$ , and  $\langle\tau\rangle = A_1$ . The latter is the finite mean waiting time as expected for this normal case.

Multiplying Eq. (17) by  $A_\alpha^{-1} u^{-\alpha}$  and using the limiting procedure defined in Eq. (18) we find that

$$\hat{P}(x,u) - \delta(x)/u = K_\alpha u^{-\alpha} L_{fp} \hat{P}(x,u), \quad (19)$$

where

$$L_{fp} \equiv \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{F(x)}{k_b T} \quad (20)$$

is the well known Fokker-Planck operator. Eq. (19) can be rewritten in  $t$  space in terms of the fractional Riemann-Liouville operator [30] as

$$P(x,t) - \delta(x) = {}_0D_t^{-\alpha} L_{fp} P(x,t). \quad (21)$$

The fractional operator in Eq. (21) is defined by

$${}_0D_t^{-\alpha} Z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t dt' \frac{Z(t')}{(t-t')^{1-\alpha}} \quad (22)$$

for  $0 < \alpha < 1$  and in Laplace space

$$\int_0^\infty e^{-ut} [{}_0D_t^{-\alpha} Z(t)] dt = u^{-\alpha} \hat{Z}(u). \quad (23)$$

When  $F(x) = 0$  Eq. (21) reduces to the result in Ref. [20].

Differentiating Eq. (21) with respect to time gives

$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha} L_{fp} P(x,t). \quad (24)$$

with the fractional derivative  ${}_0D_t^{1-\alpha}$  defined by

$${}_0D_t^{1-\alpha} Z(t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t dt' \frac{Z(t')}{(t-t')^{1-\alpha}}. \quad (25)$$

As mentioned in the Introduction Eq. (24) is the fractional Fokker-Planck equation. It reduces to the ordinary Fokker-Planck equation (1), when  $\alpha = 1$ , while for  $\alpha < 1$  it describes subdiffusive processes.

Equations (21) and (24) are initial value problems. While Eq. (21) is defined with a single initial condition (the delta function on its left-hand side), in solving Eq. (24) two initial conditions have to be specified [30], these being  $P(x,t=0)$  and  ${}_0D_t^{-\alpha} P(x,t)|_{t=0}$  [31]. When setting  ${}_0D_t^{-\alpha} P(x,t)|_{t=0}$  to zero the two equations are equivalent. Finally, we note that the derivation of the FFPE in dimensions higher than  $D = 1$  follows exactly the same lines specified in this section.

#### IV. FFPE VS CTRW IN THE FORCE FREE CASE

In the previous section we have derived the FFPE from the CTRW. Generally the two approaches are not identical. One should expect, however, that the solutions of both the processes coincide in a certain scaling regime valid for large  $r$  and  $t$ . In this section we compare between the FFPE and the CTRW for the force free case,  $F(x) = 0$ , in dimensions  $D = 1, 2, 3$ .

Schneider and Wyss [21] have found the exact solution of the FFPE in terms of a Fox function [32,33]. Using the dimensionless equation

$$\dot{P}(\vec{r},t) = {}_0D_t^{1-\alpha} \nabla^2 P(\vec{r},t) \quad (26)$$

with the initial condition  $P(\vec{r},t=0) = \delta(\vec{r})$  the solution in  $(\vec{k},u)$  Fourier-Laplace space is

$$P(\vec{k},u) = \frac{u^{\alpha-1}}{u^\alpha + k^2}. \quad (27)$$

Using Eq. (27) it is straightforward to show that the mean square displacement of the particle follows

$$\langle r^2 \rangle = \frac{2D}{\Gamma(1+\alpha)} t^\alpha. \quad (28)$$

Using the Mellin transform it can be shown that [21]

$$P(r,t) = \alpha^{-1} \pi^{-D/2} r^{-D} H_{12}^{20} \times \left( 2^{-2/\alpha} r^{2/\alpha} t^{-1} \left| \begin{matrix} (1,1) \\ (D/2, 1/\alpha), (1, 1/\alpha) \end{matrix} \right. \right), \quad (29)$$

where  $H_{12}^{20}$  is a Fox function (for a different method of solution, see [34]). The asymptotic expression for this Green function is

$$P(r,t) \sim \kappa^\alpha r^{-D} \xi^{D/2(2-\alpha)} \exp(-\lambda_1 \xi^{1/(2-\alpha)}), \quad (30)$$

where  $\xi \equiv r^2/t^\alpha$  is the scaling variable

$$\kappa^\alpha = \pi^{-d/2} 2^{-d/(2-\alpha)} (2-\alpha)^{-1/2} \alpha^{[\alpha(d+1)/2-1]/(2-\alpha)} \quad (31)$$

and

$$\lambda_1 = (2-\alpha) \alpha^{\alpha/(2-\alpha)} 2^{-2/(2-\alpha)}. \quad (32)$$

Equation (30) is valid for  $\xi \gg 1$ . The behavior of  $P(r=0,t)$  valid for  $\xi \ll 1$  is

$$P(r,t) \sim \begin{cases} \frac{1}{\Gamma(1-a/2) t^{\alpha/2}} & D=1 \\ \frac{1}{\pi \Gamma(1-\alpha) t^\alpha} \ln(t^{\alpha/2}/r) & D=2 \\ r^{-1} & D=3 \\ \frac{1}{4\pi \Gamma(1-\alpha) t^\alpha} & \end{cases} \quad (33)$$

We see that in two and three dimensions the FFPE solution is singular on the origin. Equation (33) was derived independently by Saichev [35]. A small  $\xi$  expansion of the  $P(r,t)$  is given in the Appendix for  $D=1$ .

Let us now analyze the CTRW result. We consider the  $D$  dimensional CTRW in continuum space (the extension to lattice walks is straightforward). The probability density function of jump lengths is denoted  $f(\vec{r})$  and its Fourier transform by  $f(\vec{k})$ . We use an unbiased CTRW and assume an existing variance  $\sigma$  of the jump length distribution. In this case  $f(\vec{k}) = 1 - \sigma^2 k^2/2 + \dots$  for small  $k$ . The CTRW solution in  $(\vec{r}, u)$  space is written as a sum of two terms

$$P(\vec{r}, u) = \frac{1 - \hat{\psi}(u)}{u} \delta(\vec{r}) + \frac{1 - \hat{\psi}(u)}{u} \frac{1}{(2\pi)^D} \times \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{f(\vec{k}) \exp(-i\vec{r} \cdot \vec{k})}{1 - f(\vec{k}) \hat{\psi}(u)} d^D k. \quad (34)$$

The first term on the right-hand side (RHS) of this equation is a result of random walks where the particle is trapped on its initial location during the time interval  $(0, t)$ . No such singular component appears in the solution of the FFPE Eq. (29). Since the Fokker-Planck operator contains derivatives of finite order, the CTRW singularity at the origin does not appear in the FFPE solution.

According to Eqs. (3), (4), and (16) the inverse Laplace transform of the first singular term in Eq. (34) is

$$W(t) \delta(\vec{r}) \sim \frac{A_\alpha t^{-\alpha}}{\Gamma(1-\alpha)} \delta(\vec{r}). \quad (35)$$

Clearly only for times  $A_\alpha t^{-\alpha}/\Gamma(1-\alpha) \ll 1$  can we expect the FFPE solution and the CTRW solution to coincide. Also notice that within the FFPE framework and for  $D=2,3$ ,  $P(r,t)$  for  $\xi \ll 1$  Eq. (33), decays not faster than the singular term in the CTRW solution. Therefore for on the origin and for  $D>1$  the two solutions behave differently even when  $t \rightarrow \infty$ . In contrast, for normal random walks, the singular term decays exponentially with time and then the diffusion approximation works well already after an exponentially short time.

The CTRW singular term is especially important for problems with a boundary condition [36]. This term can be used to find a lower bound on  $S(t)$ —the probability that a particle which at  $t=0$  was at the origin has not crossed a closed boundary until time  $t$ ; clearly

$$S(t) \geq W(t) \sim \frac{A_\alpha t^{-\alpha}}{\Gamma(1-\alpha)}. \quad (36)$$

Notice that this simple relation is valid for all dimensions  $D$  and is independent of the shape of the boundary of the domain. Such a result cannot be derived based upon the FFPE modeling. In the FFPE there is only a single parameter, the diffusion coefficient  $K_\alpha$ , and so based on dimensional analysis one can easily see that a bound like Eq. (36) cannot be found based on this approach.

We now consider the second term on the RHS of (34). For large  $\vec{r}$  and  $t$  one can use the small  $\vec{k}$  and  $u$  values to find an approximate solution of the CTRW. For convenience and without loss of generality we use  $\sigma^2=2$  and  $A_\alpha=1$  then

$$P(\vec{r}, u) \sim \frac{u^{\alpha-1}}{(2\pi)^d} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\exp(-i\vec{r} \cdot \vec{k})}{u^\alpha + k^2} d^D \vec{k}. \quad (37)$$

As noted already by Compte [37,38] the small  $(\vec{k}, u)$  behavior of the CTRW, Eq. (37) is identical to the solution of the FFPE Eq. (27). The exact solution of this equation is given in Eq. (29). An approximate normalized solution of the integral Eq. (37) has been given in Refs. [6,23]

$$P(r,t) \approx N r^{-D} (r^{D-1}) \xi^{D/2(2-\alpha)} \exp(-\lambda_1 \xi^{1/(2-\alpha)}) \quad (38)$$

and  $N$  is a normalization coefficient. This result [6] was derived based upon the steepest descent method for  $\xi \gg 1$ .

The approximate solution, Eq. (38), derived within the CTRW framework, is compared with the exact solution (29) found within the framework of the FFPE. First we notice that the stretched exponential term in Eq. (38) is identical to the stretched exponential term in Eq. (30). The exact FFPE result has a scaling form in terms of

$$P(r,t) r^D \sim G_1(\xi) \quad (39)$$

while the CTRW approximation Eq. (38), for  $D \neq 1$ , has a different scaling form

$$P(r,t) r^D (r^{1-D}) \sim N G_2(\xi). \quad (40)$$

In the scaling regime  $\xi \rightarrow \infty$  the two functions  $G_1(\xi)$  and  $G_2(\xi)$  coincide, however, on the LHS of Eq. (40) appears a prefactor  $r^{1-D}$  which does not exist in Eq. (39).

Why does the approximate solution, Eq. (38) or Eq. (40), derived within the CTRW framework, deviate from scaling? The derivation in [6,23] follows two steps. First the asymptotic large  $\xi$  solution of Eq. (37),  $P_{\text{ass}}(r,t)$  is found using the method of steepest descent. This method (see details in Refs. [6,23]) gives the correct scaling result. Then a normalization condition is imposed on the asymptotic result

$$P(r,t) \simeq \frac{P_{\text{ass}}(r,t)}{\int P_{\text{ass}}(r,t) d^D r}. \quad (41)$$

This second step leads to deviation from the scaling property of the solution. Imposing a normalization condition on an asymptotic expression gives the  $r^{1-D}$  term in Eq. (40) which does not exist in the exact FFPE solution Eq. (39).

Finally we remind the reader of the asymptotic (nonsingular in  $\vec{r}$ ) behavior of the CTRW solution at the origin [6,23,24]

$$P(r=0,t) \sim \begin{cases} 1/t^{\alpha/2} & D=1, \\ \ln(t)/t^\alpha & D=2, \\ 1/t^\alpha & D=3. \end{cases} \quad (42)$$

Comparing this solution with the FFPE modeling Eq. (33) we see that for  $D \neq 1$  the behaviors of the Green functions at the origin are not identical.

## V. DISCUSSION AND CONCLUSIONS

Let us briefly discuss some of the properties of the FFPE, Eq. (24). At  $t \rightarrow \infty$  thermal equilibrium is reached and then

$$\lim_{t \rightarrow \infty} P(x,t) = N \exp\left[-\frac{V(x)}{k_b T}\right], \quad (43)$$

$N$  being the normalization and  $V(x)$  is the potential field. For a constant field  $F(x) = F$  the solution of the FFPE in  $(k,u)$  space is

$$\hat{P}(k,u) = \frac{u^{\alpha-1}}{u^\alpha + K_\alpha k^2 - iK_\alpha Fk/(k_b T)}. \quad (44)$$

The inverse Fourier-Laplace transform of this equation has been analyzed extensively within the biased CTRW [6]. We have recently shown [18] that the FFPE is consistent with the generalized Einstein relation [39–41] (i.e., linear response theory). The relaxation of modes of the FFPE follow a Mittag-Leffler decay [42], with a power-law tail, which replaces the ordinary exponential decay found in the linear Fokker-Planck equation (1). We have also found a solution for the harmonic oscillator and showed how to use techniques [5] developed for solving the ordinary Fokker-Planck equation to solve the FFPE.

From a physical point of view it has been shown that some models exhibiting anomalous diffusion (including the CTRW) are sensitive to initial conditions even in the long

time limit [43–45]. Unlike normal transport processes, transport coefficients of anomalous processes can be shown to depend on the way a system has been initially prepared. In this work we have assumed that the CTRW process has started at  $t=0$ . Hence we derived the Riemann-Liouville operator with an integral whose lower limit is  $t=0$ . One could easily imagine other processes going on for a long period of time before starting the observation at  $t=0$ . In the absence of a microscopic time scale such a process could possibly lead to a different type of fractional equation from the one derived here.

Another assumption we have used is that the waiting time density  $\psi(\tau)$  does not depend on the local field  $F(x)$ . In principal  $\psi(\tau)$  could be site dependent due to the breaking of spatial invariance. Our assumption means that the external field is weak and its influence on  $\psi(\tau)$  is negligible. The influence of an external bias on anomalous subdiffusion has been investigated for chaotic deterministic diffusion [15] and for charge carrier transport in disordered media [40]. In these models the dependence of the waiting time probability density function  $\psi(\tau)$  on the linear external field was calculated. A crossover from a power law behavior for short times to an exponential decay for long times has been found. The crossover time diverges as the field becomes weak. This transition has a rather strong influence on the dynamics which switches to a normal Gaussian behavior when the external field is finite and for long times. Thus care must be taken when assuming field dependent waiting times.

As we mentioned, the CTRW solution, Eq. (34) has a singular term which describes random walks for which a particle did not leave the origin during the observation time  $t$ . Such a singular term does not appear in the FFPE. Such a term can be important when modeling anomalous type of diffusion especially for  $D=2,3$ . In contrast, for normal random walks the diffusion approximation works very well after exponentially short times and then it is justified to neglect the singular term.

To conclude, expansion of the CTRW in an external force field leads to the familiar Fokker-Planck equation when the mean waiting time is finite. When this time diverges we obtain a fractional non-Markovian Fokker-Planck equation.

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## APPENDIX

The solution of the FFPE with  $F(x)=0$ , for  $D=1$  and  $K_\alpha=1$  in  $(x,u)$  space is

$$\hat{P}(x,u) = \frac{u^{\alpha/2-1} \exp(-|x|u^{\alpha/2})}{2}. \quad (A1)$$

The inverse Laplace transform of Eq. (A1) is

$$P(x,t) = \frac{1}{\alpha} \left( \frac{t}{|x|^{1+2/\alpha}} \right) l_{\alpha/2} \left( \frac{t}{|x|^{2/\alpha}} \right). \quad (\text{A2})$$

$l_{\alpha/2}(z)$  is a one sided Lévy stable density whose Laplace transform follows

$$l_{\alpha/2}(u) = \int_0^\infty e^{-uz} l_{\alpha/2}(z) dz = \exp(-u^{\alpha/2}). \quad (\text{A3})$$

For  $\alpha=1$

$$l_{1/2}(z) = \frac{2}{\sqrt{\pi}} \frac{1}{(2z)^{-3/2}} \exp\left(-\frac{1}{2z}\right) \quad (\text{A4})$$

and it can be checked that  $P(|x|,t)$  is Gaussian. For  $\alpha=2/3$ ,

$$P(x,t) = 3^{2/3} t^{-1/3} \text{Ai}[|x|/(3t)^{1/3}], \quad (\text{A5})$$

where  $\text{Ai}(x)$  is the Airy function. Taylor expanding the exponential in Eq. (A1), and transforming  $u \rightarrow t$ , we find

$$P(x,t) = \frac{1}{2t^{\alpha/2}} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{n/2}}{n! \Gamma[1 - \alpha(n+1)/2]}. \quad (\text{A6})$$

The expansion is valid for small  $\xi$ . In Eq. (A6) terms with  $\alpha(n+1)/2$  an integer should be omitted from the series. Hence for  $\xi \ll 1$  the Green function, for  $\alpha < 1$ , decays according to

$$P(x,t) \approx \frac{1}{2\Gamma(1 - \alpha/2)t^{\alpha/2}} \exp(-c|x|/t^{\alpha/2}), \quad (\text{A7})$$

where  $c = \Gamma(1 - \alpha/2)/\Gamma(1 - \alpha)$ .

Let us now show the relation between the solution Eq. (A1), expansion Eq. (A6), and the FFPE results obtained in the literature in terms of Fox functions. The result obtained by Schneider and Wyss in Ref. [21], for one dimension

$$P(x,t) = \frac{1}{\sqrt{\alpha^2 \pi}} \frac{1}{|x|} H_{1,2}^{2,0} \left[ \frac{|x|^{2/\alpha}}{2^{2/\alpha} t} \left| \begin{matrix} (1,1) \\ \left(\frac{1}{2}, \frac{1}{\alpha}\right), \left(1, \frac{1}{\alpha}\right) \end{matrix} \right. \right] \quad (\text{A8})$$

can be shown, by simple manipulations employing the properties of the Fox function [32], to be equivalent to

$$P(x,t) = \frac{1}{\sqrt{16\pi t^\alpha}} H_{1,2}^{2,0} \left[ \frac{|x|}{2t^{\alpha/2}} \left| \begin{matrix} \left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right) \\ \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix} \right. \right]. \quad (\text{A9})$$

By using the duplication rule of the Gamma function, one can show that Eq. (A9) can be simplified to

$$P(x,t) = \frac{1}{\sqrt{4t^\alpha}} H_{1,1}^{1,0} \left[ \frac{|x|}{t^{\alpha/2}} \left| \begin{matrix} \left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right. \right]. \quad (\text{A10})$$

Relation (A10) can also be obtained by Laplace inversion of Eq. (A1). The theorem for the series expansion of the Fox function [32] then results in Eq. (A6).

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$$\int_0^{\infty} dt e^{-ut} [{}_0D_t^\alpha Z(t), u] = u^\alpha \hat{Z}(u) - \sum_{k=0}^{n-1} u_0^k D_t^{\alpha-1-k} Z(t)|_{t=0},$$

$n$  is an integer satisfying  $n-1 < \alpha \leq n$ . Notice that for fractional integrals  $\alpha \leq 0$  the sum in the last equation vanishes.

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