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From Diagrammar to Diagrammalgebra

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Analytic and algebraic properties of Feynman integrals are investigated within the de Rham theory for twisted co-homology. Linear relations, equivalent to integration-by-parts identites, differential and difference equations, as well as quadratic relations are derived by projections, using the intersection numbers. The presented results apply to the general class of Aomoto-Gel'fand-Euler integrals.

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1. Introduction

Feynman integrals play a fundamental role in Classical and Quantum Field Theory, since they naturally emerge in the perturbative calculations of Scattering Amplitudes and related quantities - such as cross sections, impact parameters, scattering angles, interaction potential, to name a few -, relevant to physicists [1]. Mathematically, bounded integrals represent fluxes of differential forms through manifolds with boundaries and they can express invariant quantities, whose identification constitute (one of) the the main objective of the research in theoretical physics, finally yielding to conservation laws: if a quantity remains unaltered under the action of a given transformation, then the considered transformation is a *symmetry* of that quantity.

For bounded integrals there are two types of transformations that can be naturally accounted for: modifying either its integrand, by shifting a differential form, or its integration domain, by deforming its boundaries. If, upon any of these transformations the values of the integral does not change, then the original integral and its modified copies are *equivalent*, as they would produce the same result, hence they would give the same physical effect, appearing as indistinguishable to an observer. For these reasons, the properties of symmetry, or invariance of integrals are naturally related to the mathematical concept of *equivalence classes* [2]. The analytic properties of Feynman integrals, as well as of all integrals, falling in the rather wide family of the so-called Aomoto-Gel'fand integrals [3–5] - also including the Gauss hypergeometric integrals, their multi fold generalizations, and the Euler integrals - can be studied by means of techniques developed in Differential and Algebraic Topology (of compact Riemann surfaces): therein, homology and co-homology classes are the vector spaces of integration contours and of differential forms, which leave the result of the integration invariant.

During the developments of the S-Matrix theory, it was recognized that topology and co-homological methods offered a clear view on the connection between the analytic properties of Feynman integrals and the geometry dictated by their singularity structure [6–8]. The appearence of special transcendental functions, of multiple Riemann zeta values and periods of algebraic varieties, in the results of the analytic evaluations of multi-loop integrals triggered the investigation of scattering amplitudes with Number Theory based concepts, yielding the developments of a Motivic theory for Feynman integrals, based on periods of Mellin moments [9–21]. Over the years, the investigation of similarities between hypergeometric integrals and Feynman integrals has revealed interesting structures (see [22, 23] for recent reviews, and references therein).

In more recent studies, the role of co-homology has been found to be pivotal for identifying novel properties of Feynman integrals and to expose the deep connections of scattering amplitudes and graph polynomials that naturally emerge in the parametric representations of Feynman integrals [24–43], see [44, 45], for recent reviews.

In particular, the *vector space structure* of Feynman integrals has emerged through a series of articles [27, 30–35], which exploited the intersection numbers for twisted de Rham co-homology groups [3, 4, 25, 26, 46–62], see [63–68] in these Proceedings, As observed in [27], intersection numbers can be employed to build two representations of the resolution of the identity, respectively in the homology and co-homology spaces, which are the dawn of the existence of linear and quadratic relations for Feynman integrals, and more generally, for Aomoto-Gel'fand integrals. The linear relations yields *contiguity relations*, that, in the case of Feynman integrals, are equivalent to

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the known *integration-by-parts identities* (IBPs) [69, 70]: *intersection numbers* of differential forms [50] can be employed to define a *scalar product* of Feynman integrals, so that the projection of any multi-loop integral onto a basis of independent elements, called *master integrals* (MIs) becomes conceptually identical to the projection of a vector onto a basis of a vector space [27, 31, 33, 35]: within this approach, it becomes natural to interpreting the finite number of MIs [71] as the dimension of the vector space of Feynman integrals, and can be related to topological quantities such as the number of critical points [24], Euler characteristics [33, 72–75], as well as to the dimension of quotient rings of polynomials, for zero dimensional ideals, in the context of computational algebraic geometry [35].

The linear relations among integrals also allow for the construction of differential equations [76–87], as well as finite difference equations [70, 88, 89] obeyed by the MIs, and they can be used for the actual evaluation of the latter. Additionally, the quadratic relations correspond to the twisted version of the bilinear Riemann relations, therefore called *twisted Riemann periods relations* (TRPR) [50], see [64, 65] in these Proceedings. In the case of Feynman integrals, quadratic relations have been independently found within the application of number-theoretic methods to Feynman calculus [19–21, 40, 41, 90, 91], see [92] in these Proceedings, and in the study of special differential equations [93]. Understanding whether these types of bilinear relations are equivalent to the TRPR is an interesting open question. Additional applications of algebraic and differential topology to Scattering Amplitudes in QFT and String theory are presented in [68, 94–97].

In the following pages, I will briefly introduce the topic of intersection theory for twisted de Rham co-homology and its application to Feynman integrals, which will be further discussed in these Proceedings, in particular in [92, 98–101].

2. Feynman integrals and Intersection Theory: a smooth invitation

Let $\mathcal{B}_i$, with $i = 1, \ldots, m$, be complex polynomials in the variables $z = \{z_1, \ldots, z_n\}$. We introduce an oriented manifold $M = \mathbb{C}^n - \cup_{i=1}^m \mathcal{S}_i$, where the hypersurfaces $\mathcal{S}_i$ are identified by the equations $\mathcal{B}_i = 0$. We introduce the Aomoto-Gel'fand *integrals*, namely the subject of our investigation, defined as *twisted period integrals*,

$$\int_{\Gamma_n} u\, \varphi_n \,, \tag{1}$$

where: u is a multivalued function, which regulates the integral, called *twist*; Γ_n is a *regularised cycle* called *twisted or loaded cycle*, *i.e.* a n-chain with empty boundary on M (usually Γ_n is represented as $\Gamma_n \equiv \gamma_n \otimes u$, to indicate the pure integration path γ_n together with a specific choice of the branch along it, required because u is multivalued); φ_n is a meromorphic differential n-form defined on M, called *twisted cocycle*. In general u is written as product of factors, $u = \prod_{i=1}^m \mathcal{B}_i^{\alpha_i}$, raised to non-integer powers α_i, with the requirement that u vanishes on the integration boundary, say $u(\partial\Gamma_n) = 0$. The latter property ensures that the integral of any total differential vanishes:

$$0 = \int_{\Gamma_n} d(u\, \varphi_{n-1}) = \int_{\Gamma_n} u\, \nabla_\omega\, \varphi_{n-1} \,, \tag{2}$$

where we introduced the *covariant derivative*

$$\nabla_\omega = d + \omega\wedge = u^{-1} \cdot d \cdot u \,, \qquad \text{with} \quad \omega = d\log(u) \,, \tag{3}$$

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for any generic $(n-1)$-form φ_{n-1}.

Aomoto-Gel'fand *integrals* represent rather a wide class of integrals, such as Gauss hypergeometric functions, Lauricella functions, and their generalizations, Euler-type integrals, and Feynman integrals [27]. The considered class of integrals are invariant, under the following transformations:

- either shifting the differential n-form, by a term containing a covariant derivative, *i.e.* $\varphi_n \to \varphi_n + \nabla_\omega \varphi_{n-1}$;
- or shifting the integration domain, by a pure boundary term (containing no holes), *i.e.* $\Gamma_n \to \Gamma_n + \partial\Gamma_{n+1}$;

namely,

$$\int_{\Gamma_n} u\,\varphi_n = \int_{\Gamma_n} u\,(\varphi_n + \nabla_\omega\,\varphi_{n-1}) = \int_{\Gamma_n+\partial\Gamma_{n+1}} u\,\varphi_n\,, \tag{4}$$

Similar results are obtained also for the so called *dual integrals*, obtained from the integrals defined above by replacing $u \to u^{-1}$ (and correspondingly $\omega \to -\omega$, in the definition of the covariant derivative).

In the case of Feynman integrals, according to the chosen parametric representation, the factors B_i that appear in u are identified with (or built out of) the graph polynomial(s) and the denominators. For these set of functions, analyticity, unitarity, and algebraic structure are related to the geometry captured by the *Morse function* $h \equiv Re(\log(u))$.

The multivalued twist u carries informations on the *regularization*: for dimensionally regulated Feynman integrals, it depends on the integration variables as well as on external scales, such as Mandelstam invariants and masses (all appearing in the polynomials B_i), and on the space time d (appearing in the α_i). The topological information of integrals and dual integrals are contained in ω that is a differential form with *zeroes* and *poles*, collected in the respective sets,

$$\mathbb{Z}_\omega = \{\text{zeroes of } \omega\}\,, \qquad \text{and} \qquad \mathbb{P}_\omega = \{\text{poles of } \omega\} \cup \{\infty\}\,. \tag{5}$$

The invariance of integrals and dual integrals under the two types of transformation mentioned above can be exploited to expose the algebraic structure of Aomoto-Gel'fand integrals. Let us introduce four vector spaces, for twisted cycles and cocycles: the *de Rham* n-th *homology group*,

$$H_n^\omega = \frac{\mathrm{Ker}(\partial : \Gamma_{n+1} \to \Gamma_n)}{\mathrm{Im}(\partial : \Gamma_n \to \Gamma_{n-1})}\,, \tag{6}$$

and the *de Rham* n-th *co-homology group*,

$$H^n_\omega = \frac{\mathrm{Ker}(\nabla_\omega : \varphi_n \to \varphi_{n+1})}{\mathrm{Im}(\nabla_\omega : \varphi_{n-1} \to \varphi_n)}\,, \tag{7}$$

which is the quotient space of closed n-forms, $(\varphi_n \,|\, \nabla_\omega\varphi_n = 0)$ modulo exact forms $(\varphi_n \,|\, \varphi_n = \nabla_\omega\varphi_{n-1})$; and their dual spaces, $(H_n^\omega)^* = H_n^{-\omega}$, and $(H^n_\omega)^* = H^n_{-\omega}$, respectively. These spaces are isomorphic, and their dimension ν,

$$\nu \equiv \dim(H^n_{\pm\omega}) = \dim(H_n^{\pm\omega})\,, \tag{8}$$

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can be determined by counting the number of *critical points* of $\mathcal{B}$, namely $\nu = \dim(\mathbb{Z}_\omega)$ [24], or equivalently from the Euler characterisics $\chi(\mathbb{P}_\omega)$ of the projective variety generated by the poles of ω, as $\nu = (-1)^n(n+1-\chi(\mathbb{P}_\omega))$ [33], see also [74], or by the Shape Lemma [35].

The corresponding elements, generically denoted as $\langle\varphi_L| \in H^n_\omega$, $|\varphi_R\rangle \in H^n_{-\omega}$, $[C_L| \in H^n_\omega$, $|C_R] \in H^n_{-\omega}$, can be used to define four types of natural *twisted Poincarè pairings*:

- Integrals:
$$I = \langle\varphi_L|C_R] \equiv \int_{C_R} u\,\varphi_L \; ; \tag{9}$$
- Dual Integrals:
$$\tilde{I} = [C_L|\varphi_R\rangle \equiv \int_{C_L} u^{-1}\,\varphi_R \; ; \tag{10}$$
- Intersection numbers for twisted cycles (or topological intersection numbers):
$$[C_L|C_R] \; ; \tag{11}$$
- Intersection numbers for twisted cocycles
$$\langle\varphi_L|\varphi_R\rangle \equiv \int_M (u\,\varphi_L)\wedge(u^{-1}\,\varphi_R) = \int_M \varphi_L \wedge \varphi_R \; . \tag{12}$$
where $\varphi_{L,R}$ are understood to have compact support on M.

2.1 Linear and Bilinear Relations

Consider the following bases generating the four spaces introduced above: $\{\langle e_i|\}_{i=1,\dots,\nu} \in H^n_\omega$ and $\{|h_i\rangle\}_{i=1,\dots,\nu} \in H^n_{-\omega}$, respectively for the cohomology and for the dual cohomoloygy spaces; as well as, $\{[\gamma_i|\}_{i=1,\dots,\nu} \in H_n^\omega$, and $\{|\eta_i]\}_{i=1,\dots,\nu} \in H_n^{-\omega}$, respectively for the homology and for the dual homoloygy spaces. The bases of cocycles $\{\langle e_i|\}_{i=1,\dots,\nu} \in H^n_\omega$ and $\{|h_i\rangle\}_{i=1,\dots,\nu} \in H^n_{-\omega}$, can be used to express the identity operator in the cohomology space as [27, 31],

$$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j| \tag{13}$$

where we defined the *metric matrix*

$$\mathbf{C}_{ij} \equiv \langle e_i|h_j\rangle \; , \tag{14}$$

whose elements are *intersection numbers* of the twisted basic forms. Similarly, by using the bases of cycles $\{[\gamma_i|\}_{i=1,\dots,\nu} \in H_n^\omega$ and $\{|\eta_i]\}_{i=1,\dots,\nu} \in H_n^{-\omega}$, the resolution of the identity in the homology space reads as,

$$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] \left(\mathbf{H}^{-1}\right)_{ij} [\eta_j| \, , \tag{15}$$

where

$$\mathbf{H}_{ij} \equiv [\eta_i|\gamma_j] \; , \tag{16}$$

is the metric matrix, in terms of intersection numbers of the basic twisted cycles.

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The identity resolutions $\mathbb{I}_c$ and $\mathbb{I}_h$ can be derived purely algebraically, as in [27, 31]; also, in the context of differential topology, the bilinear Riemann relations for periods of holomorphic differentials, see f.i. [44], can be suitably expressed in order to identify $\mathbb{I}_h$ (for non twisted-forms), as shown later.

Linear and bilinear relations for Aomoto-Gel'fand-Feynman integrals, as well as the differential equations and the finite difference equation they obey are a consequence of the purely algebraic application of the identity operators defined above [27].

In the context of Feynman integrals calculus, the decomposition of scattering amplitudes in terms of MIs, as well as the equations obeyd by the latter, are derived by means of IBPs [69] and of the Laporta method [70]. In the following, we show how these relations emerge by employing the algebraic properties of twisted cycles and co-cycles.

2.1.1 Linear Relations

- **Decomposition of differential forms**. Generic twisted cocycles and dual twisted cocycles can be projected onto the bases in the correspsonding vector spaces as,

$$\langle\varphi_L| \;=\; \langle\varphi_L|\mathbb{I}_c = \sum_{i=1}^{\nu} c_i\,\langle e_i|\,, \quad \text{with} \quad c_i = \sum_{j=1}^{\nu}\langle\varphi_L|h_j\rangle\left(\mathbf{C}^{-1}\right)_{ji}\;; \tag{17}$$

$$|\varphi_R\rangle \;=\; \mathbb{I}_c|\varphi_R\rangle = \sum_{i=1}^{\nu}\tilde{c}_i\,|h_i\rangle\,, \quad \text{with} \quad \tilde{c}_i = \sum_{j=1}^{\nu}\left(\mathbf{C}^{-1}\right)_{ij}\langle e_j|\varphi_R\rangle\,. \tag{18}$$

  The latter two formulas, dubbed *master decomposition formulas* for (dual) twisted cocycles [27, 31], imply that the decomposition of any (dual) Aomoto-Gel'fand-Feynman integral can be expressed as linear combination of (dual) master integrals is an algebraic operation, similar to the decomposition/projection of any vector within a vector space, which can be carried out by computing intersection numbers of twisted de Rham differential forms.

- **Integral decomposition (1)**. By using the master decomposition formulas of forms and dual forms, integrals and dual integrals can be straightforwardly written as,

$$I = \langle\varphi_L|\mathcal{C}_R] = \sum_{i=1}^{\nu} c_i\,J_i\,, \qquad \text{and} \qquad \tilde{I} = [\mathcal{C}_L|\varphi_R\rangle = \sum_{i=1}^{\nu}\tilde{c}_i\,\tilde{J}_i\,, \tag{19}$$

  respectively in terms the MIs $J_i = \langle e_i|\mathcal{C}_R]$, and of the dual MIs $\tilde{J}_i = [\mathcal{C}_L|h_i\rangle$, for $i = 1,\ldots,\nu$.

- **Decomposition of integration contours**. Equivalently, using the resolution of the identity in the homology space. twisted cycles and dual twisted cycles can be projected onto the bases in the corresponding vector spaces as,

$$|\mathcal{C}_R] \;=\; \mathbb{I}_h|\mathcal{C}_R] = \sum_i a_i\,|\gamma_i]\,, \quad \text{with} \quad a_i = \sum_{j=1}^{\nu}\left(\mathbf{H}^{-1}\right)_{ij}[\eta_j|\mathcal{C}_R]\,, \tag{20}$$

$$[\mathcal{C}_L| \;=\; [\mathcal{C}_L|\,\mathbb{I}_h = \sum_i \tilde{a}_i\,[\eta_i|\,, \quad \text{with} \quad \tilde{a}_i = \sum_{i=1}^{\nu}[\mathcal{C}_L|\gamma_j]\left(\mathbf{H}^{-1}\right)_{ji} \tag{21}$$

  The latter two formulas are dubbed *master decomposition formulas* for (dual) twisted cycles, and may lead to alternative decomposition of integrals and dual integrals.

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- **Integral decomposition (2)**. By using the master decomposition formulas of contours and dual contours, integrals and dual integrals can be straightforwardly written as,

$$I = \langle \varphi_L | C_R] = \sum_{i=1}^{\nu} a_i \, J_i' \,, \qquad \text{and} \qquad \tilde{I} = [C_L | \varphi_R \rangle = \sum_{i=1}^{\nu} \tilde{a}_i \, \tilde{J}_i' \,, \tag{22}$$

respectively in terms the MIs $J_i' = \langle \varphi_L | \gamma_i]$, and of the dual MIs $\tilde{J}_i' = [\eta_i | \varphi_R \rangle$, for $i = 1, \ldots, \nu$.

In the above formulas, $\mathbf{C}$ and $\mathbf{H}$ are $(\nu \times \nu)$-matrices of intersection numbers, which, in general, differ from the identity matrix. For intersections number of orthonormal elelements they turn into unit matrices, hence simplifying the decomposition formulas. The Gram-Schmidt algorithm can be employed to build orthonormal bases from generic sets of independent elements, using the intersection numbers as scalar products. More generally the coefficients appearing in the four types of decomposition formulas for twisted cocycles and cycles and their duals given above are independent of the respective dual elements. Therefore, by exploiting the freedom in choosing the corresponding dual bases may yield striking simplifications [31, 102, 103].

Let me remark that the above discussion and the decomposition formulas defined above hold also in the case of *relative* twisted de Rham theory, namely releasing the non-integer conditions for the exponents α_i that appear in the definition of u [102–104].

2.1.2 Differential Equations

- **Differential Forms**. The identity resolution $\mathbb{I}_c$ can be used to derive the system of differential equation obeyed by the master forms $\langle e_i |$. In fact, let as assume that the u depends on an external variables, say x, then

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma_x) e_i | = \langle (\partial_x + \sigma_x) e_i | \, \mathbb{I}_c = \mathbf{\Omega}_{ij} \, \langle e_j | \,, \tag{23}$$

where the entries of the matrix of the system are $\mathbf{\Omega}_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$, and $\sigma_x \equiv \partial_x \log(u)$.

Following similar steps, the system of differential equations for the master dual forms $|h_i\rangle$ reads,

$$\partial_x | h_i \rangle = | (\partial_x - \sigma_x) h_i \rangle = \mathbb{I}_c | (\partial_x - \sigma_x) h_i \rangle = \tilde{\mathbf{\Omega}}_{ij} | h_j \rangle \,, \tag{24}$$

where the entries of the matrix $\tilde{\mathbf{\Omega}}$ are $\tilde{\mathbf{\Omega}}_{ij} = (\mathbf{C}^{-1})_{jk} \, \langle e_k | (\partial_x - \sigma_x) h_i \rangle$.

- **Master Integrals** Since integrals are obtained by pairing forms and integration contours, the matrices $\mathbf{\Omega}$ and $\tilde{\mathbf{\Omega}}$, whose entries are computed by evaluating intersection numbers, are the matrix of the system of differential equations obeyed by the master integrals J_i and by the dual master integrals $\tilde{J}$, respectively ,

$$\partial_x \, J_i = \Omega_{ij} \, J_j \,, \qquad \partial_x \, \tilde{J}_i = \Omega_{ij} \, \tilde{J}_j \,. \tag{25}$$

- **Intersection Matrices.** The systems of differential equations for forms and dual forms can be used to show that the intersection matrices $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$ satisfy differential equations, known as *secondary equations* [35, 36, 105],

$$\partial_x \mathbf{C} = \mathbf{\Omega} . \mathbf{C} + \mathbf{C} . \tilde{\mathbf{\Omega}} \,, \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} . \mathbf{C}^{-1} - \mathbf{C}^{-1} . \mathbf{\Omega} \,, \tag{26}$$

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where the product between generally non commuting matrices is understood.

Following a similar approach, in the homology space, hence using $\mathbb{I}_h$, it is possible to derive differential equations for (dual) master cycles, $|\gamma_i]$ and $[\eta_i|$, and the secondary equations obeyed the corresponding $\mathbf{H}$ intersection matrix.

2.1.3 Bilinear Relations

Riemann bilinear relations for periods of closed holomorphic (non-twisted) differentials forms, ϕ_L and ϕ_R, see [44] reads as,

$$\langle \phi_L | \phi_R \rangle = \int_\Sigma \phi_L \wedge \phi_R \quad = \quad \sum_{i=1}^{g} \left(\int_{a_i} \phi_L \int_{b_i} \phi_R - \int_{b_i} \phi_L \int_{a_i} \phi_R \right), \tag{27}$$

where Σ is an oriented Riemann surface of genus $g > 0$, built out of a $4g$-gon with edges $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$ (where the exponent ± 1 stands for clock/anticlockwise orientation) and gluing each edge with its inverse. The integration contours a_i and b_i, for $i = 1, \ldots g$, are a canonical bases of cycles, hence intersect *transversally*, *i.e.* their pairwise intersection numbers are: $a_i \cdot a_j = b_i \cdot b_j = 0$, and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$. Riemann bilinear relation can be cast as,

$$\langle \phi_L | \phi_R \rangle = \sum_{i,j}^{2g} \int_{\gamma_i} \phi_L \;\; (\mathbf{H}^{-1})_{ij} \int_{\gamma_j} \phi_R \,, \tag{28}$$

where $\{\gamma_i\}_{i=1,\ldots,g} = a_i$ and $\{\gamma_i\}_{i=g+1,\ldots,2g} = b_i$, and $\mathbf{H}_{ij} = [\gamma_i | \gamma_j]$, namely

$$\mathbf{H} = \begin{pmatrix} 0 & \mathbb{I}_{g\times g} \\ -\mathbb{I}_{g\times g} & 0 \end{pmatrix}, \qquad \text{yielding} \quad \mathbf{H}^{-1} = \begin{pmatrix} 0 & -\mathbb{I}_{g\times g} \\ \mathbb{I}_{g\times g} & 0 \end{pmatrix}, \tag{29}$$

and $\mathbb{I}_{g\times g}$ is the identity matrix in the $(g \times g)$-space.

Bilinear relations can be derived also for the cases of twisted co-cycles. The operators $\mathbb{I}_c$ and $\mathbb{I}_h$ can be inserted in the pairing between twisted (co)cyles, to obtain the following identities:

- **Twisted Riemann Periods Relations.**

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \langle \varphi_L | \mathbb{I}_h | \varphi_R \rangle = \sum_{i,j=1}^{\nu} \langle \varphi_L | \gamma_i] \left(\mathbf{H}^{-1} \right)_{ij} [\eta_j | \phi_R \rangle && (30) \\ [C_L | C_R] &= [C_L | \mathbb{I}_c | C_R] = \sum_{i,j=1}^{\nu} [C_L | h_i \rangle \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j | C_R] \,, && (31) \end{aligned}$$

which are the *Twisted Riemann Period Relations* (TRPR) [50]. TRPR relates intersection numbers for (co)-homologies to products of integrals and dual integrals.

2.1.4 Trilinear Identies

Multiple insertions of the identity resolutions $\mathbb{I}_h$ and $\mathbb{I}_c$ can generate multilinear relations.

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- **Cubic relations.** Consider a generic integral, and let us use the resolution of identity twice, as follows (summation over repeated indices understood, for notation ease)

$$I \quad = \quad \langle \varphi_L | C_R] = \langle \varphi_L | \mathbb{I}_h \, \mathbb{I}_c | C_R] = \langle \varphi_L | \gamma_j] \left(\mathbf{H}^{-1}\right)_{jk} [\eta_k | h_\ell \rangle \left(\mathbf{C}^{-1}\right)_{\ell i} \langle e_i | C_R] \,, \tag{32}$$

whose *r.h.s.* involves two integrals and one dual integral. By rewriting it as,

$$I \quad \equiv \quad g_{ij} \, \langle e_i | C_R] \, \langle \varphi_L | \gamma_j] \,, \qquad \text{with} \quad g_{ij} \equiv \left(\mathbf{H}^{-1}\right)_{jk} [\eta_k | h_\ell \rangle \left(\mathbf{C}^{-1}\right)_{\ell i} \,, \tag{33}$$

it emerges a structure similar to the general formula of *coaction* on integrals $\Delta(I)$ introduced in [106]: it would be interesting to verify if there is an actual correspondence between the two constructions (plausibly subject to a special choices the bases).

3. Intersection Numbers of Twisted Forms

The evaluation of intersection numbers for twisted cycles and cocycles is the key operation required to derive linear and quadratic relations for Aomoto-Gel'fand-Feynman integrals. The intersection numbers for twisted cycles, also known simply as *topological intersection numbers* are discussed in [47], see also [25, 107]. For the application within the linear decomposition and related (differential and finite difference) equations, it is sufficient to limiting our discussion to the intersection numbers for twisted forms. Their systematic derivation can be found in [26, 30, 50, 53], and it is recalled in these Proceedings [98, 99], see also [44, 45]. Let me hereby give a non-rigorous derivation - which I hope it might be useful for those who approach the subject for the first time. I will discuss the evaluation of intersection numbers for twisted 1-form, beginning with the case of non-twisted forms.

3.1 Closed 1-forms

Consider the wedge product of two closed 1-forms, $\phi_L$and ϕ_R, and a 1-form Ω, such that $d\Omega = \phi_L \wedge \phi_R$. Then, the intersection number $\langle \phi_L | \phi_R \rangle$ can be computed via Stokes' theorem, as

$$\langle \phi_L | \phi_R \rangle = \frac{1}{(2\pi i)} \int_M \phi_L \wedge \phi_R = \frac{1}{(2\pi i)} \int_M d\Omega = \frac{1}{(2\pi i)} \int_{\partial M} \Omega \,. \tag{34}$$

Moreover, if M contains holes, then the integration along its boundary $\partial \Sigma$ can be substituted by a sum of contour integrals around each pole, and the intersection number can be evaluated by Cauchy's residue theorem,

$$\langle \phi_L | \phi_R \rangle = \sum_{p \in \text{Poles}} \text{Res}_{z=p}(\Omega) \,. \tag{35}$$

(let us observe that the presence of poles in ϕ_L and ϕ_R makes the wedge product, hence the interesection number non-vanishing). Indeed, if we introduce the scalar potential ψ, such that

$$d\psi = \phi_L \tag{36}$$

then, we can define

$$\Omega \equiv \psi \, \phi_R \,, \tag{37}$$

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having the desired property, $d\Omega = d\psi \wedge \phi_R + \psi\, d\phi_R = \phi_L \wedge \phi_R + 0$ (the second term vanishes because of the closure).

3.2 Closed Twisted 1-forms

For Aomoto-Gel'Fand/Feynman integrals, owing to the presence of the multivalued function u, we need to consider the twisted (and dual-twisted) forms

$$\phi_L \equiv u\,\varphi_L\,, \qquad \phi_R \equiv u^{-1}\,\varphi_R\,. \tag{38}$$

In this case, we take

$$\Omega \equiv u\,\psi\,\phi_R = \psi\,\varphi_R\,, \tag{39}$$

with $d(u\,\psi) = \phi_L$, equivalently rewritten as,

$$\nabla_\omega \psi \quad = \quad \varphi_L\,, \tag{40}$$

such that $d\Omega = (u\,\varphi_L) \wedge (u^{-1}\,\varphi_R) = \varphi_L \wedge \varphi_R$. Therefore, the expression of the intersection number for twisted 1-form reads

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)} \int_M (u\varphi_L) \wedge (u^{-1}\varphi_R) = \frac{1}{(2\pi i)} \int_M \varphi_L \wedge \varphi_R = \int_M d\Omega = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\Omega)\,. \tag{41}$$

Equation (40) is the same differential equation obeyed by the potential ψ as proposed in the complete algorithm [50, 53]: actually (because of the local compactness of φ_L), it is sufficient to know the differential form Ω, hence the scalar function ψ just *locally*, around each pole $p \in \mathbb{P}_\omega$, therefore implying a local solution of the differential equation in eq. (40).

To derive relations for multivariate integrals, whose integrand contains generic meromorphic n-forms, the evaluation of the intersection numbers becomes a key task: for the case of logarithmic n-forms [53], intersection numbers can be evaluated by means of the *global residue theorem* [26]; while for the generic case, an *iterative approach* proposed in [30, 33, 35], (see [98, 99] in these Proceedings), further refined in [36, 102, 103].

Let me simply mention that the resolution of the identity has played an essential role also in the development of the iterative approach for the evaluation of the intersection numbers. In the case of n-forms $\varphi_{L,R} \in H^n_{\pm\omega}$, can be computed *by induction*, from intersection numbers of $(n-1)$-forms, as

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L | \mathbb{I}_c^{(n-1)} | \varphi_R \rangle \tag{42}$$

where $\mathbb{I}_c^{(n-1)}$ is the identity resolution within the $(n-1)$-dimensional cohomology group, namely written in terms of the generators of $H^{n-1}_{\pm\omega}$. Other algorithms for the evaluation of intersection numbers [60] have been recently proposed, and further ideas based on Stokes' theorem, extending what exposed earlier in this contribution, are in progress.

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4. Conclusion

Dimensional regularization has been a crucial concept for the development of computational methods in gauge theories. Understanding (supposedly knonwn) physics, and searching for deviations from the theoretical expectations within data requires the ability of computing Scattering Amplitudes at higher order in perturbation theory, therefore of evaluating rather complicated Feynman integrals. Along the last four decades, integration by parts identities, integrand decomposition, together with differential and difference equations for Feynman integrals, allowed the study of fundamental interactions among elementary particles at very high accuracy, when any direct attempt through explicit integration techniques becomes prohibitive, especially for those integrals that depend on many external parameters. The recent developments of solving strategies for linear systems, based on rational reconstruction over finite fields, pushed to integration-by-parts based decomposition to a very advanced level of efficiency.

Over the recent few years, theoretical physicists have been borrowing advanced concept and computational tools from Number Theory, Differential and Algebraic Topology for the study of Feynman integrals, which indicate that de Rahm theory for (relative) twisted co-homology seems to represent a complete mathematical framework to investigate the formal properties of these integrals. This research has recently culminated in the realization of the existence of a *vector space structure* controlled by *intersection numbers*, yielding the possibility of defining an *inner product* among integrals: the algebro-analytic properties of Feynman integrals, and more generally of Aomoto-Gel'fand integrals are characterized, determined by geometry – or to better say, by the topological properties of the algebraic variety identified by the zeroes of the (graph) polynomial appearing in the parametric representation of the integral, i.e. the twist. In this picture, Feynman integrals appear as *moments* of a statistical distribution identified with the twist, and related to the linear independence of the integration variables, and the master integrals become the independent moments of such distribution [108].

It is probably too early for forecasting if intersection theory based methods can lead to the development of novel more efficient computational tools than the ones currently available (in this case, we would say that math was helpful to physics), or if - by turning the arguments around - we will reach the conclusion that integration-by-parts identities are the most efficient method to evaluate intersection numbers (in this case, we would say that physics was helpful to math).

In any case, at the moment we can defintely mention that intersection theory offers a novel perspective to Feynman calculus, and allowed us to explore its underpinning mathematical structure, which we propose through the correspondence between QFT and de Rham theory, shown in Tab. 1,

The results achieved so far, and the successful outcome of the *MathemAmplitudes 2019* workshop, motivates us in further addressing open questions related to the adopted integral representation, hence to the consequent role of regularization of Feynman integrals, and of the corresponding equivalence classes, which may have a twofold impact: on the one side, it can lead to the identification of new mathematical methods for the evaluation of intersection numbers; but, more globally, on the other side, it may shed new light on the structure of radiative corrections in Quantum Field Theories, namely on a new layout of the diagrammatic contributions to the Scattering Amplitudes.

We hope that the vivid interplay between Theoretical Physics and Mathematics, as the one, sharing the common interest on Aomoto-Gel'fand integrals, which Feynman integrals are part of,

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Quantum Field Theory	de Rham Twisted Co-homology Theory
Feynman integrals	Aomoto-Gel'fand integrals
graph polynomials	Morse functions
regularisation	multivaluedness
vector space	co-homology groups
integration-by-parts identities	contiguity relations
quadratic relations	Riemann twisted periods relations
master integrals	independent forms / independent cycles
differential / difference equations	differential / difference equations

Table 1: Correspondence QFT *vs* de Rham theory.

can lead to progress in both fields, and, through it, to other areas that make use of these ubiquitous functions [2].

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