From differential and perfect differential to Roman domination and perfect Roman domination in probabilistic neural networks

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Research Article

Keywords: Differential, perfect differential, Roman domination, perfect Roman domination, probabilistic neural networks

Posted Date: January 12th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2457799/v1

Additional Declarations: No competing interests reported.

From differential and perfect differential to Roman domination and perfect Roman domination in probabilistic neural networks

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Abstract Let $G=(V,E)$ be a graph of order n. Let $B(S)$ be the set of vertices in $V\setminus S$ that have a neighbor in the vertex set S. The differential of a vertex set S is defined as $\partial(S)=|B(S)|-|S|$ and the maximum value of $\partial(S)$ for any subset S of V is the differential of G. For $S\subseteq V(G)$, the set $N_p(S)$ is defined as the perfect neighborhood of S such that all vertices in $V(G)\setminus S$ have exactly one neighbor in S. The perfect differential of S is defined to be $\partial_p(S)=|N_p(S)|-|S|$ and the perfect differential of a graph is defined as $\partial_p(G)=\max\{\partial_p(S):S\subseteq V(G)\}$. A Roman dominating function of G is a function $f:V\to\{0,1,2\}$ such that every vertex v for which $f(v)=0$ has a neighbor u with $f(u)=2$. The weight of a Roman dominating function f is $w(f)=\sum_{v\in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions. A perfect Roman dominating function is defined as an Roman dominating function f satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to exactly one vertex v for which $f(v)=2$. The perfect Roman domination number, denoted by $\gamma_R^p(G)$, is the minimum weight among all perfect Roman dominating functions on G, that is $\gamma_R^p(G)=\min\{w(f):f \text{ is a perfect Roman dominating function on } G\}$. This paper is devoted to the computation of differential, perfect differential and Roman domination, perfect Roman domination of probabilistic neural networks by the use of the proven Gallai-type results $\gamma_R(G)=n-\partial(G)$, $\gamma_R^p(G)+\partial_p(G)=n$. Besides, existing Roman and perfect Roman graph classes of probabilistic neural networks are characterized.

2020 Mathematics Subject Classification: 05C69, 05C05, 05C82

Keywords: Differential; perfect differential; Roman domination; perfect Roman domination; probabilistic neural networks

1. Introduction

In this paper, simple, finite and undirected graphs without loops and multiple edges are considered. Let $G=(V,E)$ be a graph with vertex set V and edge set E. The *order* of G is given by $|V|=n$ and the *size* is defined as $|E|=m$ where $|*|$ denotes the number of elements in the set (i.e. the cardinality). The *neighborhood* of a vertex $v \in V$ is the set of vertices adjacent to v, denoted $N_G(v)$ or just $N(v)$, and the *closed neighborhood* of v is given by $N[v]=N(v)\cup\{v\}$. Thus, $N(v)=\{u \in V | uv \in E\}$ and $N(v)$ is referred to as the *open neighborhood* of v. For a set $S \subseteq V$, $N(S)=\bigcup_{v\in S} N(v)$ and $N[S]=N(S)\cup S$. For $S \subseteq V$, the subgraph of G induced by S is denoted by $G[S]$ [4].

The differential of a graph was introduced to model ways of influencing a network. The differential in graphs is a subject of increasing interest, both in pure and applied mathematics. The research and application of the $\partial(G)$ appears mainly in computational mathematics. The differential of a graph was introduced in [10], and studied by several authors [2,9,11-12,14,24,26-33,39], motivated by its applications to information diffusion in social networks. The study of the mathematical properties of the differential in graphs stated in [2,9,10-12,14,24,26-33,39]. This parameter has been studied by many authors, both from the viewpoint of combinatorics and from the viewpoint of the algorithmic complexity. We refer to the papers [2,9,10-12,14,24,26-33,39] and the literature quoted therein.

Let $G=(V,E)$ be a graph of order n, for every set $D \subseteq V$ let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D. The *differential of* D is defined as $\partial(D)=|B(D)|-|D|$ and the *differential of a graph* G, written $\partial(G)$, is equal to $\max\{\partial(D): D \subseteq V\}$. A $\partial(G)$-set is a set $D \subseteq V$ such that $\partial(D)=\partial(G)$.

The *perfect neighborhood of a set* $S \subseteq V$ is defined to be $N_p(S)=\{v \in V \setminus S : |N(v)\cap S|=1\}$. The *perfect differential of a set* $S \subseteq V$ is defined as $\partial_p(S)=|N_p(S)|-|S|$ and the *perfect differential of a graph* is defined as $\partial_p(G)=\max\{\partial_p(S): S \subseteq V\}$. A $\partial_p(G)$-set is a set $S \subseteq V$ such that $\partial_p(S)=\partial_p(G)$ [1].

A subset $S \subseteq V$ is a *dominating set* of G, if for any vertex $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$. The *domination number* of G, denoted by $\gamma(G)$, equals the minimum cardinality of a dominating set [35]. A dominating set S of G is a *perfect dominating set* if for every $v \in V \setminus S$, there exists exactly one $u \in S$ for which $uv \in E$. The minimum cardinality of a dominating set is the *perfect domination number* of G, which is denoted by $\gamma^p(G)$. Since perfect dominating sets are dominating sets, $\gamma(G) \leq \gamma^p(G)$ for any graph G. The theory of perfect domination was introduced in [ ], and has been studied by several authors [ ]. The Roman domination number is defined as a variant of domination number in [7]. A *Roman dominating function* on graph $G = (V, E)$ is defined as a function $f : V \to \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of a Roman dominating function is the value $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions on G. A graph G is a *Roman graph* or *Roman* if $\gamma_R(G) = 2\gamma(G)$. We refer to the papers [5,15,20-23,28,33,36-38] and the literature quoted therein. A perfect version of Roman domination was introduced in [16]. A *perfect Roman dominating function* is defined as an Roman dominating function f satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The *perfect Roman domination number*, denoted by $\gamma_R^p(G)$, is the minimum weight among all perfect Roman dominating functions on G, that is $\gamma_R^p(G) = \min\{w(f) : f \text{ is a perfect Roman dominating function on } G\}$. Obviously, $\gamma_R(G) \leq \gamma_R^p(G) \leq 2\gamma^p(G)$ for every graph G, and those graphs attaining the equality $\gamma_R^p(G) = 2\gamma^p(G)$ are called *perfect Roman graphs*. For recent results on perfect Roman domination in graphs, we refer to the cited papers [13,17,18,25].

The domination based parameters reveal an underlying effcient and stable communication network. The study of domination in graphs is an important research area, and also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research. Research on domination in graphs has not only important theoretical signification, but also varied application in such fields as computer science, communication networks, ad hoc networks, biological and social networks, distributed computing, coding theory, and web graphs. Domination and its variations have been extensively studied [3,6,8,34-35,40]. In general, the concept of dominating sets in graph theory finds wide applications in different types of communication networks. A broadcast from a communication vertex is received by all its neighbors. This is captured by the notion of domination in a graph. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. A thorough study of domination appears in [34-35].

The differential set problem was proved to be an NP-complete problem [27]. Since also the problem of computing the Roman domination number of an arbitrary graph is NP-complete [22], it is worthwhile to compute the differential and Roman domination number of various classes of graphs [5,15,20-21,23,36-38]. In [25], it was shown that the perfect Roman domination problem is NP-complete for chordal graphs, planar graphs, and bipartite graphs. In [1], it was also shown that perfect differential problem is NP-complete, even for chordal graphs, planar graphs, and bipartite graphs. Since both the problem of computing the perfect Roman domination number and the perfect differential of a graph is NP-complete in general, it is worthwhile to compute the perfect Roman domination number and the perfect differential of various classes of graphs.

The theory of differential and perfect differential in graphs can be seen as novel approaches to the theory of Roman domination and perfect Roman domination. In [28] and [1], Gallai-type theorems are established which state the relationship between the differential and the Roman domination number, and the relationship between the perfect differential and the perfect Roman domination number. Allowing to study the Roman domination number and perfect Roman domination number of a graph without the use of functions are the advantages of these approaches. The perfect differential sets play an important role in the theory of perfect Roman domination. The aforementioned Gallai-type theorems allow us to derive results on the Roman domination number and perfect Roman domination number from results on the differential and perfect differential and vice versa.

A neural network is a computer system modeled on the nerve tissue and nervous system. Neural networks are not only studied in Neurochemistry but also in many applications in different areas of studies such as intrusion detection system, image processing, artificial intelligence localization, medicine, chemical and environmental sciences [19].

The 3-layered probabilistic neural network consists of three layers of nodes. The first layer, second layer and third layer are known as input layer, hidden layer and output layer, respectively. The first layer, second layer and third layer contain n nodes, k classes with m nodes in each class and k nodes, respectively. In the architecture of a 3-layered probabilistic neural network, each node of input layer is connected to all the nodes of each class of the hidden layer and all the nodes of each class of the hidden layer are connected to a unique node of the output layer. Thus, a 3-layered probabilistic neural network denoted by $PNN(n,k,m)$ has $|V(PNN(n,k,m))| = n + k(m+1)$ vertices where $n,k,m \in \mathbb{Z}^+$. Let the vertices of the 3-layered probabilistic neural network of the input layer be $v_1,\ldots,v_n$, the vertices of the hidden layer be $u_{11},\ldots,u_{1m},\ldots,u_{k1},\ldots,u_{km}$, the vertices of the output layer be $z_1,\ldots,z_k$. The graphical representation of the 3-layered probabilistic neural network $PNN(n,k,m)$ is shown in Figure 1.1.

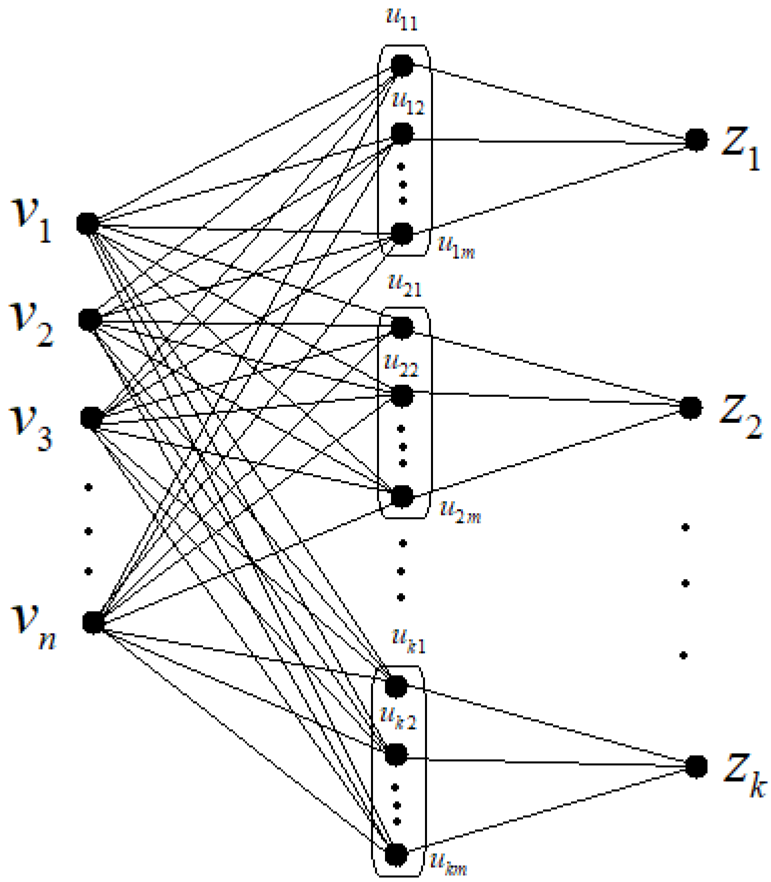

Figure 1.1. The 3-layered probabilistic neural network $PNN(n,k,m)$

The 4-layered probabilistic neural network consists of four layers of nodes. The first layer, second layer, third layer and fourth layer are known as input layer, hidden layer, summation layer, and output layer, respectively. The first layer, second layer, third layer and fourth layer contain n nodes, k classes with m nodes in each class, k nodes and one node, respectively. Each node of input layer is connected with every node in hidden layer, each node of a class in the hidden layer is connected to a unique node in the summation layer and all the nodes of summation layer are connected to the only node of the output layer. Thus, a 4-layered probabilistic neural network denoted by $PNN(n,k,m,1)$ has $|V(PNN(n,k,m,1))| = n + k(m+1) + 1$ vertices where $n,k,m \in \mathbb{Z}^+$. Let the vertices of the input layer be $v_1,\ldots,v_n$, the vertices of the hidden layer be $u_{11},\ldots,u_{1m},\ldots,u_{k1},\ldots,u_{km}$, the vertices of the summation layer be $z_1,\ldots,z_k$, and the vertex of the output layer be w. The graphical representation of the 4-layered probabilistic neural network $PNN(n,k,m,1)$ is shown in Figure 1.2.

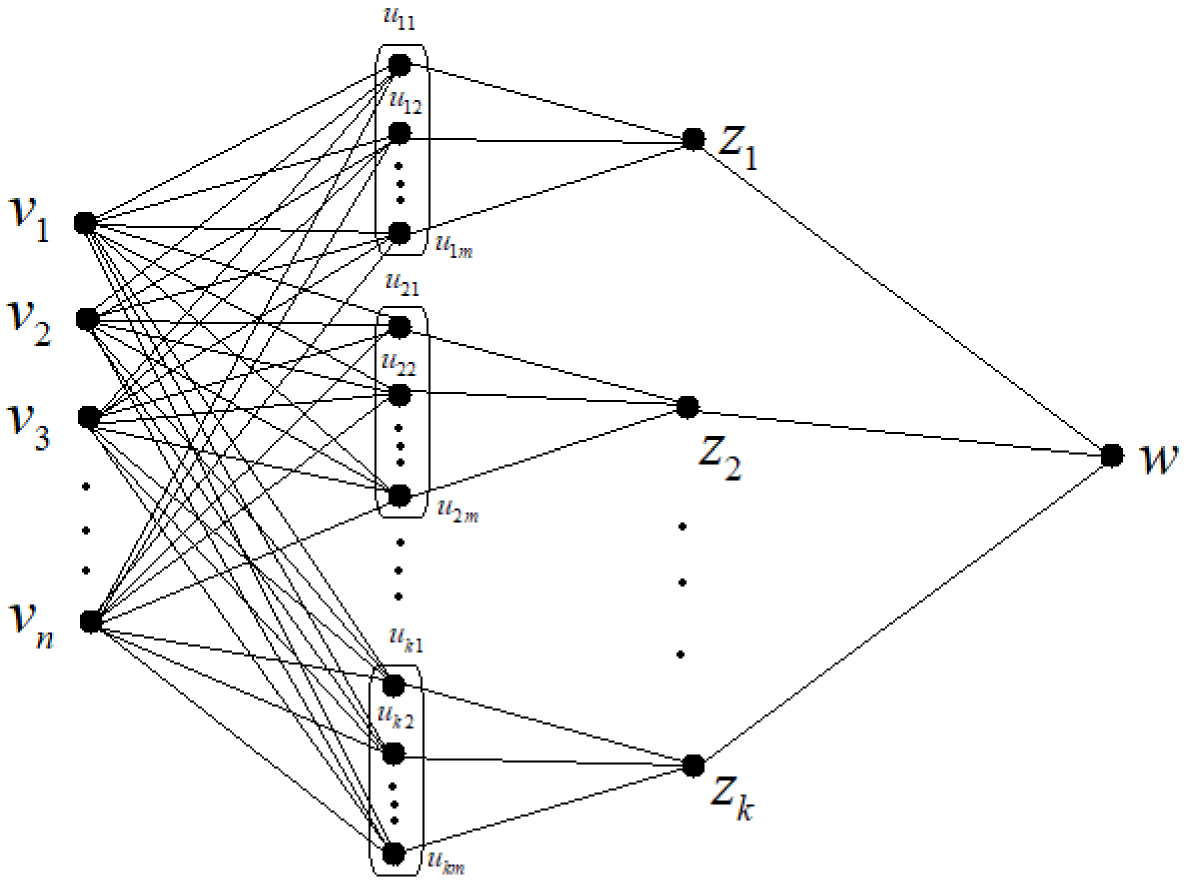

Figure 1.2. The 4-layered probabilistic neural network $PNN(n,k,m,1)$

The paper proceeds as follows. In section 2 and 3, average lower independence and domination numbers of 3 and 4-layered probabilistic neural networks are computed. Domination and independent domination numbers 3 and 4-layered probabilistic neural networks are given as immediate results.

2. Differential and Roman domination of probabilistic neural networks

Theorem 2.1 [9] *Let G be a graph of order n. Then, a graph G is dominant differential iff $\partial(G)=n-2\gamma(G)$.*

Theorem 2.2 [28] *If G is a graph of order n, then $\gamma_R(G)=n-\partial(G)$.*

Theorem 2.3 *Let $PNN(n,k,m)$ be a 3-layered probabilistic neural network with $n+k(m+1)$ vertices. Then, the differential of $PNN(n,k,m)$ is*

$$\partial(PNN(n,k,m))=\begin{cases} n+m-1, if\ \ k=1\ or\ n\geq 2, k=2, m=1; \\ km-1, if\ \ n=1, k>1; \\ n+km-3, if\ \ n\geq 2, k>1, m>1\ or\ n\geq 2, k\geq 3, m=1. \end{cases}$$

Proof. If we take a vertex v_i $(1\leq i\leq n)$ of the input layer of $PNN(n,k,m)$ and so $D_1=\{v_i\}$, then we have that $B(D_1)=N_{PNN(n,k,m)}(v_i)=\bigcup_{x=1}^{k}\bigcup_{y=1}^{m}u_{xy}$ and so $\partial(D_1)=km-1$, and adding either any vertex v_j $(1\leq j\leq n, j\neq i)$ of the input layer or any vertex z_t $(1\leq t\leq k)$ of the output layer of $PNN(n,k,m)$ to the set D_1 yields $\partial(D_1)<km-1$.

If we add a vertex u_{xy} $(1\leq x\leq k, 1\leq y\leq m)$ of the hidden layer of $PNN(n,k,m)$ to the set D_1, that is the set $D_2=\{v_i,u_{xy}\}$, we have $B(D_2)=\{v_1,\ldots,v_n\}\setminus\{v_i\}\cup\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\setminus\{u_{xy}\}\cup\{z_x\}$ yielding $\partial(D_2)=n+km-3$, and taking any other subset of $V(PNN(n,k,m))$ to the set D_2 yields $\partial(D_2)<n+km-3$.

If we take a vertex u_{xy} $(1 \le x \le k, 1 \le y \le m)$ of the hidden layer and so $D_3 = \{u_{xy}\}$, then we have that $B(D_3) = N_{PNN(n,k,m)}(u_{xy}) = \{v_1, \ldots, v_n\} \cup \{z_x\}$ and so $\partial(D_3) = n$, and adding any other vertex subset of the hidden layer to the set D_3 yields $\partial(D_3) \le n$.

If we add the vertex z_x of the output layer to the set D_3, that is the set $D_4 = \{u_{xy}, z_x\}$, then we have $B(D_4) = \{v_1, \ldots, v_n\} \cup \bigcup_{b=1}^{m} u_{xb} \setminus \{u_{xy}\}$ yielding $\partial(D_4) = n + m - 3$.

If we add all of the vertices of the output layer except the vertex z_x to the set D_3, that is the set $D_5 = \{u_{xy}\} \cup \{z_1, \ldots, z_k\} \setminus \{z_x\}$, then we have $B(D_5) = B(D_3) \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{x1}, \ldots, u_{xm}\}$ yielding $\partial(D_5) = n + (k-1)(m-1)$.

If we add the vertex z_x of the output layer to the set D_5, that is the set $D_6 = \{u_{xy}\} \cup \{z_1, \ldots, z_k\}$, then we have $B(D_6) = \{v_1, \ldots, v_n\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\}$ yielding $\partial(D_6) = n + k(m-1) - 2$, and adding any other subset of $V(PNN(n,k,m))$ to the set D_6 yields $\partial(D_6) < n + k(m-1) - 2$.

If we add a vertex v_i $(1 \le i \le n)$ of the input layer to the set D_5, that is the set $D_7 = \{v_i, u_{xy}\} \cup \{z_1, \ldots, z_k\} \setminus \{z_x\}$, then we have $B(D_7) = \{v_1, \ldots, v_n\} \setminus \{v_i\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\} \cup \{z_x\}$ yielding $\partial(D_7) = n + k(m-1) - 2$ and adding any other subset of $V(PNN(n,k,m))$ to the set D_7 yields $\partial(D_7) < n + k(m-1) - 2$.

If we take a vertex z_t $(1 \le t \le k)$ of the output layer, then we have $D_8 = \{z_t\}$ and $B(D_8) = \{u_{t1}, \ldots, u_{tm}\}$ yielding $\partial(D_8) = m - 1$.

If we add a vertex v_i $(1 \le i \le n)$ of the input layer to the set D_8, that is the set $D_9 = \{z_t, v_i\}$, then we have $B(D_9) = \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab}$ yielding $\partial(D_9) = km - 2$, and adding any other vertex subset of the input and output layers to the set D_9 yields $\partial(D_9) < km - 2$.

If we add a vertex u_{xy} $(1 \le x \le k, x \ne t, 1 \le y \le m)$ of the hidden layer to the set D_9, that is the set $D_{10} = \{z_t, v_i, u_{xy}\}$, we have $B(D_{10}) = \{v_1, \ldots, v_n\} \setminus \{v_i\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\} \cup \{z_x\}$ yielding

$\partial(D_{10}) = n + km - 4$, and adding any other subset of $V(PNN(n,k,m))$ to the set D_{10} yields $\partial(D_{10}) < n + km - 4$.

If we add a vertex u_{xy} $(1 \le x \le k, x \ne t, 1 \le y \le m)$ of the hidden layer to the set D_8, that is the set $D_{11} = \{z_t, u_{xy}\}$, then we get $B(D_{11}) = \{v_1, \dots, v_n\} \cup \{u_{t1}, \dots, u_{tm}\} \cup \{z_x\}$ and so $\partial(D_{11}) = n + m - 1$.

If we take all vertices of the output layer to the set D_{12}, then we have $D_{12} = \{z_1, \dots, z_k\}$ and $B(D_{12}) = \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab}$ yielding $\partial(D_{12}) = k(m-1)$.

If we add a vertex u_{xy} $(1 \le x \le k, 1 \le y \le m)$ of the hidden layer to the set D_{12}, that is the set $D_{13} = \{z_1, \dots, z_k\} \cup \{u_{xy}\}$, then we have $B(D_{13}) = \{v_1, \dots, v_n\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\}$ yielding $\partial(D_{13}) = n + k(m-1) - 2$, and adding any other subset of $V(PNN(n,k,m))$ to the set D_{13} yields $\partial(D_{13}) < n + k(m-1) - 2$.

By the definition of graph differential, among all the cardinalities of differential sets, we have that

$$\partial(PNN(n,k,m)) = \max\{\partial(D_p)\} \quad (1 \le p \le 13)$$

$$\partial(PNN(n,k,m)) = \begin{cases} n + m - 1, \text{ if } k = 1 \text{ or } n \ge 2, k = 2, m = 1; \\ km - 1, \text{ if } n = 1, k > 1; \\ n + km - 3, \text{ if } n \ge 2, k > 1, m > 1 \text{ or } n \ge 2, k \ge 3, m = 1. \end{cases}$$

The theorem is thus proved. ■

Lemma 2.1 [40] *The domination number of a 3-layered probabilistic neural network $PNN(n,k,m)$ is $\gamma(PNN(n,k,m)) = \begin{cases} k, \text{ if } m = 1 \text{ and } k > 1; \\ k + 1, \text{ otherwise.} \end{cases}$*

Remark 2.1 In [40], the domination number of a 3-layered probabilistic neural network is computed as $\gamma(PNN(n,k,m)) = k$, if $m = 1$ and $k > 1$; $k + 1$, otherwise. Then, by Theorem 2.1 and 2.3, we can easily observe that if $n = 1, k = 2, m = 1$ or $n \ge 2, k = 2, 3, m = 1$, then a 3-layered probabilistic neural network $PNN(n,k,m)$ is a dominant differential graph; that is the maximal ∂-sets of those networks are also dominating sets.

Since the order of a 3-layered probabilistic neural network is $\left|V\left(PNN\left(n,k,m\right)\right)\right| = n + k\left(m+1\right)$, by the use of Theorem 2.2, the result in Corollary 2.1 is derived directly as a consequence of Theorem 2.3.

Corollary 2.1 *The Roman domination number of a 3-layered probabilistic neural network* $PNN\left(n,k,m\right)$ *with* $n+k\left(m+1\right)$ *vertices is*

$$\gamma_R\left(PNN\left(n,k,m\right)\right) = \begin{cases} k\left(m+1\right) - m + 1, if \; k = 1 \; or \; n \geq 2, k = 2, m = 1; \\ k+2, if \; n = 1, k > 1; \\ k+3, if \; n \geq 2, k > 1, m > 1 \; or \; n \geq 2, k \geq 3, m = 1. \end{cases}$$

Remark 2.2 A graph G is a Roman graph or Roman if $\gamma_R\left(G\right) = 2\gamma\left(G\right)$. Then, by the use of Corollary 2.1 and Lemma 2.1, Roman 3-layered probabilistic neural networks are identified. If $n = 1, k = 2, m = 1$ or $n \geq 2, k = 2, 3, m = 1$, then the 3-layered probabilistic neural network $PNN\left(n,k,m\right)$ is a Roman tree.

Theorem 2.4 *Let* $PNN\left(n,k,m,1\right)$ *be a 4-layered probabilistic neural network with* $n+k\left(m+1\right)+1$ *vertices. Then, the differential of* $PNN\left(n,k,m,1\right)$ *is*

$$\partial\left(PNN\left(n,k,m,1\right)\right) = \begin{cases} n+m, if \; k = 1 \; or \; n > 2, k > 2, m \leq 2 \; or \; n = 2, k = 2, m = 1; \\ k\left(m+1\right) - 2, if \; n = 1, k > 1 \; or \; n = 2, k > 1, m > 2 \; or \; n = 2, k > 2, m = 1; \\ n + k\left(m+1\right) - 5, if \; n > 2, k > 1, m + 2k \geq 7. \end{cases}$$

Proof. If we take a vertex v_i $\left(1 \leq i \leq n\right)$ of the input layer of $PNN\left(n,k,m,1\right)$ and so $D_1 = \left\{v_i\right\}$, we have $B\left(D_1\right) = N_{PNN(n,k,m,1)}\left(v_i\right) = \bigcup_{a=1}^{k}\bigcup_{b=1}^{m} u_{ab}$ yielding $\partial\left(D_1\right) = km - 1$.

If we add the vertex w of the output layer to the set D_1, that is the set $D_2 = \left\{v_i, w\right\}$, then we get $B\left(D_2\right) = B\left(D_1\right) \cup \left\{z_1, \ldots, z_k\right\}$ and so $\partial\left(D_2\right) = k\left(m+1\right) - 2$.

If we add a vertex u_{xy} $\left(1 \leq x \leq k, 1 \leq y \leq m\right)$ of the hidden layer to the set D_2, that is the set $D_3 = \left\{v_i, w, u_{xy}\right\}$, then we get $B\left(D_3\right) = \left\{v_1, \ldots, v_n\right\} \setminus \left\{v_i\right\} \cup \bigcup_{a=1}^{k}\bigcup_{b=1}^{m} u_{ab} \setminus \left\{u_{xy}\right\} \cup \left\{z_1, \ldots, z_k\right\}$ and so

$\partial(D_3)=n+k(m+1)-5$, and adding any other subset of $V(PNN(n,k,m,1))$ to the set D_3 yields $\partial(D_3)<n+k(m+1)-5$.

If we take a vertex u_{xy} $(1\le x\le k, 1\le y\le m)$ of the hidden layer to the set D_4, then we have $D_4=\{u_{xy}\}$ and $B(D_4)=N_{PNN(n,k,m,1)}(u_{xy})=\{v_1,\dots,v_n\}\cup\{z_x\}$ yielding $\partial(D_4)=n$.

If we add the vertex w of the output layer to the set D_4, that is the set $D_5=\{u_{xy},w\}$, then we get $B(D_5)=\{v_1,\dots,v_n\}\cup\{z_1,\dots,z_k\}$ yielding $\partial(D_5)=n+k-2$.

If we add all vertices of the summation layer except the vertex z_x to the set D_4, that is the set $D_6=\{z_1,\dots,z_k\}\setminus\{z_x\}\cup\{u_{xy}\}$, then we have $B(D_6)=B(D_4)\cup\bigcup_{a=1,a\ne x}^{k}\bigcup_{b=1}^{m}u_{ab}\cup\{w\}$ and so $\partial(D_6)=n+(k-1)(m-1)+1$.

If we add the vertex z_x of the summation layer to the set D_6, that is the set $D_7=\{u_{xy}\}\cup\{z_1,\dots,z_k\}$, then we have $B(D_7)=\{v_1,\dots,v_n\}\cup\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\setminus\{u_{xt}\}\cup\{w\}$ $(1\le t\le m)$ and so $\partial(D_7)=n+k(m-1)-1$, and adding any other subset of $V(PNN(n,k,m,1))$ to the set D_7 yields $\partial(D_7)<n+k(m-1)-1$.

If we take a vertex z_p $(1\le p\le k)$ of the summation layer to the set D_8, then we get $D_8=\{z_p\}$, $B(D_8)=N_{PNN(n,k,m,1)}(z_p)=\{u_{p1},\dots,u_{pm}\}\cup\{w\}$, and so $\partial(D_8)=m$.

If we take all of the vertices of the summation layer as a differential set $D_9=\{z_1,\dots,z_k\}$, then we receive $B(D_9)=\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\cup\{w\}$ and so $\partial(D_9)=k(m-1)+1$, and taking any other vertex subset of the input layer to the set D_9 yields $\partial(D_9)<k(m-1)+1$.

If we add a vertex v_i $(1\le i\le n)$ of the input layer to the set D_8, that is the set $D_{10}=\{z_p,v_i\}$, then we get $B(D_{10})=\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\cup\{w\}$ and so $\partial(D_{10})=km-1$.

If we add a vertex u_{xy} $(1\le x\le k, x\ne p, 1\le y\le m)$ of the hidden layer to the set D_{10}, that is the set $D_{11}=\{z_p,v_i,u_{xy}\}$, then we have $B(D_{11})=\{v_1,\dots,v_n\}\setminus\{v_i\}\cup\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\setminus\{u_{xy}\}\cup\{w\}$ and so $\partial(D_{11})=n+km-3$, and adding any other subset of $V(PNN(n,k,m,1))$ to the set D_{11} yields $\partial(D_{11})<n+km-3$.

If we add a vertex u_{xy} $\left(1\le x\le k, x\ne p, 1\le y\le m\right)$ of the hidden layer to the set D_8, that is the set $D_{12}=\left\{z_p,u_{xy}\right\}$, then we have $B\left(D_{12}\right)=\{v_1,\dots,v_n\}\cup\left\{u_{p1},\dots,u_{pm}\right\}\cup\left\{z_x\right\}\cup\{w\}$ and so $\partial\left(D_{12}\right)=n+m$.

By the definition of graph differential, among all cardinalities of differential sets, we receive

$$\partial\left(PNN\left(n,k,m,1\right)\right)=\max\left\{\partial\left(D_l\right)\right\}\ \left(1\le l\le 12\right)$$

$$\partial\left(PNN\left(n,k,m,1\right)\right)=\begin{cases} n+m, \text{ if } k=1 \text{ or } n>2,k>2,m\le 2 \text{ or } n=2,k=2,m=1; \\ k\left(m+1\right)-2, \text{ if } n=1,k>1 \text{ or } n=2,k>1,m>2 \text{ or } n=2,k>2,m=1; \\ n+k\left(m+1\right)-5, \text{ if } n>2,k>1,m+2k\ge 7. \end{cases}$$

The theorem is thus proved. ■

Lemma 2.2 [40] *The domination number of a 4-layered probabilistic neural network $PNN\left(n,k,m,1\right)$ for $n>1,\ k>2,\ m>1$ is $\gamma\left(PNN\left(n,k,m,1\right)\right)=3$.*

Remark 2.3 In [40], the domination number of a 4-layered probabilistic neural network is computed as $\gamma\left(PNN\left(n,k,m,1\right)\right)=3$ for $n>1,k>2,m>1$. In addition, it can be easily observed that the domination number of a 4-layered probabilistic neural network $\gamma\left(PNN\left(n,k,m,1\right)\right)=2$ or 3. Then, by the use of Theorem 2.1 and 2.4, we can conclude that if $n=2,k=2,m=1$ or $n=1,k>1$ or $n>2,k>1,m+2k\ge 7$, then a 4-layered probabilistic neural network $PNN\left(n,k,m,1\right)$ is dominant differential, that is the maximal ∂-sets of those networks are also dominating sets.

Since the order of a 4-layered probabilistic neural network is $\left|V\left(PNN\left(n,k,m,1\right)\right)\right|=n+k\left(m+1\right)+1$, by the use of Theorem 2.2, the result in Corollary 2.2 is derived directly as a consequence of Theorem 2.4.

Corollary 2.2 *The Roman domination number of a 4-layered probabilistic neural network $PNN\left(n,k,m,1\right)$ with $n+k\left(m+1\right)+1$ vertices is*

$$\gamma_R\left(PNN\left(n,k,m,1\right)\right)=\begin{cases} k\left(m+1\right)-m+1, if\ k=1\ or\ n>2,k>2,m\le 2\ or\ n=2,k=2,m=1; \\ n+3, if\ n=1,k>1\ or\ n=2,k>1,m>1\ or\ n=2,k>2,m=1; \\ 6, if\ n>2,k>1,m+2k\ge 7. \end{cases}$$

Remark 2.4 By the use of Corollary 2.2 and Lemma 2.2., Roman 4-layered probabilistic neural networks are identified. Also, we emphasize that the domination number of a 4-layered probabilistic neural network is $\gamma\left(PNN(n,k,m,1)\right)=2$ or 3. Then, we can easily observe that for $n=1,k>1$ or $n>2,k>1,m+2k\geq 7$, the 4-layered probabilistic neural network $PNN(n,k,m,1)$ is a Roman graph.

3. Perfect differential and perfect Roman domination of probabilistic neural networks

Theorem 3.1 [1] *If G is a graph of order n, then $\gamma_R^p(G)+\partial_p(G)=n$.*

Theorem 3.2. *Let $PNN(n,k,m)$ be a 3-layered probabilistic neural network with $n+k(m+1)$ vertices. Then, the perfect differential of $PNN(n,k,m)$ is*

$$\partial_p\left(PNN(n,k,m)\right)=\begin{cases} n+k(m-1), if\ n=1,k=1\ or\ n>1,k\leq 2; \\ km-1, if\ n=1,k\geq 2; \\ n+km-3, if\ n>1,k>2. \end{cases}$$

Proof. If we take a vertex v_i $(1\leq i\leq n)$ of the input layer of $PNN(n,k,m)$ to the set S_1, then $S_1=\{v_i\}$, the perfect neighborhood of S_1 is $N_p(S_1)=\bigcup_{x=1}^{k}\bigcup_{y=1}^{m}u_{xy}$ with cardinality $|N_p(S_1)|=km$ and so $\partial_p(S_1)=km-1$.

If we add a vertex u_{xy} $(1\leq x\leq k, 1\leq y\leq m)$ of the hidden layer to the set S_1, that is the set $S_2=\{v_i,u_{xy}\}$, then being $N_p(S_2)=\{v_1,\ldots,v_n\}\setminus\{v_i\}\cup\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\setminus\{u_{xy}\}\cup\{z_x\}$ with cardinality $|N_p(S_2)|=n+km-1$ yields $\partial_p(S_2)=n+km-3$, and adding any other possible subset of $V\left(PNN(n,k,m)\right)$ to the set S_2 yields $\partial_p(S_2)<n+km-3$.

If we take a vertex u_{xy} $(1\leq x\leq k, 1\leq y\leq m)$ of the hidden layer to the set S_3, then $S_3=\{u_{xy}\}$, the perfect neighborhood of S_3 is $N_p(S_3)=\{v_1,\ldots,v_n\}\cup\{z_x\}$ with cardinality $|N_p(S_3)|=n+1$ and so $\partial_p(S_3)=n$.

If we add all vertices of the output layer except the vertex z_x to the set S_3, that is the set $S_4 = \{u_{xy}\} \cup \{z_1, \ldots, z_k\} \setminus \{z_x\}$, then being $N_p(S_4) = \{v_1, \ldots, v_n\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\} \cup \{z_x\}$ with cardinality $|N_p(S_4)| = n + km$ yields $\partial_p(S_4) = n + k(m-1)$.

If we add all of the vertices of the output layer to the set S_3, that is the set $S_5 = \{u_{xy}\} \cup \{z_1, \ldots, z_k\}$, then being $N_p(S_5) = \{v_1, \ldots, v_n\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\}$ with cardinality $|N_p(S_5)| = n + km - 1$ yields $\partial_p(S_5) = n + k(m-1) - 2$, and any other subset of $V(PNN(n,k,m))$ cannot be added to the set S_5.

If we take a vertex z_t $(1 \le t \le k)$ of the output layer to the set S_6, then $S_6 = \{z_t\}$, the perfect neighborhood of S_6 is $N_p(S_6) = \{u_{t1}, \ldots, u_{tm}\}$ with cardinality $|N_p(S_6)| = m$ and so $\partial_p(S_6) = m - 1$.

If we take all the other vertices of the output layer to the set S_6, that is the set $S_7 = \{z_1, \ldots, z_k\}$, then being $N_p(S_7) = \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab}$ with cardinality $|N_p(S_7)| = km$ yields $\partial_p(S_7) = k(m-1)$.

By the definition of perfect differential of a graph, among all the perfect differential sets, we receive

$$\partial_p(PNN(n,k,m)) = \max\{\partial_p(S_l)\} \quad (1 \le l \le 7)$$

$$\partial_p(PNN(n,k,m)) = \begin{cases} n + k(m-1), \text{if } n = 1, k = 1 \text{ or } n > 1, k \le 2; \\ km - 1, \text{if } n = 1, k \ge 2; \\ n + km - 3, \text{if } n > 1, k > 2. \end{cases}$$

Thus, the proof holds. ■

Theorem 3.1 allows us to derive results on the perfect Roman domination number from results on the perfect differential and vice versa. The next result is a direct consequence of combining Theorem 3.1 and 3.2.

Corollary 3.1. *The perfect Roman domination number of a 3-layered probabilistic neural network $PNN(n,k,m)$ with $n + k(m+1)$ is*

$$\gamma_R^p\left(PNN\left(n,k,m\right)\right)=\begin{cases}2k, if\ k=1\ or\ n\geq 2, m=1, k=2;\\ n+k+1, if\ n=1, k>1;\\ k+3, if\ n\geq 2, m>1, k>1\ or\ n\geq 2, m=1, k\geq 3.\end{cases}$$

Theorem 3.3. *Let $PNN(n,k,m,1)$ be a 4-layered probabilistic neural network with $n+k(m+1)+1$ vertices. Then, the perfect differential of $PNN(n,k,m,1)$ is*

$$\partial_p\left(PNN\left(n,k,m,1\right)\right)=\begin{cases}n+m, if\ k=1\ or\ k=2, m=1\ or\ k=2, m=2, n>2\ or\ k=3, m=1, n>2;\\ n+km-3, if\ k=2, m\geq 3, n\geq 3\ or\ k=3, m>1, n\geq 4\ or\ k>3, n\geq k+1;\\ k\left(m+1\right)-2, if\ k=2, m>1, n<3\ or\ k=3, n\leq 3\ or\ k>3, n<k+1.\end{cases}$$

Proof. If we take a vertex v_i $(1\leq i\leq n)$ of the input layer of $PNN(n,k,m,1)$ to the set S_1, then $S_1=\{v_i\}$, the perfect neighborhood of S_1 is $N_p(S_1)=\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}$ with cardinality $|N_p(S_1)|=km$ and so $\partial_p(S_1)=km-1$.

If we add a vertex u_{xy} $(1\leq x\leq k, 1\leq y\leq m)$ of the hidden layer to the set S_1, that is the set $S_2=\{v_i,u_{xy}\}$, then being $N_p(S_2)=\{v_1,\ldots,v_n\}\setminus\{v_i\}\cup\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\setminus\{u_{xy}\}\cup\{z_x\}$ with cardinality $|N_p(S_2)|=n+km-1$ yields $\partial_p(S_2)=n+km-3$, and adding any other possible subset of $V(PNN(n,k,m,1))$ to the set S_2 yields $\partial_p(S_2)<n+km-3$.

If we add the vertex w of the output layer to the set S_1, that is the set $S_3=\{v_i,w\}$, then being $N_p(S_3)=\bigcup_{a=1}^{k}\bigcup_{b=1}^{m}u_{ab}\cup\{z_1,\ldots,z_k\}$ with cardinality $|N_p(S_3)|=k(m+1)$ yields $\partial_p(S_3)=k(m+1)-2$, and adding any other possible subset of $V(PNN(n,k,m,1))$ to the set S_3 yields $\partial_p(S_3)<k(m+1)-2$.

If we take a vertex u_{xy} $(1\leq x\leq k, 1\leq y\leq m)$ of the hidden layer to the set S_4, then $S_4=\{u_{xy}\}$, the perfect neighborhood of S_4 is $N_p(S_4)=\{v_1,\ldots,v_n\}\cup\{z_x\}$ with cardinality $|N_p(S_4)|=n+1$ and so $\partial_p(S_4)=n$.

If we add a vertex z_t $(1 \le t \le k, t \ne x)$ of the summation layer to the set S_4, that is the set $S_5 = \{u_{xy}, z_t\}$, then being $N_p(S_5) = \{v_1, \dots, v_n\} \cup \{u_{t1}, \dots, u_{tm}\} \cup \{z_x\} \cup \{w\}$ with cardinality $|N_p(S_5)| = n + m - 2$ yields $\partial_p(S_5) = n + m$.

If we add the vertex z_x of the summation layer to the set S_4, that is the set $S_6 = \{u_{xy}, z_x\}$, then being $N_p(S_6) = \{v_1, \dots, v_n\} \cup \{u_{x1}, \dots, u_{xm}\} \setminus \{u_{xy}\} \cup \{w\}$ with cardinality $|N_p(S_6)| = n + m$ yields $\partial_p(S_6) = n + m - 2$.

If we add the vertex w of the output layer to the set S_6, that is the set $S_7 = \{u_{xy}, z_x, w\}$, then being $N_p(S_7) = \{v_1, \dots, v_n\} \cup \{u_{x1}, \dots, u_{xm}\} \setminus \{u_{xy}\} \cup \{z_1, \dots, z_k\} \setminus \{z_x\}$ with cardinality $|N_p(S_7)| = n + k + m - 2$ yields $\partial_p(S_7) = n + k + m - 5$.

If we add all the other vertices of the summation layer to the set S_7, that is the set $S_8 = \{u_{xy}, w, z_1, \dots, z_k\}$, then being $N_p(S_8) = \{v_1, \dots, v_n\} \cup \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab} \setminus \{u_{xy}\}$ with cardinality $|N_p(S_8)| = n + km - 1$ yields $\partial_p(S_8) = n + k(m-1) - 3$, and adding any other possible subset of $V(PNN(n,k,m,1))$ to the set S_8 yields $\partial_p(S_8) < n + k(m-1) - 3$.

If we take a vertex z_t $(1 \le t \le k)$ of the summation layer to the set S_9, then $S_9 = \{z_t\}$, the perfect neighborhood of S_9 is $N_p(S_9) = \{u_{t1}, \dots, u_{tm}\} \cup \{w\}$ with cardinality $|N_p(S_9)| = m + 1$ and so $\partial_p(S_9) = m$.

If we add the vertex w of the output layer to the set S_9, that is the set $S_{10} = \{z_t, w\}$, then being $N_p(S_{10}) = \{u_{t1}, \dots, u_{tm}\} \cup \{z_1, \dots, z_k\} \setminus \{z_x\} \cup \{w\}$ with cardinality $|N_p(S_{10})| = k + m - 1$ yields $\partial_p(S_{10}) = k + m - 3$.

If we add all the other vertices of the summation layer to the set S_{10}, that is the set $S_{11} = \{w, z_1, \dots, z_k\}$, then being $N_p(S_{11}) = \bigcup_{a=1}^{k} \bigcup_{b=1}^{m} u_{ab}$ with cardinality $|N_p(S_{11})| = km$ yields $\partial_p(S_{11}) = k(m-1) - 1$.

If we take the vertex w of the output layer to the set S_{12}, then $S_{12}=\{w\}$, the perfect neighborhood of S_{12} is $N_p(S_{12})=\{z_1,\ldots,z_k\}$ with cardinality $|N_p(S_{12})|=k$ and so $\partial_p(S_{12})=k-1$.

By the definition of perfect differential of a graph, among all the perfect differential sets, we receive

$$\partial_p(PNN(n,k,m,1))=\max\{\partial_p(S_l)\}\ (1\le l\le 12)$$

$$\partial_p(PNN(n,k,m,1))=\begin{cases} n+m, \text{if } k=1 \text{ or } k=2, m=1 \text{ or } k=2, m=2, n>2 \text{ or } k=3, m=1, n>2; \\ n+km-3, \text{if } k=2, m\ge 3, n\ge 3 \text{ or } k=3, m>1, n\ge 4 \text{ or } k>3, n\ge k+1; \\ k(m+1)-2, \text{if } k=2, m>1, n<3 \text{ or } k=3, n\le 3 \text{ or } k>3, n<k+1. \end{cases}$$

The theorem is thus proved. ■

The next result is a direct consequence of combining Theorem 3.1 and 3.3.

Corollary 3.2. *The perfect Roman domination number of a 4-layered probabilistic neural network $PNN(n,k,m,1)$ with $n+k(m+1)+1$ is*

$$\gamma_R^p(PNN(n,k,m,1))=\begin{cases} k(m+1)-m+1, if\ k=1\ or\ k=2, m=1\ or\ k=2, m=2, n>2\ or\ k=3, m=1, n>2; \\ k+4, if\ k=2, m\ge 3, n\ge 3\ or\ k=3, m>1, n\ge 4\ or\ k>3, n\ge k+1; \\ n+3, if\ k=2, m>1, n<3\ or\ k=3, n\le 3\ or\ k>3, n<k+1. \end{cases}$$

As we proceed to show, for the case of perfect Roman graphs, Corollary 3.1 and 3.2 lead to the following remark.

Remark 3.1. Obviously $\gamma_R^p(G)\le 2\gamma^p(G)$ for every graph G, and those graphs attaining the equality $\gamma_R^p(G)=2\gamma^p(G)$ are called perfect Roman graphs. It can be easily detected that for a 3-layered probabilistic neural network $PNN(n,k,m)$, $\gamma^p(PNN(n,k,m))=k+1$, and for a 4-layered probabilistic neural network $PNN(n,k,m,1)$, $\gamma^p(PNN(n,k,m,1))=\begin{cases} 3, \text{ if } k=1; \\ 4, \text{ otherwise.} \end{cases}$

By Corollary 3.1, we deduce that 3-layered probabilistic neural networks are not a class of perfect Roman graphs. However, by Corollary 3.2, we characterize a class of perfect Roman 4-layered probabilistic neural networks. If $k=4$ and $n\ge 5$ or $k>4$ and $n=5$, a 4-layered probabilistic neural network is a perfect Roman tree.

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