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From Dynkin Diagram Symmetries to Fixed Point Structures

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Abstract: Any automorphism of the Dynkin diagram of a symmetrizable Kac–Moody algebra $\mathfrak{g}$ induces an automorphism of $\mathfrak{g}$ and a mapping τ_ω between highest weight modules of $\mathfrak{g}$. For a large class of such Dynkin diagram automorphisms, we can describe various aspects of these maps in terms of another Kac–Moody algebra, the "orbit Lie algebra" $\breve{\mathfrak{g}}$. In particular, the generating function for the trace of τ_ω over weight spaces, which we call the "twining character" of $\mathfrak{g}$ (with respect to the automorphism), is equal to a character of $\breve{\mathfrak{g}}$. The orbit Lie algebras of untwisted affine Lie algebras turn out to be closely related to the fixed point theories that have been introduced in conformal field theory. Orbit Lie algebras and twining characters constitute a crucial step towards solving the fixed point resolution problem in conformal field theory.

1. Introduction

In this paper we associate algebraic structures to automorphisms of Dynkin diagrams and study some of their interrelations. The class of Dynkin diagrams we consider are those of symmetrizable Kac–Moody algebras [1]. These are those Lie algebras which possess both a Cartan matrix and a Killing form, which includes in particular the simple, affine, and hyperbolic Kac–Moody algebras.

An automorphism of a Dynkin diagram is a permutation of its nodes which leaves the diagram invariant. Any such map divides the set of nodes of the diagram into invariant subsets, called the orbits of the automorphism. We focus our attention on two main types of orbits, namely those where each of the nodes on an orbit is either connected by a single link to precisely one node on the same orbit or not linked to any other node on the same orbit. If all orbits of a given Dynkin diagram automorphism are of one of these two types, we say that the automorphism satisfies the *linking condition*. Except for the order N automorphisms of the affine Lie algebras $A_{N-1}^{(1)}$, all diagram automorphisms of simple and affine Lie algebras belong to this class.

* Heisenberg fellow

1.1. Orbit Lie algebras. In Sect. 2 we show that for any automorphism satisfying the linking condition one can define a "folded" Dynkin diagram (and associated Cartan matrix) which is again the Dynkin diagram of a symmetrizable Kac-Moody algebra. The folded Dynkin diagram has one node for each orbit of the original diagram, and there is a definite prescription for the number of links between any two nodes of the folded diagram. If the Kac-Moody algebra corresponding to the original Dynkin diagram is g, we denote the algebra corresponding to the folded Dynkin diagram by $\breve{\mathsf{g}}$ and call it the *orbit Lie algebra*. We show that the folding procedure preserves the "type" of the Kac-Moody algebra, where the type of a symmetrizable Kac-Moody algebra can be either "simple", "affine", "hyperbolic", or "non-hyperbolic indefinite". (However, "untwisted affine" and "twisted affine" are not separately preserved.)

Any automorphism of a Dynkin diagram (not necessarily satisfying the linking condition) induces an outer automorphism of the associated Kac-Moody algebra g. This is described in Sect. 3. For simple, affine and hyperbolic algebras the induced automorphism is unique. In the case of simple Lie algebras, these outer automorphisms are well known; they correspond to charge conjugation (for $\mathsf{g} = A_n$ $(n > 1)$, D_{2n+1} $(n > 1)$, and E_6), to the spinor conjugation of D_{2n} $(n > 2)$, and to the triality of D_4. The induced outer automorphisms of untwisted affine Lie algebras are either the aforementioned ones (inherited from the simple horizontal subalgebra), or certain automorphisms related to simple currents [2] of WZW theories (i.e. conformal field theories for which the chiral symmetry algebra is the semidirect sum of the untwisted affine Lie algebra and the Virasoro algebra), or combinations thereof.

1.2. Twining characters. The automorphism of g induces a natural map on the weight space of g. We can also employ the action on the algebra, in a less straightforward manner, to obtain an action, compatible with the action on the weight space, on the states of any highest weight module of g. We can therefore define a new type of character-like quantities for these modules by inserting the generator of the automorphism into the trace that defines the ordinary character. We call the object constructed in this manner the *twining character* of the highest weight module; its precise definition is presented in Sect. 4.

Trivially, the twining character vanishes whenever the highest weight is changed by the automorphism. As a consequence, our interest is in those highest weight modules whose highest weight is not changed by the automorphism; we call these special modules the *fixed point modules* of the automorphism and refer to their highest weights as *symmetric* g-weights. For fixed point modules, the twining character receives a non-vanishing contribution from at least one state, namely the highest weight state, but it is far from obvious what happens for all the other states of the module. Note that the weight of a state does not provide sufficient information for answering this question. Rather, the action of the automorphism also depends on the specific way in which the state is obtained from the highest weight state by applying step operators, as the automorphism acts non-trivially on these step operators.

There is one interesting class of automorphisms for which *only* the highest weight state of a fixed point module contributes to the twining character. These are the order N automorphisms of the affine Lie algebras $A^{(1)}_{N-1}$. In this particular case the Serre relations among the commutators of step operators conspire in such a way that all other states in the Verma module (and hence also in the irreducible module) cancel each others' contributions to the twining character. This statement will be proven in Sect. 7 (following a route that does not rely on the explicit use of the Serre relations and hence avoids various technical complications). Note that these automorphisms do *not* satisfy the linking condition. Rather, all N nodes of the Dynkin diagram of $A_{N-1}^{(1)}$ lie on the same orbit of this automorphism, and hence each node on the orbit is connected to *two* other nodes on the orbit. Correspondingly, there is no associated orbit Lie algebra (formally one obtains the "Lie algebra" which has the 1×1 Cartan matrix $A = (0)$).

The main result of this paper, proved in Sect. 5, concerns the fixed point modules of Dynkin diagram automorphisms which do satisfy the linking condition. We prove that these modules are in one-to-one correspondence with the highest weight modules of the orbit Lie algebra $\breve{\mathfrak{g}}$, and that the twining characters of the fixed point modules (both for Verma modules and for their irreducible quotients) coincide with the ordinary characters of the highest weight modules of $\breve{\mathfrak{g}}$. Note that we are not claiming that the orbit Lie algebra $\breve{\mathfrak{g}}$ is embedded in the original algebra $\mathfrak{g}$. We can show, however, that the Weyl group of $\breve{\mathfrak{g}}$ is isomorphic to a subgroup of the Weyl group of $\mathfrak{g}$. This observation plays a key rôle in the proof, as it enables us to employ constructions that are analogous to those used by Kac in his proof of the Weyl-Kac character formula.

In Sects. 6, 8 and 9, we specialize to the case of untwisted affine Lie algebras and those automorphisms which correspond to the action of simple currents. In Sect. 6 the action of such automorphisms is described in some detail, using the realization of affine Lie algebras as centrally extended loop algebras. We find that for this special class of automorphisms the characters of $\breve{\mathfrak{g}}$, and hence also the twining characters, have nice modular transformation properties. In Sect. 8 it is shown that the modification of the irreducible characters of $\breve{\mathfrak{g}}$, and hence of the irreducible twining characters of $\mathfrak{g}$, that is required in order to obtain these nice modular transformation properties, differs from the modification of the irreducible characters of $\mathfrak{g}$ only by an overall constant. Finally, in Sect. 9 we comment on those cases where the orbit Lie algebra is one of the twisted affine Lie algebras $\tilde{B}_n^{(2)}$ rather than an untwisted affine algebra.

1.3. Fixed point resolution. Our main motivation for introducing and studying twining characters stems from a long-standing problem in conformal field theory, namely the "resolution of fixed points". Twining characters and orbit Lie algebras constitute important progress towards solving this problem. This will not be discussed further in the present paper, except for the following brief explanation of the relation between the two issues.

The fixed point resolution problem can be divided into two aspects. The first aspect is the construction of representations of the modular group; the second is the description of representation spaces of the chiral symmetry algebra whose characters transform in these representations of the modular group. For theories with an extended chiral symmetry algebra, one tries to achieve the construction of representations of the modular group by starting from the modular transformation matrices S and T of the original, unextended chiral algebra. One then typically finds that certain irreducible modules appear in the spectrum more than once or appear only in reducible linear combinations; it follows in particular that the original matrix S does not contain enough information to derive the matrix S_{ext} of the extended theory. If the extension of the chiral algebra is by simple currents (the corresponding modular invariants are often referred to as "D-type invariants"), these reducible modules originate from fixed points of these simple currents.

By requiring the characters of the extended theory to have the correct modular transformation properties, one learns that the missing information is supplied by another matrix $\mathring{S}$, which is defined only on the fixed point representations and together with a diagonal matrix $\mathring{T}$ again generates a representation of the modular group; $\mathring{T}$ is simply T restricted to the fixed points. By studying the spectrum of $\mathring{T}$ and comparing it to known conformal field theories, conjectures regarding $\mathring{S}$ could be made for most, though not all, simple current invariants for untwisted affine Lie algebras. Indeed, it was found in [3, 4] that in all cases except $B_n^{(1)}$ and $C_{2n}^{(1)}$ at even levels, $\mathring{T}$ is equal up to an overall phase to the T-matrix of another untwisted affine algebra. One may call this the "fixed point algebra" (as we will see in a moment, this is a more appropriate name than the term "fixed point conformal field theory" that was chosen in [3, 4]).

The second aspect of the fixed point resolution problem is closely related to the "field identification" in coset conformal field theories. From the point of view of the modular group, this can be described in terms of an "extension of the chiral algebra by spin-zero currents". As far as the matrix S is concerned we are then in exactly the same situation as discussed above. However, if the field identification currents have fixed points, then there is an additional problem: formally one either obtains a partition function with more than one vacuum state, or, if one normalizes it, a partition function with fractional multiplicities for the fixed point states. The solution to the latter problem is that the various irreducible components of the reducible module that is associated to the fixed point possess in fact different characters. The difference of these characters must then transform like a character with respect to the new modular matrix $\mathring{S}$.

This implies that for field identification fixed points the characters of the coset theory are not simply equal to the branching functions of the embedding of affine Lie algebras, which are merely sums over the characters of the irreducible components. It may seem that writing down the correct irreducible characters requires additional information that is not directly provided by the Lie algebras g and h defining a coset theory $\mathscr{C}(\mathsf{g}/\mathsf{h})$. This additional information is contained in the matrix $\mathring{S}$ and in the character modifications.

As already mentioned, some of the diagram automorphisms introduced above are closely related to the action of simple currents. Simple currents act as a permutation on the modules of the chiral symmetry algebra. More precisely, their action is defined via the fusion rules of the conformal field theory. On the other hand, an action of simple currents on the Hilbert space of states of the theory could so far not be defined, nor was it required for the purpose for which the simple currents were used, namely the construction of modular invariants. In the special case of WZW models, simple currents act by permuting the integrable highest weight modules of the underlying affine Lie algebra. Since the action of some of the diagram automorphisms on highest weights is identical to this simple current action, and since the action of diagram automorphism is defined on *individual* states, the results of the present paper provide a natural definition of the simple current action on the entire Hilbert space.

In the application to field identification in coset models, this should enable us to prove that identified fields are really identical as modules of the chiral algebra. The action on fixed point modules is more interesting still. In this case the module is mapped to itself, and as the mapping has finite order N, the module splits into invariant subspaces of eigenvectors with an N^{th} root of unity as eigenvalue. These eigenspaces are natural candidates for the irreducible modules. The twining characters are then natural ingredients for the character modifications. Note, however, that although the twining characters of untwisted affine Lie algebras are non-trivial, no character modifications (for the Virasoro-specialized characters) are required for the D-type modular invariants of the WZW theories which are associated to the affine Lie algebras. This already shows that more work will be needed to make all this precise; we plan to analyze this situation in detail in a separate publication. The effort will be worthwhile, however, since in this formalism we should be able to derive the character modifications for a coset theory $\mathscr{C}(\mathsf{g}/\mathsf{h})$ in terms of the twining characters of the Kac–Moody algebras g and h rather than having to introduce them as extraneous objects as was done in [3, 4].

An obvious candidate for the fixed point resolution matrix $\mathring{S}$ is the modular transformation matrix $\breve{S}$ of the orbit Lie algebra $\breve{\mathsf{g}}$. Indeed, in [3, 4] the relation between the matrix $\mathring{T}$ and the spectrum of the WZW theory based on an affine algebra g was proved by applying a folding procedure to the weight space metric (the inverse of the symmetrized Cartan matrix) of the horizontal Lie algebra $\bar{\mathsf{g}}$. For all simply-laced algebras and also for C_{2n+1} at even levels this folding procedure is equivalent to the one discussed here, and consequently $\mathring{S} = \breve{S}$ in these cases. In other words, the fixed point algebra is equal to the orbit Lie algebra defined here.

As remarked above, the folding discussed here does not necessarily map untwisted to untwisted affine algebras. This turns out to be relevant for the remaining cases, i.e. for $B_{n+1}^{(1)}$ and $C_{2n}^{(1)}$. Here a "fixed point conformal field theory" could only be identified for odd levels $k = 2p + 1$, namely the WZW theory based on $C_n^{(1)}$ at level p in both cases. For even levels $k = 2p$, the fixed point spectra for $B_{n+1}^{(1)}$ and $C_{2n}^{(1)}$ were shown to differ by an overall constant, but they could not be identified with any known conformal field theory, apart from a few special cases. (These spectra were denoted as $\mathscr{B}_{n,p}$ in [3, 4]. Meanwhile, the matrix $\mathring{S}$ has also been constructed in an indirect manner, using rank-level duality in $N = 2$ supersymmetric coset models [6].) A natural solution now suggests itself, namely that again the fixed point algebra is equal to the orbit Lie algebra, just as in all other cases. Applying our folding procedure, we find that the orbit Lie algebra is in fact a twisted algebra, namely $A_1^{(2)}$ (for $n = 1$) or $\tilde{B}_n^{(2)}$.[1] The fixed points of $B_{n+1}^{(1)}$ and $C_{2n}^{(1)}$ at level $k^\vee$ are in one-to-one correspondence with the representations of $\tilde{B}_n^{(2)}$ at level $k^\vee$.

The modular transformations for characters of twisted algebras do not always close within a given algebra. Rather, typically the characters of one algebra are mapped to those of a different algebra. In fact, $A_1^{(2)}$ and $\tilde{B}_n^{(2)}$ are precisely those twisted affine Lie algebras whose characters possess well-defined modular transformations among themselves, and just as for untwisted algebras, these transformations preserve the level of a module [1]. Remarkably, the modular matrix $\breve{S}$ of $\tilde{B}_n^{(2)}$, respectively $A_1^{(2)}$, appears to provide the correct fixed point resolution for even as well as odd levels. Indeed, $\breve{S}$ at level k is related in the correct way to the matrices S of $C_n^{(1)}$ at level p (for $k = 2p + 1$), respectively $\mathscr{B}_{n,p}$ (for $k = 2p$). At present we do not have a general proof that these matrices resolve the fixed points correctly, but we have checked it for algebras of low rank at low level.

1.4. Organization. Let us briefly summarize how this paper is organized. There are two main results, which concern the automorphisms of Dynkin diagrams satisfying the linking condition and the order N automorphisms of the affine $A_{N-1}^{(1)}$ Dynkin diagrams, respectively. These two theorems are stated at the end of Sect. 4; the

[1] Here we use the notation of [5]; in the notation of [1], these algebras are called $A_{2n}^{(2)}$.

former is proven in Sect. 5 (some details are deferred to Appendix A), and the latter in Sect. 7. In the earlier sections, various concepts which are necessary for being able to formulate our theorems are introduced, such as the folding of Cartan matrices (Sect. 2), the induced automorphisms of Lie algebras and the concept of an orbit Lie algebra (Sect. 3), and the maps induced on Verma and irreducible modules as well as the concept of their twining characters (Sect. 4). The remaining Sects. 6, 8 and 9 contain further details about the special case of affine Kac-Moody algebras, in particular about modular transformation properties, which are relevant for applications in conformal field theory.

2. Folding Cartan Matrices

2.1. Dynkin diagram automorphisms. In this paper we consider symmetries of indecomposable symmetrizable Cartan matrices. A symmetrizable Cartan matrix is by definition a square matrix $A = (A^{i,j})_{i,j \in I}$, where $I \subset \mathbf{Z}$ is some finite index set, satisfying the properties $A^{i,j} \in \mathbf{Z}$, $A^{i,i} = 2$, $A^{i,j} \leq 0$ for $i \neq j$, $A^{i,j} = 0$ iff $A^{j,i} = 0$, and that there is a non-singular diagonal matrix D such that DA is symmetric. To any symmetrizable Cartan matrix there is associated a unique Lie algebra g with an invariant bilinear form $(\cdot|\cdot)\colon \mathsf{g} \times \mathsf{g} \to \mathsf{g}$ (see [1] and Sect. 3). The Dynkin diagram of g is defined as the graph with $|I|$ vertices which has coincidence matrix $2 \cdot \mathbb{1} - A$, with $\mathbb{1}$ the identity matrix. The Dynkin diagram is connected iff A is indecomposable.

By an automorphism of the Dynkin diagram of g (or of the associated Cartan matrix) we mean a bijective mapping $\dot\omega\colon\ I \to I$ satisfying

$$A^{\dot\omega i,\dot\omega j} = A^{i,j} \tag{2.1}$$

for all $i, j \in I$. We denote by N the order of $\dot\omega$, i.e. the smallest positive integer such that $\dot\omega^N = id$ (N is finite since I is finite), and by

$$N_i := |\,\{i, \dot\omega i, \dots, \dot\omega^{N-1} i\}\,| \tag{2.2}$$

the length of the $\dot\omega$-orbit through i. Also, let $\breve I$ denote a set of representatives for the orbits of $\dot\omega$. It will be convenient to fix the choice of these representatives once and for all; for definiteness we choose the smallest representatives of the orbits (for a given labelling by $I \subset \mathbf{Z}$),

$$\breve I := \{i \in I \mid i \leq \dot\omega^n i \ \text{ for } 1 \leq n \leq N-1\}\,. \tag{2.3}$$

We will now show that a large class of automorphisms $\dot\omega$ of symmetrizable Cartan matrices can be used to "fold" the Cartan matrix A such as to obtain another matrix $\breve A$ which is again a symmetrizable Cartan matrix. In particular, with the exception of the automorphism of order N of $A^{(1)}_{N-1}$, all diagram automorphisms of all simple and affine Lie algebras belong to this class.

2.2. The folded Cartan matrix. For any given automorphism $\dot\omega$ of an indecomposable symmetrizable Cartan matrix A and any $i \in I$ let us define the integer

$$s_i := 3 - \frac{N_i}{N} \sum_{l=0}^{N-1} A^{\dot\omega^l i,i} = 1 - \sum_{l=1}^{N_i-1} A^{\dot\omega^l i,i}\,. \tag{2.4}$$

In the following we restrict our attention to the class of those automorphisms $\dot\omega$ which satisfy the relation

$$s_i \leq 2 \quad \text{for all } i \in I\,, \tag{2.5}$$

to which we will refer as the *linking condition*. As the last sum in (2.4) is non-positive and integer, this means that s_i is either 1 or 2. Since each contribution to that sum is non-positive, in the case $s_i = 1$ we have

$$A^{i,\dot\omega^l i} = 0 \tag{2.6}$$

whenever $l \not\equiv 0 \bmod N_i$, and accordingly the restriction of the Dynkin diagram of $\mathfrak{g}$ to the orbit of i is isomorphic to the Dynkin diagram of the direct sum of N_i copies of A_1. For $s_i = 2$, there is exactly one m, $1 \leq m \leq N_i - 1$, such that $A^{\dot\omega^m i,i} = -1$. In this situation we have $\dot\omega^m i = \dot\omega^{-m} i$ (otherwise $A^{\dot\omega^{-m} i,i} = A^{i,\dot\omega^m i}$ would be negative as well, leading to a contradiction with the assumption (2.5)). This implies that in this case N_i and hence also N are even; the restriction of the Dynkin diagram of $\mathfrak{g}$ to the orbit of i is then isomorphic to the Dynkin diagram of the direct sum of $N_i/2$ copies of A_2.

Next we introduce a $|\breve I|\times|\breve I|$-matrix $\breve A$ that is obtained from A by folding it in the sense of summing up the rows of A that are related by $\dot\omega$ and multiplying them by s_i, and afterwards eliminating redundant columns; thus we define

$$\breve A^{i,j} := s_i \frac{N_i}{N} \sum_{l=0}^{N-1} A^{\dot\omega^l i,j} \tag{2.7}$$

for $i,j \in \breve I$. From the indecomposability of A it is obvious that $\breve A$ is indecomposable as well. We claim that $\breve A$ is also again a symmetrizable Cartan matrix, i.e. that it satisfies the following five properties:

$$\begin{aligned} &\text{a)} \quad \breve A^{i,i} = 2 \quad \text{for all } i \in \breve I\,,\\ &\text{b)} \quad \breve A^{i,j} \in \mathbf{Z} \quad \text{for all } i,j \in \breve I\,,\\ &\text{c)} \quad \breve A^{i,j} \leq 0 \quad \text{for all } i,j \in \breve I,\ i \neq j\,,\\ &\text{d)} \quad \breve A^{i,j} = 0 \iff \breve A^{j,i} = 0\,,\\ &\text{e)} \quad \text{there is a non-singular diagonal matrix } \breve D\\ &\qquad \text{such that } \breve B := \breve D \breve A \text{ is symmetric.} \end{aligned} \tag{2.8}$$

Let us prove the relations (2.8) consecutively. First, under the assumption that the linking condition (2.5) holds, we have

$$\breve A^{i,i} = s_i\,(3 - s_i) = 2\,, \tag{2.9}$$

which proves (2.8a). The property (2.8b) is fulfilled because in fact we only add up and multiply integers, as is made manifest by rewriting (2.7) as

$$\breve A^{i,j} = s_i \sum_{l=0}^{N_i-1} A^{\dot\omega^l i,j}\,. \tag{2.10}$$

Next we observe that if $i,j \in \breve I$ are different, then $i,j \in I$ lie on different orbits of $\dot\omega$. As a consequence, $\dot\omega^l i \neq \dot\omega^m j$ for all l,m, and hence, as A is a Cartan matrix,

$A^{\dot\omega^l i,\dot\omega^m j} \le 0$ for all l,m. Thus in particular the sum on the right-hand side of (2.10) is also smaller than zero, which proves (2.8c). Further, since all terms in the sum (2.10) are non-positive, $\breve A^{i,j}=0$ implies that $A^{\dot\omega^l i,j}=0$ for all l. Because of (2.1) this means that also $A^{i,\dot\omega^l j}=0$ for all l. Since A is itself a Cartan matrix, this in turn implies that also $A^{\dot\omega^l j,i}$ vanishes for all l. Thus

$$\breve A^{j,i} = \sum_{l=0}^{N_j-1} A^{\dot\omega^l j,i} = 0\,, \tag{2.11}$$

and hence we obtain the property (2.8d).

Finally, we know that there is a non-singular diagonal matrix $D=\mathrm{diag}(d_i)$ such that $B:=DA$ is symmetric. This matrix is unique up to scalar multiplication, and we can choose $d_i>0$ for all $i\in I$. To verify (2.8e), we first show that $d_i=d_{\dot\omega i}$ for all $i\in I$. To this end suppose that we are given a matrix D which has the required properties. Then we define the "orbit average" $\tilde D$ of D as follows. For any $l=0,1,\ldots,N-1$ we set $D_{(l)}:=\mathrm{diag}(d_{\dot\omega^l i})$, and then define $\tilde D:=\sum_{l=0}^{N-1}D_{(l)}$. The automorphism property of ω implies that $B_{(l)}:=D_{(l)}A$ satisfies

$$B_{(l)}^{i,j} = d_{\dot\omega^l i}A^{i,j} = d_{\dot\omega^l i}A^{\dot\omega^l i,\dot\omega^l j} = B^{\dot\omega^l i,\dot\omega^l j}\,. \tag{2.12}$$

This shows that $B_{(l)}$ is symmetric, and hence $\tilde B:=\tilde DA=\sum_{l=0}^{N-1}B_{(l)}$ is symmetric as well. Thus $\tilde D$ possesses all the properties required for D, so that by the uniqueness of D it follows that $D\propto\tilde D$. This proves that $d_i=d_{\dot\omega i}$ for all $i\in I$, as claimed. Next we define $\breve D$ as

$$\breve D = \mathrm{diag}(\breve d_i)\,, \qquad \breve d_i := \frac{1}{s_i}\,\frac{N}{N_i}\,d_i\,. \tag{2.13}$$

Clearly, $\breve D$ is a non-degenerate diagonal matrix with positive diagonal entries. Further, the entries of $\breve B:=\breve D\breve A$ read

$$\breve B^{i,j} = \frac{N}{s_iN_i}\,d_i\breve A^{i,j} = d_i\sum_{l=0}^{N-1}A^{\dot\omega^l i,j} = \frac1N\sum_{l,l'=0}^{N-1}d_{\dot\omega^l i}A^{\dot\omega^l i,\dot\omega^{l'}j} = \frac1N\sum_{l,l'=0}^{N-1}B^{\dot\omega^l i,\dot\omega^{l'}j}\,. \tag{2.14}$$

This shows that the matrix $\breve B$ is symmetric, and hence completes the proof of (2.8e). As we will see below, the formula (2.14) encountered in this proof is also interesting in its own right; it describes the relation between the invariant bilinear form of $\mathfrak g$ and that of the orbit Lie algebra $\breve{\mathfrak g}$ that will be defined in Subsect. 3.3.

2.3. Type conservation. Symmetrizable Cartan matrices belong to one of the following three classes (compare e.g. [1, §4.3]: they are either of finite, affine or indefinite *type*. We are now going to show that $\breve A$ as obtained from A by the prescription (2.7) is of the same type as A.

If A is symmetrizable and the bilinear form given by $B=DA$ is positive definite, then A is said to be of *finite* (or simple) type. Now for any vector $\breve u=(\breve u_i)_{i\in\breve I}$ we have by (2.14) the relation

$$\sum_{i,j\in\breve I}\breve B^{i,j}\breve u_i\breve u_j = \frac{N}{N_iN_j}\sum_{i,j\in\breve I}\sum_{l=0}^{N_i-1}\sum_{l'=0}^{N_j-1}B^{\dot\omega^l i,\dot\omega^{l'}j}\breve u_i\breve u_j = N\sum_{i,j\in I}B^{i,j}u_iu_j\,, \tag{2.15}$$

where $u_i = \breve{u}_{\dot\omega^{m}i}/N_i$ with m chosen such that $\dot\omega^m i \in \breve{I}$. As a consequence, if B is positive definite, then so is $\breve{B}$, and hence $\breve{A}$ is of finite type as well.

If A is symmetrizable and the bilinear form B is positive semidefinite and has exactly one eigenvector with eigenvalue zero, then A is of *affine* type. The components of the left respectively right eigenvector of A with eigenvalue zero,

$$\sum_{i=0}^{r} a_i A^{i,j} = 0 = \sum_{j=0}^{r} A^{i,j} a_j^{\vee} \tag{2.16}$$

are thus unique once the normalization of the eigenvector is specified (in (2.16), we set $I \equiv \{0, 1, \dots, r\}$, which is the conventional labelling in the affine case). Fixing the normalization in such a way that the minimal value (denoted by a_0 and $a_0^{\vee}$, respectively) is equal to 1, the components a_i and $a_i^{\vee}$ are called the Coxeter labels and dual Coxeter labels of A, respectively. In the affine case (2.14) implies that $\breve{B}$ is either positive definite or positive semidefinite.

Now by (2.16) and the invariance (2.1) of the Cartan matrix, the vector with i^{th} component $a_{\dot\omega i}^{\vee}$ is also a right eigenvector with eigenvalue zero and hence is proportional to the vector of dual Coxeter labels, and an analogous statement holds for the Coxeter labels. The fact that $\dot\omega$ has finite order (together with the positivity of the (dual) Coxeter labels) then implies that

$$a_{\dot\omega i} = a_i\,, \qquad a_{\dot\omega i}^{\vee} = a_i^{\vee} \tag{2.17}$$

for all $i \in I$. It follows that the vectors with entries

$$\breve{a}_i := \frac{s}{s_i}\, a_i\,, \qquad \breve{a}_i^{\vee} := \frac{N_i}{N}\, a_i^{\vee} \qquad \text{for } i \in \breve{I}\,, \tag{2.18}$$

with $s \equiv \max_{j\in I}\{s_j\}$, satisfy

$$\sum_{i\in\breve{I}} \breve{a}_i \breve{A}^{i,j} = \sum_{i\in\breve{I}} \sum_{l=0}^{N-1} s\, \frac{N_i}{N}\, a_i A^{\dot\omega^l i,j} = s \sum_{i\in I} a_i A^{i,j} = 0 \tag{2.19}$$

and [2]

$$\sum_{j\in\breve{I}} \breve{A}^{i,j} \breve{a}_j^{\vee} = \sum_{j\in\breve{I}} \sum_{l=0}^{N-1} s_i\, \frac{N_i N_j}{N^2}\, A^{\dot\omega^l i,j}\, a_j^{\vee} = s_i\, \frac{N_i}{N} \sum_{j\in I} A^{i,j}\, a_j^{\vee} = 0\,. \tag{2.20}$$

In particular there is an eigenvector of $\breve{B}$ with eigenvalue zero, i.e. $\breve{B}$ is positive semidefinite rather than positive definite. Also, the eigenspace of $\breve{B}$ to the eigenvalue zero is one-dimensional, since if $\breve{v}$ with entries $\breve{v}_i$, $i \in \breve{I}$, is an eigenvector of $\breve{B}$ to the eigenvalue zero, then the vector v with entries $v_i = \frac{1}{N_i}\breve{v}_{\dot\omega^l i}$, with l such that $\dot\omega^l i \in \breve{I}$, is an eigenvector of B to the eigenvalue zero, which is, however, unique up to normalization.

Finally, a symmetrizable Cartan matrix is said to be of *indefinite* type if it is neither of finite nor of affine type. Let us show that if A is of indefinite type, then $\breve{A}$ is also of indefinite type. By Theorem 4.3 of [1], A is of indefinite type iff

[2] The chosen normalization of $\breve{a}_i^{\vee}$ proves to be convenient for the treatment of the affine case. In the general case, with this specific normalization the coefficients $\breve{a}_i^{\vee}$ are, however, not necessarily integral.

1. there is a vector u with strictly positive components, such that uA has strictly negative components, and
2. the fact that a vector v and vA both have positive components implies that v is the zero vector.

To show that the first condition is fulfilled for the folded Cartan matrix $\breve{A}$, assume that u is a vector with strictly positive components for which uA has strictly negative components. Clearly, the vector u' with i^{th} component $u'_i = u_{\dot\omega i}$ shares these properties of u, and hence we can assume without loss of generality that $u_i = u_{\dot\omega i}$. We then define $\breve{u}_i := u_i/s_i$; this is positive as well, and also obeys

$$\sum_{i\in\breve{I}} \breve{u}_i \breve{A}^{i,j} = \sum_{i\in I} u_i A^{i,j} < 0 \quad \text{for all } j \in \breve{I}\,. \tag{2.21}$$

To show that the second condition is fulfilled, we assume that $\breve{v}$ is a vector such that

$$\breve{v}\breve{A} \geq 0 \quad \text{and} \quad \breve{v} \geq 0\,. \tag{2.22}$$

Define a vector v by $v_i := s_i \breve{v}_{\dot\omega^{l_i}}$ for $i \in I$, where l is chosen such that $\dot\omega^l i \in \breve{I}$. Then v fulfills the conditions (2.22) with $\breve{A}$ replaced by A. Since A is by assumption of indefinite type, v and hence also $\breve{v}$ have to vanish. Together, these results imply that $\breve{A}$ is of indefinite type as well.

We have thus shown that the matrix $\breve{A}$ that is obtained by the folding prescription (2.7) is always of the same type as the Cartan matrix A.

A particularly interesting subclass among the Cartan matrices of indefinite type is given by the *hyperbolic* Cartan matrices. These are characterized by the additional property that any indecomposable submatrix of the Cartan matrix A that is obtained by deleting any row and the corresponding column of A is of finite or affine type. Again one can show that if A is hyperbolic then the same is true for $\breve{A}$. Namely, the pre-image (under the folding) of any proper subdiagram of the Dynkin diagram of $\breve{A}$ is a subdiagram of the Dynkin diagram of A, which, as A is assumed to be hyperbolic, is of finite or affine type. But as we have just seen, these diagrams are mapped to diagrams of affine or finite type, and hence the subdiagram of $\breve{A}$ has to be of affine or finite type as well. This shows that also $\breve{A}$ is hyperbolic.

2.4. Simple Cartan matrices. In the next two subsections we will list all automorphisms of all simple and affine Dynkin diagrams explicitly. The numbering of the nodes of the Dynkin diagrams is taken from [5, p. 43]. Below we write $\mathfrak{g}$ and $\breve{\mathfrak{g}}$ for the Kac–Moody algebras which have Cartan matrix A and $\breve{A}$, respectively ($\breve{\mathfrak{g}}$ will be called the orbit Lie algebra associated to $\mathfrak{g}$ and ω, see Subsect. 3.3 below).

The non-trivial automorphisms of the Dynkin diagrams of simple Lie algebras are as follows. For A_r, D_r and E_6 there is a reflection which we denote by γ; it acts as $i \to r+1-i$ for A_r, as $r-1 \leftrightarrow r$, $i \mapsto i$ else, for D_r, and as $1 \leftrightarrow 5,\ 2 \leftrightarrow 4,\ 3 \mapsto 3,\ 6 \mapsto 6$ for E_6. In addition, for D_4 there is the triality ρ_3, an order three rotation which acts as $2 \mapsto 2,\ 1 \mapsto 3 \mapsto 4 \mapsto 1$.

The algebras $\breve{\mathfrak{g}}$ for the cases when $\mathfrak{g}$ is simple are listed in Table (2.23); this is in fact a well known list, as $\breve{\mathfrak{g}}$ plays an important role in the realization of the twisted affine Lie algebras as centrally extended twisted loop algebras [1].

$\mathfrak{g}$	ω	N	$\breve{\mathfrak{g}}$
$A_{2n+1},\ n>0$	γ	2	B_{n+1}
D_4	ρ_3	3	G_2
$D_n,\ n\geq 4$	γ	2	C_{n-1}
E_6	γ	2	F_4
A_{2n}	γ	2	C_n

(2.23)

In this table we have separated the A_{2n} case from the others because in this case we have $s_n = 2$ (and $s_i = 1$ else), whereas in all other cases all the s_i are equal to 1.

2.5. Affine Cartan matrices. The relevant automorphisms $\dot\omega$ for affine Lie algebras are the following. For $\mathfrak{g} = A_n^{(1)}$, the automorphism group of the Dynkin diagram is the dihedral group $\mathscr{D}_{n+1}$ which is generated by the reflection $\gamma\colon\ i \mapsto n+1-i \bmod n+1$ and the rotation $\sigma_{n+1}:\ i \mapsto i+1 \bmod n+1$ which is of order $n+1$. Among the powers σ_{n+1}^{l}, only those need to be considered for which l is a divisor of $n+1$ so that the order is $N = (n+1)/l$.

For $\mathfrak{g} = D_r^{(1)}$ the automorphism group is generated by the "vector automorphism" $\sigma_{\rm v}$, the "spinor automorphism" $\sigma_{\rm s}$ and a conjugation γ. $\sigma_{\rm v}$ acts as $0 \leftrightarrow 1,\ r \leftrightarrow r-1$ and $i \mapsto i$ else, and hence is of order two; the map γ acts as $r \leftrightarrow r-1$ and $i \mapsto i$ else. If r is even, $\sigma_{\rm s}$ acts as $i \mapsto r-i$ and hence has order two, while for odd r the prescription $i \mapsto r-i$ only holds for $2 \leq i \leq r-2$ and is supplemented by $0 \mapsto r \mapsto 1 \mapsto r-1 \mapsto 0$, so that $\sigma_{\rm s}$ has order 4. If $r = 4$, then the automorphism group is larger, namely the symmetric group $\mathscr{S}_4$; it contains as additional symmetries an order four rotation ρ_4, which acts as $0 \mapsto 1 \mapsto 3 \mapsto 4 \mapsto 0,\ 2 \mapsto 2$, and an order three permutation ρ_3, acting like $1 \mapsto 3 \mapsto 4 \mapsto 1,\ 2 \mapsto 2$ and $0 \mapsto 0$.

For the untwisted algebras $\mathfrak{g} = B_r^{(1)}$, $C_r^{(1)}$ and $E_7^{(1)}$ and for the twisted algebras $\mathfrak{g} = B_r^{(2)}$ and $C_r^{(2)}$, there is only a single non-trivial automorphism γ which is a reflection. For $\mathfrak{g} = E_6^{(1)}$ the automorphism group of the Dynkin diagram is the symmetric group $\mathscr{S}_3$; it is generated by the order three rotation $\sigma\colon\ 1 \mapsto 5 \mapsto 0 \mapsto 1,\ 2 \mapsto 4 \mapsto 6 \mapsto 2,\ 3 \mapsto 3$ and the reflection $\gamma\colon\ 1 \mapsto 5,\ 2 \mapsto 4,\ 3 \mapsto 3,\ 6 \mapsto 6,\ 0 \mapsto 0$. Finally, for $\mathfrak{g} = E_8^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ and for the remaining twisted algebras, there are no non-trivial Dynkin diagram automorphisms at all.

$\mathfrak{g}$	ω	N	$\breve{\mathfrak{g}}$
$A_n^{(1)}$	$(\sigma_{n+1})^{(n+1)/N}$	$N<n+1$	$A_{((n+1)/N)-1}^{(1)}$
$A_{2n+1}^{(1)}$	γ	2	$B_{n+1}^{(2)}$
$B_n^{(1)}$	$\sigma_{\rm v}$	2	$\tilde{B}_{n-1}^{(2)}$
$B_{2n}^{(2)}$	γ	2	$B_n^{(2)}$
$C_{2n}^{(1)}$	σ	2	$\tilde{B}_n^{(2)}$
$C_2^{(1)}$	σ	2	$A_1^{(2)}$
$C_n^{(2)}$	γ	2	$C_{n-1}^{(1)}$
$D_4^{(1)}$	ρ_4	4	$A_1^{(2)}$
$D_4^{(1)}$	ρ_3	3	$G_2^{(3)}$
$D_n^{(1)}$	$\sigma_{\rm v}$	2	$C_{n-2}^{(1)}$
$D_n^{(1)}$	γ	2	$C_{n-1}^{(2)}$
$D_{2n}^{(1)}$	$\sigma_{\rm s}$	2	$B_n^{(1)}$
$D_{2n}^{(1)}$	$\sigma_{\rm s}\gamma$	4	$\tilde{B}_{n-1}^{(2)}$
$E_6^{(1)}$	σ	3	$G_2^{(1)}$
$E_6^{(1)}$	γ	2	$F_4^{(2)}$
$E_7^{(1)}$	σ	2	$F_4^{(1)}$
$A_{2n}^{(1)}$	γ	2	$\tilde{B}_n^{(2)}$
$A_{2n+1}^{(1)}$	$\sigma_{n+1}\gamma$	2	$C_n^{(1)}$
$B_{2n+1}^{(2)}$	γ	2	$\tilde{B}_n^{(2)}$
$C_{2n+1}^{(1)}$	σ	2	$C_n^{(1)}$
$D_{2n+1}^{(1)}$	$\sigma_{\rm s}$	4	$C_{n-1}^{(1)}$
$D_{2n+1}^{(1)}$	$\sigma_{\rm s}\gamma$	2	$C_n^{(2)}$
$A_n^{(1)}$	σ_{n+1}	$n+1$	$\{0\}$

(2.24)

Notation of [1]	$A_2^{(2)}$	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
Notation of [5]	$A_1^{(2)}$	$\tilde{B}_n^{(2)}$	$C_n^{(2)}$	$B_n^{(2)}$	$F_4^{(2)}$	$G_2^{(3)}$

Let us remark that the notation has been chosen such that for the untwisted affine algebras, γ implements charge conjugation, while σ corresponds, in conformal field theory terms, to a simple current [2].

We list $\breve{\mathsf{g}}$ for these automorphisms in Table (2.24). In this table we again separate the cases where all the s_i are equal to 1 from the others. Also, there is a single series of automorphisms which do not obey the linking condition (2.5), namely for any $N \geq 2$ the automorphism of the Dynkin diagram of $A^{(1)}_{N-1}$ that has order N; this series is displayed in the last row of the table. In this case, which will be treated separately in Sect. 7, there is only a single s, which takes the value zero, and the prescription (2.7) formally yields a one-by-one matrix with entry zero.

Also, in the table we use the notations of [5, p. 95] for twisted affine algebras; the relation with the notation of [1, p. 55] is indicated below the table.

2.6. Hyperbolic Cartan matrices. One can, of course, compile an analogous list for the hyperbolic Lie algebras as well. However, the number of these algebras and their automorphisms (satisfying the linking condition) is rather large, and hence we refrain from presenting this list here. Let us just mention that the result that along with g also $\breve{\mathsf{g}}$ is a hyperbolic Lie algebra may be easily verified case by case. As a by-product, this provides a check on the completeness of the list of hyperbolic Lie algebras that has been given in the literature. [3]

3. Lie Algebra Automorphisms

In this section we show that any automorphism $\dot{\omega}$ of finite order of the Cartan matrix A induces an automorphism of the same order of the Kac-Moody algebra g which has Cartan matrix A. To this end we first sketch how g can be constructed from the Cartan matrix [1]. Then we show how $\dot{\omega}$ induces an automorphism ω of g and investigate to what extent this automorphism is unique.

3.1. Symmetrizable Kac-Moody algebras. To any symmetrizable Cartan matrix A there is associated a Lie algebra, denoted by $\mathsf{g} \equiv \mathsf{g}(A)$ and called a symmetrizable Kac-Moody algebra, which is unique up to isomorphism [1, Prop. 1.1]. g is constructed from A as follows. Denote by n the dimension and by r the rank of the matrix A. We introduce a complex vector space $\mathsf{g}_\circ$ of complex dimension $2n - r$. Next we choose n linearly independent elements H^i of $\mathsf{g}_\circ$ and n linearly independent functionals $\alpha^{(i)} \in \mathsf{g}_\circ^\star$ (called the simple roots of g), such that $\alpha^{(i)}(H^j) = A^{i,j}$ for $i, j = 1, 2, \ldots, n$. This choice is unique up to isomorphism.

The Kac-Moody algebra g is then the Lie algebra that is generated freely by the elements of $\mathsf{g}_\circ$ and $2n$ further elements $E^i_\pm \equiv E^{\pm\alpha^{(i)}}$, with $i \in I \equiv \{1, 2, \ldots, n\}$, modulo the relations

$$\begin{aligned} &[x, y] = 0 \qquad \text{for all } x, y \in \mathsf{g}_\circ\,, \\ &[x, E^i_\pm] = \pm\alpha^{(i)}(x)\, E^i_\pm \quad \text{for all } x \in \mathsf{g}_\circ\,, \\ &[E^i_+, E^j_-] = \delta_{ij}\, H^j\,, \\ &(\mathrm{ad}_{E^i_\pm})^{1-A^{j,i}} E^j_\pm = 0 \quad \text{for } i \neq j\,, \end{aligned} \tag{3.1}$$

[3] In fact, the classification of hyperbolic Lie algebras presented in [7] turns out to be not quite complete. We thank C. Saclioglu for a correspondence on this issue.

where the map ad_x is defined by $\mathrm{ad}_x(y) := [x, y]$. Thus the subspace $\mathsf{g}_\circ$ is an abelian subalgebra of g; it is called the Cartan subalgebra of g. Also, g has a triangular decomposition

$$\mathsf{g} = \mathsf{g}_+ \oplus \mathsf{g}_\circ \oplus \mathsf{g}_- \,, \tag{3.2}$$

where $\mathsf{g}_\pm$ are subalgebras and generated freely by the $E^i_\pm$ modulo the relations in the last line of (3.1), which are known as the Serre relations.

The algebra $\hat{\mathsf{g}} := [\mathsf{g}, \mathsf{g}]$ is called the derived algebra of g. It contains all central elements and has a triangular decomposition

$$\hat{\mathsf{g}} \equiv [\mathsf{g}, \mathsf{g}] = \mathsf{g}_+ \oplus \hat{\mathsf{g}}_\circ \oplus \mathsf{g}_- \,, \tag{3.3}$$

where $\hat{\mathsf{g}}_\circ$ is the span of the elements H^i, $i = 1, 2, \dots, n$. Thus $\hat{\mathsf{g}}_\circ \subseteq \mathsf{g}_\circ$ is the Cartan subalgebra of $\hat{\mathsf{g}}$, and the *derivations*, i.e. the generators of a complement of $\hat{\mathsf{g}}$ in g, span a complement of $\hat{\mathsf{g}}_\circ$ in $\mathsf{g}_\circ$. We will also denote by g_K the common kernel of all the simple roots $\alpha^{(i)}$ (and hence of all roots, since any root α is a linear combination of the $\alpha^{(i)}$). g_K is a subspace of $\hat{\mathsf{g}}_\circ$.

By definition, the non-degenerate bilinear form of g satisfies

$$(H^i \mid x) = d_i\, \alpha^{(i)}(x) \quad \text{for all } x \in \mathsf{g}_\circ \tag{3.4}$$

with d_i as defined after (2.11), and hence in particular

$$(H^i \mid H^j) = d_i\, A^{i,j} = B^{i,j} \quad \text{for } i = 1, 2, \dots, n \,. \tag{3.5}$$

3.2. Induced outer automorphisms. We are now in a position to construct an automorphism ω of g using any symmetry $\dot\omega$ of the Dynkin diagram of g. We start by defining ω on the generators $E^i_\pm$ of $\mathsf{g}_\pm$:

$$\omega(E^i_\pm) := E^{\dot\omega i}_\pm \,. \tag{3.6}$$

Because of (2.1) this mapping preserves the Serre relations, and hence it provides automorphisms of both g_+ and g_-. Further, the automorphism property of ω implies that it has to act on the H^i as

$$\omega(H^i) = \omega([E^i_+, E^i_-]) = [E^{\dot\omega i}_+, E^{\dot\omega i}_-] = H^{\dot\omega i} \,. \tag{3.7}$$

This way we have constructed a unique automorphism of the derived algebra $\hat{\mathsf{g}}$ of g. This automorphism has the same order N as the automorphism $\dot\omega$ of the Dynkin diagram. To show how ω acts on the rest of the Cartan subalgebra of g, i.e. on the derivations, requires a bit more work. To this end it is helpful to work with a special basis of $\mathsf{g}_\circ$.

Since $\alpha^{(i)}(H^j) = A^{i,j} = A^{\dot\omega i, \dot\omega j} = \alpha^{(\dot\omega i)}(H^{\dot\omega j})$ and since the H^j span $\hat{\mathsf{g}}_\circ$, for all $x \in \hat{\mathsf{g}}_\circ$ we have

$$\alpha^{(\dot\omega i)}(\omega(x)) = \alpha^{(i)}(x) \,. \tag{3.8}$$

Hence the subspace g_K is mapped by ω bijectively to itself. We can therefore diagonalize ω on g_K and choose a basis of $n - r$ eigenvectors K^a, $a = 1, 2, \dots, n - r$, such that

$$\omega(K^a) = \zeta^{n_a} K^a \,, \tag{3.9}$$

where

$$\zeta := \exp\left(\frac{2\pi\mathrm{i}}{N}\right) \tag{3.10}$$

is a primitive N^{th} root of unity. We can extend the basis of $\mathfrak{g}_K$ to a basis of $\hat{\mathfrak{g}}_\circ$ by adding further eigenvectors of ω on $\hat{\mathfrak{g}}_\circ$, which we denote by J^p, $p = 1, 2, \dots, r$. We write

$$\omega(J^p) = \zeta^{m_p} J^p \,, \tag{3.11}$$

and denote the span of all J^p by $\mathfrak{g}_J$. Clearly, the restriction of the invariant bilinear form to $\mathfrak{g}_J$ is non-degenerate. Moreover, we find that

$$(K^a \mid x) = 0 \quad \text{for all } x \in \hat{\mathfrak{g}}_\circ \,. \tag{3.12}$$

This holds because writing $K^a = \sum_{j=1}^n \kappa_j^a H^j$ we have

$$0 = \alpha^{(i)}(K^a) = \sum_{j=1}^{n} \kappa_j^a \alpha^{(i)}(H^j) = \sum_{j=1}^{n} A^{i,j} \kappa_j^a \tag{3.13}$$

for all $i = 1, 2, \dots, n$, so that the invariant bilinear form obeys

$$(H^i \mid K^a) = \sum_j \kappa_j^a \, (H^i \mid H^j) = \sum_j d_i A^{i,j} \kappa_j^a = 0 \tag{3.14}$$

for all $i = 1, 2, \dots, n$.

Using the fact that the invariant form is non-degenerate on $\mathfrak{g}_\circ$, we conclude that there are $n - r$ unique elements D^a of $\mathfrak{g}_\circ$ such that

$$\begin{aligned} &(D^a \,|\, K^b) = \delta^{ab} && \text{for all} \quad 1 \le a, b \le n - r \,, \\ &(D^a \,|\, D^b) = 0 && \text{for all} \quad 1 \le a, b \le n - r \,, \\ &(D^a \,|\, J^p) = 0 && \text{for all} \quad 1 \le a \le n - r, \ 1 \le p \le r \,. \end{aligned} \tag{3.15}$$

The elements D^a are linearly independent and span a complement $\mathfrak{g}_D$ of $\hat{\mathfrak{g}}_\circ$ in $\mathfrak{g}_\circ$.

We can now study the action of ω on the derivations D^a. To this end we first show that $\omega(D^a)$ is again an element of the Cartan subalgebra $\mathfrak{g}_\circ$. Namely, let us start from the most general ansatz $\omega(D^a) = h + \sum_{\alpha,\ell} \xi_{\alpha,\ell} E^{\alpha,\ell}$, where $h \in \mathfrak{g}_\circ$ and the elements $E^{\alpha,\ell}$ are generators of the root space for the root α (thus the number of possible values of the index ℓ equals the (finite) dimension of this root space). Assume now that $\xi_{\alpha,\ell} \neq 0$ for some root $\alpha > 0$ (the argument for α a negative root is completely parallel) and some ℓ. The step operator $E^{-\alpha,\ell}$ is an element of $\mathfrak{g}_-$, and hence so is $\omega^{-1}(E^{-\alpha,\ell})$. Because of $D^a \in \mathfrak{g}_\circ$ and the fact that ω is an automorphism of $\mathfrak{g}$, this implies that $[D^a, \omega^{-1}(E^{-\alpha,\ell})]$ and $[\omega(D^a), E^{-\alpha,\ell}] = \omega([D^a, \omega^{-1}(E^{-\alpha,\ell})])$ are elements of $\mathfrak{g}_-$, too. On the other hand, we have $[E^{\alpha,\ell}, E^{-\beta,\ell'}] = \delta_{\alpha,\beta}\delta_{\ell,\ell'} h^{\alpha,\ell}$ with non-vanishing $h^{\alpha,\ell} \in \mathfrak{g}_\circ$, so that by inserting the ansatz we made above we find that the element $[\omega(D^a), E^{-\alpha,\ell}]$ of $\mathfrak{g}$ has a component $\xi_{\alpha,\ell} h^{\alpha,\ell}$ in $\mathfrak{g}_\circ$. This is a contradiction, and hence the assumption that $\xi_{\alpha,\ell} \neq 0$ is wrong. Thus we conclude that $\omega(D^a) \in \mathfrak{g}_\circ$.

We can therefore make the general ansatz

$$\omega(D^a) = \sum_{b=1}^{n-r} (U_b^a D^b + \zeta^{n_b} V_b^a K^b) + \sum_{p=1}^{r} W_p^a J^p \,. \tag{3.16}$$

Here in the second term we have introduced an explicit phase factor, which will simplify the discussion below. We now impose the condition that ω preserves the invariant bilinear form, i.e. require that $(\omega(D^a)\,|\,\omega(J^p)) = 0 = (\omega(D^a)\,|\,\omega(D^b))$ and $(\omega(D^a)\,|\,\omega(K^b)) = \delta^{ab}$ for all $p = 1,2,\ldots,r,\ a,b = 1,2,\ldots,n-r$. Inserting the ansatz (3.16), the first of these conditions reads

$$0 = (\omega(D^a)\mid\omega(J^p)) = \zeta^{m_p}\,(\sum_{q=1}^{r} W^a_q J^q \mid J^p) \tag{3.17}$$

for all $p = 1,2,\ldots,r$. As the metric on $\mathfrak{g}_J$ is non-degenerate, this implies that W^a_q vanishes. The second requirement then amounts to

$$\delta^{ab} = (\omega(D^a)\mid\omega(K^b)) = \sum_{c=1}^{n-r} U^a_c \zeta^{n_b}(D^c \mid K^b) = U^a_b \zeta^{n_b}\,. \tag{3.18}$$

Thus the ansatz (3.16) for $\omega(D^a)$ gets reduced to

$$\omega(D^a) = \zeta^{-n_a} D^a + \sum_{b=1}^{n-r} V^a_b \zeta^{n_b} K^b\,. \tag{3.19}$$

The last requirement then constrains the matrix V; we obtain

$$0 = (\omega(D^a)\mid\omega(D^b)) = V^a_b + V^b_a\,, \tag{3.20}$$

i.e. V has to be an antisymmetric matrix.

To summarize, we have shown that the only freedom we are left with consists in adding terms proportional to central elements to $\omega(D^a)$, and that this freedom is parametrized in terms of an antisymmetric $(n-r)\times(n-r)$ matrix. In the particularly interesting cases where $\mathfrak{g}$ is simple, affine, or hyperbolic, there is thus no freedom left at all; in the simple and hyperbolic cases there are no derivations, while in the affine case no term with the central element K appears (the only antisymmetric one-by-one matrix is zero) so that just $\omega(D) = D$.

We can restrict the freedom in $\omega(D)$ even more by imposing the requirement that ω has order N also on the derivations D^a. The relation (3.19) implies that

$$\omega^l(D^a) = \zeta^{-ln_a} D^a + \sum_{b=1}^{n-r} V^a_b (\sum_{t=1}^{l} \zeta^{tn_b-(l-t)n_a}) K^b\,. \tag{3.21}$$

It follows that ω has order N if and only if V^a_b vanishes whenever $n_a = -n_b \bmod N$. It is also clear that these constraints always possess the trivial solution $V \equiv 0$.

From the invariance of the bilinear form on $\mathfrak{g}_\circ$ it follows that ω as defined above is in fact an automorphism of $\mathfrak{g}$. The only identity that still has to be shown to this end is that $\alpha^{(i)}(D^a)$ coincides with $\alpha^{(\dot\omega i)}(\omega(D^a))$; this follows by

$$\alpha^{(i)}(D^a) = \frac{1}{d_i}\,(H^i\,|\,D^a) = \frac{1}{d_{\dot\omega i}}\,(H^{\dot\omega i}\,|\,\omega(D^a)) = \alpha^{(\dot\omega i)}(\omega(D^a))\,. \tag{3.22}$$

From now on we will assume that V has been chosen such that ω has in fact order N on all of $\mathfrak{g}$. We will refer to such an automorphism which respects the triangular decomposition as a *strictly outer automorphism* or as a *diagram automorphism* of $\mathfrak{g}$.

The first of these terms is appropriate because any such automorphism is indeed outer, as can be seen e.g. by the fact that (compare Sect. 4 below) it induces a non-trivial map on the representation ring of g, whereas inner automorphisms do not change the isomorphism class of a representation.

3.3. Orbit Lie algebras. We denote by $\breve{\mathsf{g}}$ the symmetrizable Kac–Moody algebra that has $\breve{A}$ as its Cartan matrix and call $\breve{\mathsf{g}}$ the *orbit Lie algebra* that is associated to the Dynkin diagram automorphism $\dot\omega$, respectively to the automorphism ω of g that is induced by $\dot\omega$. We would like to stress that $\breve{\mathsf{g}}$ is *not* constructed as a subalgebra of g; in particular it need not be isomorphic to the subalgebra of g that consists of those elements which are mapped to themselves by ω. There does exist, however, a subalgebra $\bar{\mathsf{g}}$ of g (to be described elsewhere) which is pointwise fixed under ω and whose Cartan matrix is closely related to the Cartan matrix $\breve{A}$ of $\breve{\mathsf{g}}$; namely, the transpose of the Cartan matrix of $\bar{\mathsf{g}}$ is equal to the matrix $(A^t)^{\breve{}}$ that one obtains when applying our folding procedure to the transpose A^t of the Cartan matrix A of g.

Later on, we will use this orbit Lie algebra to describe aspects of the action that ω induces on irreducible highest weight modules of g. To this end, we need to set up some relations between g and $\breve{\mathsf{g}}$. In preparation for these considerations, we first show that there is a close relation between the Cartan subalgebra $\breve{\mathsf{g}}_\circ$ of the orbit Lie algebra and the eigenspace $\mathsf{g}_\circ^{(0)}$ of ω to the eigenvalue $\zeta^0 = 1$ in $\mathsf{g}_\circ$. This relation is described by a map P_ω which is defined as follows. First consider the subalgebra $\hat{\mathsf{g}}_\circ^{(0)} := \mathsf{g}_\circ^{(0)} \cap \hat{\mathsf{g}}_\circ$ of $\mathsf{g}_\circ^{(0)}$. The elements of $\hat{\mathsf{g}}_\circ^{(0)}$ are those elements $h = \sum_{i=1}^n v_i H^i$ of $\hat{\mathsf{g}}_\circ$ which are fixed under ω, $\omega(h) = h$, which implies that $v_i = v_{\dot\omega^l i}$ for all l. To any such h we associate the element

$$\mathrm{P}_\omega(h) := \sum_{i\in\breve{I}} \breve{v}_i \breve{H}^i \tag{3.23}$$

of $\hat{\breve{\mathsf{g}}}_\circ$, where

$$\breve{v}_i := N_i v_i \tag{3.24}$$

for all $i \in \breve{I}$. It is obvious that the map P_ω is an isomorphism between $\hat{\mathsf{g}}_\circ^{(0)}$ and $\hat{\breve{\mathsf{g}}}_\circ$.

Moreover, the invariant bilinear forms on $\hat{\mathsf{g}}_\circ^{(0)}$ and $\hat{\breve{\mathsf{g}}}_\circ$ satisfy

$$(h \mid h') = \frac{1}{N}\,(\mathrm{P}_\omega(h) \mid \mathrm{P}_\omega(h')) \tag{3.25}$$

for all $h, h' \in \hat{\mathsf{g}}_\circ^{(0)}$. To prove this relation, it is sufficient to check it on a basis of $\hat{\mathsf{g}}_\circ^{(0)}$. As a basis, we choose

$$h^i := \frac{1}{N}\sum_{l=0}^{N-1} H^{\dot\omega^l i} = \frac{1}{N_i}\sum_{l=0}^{N_i-1} H^{\dot\omega^l i} \tag{3.26}$$

for $i \in \breve{I}$. Then we have $\mathrm{P}_\omega(h^i) = \breve{H}^i$, and we can use (3.5) to find

$$(h^i \mid h^j) = \frac{1}{N^2}\sum_{l,l'=0}^{N-1} B^{\dot\omega^l i,\dot\omega^{l'} j} = \frac{1}{N}\,\breve{B}^{i,j} = \frac{1}{N}\,(\mathrm{P}_\omega(h^i) \mid \mathrm{P}_\omega(h^j))\,. \tag{3.27}$$

Next we show that P_ω yields a one-to-one correspondence between central elements in $\hat{\mathsf{g}}_\circ^{(0)}$ and central elements of $\breve{\mathsf{g}}$. First note that $K = \sum_{i=1}^n \kappa_i H^i$ is central iff

$$\sum_{i\in I} A^{j,i}\kappa_i = 0 \quad \text{for all} \quad j\in I\,. \tag{3.28}$$

Now (3.28) implies that

$$\sum_{i\in\breve I} \breve A^{j,i} N_i\kappa_i = N_j\, s_j \sum_{i\in I} A^{j,i}\kappa_i = 0 \quad \text{for all} \quad j\in \breve I\,, \tag{3.29}$$

and hence also the element $\mathrm{P}_\omega(K) = \sum_{i\in\breve I} \breve\kappa_i \breve H^i$ of $\breve{\mathfrak g}$, with $\breve\kappa_i := N_i\kappa_i$, is central. Conversely, with $\breve K$ also the pre-image $\mathrm{P}_\omega^{-1}(\breve K)$ is central. This result shows in particular that the dimension of $\mathfrak g_K^{(0)}$ is precisely $|\breve I| - \breve r$, where $\breve r$ is the rank of $\breve A$.

Finally we can continue the range of definition of P_ω to all of $\mathfrak g_\circ^{(0)}$ such that (3.25) is still valid: we use again the basis of eigenvectors of ω introduced in Subsect. 3.2. Given the derivation D^a, consider the projection $\mathrm{P}_\omega(K^a)\in\breve{\mathfrak g}$ of the corresponding central element $K^a\in\mathfrak g$. Since the bilinear form on $\breve{\mathfrak g}_\circ$ is non-degenerate, for each $D^a\in\mathfrak g_D^{(0)}$ we can define $\mathrm{P}_\omega(D^a)$ to be the unique derivation in $\breve{\mathfrak g}_\circ$ for which

$$(\mathrm{P}_\omega(D^a)\mid \mathrm{P}_\omega(D^b)) = 0\,, \qquad (\mathrm{P}_\omega(D^a)\mid \mathrm{P}_\omega(K^b)) = N\delta^{ab} \tag{3.30}$$

for all $K^b\in\mathfrak g_K^{(0)}$, and

$$(\mathrm{P}_\omega(D^a)\mid \mathrm{P}_\omega(x)) = 0 \qquad \text{for all } x\in\mathfrak g_J^{(0)} \tag{3.31}$$

(the factor of N in the second of the conditions (3.30) ensures that the relation (3.25) between the invariant bilinear forms on $\hat{\mathfrak g}_\circ^{(0)}$ and $\mathring{\breve{\mathfrak g}}_\circ$ extends to all of $\mathfrak g_\circ^{(0)}$ and $\breve{\mathfrak g}_\circ$). This completes the definition of P_ω.

We can use the action of ω to define a dual action, denoted by $\omega^\star$, on the space $\mathfrak g_\circ^\star$ that is dual to $\mathfrak g_\circ$, i.e. on the weight space of $\mathfrak g$, namely as

$$(\omega^\star\beta)(x) := \beta(\omega^{-1}x) \tag{3.32}$$

for all $\beta\in\mathfrak g_\circ^\star$ and all $x\in\mathfrak g_\circ$. The natural correspondence between $\mathfrak g_\circ^{(0)}$ and the Cartan subalgebra $\breve{\mathfrak g}_\circ$ of $\breve{\mathfrak g}$ implies a corresponding relation for the dual spaces, the weight spaces. We therefore have a bijective map

$$\mathrm{P}_\omega^\star:\quad \breve{\mathfrak g}_\circ^\star \to \mathfrak g_\circ^{\star(0)} \tag{3.33}$$

between the weights of $\breve{\mathfrak g}$ and the weights $\lambda\in\mathfrak g_\circ^{\star(0)}$, i.e. those weights of $\mathfrak g$ that are fixed under $\omega^\star$, $\omega^\star\lambda=\lambda$. We will refer to the elements of $\mathfrak g_\circ^{\star(0)}$ as *symmetric* $\mathfrak g$-weights. For brevity, we will also often denote the pre-image $\mathrm{P}_\omega^{\star-1}(\lambda)\in\breve{\mathfrak g}_\circ^\star$ of $\lambda\in\mathfrak g_\circ^{\star(0)}$ by $\breve\lambda$.

By duality, the invariant bilinear form on $\mathfrak g_\circ^{(0)}$ defines an invariant bilinear form on $\mathfrak g_\circ^{\star(0)}$, and analogously for $\breve{\mathfrak g}$. The relation (3.25) between the restriction of the invariant bilinear form on $\mathfrak g_\circ^{(0)}$ and the bilinear form on $\breve{\mathfrak g}_\circ$ therefore implies an analogous relation between the bilinear form on symmetric weights $\lambda\in\mathfrak g_\circ^{\star(0)}$ and the one on $\breve{\mathfrak g}$-weights: [4]

$$(\lambda\mid\mu) = N\cdot(\mathrm{P}_\omega^{\star-1}(\lambda)\mid \mathrm{P}_\omega^{\star-1}(\mu)) \equiv N\cdot(\breve\lambda\mid\breve\mu)\,. \tag{3.34}$$

[4] As for Eqns. (3.34) and (3.25), the following remark is in order. For an arbitrary symmetrizable Kac-Moody algebra there is no canonical normalization of the invariant bilinear symmetric form. On the other hand, in (3.34) and (3.25) the relative normalization of these forms on $\mathfrak g$ and $\breve{\mathfrak g}$ has been fixed in a convenient way. This can be in conflict with the conventional normalization as soon as $\mathfrak g$, and along with $\mathfrak g$ also $\breve{\mathfrak g}$, is simple or affine.

4. Twining Characters

4.1. The map τ_ω. Let V be a vector space and

$$R: \quad \mathfrak{g} \to End(V) \tag{4.1}$$

a representation of a Lie algebra $\mathfrak{g}$ by endomorphisms $R(x)$: $V \to V$. Any automorphism ω of $\mathfrak{g}$ induces in a natural manner a map on the $\mathfrak{g}$-module (V, R). Namely, via

$$R^\omega(x) := R(\omega(x)) \tag{4.2}$$

for all $x \in \mathfrak{g}$, the action of ω provides another representation R^ω of $\mathfrak{g}$. This is again a representation of $\mathfrak{g}$ in $End(V)$. To describe the structure of the module (V, R^ω) in more detail, we first note that the construction does not change V as a vector space. However, this identity between vector spaces in general does *not* extend to an isomorphism of $\mathfrak{g}$-modules, i.e. in general the map does change the (isomorphism class of the) module.

Here we are interested in the case where ω is a strictly outer automorphism and where the module is a highest weight module. If the highest weight module with highest weight Λ is a Verma module, we denote it by (V, R_Λ), while if it is the irreducible quotient of (V, R_Λ), we write $(\mathcal{H}, R_\Lambda)$. A natural basis of a highest weight module consists of eigenvectors of the action of the Cartan subalgebra $\mathfrak{g}_\circ \subset \mathfrak{g}$. Both for Verma and irreducible modules, the eigenspaces $W_{(\lambda)} \subset V$ of weight λ with respect to the action of $R^\omega(\mathfrak{g}_\circ)$ coincide with the eigenspaces with respect to the original action $R(\mathfrak{g}_\circ)$.

Further, recall that the action of ω preserves the triangular decomposition (3.2) of $\mathfrak{g}$, i.e. not only maps the Cartan subalgebra to the Cartan subalgebra, but also the generators for positive (negative) roots to generators for positive (negative) roots. As a consequence, (V, R^ω_Λ) is again a Verma module. Moreover, since ω maps $\mathfrak{g}_+$ to $\mathfrak{g}_+$, the sets of primitive singular vectors, i.e. those vectors which are annihilated by the enveloping algebra $\mathsf{U}(\mathfrak{g}_+)$, of (V, R_Λ) and (V, R^ω_Λ) coincide. Now an irreducible highest weight module $\mathcal{H}_\Lambda$ has a single primitive singular vector, namely its highest weight vector, and hence the previous observation implies that $(\mathcal{H}, R^\omega_\Lambda)$ is again an irreducible highest weight module.

To obtain a more detailed description of the relation between (V, R_Λ) and (V, R^ω_Λ) (respectively $(\mathcal{H}, R_\Lambda)$ and $(\mathcal{H}, R^\omega_\Lambda)$) as $\mathfrak{g}$-modules, we note that the highest weight vector in both modules is the same element of the underlying vector space V (respectively $\mathcal{H}$). However, as an element of the module, its associated weight has to be transformed by the map $\omega^\star$ defined by (3.32), so that in fact the highest weight vector $v_{\text{h.w.}} \in V$ has highest weight Λ in (V, R_Λ), but highest weight $\omega^\star\Lambda$ in (V, R^ω_Λ). We thus conclude that as a module, (V, R_Λ) is isomorphic to the abstract Verma module V_Λ (and hence $(\mathcal{H}, R_\Lambda)$ is isomorphic to the irreducible quotient $\mathcal{H}_\Lambda$, the irreducible highest weight module with highest weight Λ), while (V, R^ω_Λ) and $(\mathcal{H}, R^\omega_\Lambda)$ are isomorphic to $V_{\omega^\star\Lambda}$ and $\mathcal{H}_{\omega^\star\Lambda}$, respectively:

For simple Lie algebras and affine Lie algebras other than $\tilde{B}^{(2)}_n$ one usually fixes the normalization by requiring that the long roots have length squared 2 [1, (6.4.2)], while for $\tilde{B}^{(2)}_n$ one normalizes the bilinear form such that the roots have length squared 1, 2 or 4. If one sticks to this normalization, then the factor N in Eqns. (3.25) and (3.34) must be replaced by a different factor N' in the following cases: for $\mathfrak{g} = A_{2n}$, for $\mathfrak{g} = A^{(1)}_{2n}$ with the order two automorphism γ, and for $\mathfrak{g} = B^{(2)}_{2n+1}$ one has $N' = 2N = 4$, while for the order two automorphism of $C^{(2)}_n$ one has $N' = N/2 = 1$, and for the order four automorphism $\sigma_s\gamma$ of $D^{(1)}_{2n+1}$ one needs $N' = 2N = 8$.

$$\begin{aligned} (V, R_\Lambda) &\cong V_\Lambda , & (V, R^{\omega}_\Lambda) &\cong V_{\omega^\star\Lambda} , \\ (\mathcal{H}, R_\Lambda) &\cong \mathcal{H}_\Lambda , & (\mathcal{H}, R^{\omega}_\Lambda) &\cong \mathcal{H}_{\omega^\star\Lambda} . \end{aligned} \tag{4.3}$$

Via these isomorphisms, one and the same element v of the vector space V (respectively $\mathcal{H}$) is identified with an element v' of V_Λ and another element v'' of $V_{\omega^\star\Lambda}$ (respectively of $\mathcal{H}_\Lambda$ and $\mathcal{H}_{\omega^\star\Lambda}$). In other words, the automorphism ω induces maps $\tau_\omega:\ V_\Lambda \to V^\omega_\Lambda$ and $\tau_\omega:\ \mathcal{H}_\Lambda \to \mathcal{H}^\omega_\Lambda$ acting as $v' \mapsto v''$ (for simplicity we use the same symbol for the map on the Verma module and its restriction to the irreducible quotient). By definition, this map τ_ω thus satisfies

$$\tau_\omega(R_\Lambda(x)\cdot v) = R_{\omega^\star\Lambda}(\omega(x))\cdot \tau_\omega(v) \tag{4.4}$$

for all $x \in \mathfrak{g}$ and all $v \in V_\Lambda$ (respectively $\mathcal{H}_\Lambda$), i.e. the diagrams

$$\begin{array}{ccc} V_\Lambda & \xrightarrow{R_\Lambda(x)} & V_\Lambda \\ \tau_\omega \downarrow & & \downarrow \tau_\omega \\ V_{\omega^\star\Lambda} & \xrightarrow{R_{\omega^\star\Lambda}(\omega(x))} & V_{\omega^\star\Lambda} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{H}_\Lambda & \xrightarrow{R_\Lambda(x)} & \mathcal{H}_\Lambda \\ \tau_\omega \downarrow & & \downarrow \tau_\omega \\ \mathcal{H}_{\omega^\star\Lambda} & \xrightarrow{R_{\omega^\star\Lambda}(\omega(x))} & \mathcal{H}_{\omega^\star\Lambda} \end{array} \tag{4.5}$$

commute. As (4.4) generalizes the defining property of an intertwining map, we will refer to the relation (4.4) as the *ω-twining property* of τ_ω. Also note that for any weight λ of the module, the action of τ_ω restricts to an action

$$\tau_\omega\big|_{W_{(\lambda)}}: \quad W_{(\lambda)} \to W_{(\omega^\star\lambda)} \tag{4.6}$$

on the (finite-dimensional) weight space $W_{(\lambda)}$.

4.2. Twining characters. Of particular interest in applications, e.g. in conformal field theory, are those irreducible highest weight representations for which $\omega^\star\Lambda = \Lambda$; in the physics literature they are known as "fixed points" of the diagram automorphism [4]. While τ_ω is generically a map between two different irreducible modules, in this case it is an endomorphism of a single irreducible module.[5] In this situation the following definition makes sense. For any strictly outer automorphism ω of $\mathfrak{g}$ let us define the *automorphism-twined characters*, or, briefly, *twining characters* $\mathcal{X}^{(\omega)}_\Lambda$ of a Verma module V_Λ and $\chi^{(\omega)}_\Lambda$ of its irreducible quotient $\mathcal{H}_\Lambda$, as follows. They are (formal) functions on the Cartan subalgebra $\mathfrak{g}_\circ$, defined analogously to ordinary characters, but with an additional insertion of the map τ_ω in the trace. Thus in the case of Verma modules the twining character $\mathcal{X}^{(\omega)}_\Lambda$ reads

$$\mathcal{X}^{(\omega)}_\Lambda: \quad \begin{array}{rcl} \mathfrak{g}_\circ & \to & \mathbf{C}, \\ \mathcal{X}^{(\omega)}_\Lambda(h) & := & \mathrm{tr}_{V_\Lambda} \tau_\omega\, \mathrm{e}^{2\pi \mathrm{i} R_\Lambda(h)} , \end{array} \tag{4.7}$$

and analogously the twining character $\chi^{(\omega)}_\Lambda$ of the irreducible module is given by

$$\chi^{(\omega)}_\Lambda: \quad \begin{array}{rcl} \mathfrak{g}_\circ & \to & \mathbf{C}, \\ \chi^{(\omega)}_\Lambda(h) & := & \mathrm{tr}_{\mathcal{H}_\Lambda} \tau_\omega\, \mathrm{e}^{2\pi \mathrm{i} R_\Lambda(h)} . \end{array} \tag{4.8}$$

[5] As a consequence, we will always be dealing with a definite representation R, and correspondingly often simplify notation by writing x in place of $R(x)$.

These twining characters are majorized by the ordinary characters, and hence in particular they are convergent wherever the ordinary characters converge. Note that generically some contributions to the twining characters have non-zero phase, so that instead of using the term character one might prefer to call these objects character-valued indices. However, by the identifications (4.16) and (4.17) below, it follows that the expansion coefficients of the twining characters are still non-negative integers.

The twining character can be interpreted as the generating functional of the trace of the map τ_ω restricted to the various weight spaces. Taking the trace separately on each weight space and extending the definition of the weights as functionals on $\mathfrak{g}_\circ$ to formal exponentials, $\mathrm{e}^{2\pi\mathrm{i}\lambda}(h) := \exp(2\pi\mathrm{i}\lambda(h))$, we can rewrite the twining character $\mathscr{V}_\Lambda^{(\omega)}$ in the form

$$\mathscr{V}_\Lambda^{(\omega)} = \sum_{\lambda\leq\Lambda} m_\lambda^{(\omega)} \mathrm{e}^{2\pi\mathrm{i}\lambda}\,, \tag{4.9}$$

and analogously for the twining character $\chi_\Lambda^{(\omega)}$ of the irreducible module. Here $m_\lambda^{(\omega)}$ denotes the trace of the restriction of τ_ω to the (finite-dimensional) weight space $W_{(\lambda)}$ of weight λ, and we write $\lambda \leq \Lambda$ iff $\Lambda - \lambda$ is a non-negative linear combination of simple roots. Because of the trace operation, the coefficient $m_\lambda^{(\omega)}$ can be different from zero only for $\lambda \in \mathfrak{g}_\circ^{\star(0)}$, i.e. only if λ is a symmetric weight. Hence we can restrict the sum in (4.9) to symmetric weights.

Combining the cyclic invariance of the trace and the ω-twining property (4.4) of τ_ω, we also learn that

$$\begin{aligned} \chi_\Lambda^{(\omega)}(h) &= \mathrm{tr}_{\mathscr{H}_\Lambda} \tau_\omega \mathrm{e}^{2\pi\mathrm{i}R_\Lambda(h)} = \mathrm{tr}_{\mathscr{H}_\Lambda} \mathrm{e}^{2\pi\mathrm{i}R_\Lambda(\omega(h))}\tau_\omega \\ &= \mathrm{tr}_{\mathscr{H}_\Lambda} \tau_\omega \mathrm{e}^{2\pi\mathrm{i}R_\Lambda(\omega(h))} = \chi_\Lambda^{(\omega)}(\omega(h)) \end{aligned} \tag{4.10}$$

for the character of the irreducible module, and an analogous result holds for the character of the Verma module.

4.3. Eigenspace decompositions. In the discussion in Subsect. 3.3, the eigenspace $\mathfrak{g}_\circ^{(0)}$ in $\mathfrak{g}_\circ$ to the eigenvalue ζ^0 of ω played an important rôle. Similarly, when analyzing the properties of twining characters, it proves to be convenient to decompose elements of $\mathfrak{g}_\circ$ into their components in all eigenspaces $\mathfrak{g}_\circ^{(l)}$ (to the eigenvalue ζ^l) of ω. Also, the map $\omega^\star$ on the weight space has the same order N as ω, and hence we can decompose the weight space into eigenspaces $\mathfrak{g}_\circ^{\star(j)}$ of $\omega^\star$ to the eigenvalue ζ^j,

$$\mathfrak{g}_\circ^\star = \bigoplus_{j=0}^{N-1} \mathfrak{g}_\circ^{\star(j)}\,. \tag{4.11}$$

The elements of the subspaces of $\mathfrak{g}_\circ^{\star(j)}$ can be characterized by the fact that for any l different from $-j$ mod N, their restriction on $\mathfrak{g}_\circ^{(l)}$ vanishes. To see this, consider arbitrary elements $\beta \in \mathfrak{g}_\circ^{\star(j)}$ and $x \in \mathfrak{g}_\circ^{(l)}$. Then we have

$$\beta(x) = (\omega^\star\beta)(\omega x) = \zeta^{j+l}\beta(x)\,, \tag{4.12}$$

which shows that $\beta(x)$ has to vanish whenever $j + l \not\equiv 0$ mod N. Conversely, if an element β of $\mathfrak{g}_\circ^\star$ vanishes on all elements of $\mathfrak{g}_\circ$ except for those of $\mathfrak{g}_\circ^{(l)}$, then by decomposing any element $h \in \mathfrak{g}_\circ$ into its components in the various eigenspaces $\mathfrak{g}_\circ^{(j)}$ according to $h = \sum_j h^{(j)}$ with $\omega(h^{(j)}) = \zeta^j h^{(j)}$, we find that

$$(\omega^\star\beta)(h) = \beta(\omega^{-1}h) = \zeta^{-l}\beta(h^{(l)}) = \zeta^{-l}\beta(h)\,, \tag{4.13}$$

and hence that $\beta \in \mathsf{g}_\circ^{\star(-l \bmod N)}$.

Consider now the twining character in the formulation (4.9), i.e. $\mathcal{V}_\Lambda^{(\omega)}(h) = \sum_{\lambda\le\Lambda} m_\lambda^{(\omega)} \mathrm{e}^{2\pi\mathrm{i}\lambda}(h)$, and decompose $h \in \mathsf{g}_\circ$ into its components $h^{(j)}$ as above. As only symmetric weights contribute in (4.9), the relation (4.12) can be employed in a similar manner as above to conclude that

$$\begin{aligned}\mathcal{V}_\Lambda^{(\omega)}(h) &= \textstyle\sum_{\lambda\le\Lambda} m_\lambda^{(\omega)}\, \mathrm{e}^{2\pi\mathrm{i}\lambda}(\sum_j h^{(j)}) \\ &= \sum_{\lambda\le\Lambda} m_\lambda^{(\omega)} \exp[2\pi\mathrm{i}\sum_{j=0}^{N-1}\lambda(h^{(j)})] = \sum_{\lambda\le\Lambda} m_\lambda^{(\omega)} \exp[2\pi\mathrm{i}\,\lambda(h^{(0)})]\,.\end{aligned} \tag{4.14}$$

Thus we have

$$\mathcal{V}_\Lambda^{(\omega)}(h) = \mathcal{V}_\Lambda^{(\omega)}(h^{(0)})\,, \tag{4.15}$$

and analogously for the twining character $\chi_\Lambda^{(\omega)}$ of the irreducible module. In other words, the twining characters depend on $h \in \mathsf{g}_\circ$ non-trivially only through its component in the subspace $\mathsf{g}_\circ^{(0)}$ of the Cartan subalgebra $\mathsf{g}_\circ$ that consists of fixed points of ω. Correspondingly, from now on we will consider the twining characters just as functions on $\mathsf{g}_\circ^{(0)}$.

4.4. The main theorems. We are now in a position to state the main result of this paper. Recall that there is a natural mapping P_ω (3.23) from $\mathsf{g}_\circ^{(0)}$ to $\breve{\mathsf{g}}_\circ$, which induces a corresponding dual map $\mathrm{P}_\omega^\star$ (3.33) between the respective weight spaces. Let $\dot\omega$ satisfy the linking condition (2.5), and let Λ be a symmetric g-weight. Then we have

Theorem 1: *a) The twining character $\mathcal{V}_\Lambda^{(\omega)}$ of the Verma module of g with highest weight Λ coincides with the ordinary character of the Verma module with highest weight $\mathrm{P}_\omega^{\star-1}(\Lambda)$ of the orbit Lie algebra $\breve{\mathsf{g}}$ in the sense that*

$$\mathcal{V}_\Lambda^{(\omega)}(h) = \breve{\mathcal{V}}_{\mathrm{P}_\omega^{\star-1}(\Lambda)}(\mathrm{P}_\omega(h))\,. \tag{4.16}$$

b) The twining character $\chi_\Lambda^{(\omega)}$ of the irreducible g-module with dominant integral highest weight Λ coincides with the ordinary character of the irreducible module with highest weight $\mathrm{P}_\omega^{\star-1}(\Lambda)$ of the orbit Lie algebra $\breve{\mathsf{g}}$ in the sense that

$$\chi_\Lambda^{(\omega)}(h) = \breve{\chi}_{\mathrm{P}_\omega^{\star-1}(\Lambda)}(\mathrm{P}_\omega(h))\,. \tag{4.17}$$

As already mentioned, the linking condition (2.5) is in fact satisfied for all diagram automorphisms of all affine and simple Lie algebras with the exception of the order N automorphisms of $A_{N-1}^{(1)}$. In these exceptional cases for any value of the level there is only a single highest weight Λ on which $\mathrm{P}_\omega^{\star-1}$ is defined. These cases can still be treated with our methods; they are covered by

Theorem 2: *In the case of $\mathsf{g} = A_{N-1}^{(1)}$ and the outer automorphism of order N, the coefficients $m_\lambda^{(\omega)}$ in the expansion (4.9) for the twining character of both the irreducible and Verma modules obey*

$$m_\lambda^{(\omega)} = 0 \quad \text{for } \lambda \ne \Lambda\,, \tag{4.18}$$

i.e. except for the contribution from the highest weight vector, all contributions cancel against each other.

Theorems 1 and 2 will be proven in Sect. 5 and Sect. 7, respectively. In Sect. 7 we will also present the explicit expression for the twining character for $A_{N-1}^{(1)}$ with respect to the order N automorphism.

5. The Twining Character and the Weyl Group $\breve{W}$

Our proof of Theorem 1 proceeds in several rather distinct steps which are inspired by Kac' proof of the Weyl-Kac character formula (see e.g. [1, pp. 152, 172]). An additional crucial ingredient of the proof consists in the identification of a natural action of $\breve{W}$, the Weyl group of the orbit Lie algebra $\breve{\mathfrak{g}}$, on the twining characters.

5.1. The action of the Weyl group $\breve{W}$. We have seen that in the description (4.9) of the twining character only weights lying in $\mathfrak{g}_\circ^{\star(0)}$ contribute, and that this part of the weight space of $\mathfrak{g}$ is isomorphic to $\breve{\mathfrak{g}}_\circ^\star$ via the map $\mathrm{P}_\omega^\star$. Hence we can employ $\mathrm{P}_\omega^\star$ to push the action of $\breve{W}$ on $\breve{\mathfrak{g}}_\circ^\star$ to an action of $\breve{W}$ on $\mathfrak{g}_\circ^{\star(0)}$.

To describe the Weyl groups explicitly, we denote by w_i the fundamental reflections which generate the Weyl group W of $\mathfrak{g}$, i.e. the reflections of the weight space of $\mathfrak{g}$ with respect to the hyperplanes perpendicular to the simple roots $\alpha^{(i)}$, and analogously by $\breve{w}_i$ the fundamental reflections for $\breve{\mathfrak{g}}$. Now for any fundamental reflection $\breve{w}_i$ of $\breve{W}$ we can find an element of the Weyl group of $\mathfrak{g}$, to be denoted by $\hat{w}_i$, which acts on $\mathfrak{g}_\circ^{\star(0)}$ precisely like $\breve{w}_i$ acts on $\breve{\mathfrak{g}}_\circ^\star$, i.e. which satisfies $\mathrm{P}_\omega^{\star-1}(\hat{w}_i(\lambda)) = \breve{w}_i(\mathrm{P}_\omega^{\star-1}(\lambda))$ for all $\lambda \in \mathfrak{g}_\circ^{\star(0)}$. We will denote the mapping which maps $\breve{w}_i$ to $\hat{w}_i$ by P_W,

$$\mathrm{P}_W: \quad \breve{w}_i \mapsto \hat{w}_i\,, \qquad (\mathrm{P}_W(\breve{w}_i))(\lambda) := \hat{w}_i(\lambda) \equiv \mathrm{P}_\omega^\star(\breve{w}_i(\mathrm{P}_\omega^{\star-1}(\lambda)))\,. \tag{5.1}$$

Moreover, we will see that $\hat{w}_i$ commutes with $\omega^\star$.

Let us first deal with those fundamental reflections $\breve{w}_i$ for which the integer s_i defined in (2.4) is $s_i = 1$. In this case define

$$\hat{w}_i := \prod_{l=0}^{N_i-1} w_{\dot{\omega}^l i}\,. \tag{5.2}$$

Note that because of $s_i = 1$ we have $A^{i,\dot{\omega}^l i} = 0$ whenever $i \neq \dot{\omega}^l i$, so that $w_{\dot{\omega}^l i}$ and $w_{\dot{\omega}^{l'} i}$ commute, and hence the product in (5.2) is well-defined. The fact that $w_{\dot{\omega}^l i}$ and $w_{\dot{\omega}^{l'} i}$ commute also ensures that $\hat{w}_i^2 = id$, and that $\hat{w}_i$ commutes with the induced automorphism $\omega^\star$. This implies in particular that the action of $\hat{w}_i$ respects the orbits of $\omega^\star$. For $s_i = 1$ we also have $(\alpha^{(i)} \mid \alpha^{(\dot{\omega}^l i)}) = 0$, so that the action of $\hat{w}_i$ on $\mathfrak{g}$-weights λ reads

$$\hat{w}_i(\lambda) = \lambda - \sum_{l=0}^{N_i-1} (\lambda \mid \alpha^{(\dot{\omega}^l i)^\vee})\, \alpha^{(\dot{\omega}^l i)}\,. \tag{5.3}$$

Let us now describe how $\hat{w}_i$ acts on the positive roots of $\mathfrak{g}$. We have

$$\hat{w}_i(\alpha^{(\dot{\omega}^l i)}) = w_{\dot{\omega}^l i}(\alpha^{(\dot{\omega}^l i)}) = -\alpha^{(\dot{\omega}^l i)}\,, \tag{5.4}$$

while $\hat{w}_i$ maps any positive root which is not on the $\omega^\star$-orbit of $\alpha^{(i)}$ to a positive root which is also not on that orbit. This can be seen as follows: let $\beta = \sum_{j=1}^n n_j\alpha^{(j)}$ be a positive root which is not on the orbit of $\alpha^{(i)}$. Then there is some index j, which is not on the orbit of i, $j \neq \dot\omega^l i$, for which n_j is strictly positive. Since the only effect of $\hat{w}_i$ on β is to add terms proportional to the $\alpha^{(\dot\omega^l i)}$,

$$\hat{w}_i(\beta) = \sum_{j=1}^{n} n_j\alpha^{(j)} + \sum_{l=1}^{N_i-1} \xi_l\, \alpha^{(\dot\omega^l i)}\,, \tag{5.5}$$

the expansion of $\hat{w}_i(\beta)$ in terms of simple roots still contains the term $n_j\alpha^{(j)}$. Since $\hat{w}_i(\beta)$ is again a root of g, and since one coefficient is positive, it is again a positive root of g. Moreover, since $n_j \neq 0$ it is clear that it cannot be on the orbit of $\alpha^{(i)}$.

To deal with the case $s_i = 2$ we first recall that in this case N_i is even and that the restriction of the Dynkin diagram of g to this orbit is the Dynkin diagram of $N_i/2$ copies of the simple Lie algebra A_2. As a consequence, in the sequel we can in fact restrict ourselves to the case $N_i = 2$. Otherwise we first treat the automorphism $\omega^{N_i/2}$, which has order two and possesses $N_i/2$ orbits each of which corresponds to the Dynkin diagram of A_2. On the set of orbits of $\omega^{N_i/2}$, the automorphism ω induces an automorphism ω' of order $N_i/2$; all orbits with respect to this automorphism ω' have $s'_j = 1$.

For $N_i = 2$ and $s_i = 2$ we define

$$\hat{w}_i := w_i\, w_{\dot\omega i}\, w_i\,. \tag{5.6}$$

Clearly, $\hat{w}_i$ has order 2, $\hat{w}_i^2 = id$. Since $A^{i,\dot\omega i} = -1$, we also have $(w_i w_{\dot\omega i})^3 = id$ and hence

$$\hat{w}_{\dot\omega i} = w_{\dot\omega i} w_i w_{\dot\omega i} = w_i w_{\dot\omega i} w_i = \hat{w}_i\,. \tag{5.7}$$

This implies that again $\hat{w}_i$ and $\omega^\star$ commute. The action of $\hat{w}_i$ on the roots of our main interest reads

$$\hat{w}_i(\alpha^{(i)}) = -\alpha^{(\dot\omega i)}\,, \quad \hat{w}_i(\alpha^{(\dot\omega i)}) = -\alpha^{(i)}\,, \quad \hat{w}_i(\alpha^{(i)} + \alpha^{(\dot\omega i)}) = -(\alpha^{(i)} + \alpha^{(\dot\omega i)})\,, \tag{5.8}$$

while any other positive root is again mapped on a positive root different from $\alpha^{(i)}$, $\alpha^{(\dot\omega i)}$ and $\alpha^{(i)} + \alpha^{(\dot\omega i)}$. This can be checked explicitly by using arguments which are completely parallel to those used in the case $s_i = 1$.

Finally, we again compute the action of $\hat{w}_i$ on weights in $\mathsf{g}_\circ^{\star(0)}$. For any such g-weight we have $(\lambda\,|\,\alpha^{(i)\vee}) = (\lambda\,|\,\alpha^{(\dot\omega i)\vee}) =: l$, and hence

$$\begin{aligned}\hat{w}_i(\lambda) &= w_i w_{\dot\omega i} w_i(\lambda) = w_i w_{\dot\omega i}(\lambda - l\alpha^{(i)}) \\ &= w_i(\lambda - l\alpha^{(i)} - 2l\alpha^{(\dot\omega i)}) = \lambda - 2l\cdot(\alpha^{(i)} + \alpha^{(\dot\omega i)})\,.\end{aligned} \tag{5.9}$$

We can summarize the formulæ (5.3) and (5.9) by

$$\hat{w}_i(\lambda) = \lambda - s_i \cdot \sum_{l=0}^{N_i-1} (\lambda \mid \alpha^{(\dot\omega^l i)^\vee})\, \alpha^{(\dot\omega^l i)}\,. \tag{5.10}$$

Let us check that the prescription (5.10) indeed describes the mapping P_W defined by (5.1). Knowing how P_ω acts on $\mathsf{g}_\circ$, it is straightforward to determine how $\mathrm{P}_\omega^\star$ acts on $\breve{\mathsf{g}}_\circ^\star$. Let us first compute the action of $\mathrm{P}_\omega^\star$ on the simple coroots $\breve\alpha^{(i)^\vee} := d_i\, \breve\alpha^{(i)}$. We observe that the invariant bilinear form on the orbit Lie algebra $\breve{\mathfrak{g}}$ identifies $\breve{\mathfrak{g}}_\circ$ with its weight space $\breve{\mathfrak{g}}_\circ^\star$ in such a way that $\breve{\alpha}^{(i)^\vee}$ corresponds to $\breve{H}^i$. Also, since ω leaves the bilinear form invariant and $(\mathrm{P}_\omega h \,|\, \mathrm{P}_\omega h') = N(h\,|\,h')$, the identification of $\breve{\mathfrak{g}}_\circ$ with $\breve{\mathfrak{g}}_\circ^\star$ corresponds to identifying the maps P_ω and $\mathrm{P}_\omega^{\star-1}$ up to a rescaling by N. As a consequence, the dualization of the identity $\mathrm{P}_\omega(\sum_{l=1}^{N_i} H^{\dot\omega^l i}) = N_i \breve{H}^i$ reads

$$\mathrm{P}_\omega^\star(\breve{\alpha}^{(i)^\vee}) = \frac{N}{N_i} \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)^\vee} . \tag{5.11}$$

Using the relation (2.13) between d_i and $\breve{d}_i$ and the fact that $\mathrm{P}_\omega^\star$ is a linear map, we can also compute the action on the simple roots,

$$\mathrm{P}_\omega^\star(\breve{\alpha}^{(i)}) = \frac{1}{\breve{d}_i} \mathrm{P}_\omega^\star(\breve{\alpha}^{(i)^\vee}) = s_i \cdot \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)} . \tag{5.12}$$

With these results, the formula (5.1) for $\hat{w}_i(\lambda)$ becomes

$$\begin{aligned} \hat{w}_i(\lambda) \equiv \mathrm{P}_\omega^\star(\breve{w}_i(\breve\lambda)) &= \mathrm{P}_\omega^\star(\breve\lambda - (\breve\lambda \,|\, \breve{\alpha}^{(i)^\vee})\, \breve{\alpha}^{(i)}) \\ &= \lambda - \tfrac{1}{N} (\lambda \,|\, \mathrm{P}_\omega^\star \breve{\alpha}^{(i)^\vee}) \cdot s_i \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)} \\ &= \lambda - \tfrac{1}{N} \sum_{l=0}^{N-1} \tfrac{N}{N_i} (\lambda \,|\, \alpha^{(\dot\omega^l i)^\vee})\, s_i \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)} \\ &= \lambda - s_i \cdot \sum_{l=0}^{N_i-1} (\lambda \,|\, \alpha^{(\dot\omega^l i)^\vee})\, \alpha^{(\dot\omega^l i)} . \end{aligned} \tag{5.13}$$

Thus $\hat{w}_i(\lambda)$ as defined in (5.1) coincides with (5.10), as promised. In short, both for $s_i = 1$ and for $s_i = 2$ we have shown that we can represent the generators of the Weyl group $\breve{W}$ by elements of W which commute with $\omega^\star$.

Of particular interest is the case where the $\mathfrak{g}$-weight on which $\hat{w}_i$ acts is a Weyl vector of $\mathfrak{g}$, i.e. an element ρ of $\mathfrak{g}^\star$ which obeys $\rho(H^i) = 1$ for all $i \in I$. In this case (5.10) reads

$$\hat{w}_i(\rho) = \rho - s_i \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)} . \tag{5.14}$$

5.2. $\hat{W}$ as a subgroup of W. We can now define $\hat{W}$ as the subgroup of W that is generated by the elements $\hat{w}_i$ of W that are defined by (5.10). In this section we show that $\hat{W}$ is in fact isomorphic to $\breve{W}$, the Weyl group of the orbit Lie algebra $\breve{\mathfrak{g}}$, or in other words, that the map P_W which maps $\breve{w}_i$ to $\hat{w}_i$ as defined in (5.1) extends to an isomorphism of the groups $\breve{W}$ and $\hat{W}$. The proof involves a few lengthy calculations which will be described in detail in Appendix A.

First recall that the Weyl group $\breve{W}$ can be described as a Coxeter group, namely as the group that is freely generated by the generators $\breve{w}_i$ modulo the relations

$$\begin{aligned} &(\breve{w}_i)^2 = id && \text{for all} \quad i \in \breve{I}\,, \\ &(\breve{w}_i \breve{w}_j)^{\breve{m}_{ij}} = id && \text{for all} \quad i,j \in \breve{I},\ i \neq j\,. \end{aligned} \tag{5.15}$$

The integers $\breve{m}_{ij}$ take the specific values $\breve{m}_{ij} = 2,3,4,6$ for $\breve{A}^{i,j}\breve{A}^{j,i} = 0,1,2,3$, while for $\breve{A}^{i,j}\breve{A}^{j,i} \geq 4$ one puts $\breve{m}_{ij} = \infty$ (and uses the convention that $x^\infty = id$ for all x).

We have to show that the generators $\hat{w}_i$ obey exactly the same relations. Above we have already seen that the $\hat{w}_i$ square to the identity; thus, denoting by $\hat{m}_{ij}$ the order of $\hat{w}_i\hat{w}_j$ in W, it remains to be shown that $\hat{m}_{ij} = \breve{m}_{ij}$. To see this, we first prove that $\breve{m}_{ij}$ is a divisor of $\hat{m}_{ij}$ (and hence a fortiori $\hat{m}_{ij} \geq \breve{m}_{ij}$). Namely, assume that $(\hat{w}_i\hat{w}_j)^{\hat{m}_{ij}} \in W$ is the identity element of W; then in particular it acts as the identity on the subspace $\mathsf{g}_\circ^{\star(0)}$ of $\mathsf{g}_\circ^\star$. Hence by construction also $(\breve{w}_i\breve{w}_j)^{\hat{m}_{ij}} \in \breve{W}$ acts as the identity on the weight space of $\breve{\mathsf{g}}$; this means that it is the identity element of $\breve{W}$, which in turn by (5.15) implies that $\hat{m}_{ij}$ must a divisible by $\breve{m}_{ij}$.

The inequality $\hat{m}_{ij} \geq \breve{m}_{ij}$ automatically proves our assertion for $\breve{A}^{i,j}\breve{A}^{j,i} \geq 4$. In the remaining cases, one can show in a case by case study that in fact already $(\hat{w}_i\hat{w}_j)^{\breve{m}_{ij}} = id$, which then concludes the proof of the isomorphism property of P_W. These calculations are straightforward, but somewhat lengthy, and accordingly we present them in Appendix A.

For later convenience, we also introduce the homomorphism $\hat{\epsilon}$ from $\hat{W}$ to $\mathbf{Z}_2 = \{\pm 1\}$ that is induced by the sign function $\breve{\epsilon}$ on $\breve{W}$,

$$\hat{\epsilon}(\hat{w}) := \breve{\epsilon}(\mathrm{P}_W^{-1}(\hat{w}))\,. \tag{5.16}$$

Note that $\hat{\epsilon}$ is typically different from the sign function that $\hat{W}$ inherits as a subgroup from the sign function ϵ of W.

5.3. The action of $\hat{W}$ on the twining character of the Verma module. We now consider the action of $\hat{W}$ on the twining characters. As $\hat{W}$ is a subgroup of W, its action on the twining character is defined in the same way as the action of W on the ordinary characters is, i.e. via the action (5.10) on g-weights. In this subsection we show that the function

$$\mathscr{V}^{(\omega)} := \mathrm{e}^{-\rho-\Lambda}\mathscr{V}_\Lambda^{(\omega)}\,, \tag{5.17}$$

with $\mathscr{V}_\Lambda^{(\omega)}$ the twining character of the Verma module with highest weight Λ, is odd under the action (5.10) of $\hat{W}$, i.e.

$$\hat{w}(\mathscr{V}^{(\omega)}) = \hat{\epsilon}(\hat{w})\,\mathscr{V}^{(\omega)}\,. \tag{5.18}$$

Note that here the sign function $\hat{\epsilon}$ on $\hat{W}$ defined in (5.16) appears, rather than the sign function ϵ of W. We also remark that, since the only dependence of the twining character $\mathscr{V}_\Lambda^{(\omega)}$ of the Verma module on the specific highest weight Λ is by a multiplicative factor of e^Λ, the quantity $\mathscr{V}^{(\omega)}$ is independent of the choice of Λ. It is sufficient to check (5.18) for the fundamental reflections $\hat{w}_i$ which generate $\hat{W}$. Thus in the sequel we consider a reflection $\hat{w}_i$ with fixed $i \in \breve{I}$, for which we have to show that

$$\hat{w}_i(\mathscr{V}^{(\omega)}) = -\mathscr{V}^{(\omega)}\,. \tag{5.19}$$

To prove this, we make use of the Poincaré-Birkhoff-Witt theorem. To this end we must first choose a basis $\mathscr{B}_-$ of g_-, including some enumeration of the elements of $\mathscr{B}_-$. Let us first deal with the case $s_i = 1$. In this case we choose as the first N_i elements of $\mathscr{B}_-$ the step operators $E_-^{\dot\omega^l i}$ for $l = 0, 1, \ldots, N_i - 1$ (the root spaces corresponding to simple roots are one-dimensional so that this prescription makes sense), and then the step operators associated to all other negative roots in an arbitrary ordering. The Poincaré-Birkhoff-Witt theorem then asserts that the set of all products

$$\mathscr{E}^{(\mathbf{n},\mathbf{m})} = \mathscr{E}_1^{(\mathbf{n})} \cdot \mathscr{E}_2^{(\mathbf{m})}\,, \quad \mathscr{E}_1^{(\mathbf{n})} := (E^{-\alpha^{(i)}})^{n_0}\,(E^{-\alpha^{(\dot\omega i)}})^{n_1}\,\ldots(E^{-\alpha^{(\dot\omega^{N_i-1} i)}})^{n_{N_i-1}}\,,$$
$$\mathscr{E}_2^{(\mathbf{m})} := (E^{-\beta_1})^{m_1}(E^{-\beta_2})^{m_2}\ldots\,, \tag{5.20}$$

forms a basis of the universal enveloping algebra $\mathsf{U}(\mathsf{g}_-)$; [6] here the exponents n_i and m_i can take all values in the non-negative integers in such a way that only finitely many of them are different from zero. As the elements $\mathscr{E}^{(\mathbf{n},\mathbf{m})}$ are linearly independent, commutator terms that arise when reshuffling the products of generators of g_- can never give rise to a non-zero contribution to the twining character $\mathscr{V}^{(\omega)}$; furthermore, an element $v^{(\mathbf{n},\mathbf{m})} = \mathscr{E}^{(\mathbf{n},\mathbf{m})} \cdot v_\Lambda$ of the Verma module can contribute to $\mathscr{V}^{(\omega)}$ only if $n_0 = n_1 = \ldots = n_{N_i-1} =: n$.

The Poincaré-Birkhoff-Witt theorem also implies that the contributions to $\mathscr{V}^{(\omega)}$ stemming from the products $\mathscr{E}_1^{(\mathbf{n})}$ and $\mathscr{E}_2^{(\mathbf{m})}$ in (5.20) factorize, so that we can investigate their transformation properties under $\hat{w}_i$ separately. First, $\hat{w}_i$ commutes with $\omega^\star$ and maps any $\omega^\star$-orbit of negative roots to some other orbit of negative roots, i.e. only permutes the orbits that contribute to the second factor. Thus, by the fact that commutator terms are irrelevant for the trace, the contribution to $\mathscr{V}^{(\omega)}$ coming from operators of the type $\mathscr{E}_2^{(\mathbf{m})}$ is invariant under $\hat{w}_i$. On the other hand, the contribution $(\mathscr{V}^{(\omega)})_1$ of operators of the type $\mathscr{E}_1^{(\mathbf{n})}$ to $\mathscr{V}^{(\omega)} = \mathrm{e}^{-\rho-\Lambda}\mathscr{V}_\Lambda^{(\omega)}$ can be computed explicitly as

$$\begin{aligned}(\mathscr{V}^{(\omega)})_1 &= \mathrm{e}^{-\rho}\textstyle\sum_{n=0}^\infty \exp[n(-\alpha^{(i)} - \alpha^{(\dot\omega i)} - \ldots - \alpha^{(\dot\omega^{N_i-1} i)})] \\ &= \frac{\mathrm{e}^{-\rho}}{1-\exp[-\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}]}\,.\end{aligned} \tag{5.21}$$

Acting on this expression with $\hat{w}_i$, with the help of (5.4) and (5.14) we obtain

$$\hat{w}_i((\mathscr{V}^{(\omega)})_1) = \frac{\exp[-\rho + \sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}]}{1-\exp[\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}]} = -(\mathscr{V}^{(\omega)})_1\,. \tag{5.22}$$

Combining the two factors of the product (5.20), we thus arrive at the desired result (5.19).

Next consider the case $s_i = 2$. In this case we choose a different basis $\mathscr{B}_-$ of g_- in order to obtain a decomposition analogous to (5.20). As the first three elements of $\mathscr{B}_-$ we take the step operators $E^{-\alpha^{(i)}}$, $E^{-\alpha^{(\dot\omega i)}}$ and $E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}$, and then again the step operators corresponding to all other negative roots in an arbitrary ordering. A basis of $\mathsf{U}(\mathsf{g}_-)$ is then given by (5.20) with the first factor $\mathscr{E}_1^{(\mathbf{n})}$ replaced by

[6] The automorphism ω of g extends to an automorphism of the universal enveloping algebra $\mathsf{U}(\mathsf{g})$ by simply defining $\omega(xx') = \omega(x)\omega(x')$ for all $x, x' \in \mathsf{g}$ as well as $\omega(\mathbf{1}) = \mathbf{1}$.

$$\mathscr{E}_1^{(n_0,n_1,n')} := (E_-^i)^{n_0}(E_-^{\dot\omega i})^{n_1}(E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}})^{n'} . \tag{5.23}$$

The same type of arguments as in the previous case then shows that the contribution to $\mathscr{T}^{(\omega)}$ from operators of the type $\mathscr{E}_2^{(\mathbf{m})}$ again transforms trivially under $\hat{w}_i$. Further, in order to have a contribution $(\mathscr{T}^{(\omega)})_1$ from operators of the type $\mathscr{E}_1^{(n_0,n_1,n')}$, we need again $n_0 = n_1 =: n$. Now the transformation properties $\omega(E_-^i) = E_-^{\dot\omega i}$ and $\omega(E_-^{\dot\omega i}) = E_-^i$ imply that

$$\omega(E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}) = \omega([E_-^i, E_-^{\dot\omega i}]) = [E_-^{\dot\omega i}, E_-^i] = -E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}} . \tag{5.24}$$

This allows us to compute the contribution $(\mathscr{T}^{(\omega)})_1$ to the twining character as

$$\begin{aligned}(\mathscr{T}^{(\omega)})_1 &= \mathrm{e}^{-\rho} \sum_{n=0}^{\infty} \left[\mathrm{e}^{n(-\alpha^{(i)})} \cdot \mathrm{e}^{n(-\alpha^{(\dot\omega i)})}\right] \cdot \sum_{n'=0}^{\infty} (-1)^{n'} \mathrm{e}^{n'(-\alpha^{(i)}-\alpha^{(\dot\omega i)})} \\ &= \mathrm{e}^{-\rho}(1-\mathrm{e}^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}})^{-1}(1+\mathrm{e}^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}})^{-1} \\ &= \mathrm{e}^{-\rho}(1-\mathrm{e}^{-2\alpha^{(i)}-2\alpha^{(\dot\omega i)}})^{-1} .\end{aligned} \tag{5.25}$$

Using the transformation properties (5.8) and (5.14) (note the additional factor of $s_i = 2$ in the transformation law (5.14) of ρ) we find that this contribution to the twining character $\mathscr{T}^{(\omega)}$ changes sign under the action of $\hat{w}_i$. Hence again we obtain (5.19). This completes the proof of (5.19), and hence of (5.18).

5.4. The action of $\hat{W}$ on the irreducible twining character $\chi_\Lambda^{(\omega)}$. In this subsection we show that the twining character $\chi_\Lambda^{(\omega)}$ of an irreducible highest weight module with dominant integral highest weight Λ is even under the action of $\hat{W}$, i.e.

$$\hat{w}(\chi_\Lambda^{(\omega)}) = \chi_\Lambda^{(\omega)} . \tag{5.26}$$

Again, it is sufficient to check this for all generators $\hat{w}_i$ of $\hat{W}$. Thus for all $i \in \breve{I}$ we have to show that any weight $\lambda \in \mathfrak{g}_\circ^{\star(0)}$ contributes in the same way to the twining character as the weight $\hat{w}_i(\lambda)$.

Let us first deal with the case $s_i = 1$. Then the subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$ that is spanned by the generators $E_\pm^{\dot\omega^l i}$ and $H^{\dot\omega^l i}$,

$$\mathfrak{g}_i := \langle E_\pm^{\dot\omega^l i}, H^{\dot\omega^l i} \mid l = 0, 1, \ldots, N_i - 1 \rangle , \tag{5.27}$$

is isomorphic to a direct sum of N_i copies of A_1 algebras,

$$\mathfrak{g}_i \cong \underbrace{A_1 \oplus A_1 \oplus \ldots \oplus A_1}_{N_i \text{ summands}} . \tag{5.28}$$

Now consider the decomposition

$$\mathscr{H}_\Lambda = \bigoplus \mathscr{H}_{(L_k)} \tag{5.29}$$

of the irreducible module $\mathscr{H}_\Lambda$ of $\mathfrak{g}$ into irreducible modules $\mathscr{H}_{(L_k)}$ of $\mathfrak{g}_i$. As the highest weight Λ is dominant integral, each of the modules $\mathscr{H}_{(L_k)}$ has dominant integral highest weight, i.e. for any value of k each of the N_i numbers L_k, $k = 1, 2, \dots, N_i$, is a non-negative integer. Also, according to the representation theory of A_1, any weight of such a module with respect to the subalgebra $\mathfrak{g}_i$ is then a sequence of N_i integers ℓ_k, $k = 1, 2, \dots, N_i$, and all these weights are non-degenerate.

A module $\mathscr{H}_{(L_k)}$ of $\mathfrak{g}_i$ can of course only contribute to the twining character if it is mapped onto itself by τ_ω. For the rest of the discussion we will assume that the module under consideration fulfills this condition (otherwise no state of the module contributes to the trace, so that the contribution is trivially symmetric under $\hat{W}$). Now to the twining character only those states can contribute for which $\ell_1 = \ell_2 = \dots = \ell_{N_i}$. Thus we have to show that the unique state v in $\mathscr{H}_{(L_k)}$ with $\ell_1 = \ell_2 = \dots = \ell_{N_i} =: l > 0$ contributes precisely with the same phase to $\chi_\Lambda^{(\omega)}$ as the unique state v' with $\ell'_1 = \ell'_2 = \dots = \ell'_{N_i} =: -l$. Now v' can be obtained by acting on v as

$$v' = (E_-^i E_-^{\dot\omega i} \dots E_-^{\dot\omega^{N_i-1} i})^l \, v \,. \tag{5.30}$$

We now combine the identity $\omega(E_-^i E_-^{\dot\omega i} \dots E_-^{\dot\omega^{N_i-1} i}) = E_-^i E_-^{\dot\omega i} \dots E_-^{\dot\omega^{N_i-1} i}$ and the ω-twining property (4.4) of the map τ_ω to find that the eigenvalue equation $\tau_\omega(v) = \zeta^k v$ implies

$$\tau_\omega(v') = \tau_\omega((E_-^i E_-^{\dot\omega i} \dots E_-^{\dot\omega^{N_i-1} i})^l v) = \omega((E_-^i E_-^{\dot\omega i} \dots E_-^{\dot\omega^{N_i-1} i})^l)\tau_\omega(v) = \zeta^k (v') \,. \tag{5.31}$$

Thus v and v' contribute the same phase ζ^k, which proves our claim (5.26) in the case $s_i = 1$.

To deal with the case $s_i = 2$ we can again assume that $N_i = 2$. In this case we define $\mathfrak{g}_i$ as the subalgebra

$$\mathfrak{g}_i := \langle E^{\pm\alpha^{(i)} \pm \alpha^{(\dot\omega i)}}, H^i + H^{\dot\omega i} \rangle \tag{5.32}$$

of $\mathfrak{g}$, which is isomorphic to A_1. The automorphism ω acts on $\mathfrak{g}_i$ as (compare (5.24))

$$\omega(E^{\pm\alpha^{(i)} \pm \alpha^{(\dot\omega i)}}) = -E^{\pm\alpha^{(i)} \pm \alpha^{(\dot\omega i)}}, \qquad \omega(H^i + H^{\dot\omega i}) = H^i + H^{\dot\omega i} \,. \tag{5.33}$$

Again we decompose the irreducible module $\mathscr{H}_\Lambda$ of $\mathfrak{g}$ into irreducible modules $\mathscr{H}_\ell$ of $\mathfrak{g}_i$, for which again the weights are non-degenerate. Only states which have the same eigenvalue for H^i and $H^{\dot\omega i}$ can contribute to the twining character. Thus we have to show that the unique state v with $H^i v = H^{\dot\omega i} v = lv$ ($l > 0$) contributes the same phase as the unique state v' obeying $H^i v' = H^{\dot\omega i} v' = -lv'$. Now we have

$$v' = (E^{-\alpha^{(i)} - \alpha^{(\dot\omega i)}})^{2l} v \,, \tag{5.34}$$

where the factor of two arises because the H^i- and $H^{\dot\omega i}$-eigenvalues are added up. Thus only even powers of the step operator $E^{-\alpha^{(i)} - \alpha^{(\dot\omega i)}}$ occur. Because of (5.33) the vectors v and v' therefore contribute with the same phase (which for $N_i = 2$ is a sign), and hence the claim (5.26) is again proven.

5.5. The linear relation between irreducible and Verma characters. In this subsection we show that the twining character $\chi_\Lambda^{(\omega)}$ of the irreducible module $\mathscr{H}_\Lambda$ can be written as an (infinite) linear combination, with complex coefficients, of the twining characters of certain Verma modules. We first need to introduce some notation; as in (4.9), for two weights $\lambda, \mu \in \mathsf{g}_\circ^\star$ we write $\mu \le \lambda$ iff the difference $\lambda - \mu$ is a non-negative linear combination of the simple roots of g, i.e. iff

$$\lambda - \mu = \sum_{i\in I} n_i\, \alpha^{(i)} \qquad \text{with} \qquad n_i \in \mathbf{Z}_{\ge 0}\,. \tag{5.35}$$

To any such pair of weights we associate a non-negative integer, the depth, by

$$\mathrm{dp}_\lambda(\mu) := \sum_{i\in I} n_i\,. \tag{5.36}$$

Let us assume that λ is a symmetric weight, $\lambda \in \mathsf{g}_\circ^{\star(0)}$. We claim that for any such λ we can find complex numbers $\tilde c_{\lambda\mu}$ with $\tilde c_{\lambda\lambda} = 1$ such that

$$\mathscr{V}_\lambda^{(\omega)} = \sum_{\mu\le\lambda} \tilde c_{\lambda\mu}\, \chi_\mu^{(\omega)}\,, \tag{5.37}$$

where $\mathscr{V}_\lambda^{(\omega)}$ denotes the twining character of the Verma module with highest weight λ and $\chi_\mu^{(\omega)}$ the twining character of the irreducible module with highest weight μ. Note that the weights λ, μ need not be dominant integral. (Also, for non-symmetric weights $\mathscr{V}_\lambda^{(\omega)}$ vanishes, so that the assertion is trivially true.)

To prove (5.37) we define inductively a sequence $\mathscr{V}_\lambda^{(\omega)[n]}$ of (finite) linear combinations of twining characters of irreducible modules,

$$\mathscr{V}_\lambda^{(\omega)[n]} := \sum_{\substack{\mu\le\lambda \\ \mathrm{dp}_\lambda(\mu)\le n}} \tilde c_{\lambda\mu}^{[n]}\, \chi_\mu^{(\omega)}\,, \tag{5.38}$$

such that the coefficient of e^μ in $\mathscr{V}_\lambda^{(\omega)} - \mathscr{V}_\lambda^{(\omega)[n]}$ vanishes for any μ with $\mathrm{dp}_\lambda(\mu) \le n$ and that

$$\tilde c_{\lambda\mu}^{[n]} = \tilde c_{\lambda\mu}^{[\mathrm{dp}_\lambda(\mu)]} \qquad \text{for all} \;\; n \ge \mathrm{dp}_\lambda(\mu)\,. \tag{5.39}$$

At depth zero, there is a single state we have to take into account, namely just the highest weight vector, and hence we define $\tilde c_{\lambda\mu}^{[0]} := \delta_{\lambda,\mu}$. Next, suppose that we have already defined $\mathscr{V}_\lambda^{(\omega)[n]}$ for some value of $n > 0$. Then the difference $\mathscr{V}_\lambda^{(\omega)} - \mathscr{V}_\lambda^{(\omega)[n]}$ is of the form

$$\mathscr{V}_\lambda^{(\omega)} - \mathscr{V}_\lambda^{(\omega)[n]} = \sum_{\substack{\mu\le\lambda \\ \mathrm{dp}_\lambda(\mu)=n+1}} d_\mu^{[n]}\, \mathrm{e}^\mu + \sum_{\substack{\mu\le\lambda \\ \mathrm{dp}_\lambda(\mu)>n+1}} d_\mu^{[n]}\, \mathrm{e}^\mu\,, \tag{5.40}$$

where $d_\mu^{[n]}$ are some complex numbers. We then define

$$\mathscr{V}_\lambda^{(\omega)[n+1]} := \mathscr{V}_\lambda^{(\omega)[n]} + \sum_{\substack{\mu\le\lambda \\ \mathrm{dp}_\lambda(\mu)=n+1}} d_\mu^{[n]}\, \chi_\mu^{(\omega)}\,. \tag{5.41}$$

This way we only add terms proportional to e^{ν} with $dp_{\lambda}(\nu) \geq n+1$ to $\mathscr{X}_{\lambda}^{(\omega)[n]}$, so that $\tilde{c}_{\lambda\mu}^{[n+1]} = \tilde{c}_{\lambda\mu}^{[n]} = \tilde{c}_{\lambda\mu}^{[dp_{\lambda}(\mu)]}$ for $dp_{\lambda}(\mu) \leq n$. Moreover, all terms proportional to e^{μ} with $dp_{\lambda}(\mu) = n+1$ are removed from the difference $\mathscr{X}_{\lambda}^{(\omega)} - \mathscr{X}_{\lambda}^{(\omega)[n]}$, because any irreducible twining character contributes at $dp(\mu)$ only through the highest weight vector, while all other contributions are at higher depths. This shows that the quantities $\mathscr{X}_{\lambda}^{(\omega)[n]}$ possess the properties stated above. From (5.39) and the properties of the $\tilde{c}_{\lambda\mu}^{[n]}$ described above it then follows immediately that (5.37) holds, with the coefficients $\tilde{c}_{\lambda\mu}$ given by

$$\tilde{c}_{\lambda\mu} := \tilde{c}_{\lambda\mu}^{[dp_{\lambda}(\mu)]} \,. \tag{5.42}$$

The weights μ which give a non-vanishing contribution to the sum (5.37) have to obey further requirements in addition to $\mu \leq \lambda$. First note that the twining character $\chi_{\mu}^{(\omega)}$ vanishes unless $\mu \in \mathsf{g}_{\circ}^{\star(0)}$. Writing $\lambda - \mu$ as in (5.35), the fact that both λ and μ are fixed under $\omega^{\star}$ implies that $n_{\dot{\omega}i} = n_i$ for all $i \in I$, so that

$$\breve{\lambda} - \breve{\mu} = \sum_{i \in \breve{I}} n_i \breve{\alpha}^{(i)} \,, \tag{5.43}$$

with $\breve{\lambda} = P_{\omega}^{\star -1}(\lambda)$, $\breve{\mu} = P_{\omega}^{\star -1}(\mu)$, and hence $\breve{\mu} \leq \breve{\lambda}$.

Next, we consider the generalized second order Casimir operator of g, defined as [1]

$$\mathscr{C}_2 := 2\,(\rho\,|\,H) + \sum_{I=1}^{2n-r} (u^I\,|\,u_I) + 2 \sum_{\alpha>0} \sum_{\ell} E_-^{\alpha,\ell} E_+^{\alpha,\ell}. \tag{5.44}$$

Here $\{u^I\}$ and $\{u_I\}$ denote any two dual bases of $\mathsf{g}_{\circ}$, the sum over ℓ in the last term takes care of the possible (finite) degeneracies of roots, and in the first term we implicitly identify $\mathsf{g}_{\circ}$ with its dual space $\mathsf{g}_{\circ}^{\star}$ with the help of the invariant bilinear form. Finally, the g-weight ρ is a Weyl vector of g, i.e. a weight which obeys $\rho(H^i) = 1$ for all $i \in I$ (if the determinant of the Cartan matrix A vanishes, then this element is not unique; in this case we make some arbitrary, but definite choice for ρ). For any Weyl vector of g, the projected $\breve{\mathsf{g}}$-weight $\breve{\rho} = P_{\omega}^{\star -1}(\rho)$ is a Weyl vector of $\breve{\mathsf{g}}$. With the above results, we can then relate the eigenvalues of the generalized second order Casimir operators of g and $\breve{\mathsf{g}}$. The operator $\mathscr{C}_2$ has the constant value

$$C_2(\lambda) = (\lambda + 2\rho\,|\,\lambda) = |\lambda + \rho|^2 - |\rho|^2 \tag{5.45}$$

(with $|\mu|^2 \equiv (\mu \mid \mu)$) on V_{λ}; taking into account the relation (3.34) between the invariant bilinear forms of g and $\breve{\mathsf{g}}$, it therefore follows that

$$N \cdot |\breve{\lambda} + \breve{\rho}|^2 = |\lambda + \rho|^2 = |\mu + \rho|^2 = N \cdot |\breve{\mu} + \breve{\rho}|^2 \tag{5.46}$$

for all weights μ which appear in (5.37).

In summary, for any g-weight Λ all weights appearing in the decomposition of the twining character of the Verma module V_{Λ} that is analogous to (5.37) are contained in the subset

$$\hat{B}(\Lambda) := \{\, \lambda = P_{\omega}^{\star}(\breve{\lambda}) \mid \breve{\lambda} \leq \breve{\Lambda},\ |\breve{\lambda} + \breve{\rho}|^2 = |\breve{\Lambda} + \breve{\rho}|^2 \,\} \tag{5.47}$$

of the weight space of $\mathfrak{g}$. Also, we can assume that the elements of $\hat{B}(\Lambda)$ are indexed by the positive integers, $\hat{B}(\Lambda) = \{\lambda_i \mid i \in \mathbf{N}\}$, in such a way that $\breve{\lambda}_j \leq \breve{\lambda}_i$ implies that i is smaller than j. Applying the formula (5.37) to all elements of $\hat{B}(\Lambda)$, it then follows that for all λ_i we have

$$\mathscr{V}^{(\omega)}_{\lambda_i} = \sum_{\lambda_j \in \hat{B}(\Lambda)} \tilde{c}_{ij}\, \chi^{(\omega)}_{\lambda_j} \tag{5.48}$$

with complex coefficients $\tilde{c}_{ij}$.

Moreover, by construction we have $\tilde{c}_{ii} = 1$, and $\tilde{c}_{ij}$ can be non-zero only if $\breve{\lambda}_j \leq \breve{\lambda}_i$, which due to the chosen ordering in $\hat{B}(\Lambda)$ implies that $i \leq j$. Hence the (infinite) matrix $\tilde{c} = (\tilde{c}_{ij})$ is upper triangular so that it can be inverted; its inverse $c = (c_{ij})$ is upper triangular as well and obeys $c_{ii} = 1$. This shows that the following kind of inverse of the formula (5.37) holds: the twining character $\chi^{(\omega)}_{\Lambda}$ of the irreducible module with highest weight Λ can be written as an (infinite) linear combination

$$\chi^{(\omega)}_{\Lambda} = \sum_{\lambda \in \hat{B}(\Lambda)} c_\lambda\, \mathscr{V}^{(\omega)}_{\lambda} \tag{5.49}$$

of the twining characters of Verma modules with highest weights in $\hat{B}(\Lambda)$, where the c_λ are complex numbers such that $c_\Lambda = 1$.

5.6. The character formula. We are now in a position to complete the proof of Theorem 1. Assume that the highest weight Λ is dominant integral, and let us write the linear relation (5.49) as

$$\chi^{(\omega)}_{\Lambda} = \sum_{\lambda \in \hat{B}(\Lambda)} c_\lambda\, \mathrm{e}^{\rho+\lambda} \cdot \mathrm{e}^{-\rho-\lambda} \mathscr{V}^{(\omega)}_{\lambda} = \Big(\sum_{\lambda \in \hat{B}(\Lambda)} c_\lambda\, \mathrm{e}^{\lambda+\rho} \Big) \cdot \mathscr{V}^{(\omega)}, \tag{5.50}$$

where we used the fact that $\mathscr{V}^{(\omega)} \equiv \mathrm{e}^{-\rho-\lambda} \mathscr{V}^{(\omega)}_{\lambda}$ is independent of λ. The results of the two preceding subsections show that $\mathscr{V}^{(\omega)}$ is odd under the action of $\hat{W}$ (with respect to the sign function $\hat{\epsilon}$ inherited from $\breve{W}$), while the left-hand side of (5.50) is even under $\hat{W}$. This implies that the sum in brackets on the right-hand side must be odd under $\hat{W}$, which means that $c_\lambda = \hat{\epsilon}(\hat{w}) c_\mu$ whenever there is an element $\hat{w} \in \hat{W}$ such that $\hat{w}(\lambda + \rho) = \mu + \rho$. Thus for all $\hat{w} \in \hat{W}$ we have

$$c_\lambda = \hat{\epsilon}(\hat{w})\, c_{\hat{w}(\lambda+\rho)-\rho} \tag{5.51}$$

with $\hat{\epsilon}(\hat{w})$ as defined in (5.16).

Moreover, as Λ (and hence also $\breve{\Lambda} = \mathrm{P}^{\star -1}_{\omega}(\Lambda)$) is dominant integral, with any weight λ the weight system of the irreducible module already contains the full $\hat{W}$-orbit of λ. As a consequence, we actually need to know c_λ only for a single element of each orbit of the action of $\hat{W}$; moreover, only weights in $\mathfrak{g}^{\star(0)}_{\circ}$ contribute. Now $\mathrm{P}^{\star -1}_{\omega}$ intertwines the action of $\breve{W}$ on $\breve{\mathfrak{g}}^{\star}_{\circ}$ and the action of $\hat{W}$ on $\mathfrak{g}^{\star(0)}_{\circ}$; any orbit of the $\breve{W}$-action contains a unique representative in the fundamental Weyl chamber of $\breve{\mathfrak{g}}$. Since the only weight $\breve{\lambda}$ in the fundamental Weyl chamber of $\breve{\mathfrak{g}}$ with $\breve{\lambda} \leq \breve{\Lambda} = \mathrm{P}^{\star -1}_{\omega}(\Lambda)$ for which $\breve{\lambda} + \breve{\rho}$ has the same length as $\breve{\Lambda} + \breve{\rho}$ is the highest weight $\breve{\Lambda}$ itself, we learn that $\hat{B}(\Lambda)$ contains only a single orbit of $\hat{W}$, namely that of the highest weight Λ. Together with $c_\Lambda = 1$ this implies that (5.50) can be rewritten as

$$\chi_\Lambda^{(\omega)} = \Big(\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\Lambda+\rho)}\Big)\cdot\mathscr{D}^{(\omega)}\,. \tag{5.52}$$

We can now use the fact that any symmetrizable Kac–Moody algebra g possesses the trivial one-dimensional irreducible module with highest weight $\Lambda = 0$. This weight is obviously a symmetric weight; also, by definition, τ_ω leaves the highest weight vector fixed, and hence in this special case the twining character is constant, $\chi_0^{(\omega)} = 1$. Evaluating (5.52) for this case we find

$$1 = \chi_0^{(\omega)} = \Big(\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\rho)}\Big)\cdot\mathscr{D}^{(\omega)}\,. \tag{5.53}$$

This allows us to read off the explicit expression

$$\mathscr{D}^{(\omega)} \equiv \Big(\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\rho)}\Big)^{-1} \tag{5.54}$$

for $\mathscr{D}^{(\omega)}$. When inserted into (5.17) and (5.52), this yields the explicit expressions

$$\mathscr{V}_\Lambda^{(\omega)} = \mathrm{e}^{\Lambda+\rho}\mathscr{D}^{(\omega)} = \mathrm{e}^{\Lambda+\rho}\Big(\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\rho)}\Big)^{-1} \tag{5.55}$$

for the twining character of the Verma module, and

$$\chi_\Lambda^{(\omega)} = \frac{\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\Lambda+\rho)}}{\sum_{\hat{w}\in\hat{W}} \hat{\epsilon}(\hat{w})\,\mathrm{e}^{\hat{w}(\rho)}} \tag{5.56}$$

for the twining character of the irreducible module.

We now observe that for any $h \in \mathsf{g}_\circ^{(0)}$ we have

$$\begin{aligned}(\hat{w}(\Lambda+\rho))(h) &= ([\mathrm{P}_W(\breve{w})](\Lambda+\rho))(h) = (\mathrm{P}_\omega^\star\breve{w}\mathrm{P}_\omega^{\star-1}(\Lambda+\rho))(h)\\ &= (\breve{w}(\breve{\Lambda}+\breve{\rho}))(\mathrm{P}_\omega h)\end{aligned} \tag{5.57}$$

with $\breve{\Lambda} = \mathrm{P}_\omega^{\star-1}(\Lambda)$. When combined with (5.56), this implies that

$$\chi_\Lambda^{(\omega)}(h) = \frac{\sum_{\breve{w}\in\breve{W}} \breve{\epsilon}(\breve{w})\,\mathrm{e}^{(\breve{w}(\breve{\Lambda}+\breve{\rho}))(\mathrm{P}_\omega h)}}{\sum_{\breve{w}\in\breve{W}} \breve{\epsilon}(\breve{w})\,\mathrm{e}^{(\breve{w}(\breve{\rho}))(\mathrm{P}_\omega h)}} \tag{5.58}$$

for all $h \in \mathsf{g}_\circ^{(0)}$. By the usual Weyl-Kac character formula for the integrable highest weight module with highest weight $\breve{\Lambda}$, this means that

$$\chi_\Lambda^{(\omega)}(h) = \chi_{\breve{\Lambda}}(\mathrm{P}_\omega h)\,. \tag{5.59}$$

This completes the proof of part b) of Theorem 1. Analogously, part a) of Theorem 1 follows by comparing (5.55) with the formula for the Verma module characters of $\breve{\mathsf{g}}$ (since $\mathscr{D}^{(\omega)}$ is independent of Λ, this result holds for arbitrary highest weights, not just for dominant integral ones).

6. Simple Current Automorphisms of Untwisted Affine Lie Algebras

6.1. Centrally extended loop algebras. Let us now specialize to the case where g is an untwisted affine Lie algebra, which is relevant for applications in conformal field theory. Among the diagram automorphisms of the untwisted affine Lie algebras, there is a particularly interesting subclass which corresponds to the action of simple currents in the WZW models of conformal field theory. From now on we will also restrict to these specific diagram automorphisms; abstractly, they can be characterized as the elements of the unique maximal abelian normal subgroup $\mathscr{Z}(\mathsf{g})$ of the group $\Gamma(\mathsf{g})$ of diagram automorphisms; this abelian subgroup is isomorphic to the center of the universal covering Lie group that has the horizontal subalgebra $\bar{\mathsf{g}} \subset \mathsf{g}$ as its Lie algebra. Also, the remark at the end of Sect. 3 shows that in this situation the Eqns. (3.25) and (3.34) are valid in the conventional normalization of the invariant symmetric bilinear form.

In the affine case, the rank r of the $n \times n$ Cartan matrix A is $n-1$. Hence the space g_D of derivations is one-dimensional. However, one usually does not choose the derivation D in the way we did in the general case, i.e. such that $\omega(D) = D$, but rather in a way which is suggested by the realization of affine Lie algebras in terms of centrally extended loop algebras. We will denote the latter derivation by L_0.

In the description of untwisted affine Lie algebras via loop algebras, a basis of generators of g is given by H^i_m and $E^{\bar\alpha}_m$ together with the canonical central element K and the derivation L_0. Here m takes values in $\mathbf{Z}$, i takes values in the index set $\bar I$ that corresponds to the horizontal subalgebra $\bar{\mathsf{g}}$ of g, and $\bar\alpha$ is a root of $\bar{\mathsf{g}}$. The horizontal subalgebra is a simple Lie algebra; for $\mathsf{g} = X_r^{(1)}$ it is given by $\bar{\mathsf{g}} = X_r$. The rank of $\bar{\mathsf{g}}$ is equal to the rank $r = n-1$ of A, so that it is natural to write the index set I as [7]

$$I := \bar I \cup \{0\} = \{0, 1, 2, \ldots, r\}\,. \tag{6.1}$$

In this basis the Lie brackets of g read

$$\begin{aligned}
&[H^i_m, H^j_n] = m\, \bar B^{i,j}\, \delta_{m+n,0}\, K\,,\\
&[H^i_m, E^{\bar\alpha}_n] = \bar\alpha^i E^{\bar\alpha}_{m+n}\,,\\
&[E^{\bar\alpha}_m, E^{\bar\alpha'}_n] = e_{\bar\alpha,\bar\alpha'}\, E^{\bar\alpha+\bar\alpha'}_{m+n}\,,\\
&[E^{\bar\alpha}_m, E^{-\bar\alpha}_n] = (\bar\alpha^\vee, H_{m+n}) + m\delta_{m+n,0}K\,,\\
&[L_0, H^i_m] = -m\, H^i_m\,, \qquad [L_0, E^{\bar\alpha}_m] = -m\, E^{\bar\alpha}_m\,,
\end{aligned} \tag{6.2}$$

together with $[\,\cdot, K] = 0$. It is implicit in (6.2) that $e_{\bar\alpha,\bar\alpha'} = 0$ if $\bar\alpha + \bar\alpha'$ is not a $\bar{\mathsf{g}}$-root. Further, $\bar B$ is the symmetrized Cartan matrix of $\bar{\mathsf{g}}$,

$$\bar B^{i,j} = (\bar\alpha^{(i)^\vee}, \bar\alpha^{(j)^\vee}) = \frac{2}{(\bar\alpha^{(i)}, \bar\alpha^{(i)})}\, \bar A^{i,j} \tag{6.3}$$

for $i,j \in \bar I$, with $\bar\alpha^{(i)}$ the i^{th} simple root and $\bar\alpha^\vee \equiv 2\bar\alpha/(\bar\alpha,\bar\alpha)$, and we implicitly identify the Cartan subalgebra of $\bar{\mathsf{g}}$ with the weight space inasfar as we use the notation $(\bar\lambda, H_m) = \sum_{i=1}^r \bar\lambda_i H^i_m \equiv \sum_{i,j=1}^r \bar G_{i,j} \bar\lambda^i H^j_m$, with $\bar G$ the inverse of $\bar B$, for any $\bar{\mathsf{g}}$-weight $\bar\lambda$. The relation between the inner product $(\cdot\,,\cdot)$ on $\bar{\mathsf{g}}$ and the invariant bilinear form $(\cdot\,|\,\cdot)$ on g is

[7] Note, however, that by construction the index set $\breve I$ is then generically *not* the set $\{0, 1, \ldots, \text{rank}\,\bar{\mathsf{g}}\}$.

$$(h \mid h') = (\bar{h}, \bar{h}') + \xi\eta' + \xi'\eta \tag{6.4}$$

for $h = \bar{h} + \xi K - \eta L_0$ with $\bar{h} \in \bar{\mathfrak{g}}_\circ$. Note that the normalization of $\bar{G}$, or equivalently the normalization of $(\cdot\,,\cdot)$ is arbitrary; we fix this freedom such that the highest $\bar{\mathfrak{g}}$-root $\bar{\theta}$ has length squared 2. Then in particular the *level* $k^\vee_\lambda$, defined for any vector of $\mathfrak{g}$-weight λ by $k^\vee_\lambda = 2k_\lambda/(\bar{\theta},\bar{\theta})$, is equal to the eigenvalue k_λ of the canonical central element K.

The step operators associated to the simple roots $\alpha^{(i)}$ with $i \in \bar{I}$ are given by $E^i_\pm \equiv E^{\pm\alpha^{(i)}} = E_0^{\pm\bar{\alpha}^{(i)}}$ and the corresponding Cartan subalgebra elements by $H^i \equiv [E^i_+, E^i_-] = H^i_0$, while the step operators associated to the zeroth simple root $\alpha^{(0)}$ read $E^0_\pm = E^{\mp\bar{\theta}}_{\pm 1}$, and the corresponding Cartan subalgebra element is $H^0 \equiv [E^0_+, E^0_-] = H^0_0$. More generally, we introduce the elements H^0_n as the linear combinations

$$H^0_n = K\,\delta_{n,0} - \sum_{j=1}^{r} a^\vee_j H^j_n\,. \tag{6.5}$$

The level of a weight λ is then related by

$$k^\vee_\lambda = \sum_{i=0}^{r} a^\vee_i \lambda^i \tag{6.6}$$

to its Dynkin components λ^i. Further, for $i \in \bar{I}$ the Coxeter and dual Coxeter labels coincide with the expansion coefficients of the highest $\bar{\mathfrak{g}}$-root θ in the basis of simple roots and the basis of simple coroots, respectively, so that (6.5) may be rewritten as $H^0_n = K\,\delta_{n,0} - (\bar{\theta}, H_n)$. We also note that according to (6.6) the component λ^0 of a weight is redundant if only weights at a fixed level are considered.

6.2. The derived algebra. We will again first describe how the automorphism ω acts on the derived algebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ and later study the action on the derivation. On the generators H^i_m and $E^{\bar{\alpha}}_m$, which together with K span $\hat{\mathfrak{g}}$, the diagram automorphism ω acts as

$$\omega(H^i_n) = H^{\dot{\omega}i}_n\,, \qquad \omega(E^{\bar{\alpha}}_n) = \eta_{\bar{\alpha}}\, E^{\bar{\omega}^\star\bar{\alpha}}_{n+\ell_{\bar{\alpha}}}\,, \tag{6.7}$$

while it leaves the canonical central element K fixed,

$$\omega(K) = K\,. \tag{6.8}$$

Here we use the following notation. In (6.7) the prefactors $\eta_{\bar{\alpha}}$ are signs which are +1 for the simple roots and can be deduced for all other roots by writing the step operator for a non-simple root as a (multiple) commutator of step operators for simple roots and then using the automorphism property of ω to extend the action (3.6) of ω on the step operators for simple roots. (We have already encountered an example with $\eta_{\bar{\alpha}} = -1$ when we calculated $\omega(E^{\alpha^{(i)}+\alpha^{(\dot{\omega}i)}})$ in Subsect. 5.3, cf. Eqn (5.24).) Also, the index i in principle only takes values in the unextended index set $\bar{I}$, and for $i = \dot{\omega}^{-1}0$ the identity (6.5) is implicit on the right-hand side (note that $\dot{\omega}^{-1}0 \in \bar{I}$ for all $\omega \in \mathscr{Z}(\mathfrak{g})$). However, owing to this same identity and the invariance (6.8) of K, the relations (6.7) are still valid if one allows for i to lie in the extended index set I, i.e. including $i = 0$. Further, in (6.7) we introduced for any $\bar{\mathfrak{g}}$-root $\bar{\alpha}$ the number $\ell_{\bar{\alpha}}$ defined by

$$\ell_{\bar\alpha} := (\bar\alpha, \bar\Lambda_{(\dot\omega^{-1}0)}) , \tag{6.9}$$

where we denote by $\bar\Lambda_{(i)}$, $i = 1, 2, \dots, r$, the *horizontal fundamental weights*, i.e. the fundamental weights of the horizontal subalgebra $\bar{\mathfrak g}$. Finally, we introduced a map $\bar\omega^\star$ on the weight space of $\bar{\mathfrak g}$; it is defined by the following action on the Dynkin components $\bar\lambda^j$ of the weight $\bar\lambda$:

$$(\bar\omega^\star\bar\lambda)^j = \bar\lambda^{\dot\omega^{-1}j} \quad \text{for } j \neq \dot\omega 0 , \tag{6.10}$$

while

$$(\bar\omega^\star\bar\lambda)^{\dot\omega 0} = \lambda^0 \equiv k^\vee_\lambda - \sum_{j=1}^{r} a^\vee_j \bar\lambda^j . \tag{6.11}$$

Hence

$$\bar\omega^\star\bar\lambda = k^\vee_\lambda \bar\Lambda_{(\dot\omega 0)} + \sum_{\substack{j=1\\ j\neq\dot\omega 0}}^{r} \bar\lambda^{\dot\omega^{-1}j}\bar\Lambda_{(j)} - (\sum_{j=1}^{r} a^\vee_j \bar\lambda^j)\,\bar\Lambda_{(\dot\omega 0)} . \tag{6.12}$$

Note that $\bar\omega^\star$ is an affine mapping on the weight space of $\bar{\mathfrak g}$.

As the components of the simple $\bar{\mathfrak g}$-roots $\bar\alpha^{(i)}$ in the Dynkin basis are just the rows of the Cartan matrix of $\bar{\mathfrak g}$, the definition of $\bar\omega^\star$ implies in particular that

$$\bar\omega^\star\bar\alpha^{(i)} = \bar\alpha^{(\dot\omega i)} \;\text{ for } i \neq \dot\omega^{-1}0 , \qquad \bar\omega^\star(\bar\alpha^{(\dot\omega^{-1}0)}) = -\bar\theta . \tag{6.13}$$

By making use of the Serre relations and the invariance property of the Cartan matrix, it then follows that $\bar\omega^\star\bar\alpha$ is a $\bar{\mathfrak g}$-root whenever $\bar\alpha$ is (and hence the notation $E^{\bar\omega^\star\bar\alpha}_{n+\ell_{\bar\alpha}}$ introduced in (6.7) indeed makes sense).

The action of $\bar\omega^\star$ on the roots of $\bar{\mathfrak g}$ can be described more concretely as follows. We write any root $\bar\beta$ of $\bar{\mathfrak g}$ as a linear combination of simple coroots as $\bar\beta = \sum_{i=1}^r \beta_i \bar\alpha^{(i)^\vee}$, and for simplicity set $\beta_0 = 0$ for all roots $\bar\beta$ as well as $\bar\alpha^{(0)^\vee} = -\bar\theta$. With these conventions we have

$$\ell_{\bar\alpha} = (\bar\alpha, \bar\Lambda_{(\dot\omega^{-1}0)}) = \alpha_{\dot\omega^{-1}0} , \tag{6.14}$$

while the action of $\bar\omega^\star$ on the roots is

$$\bar\omega^\star\bar\alpha = \sum_{j=0}^{r} \alpha_j \bar\alpha^{(\dot\omega j)^\vee} = -\ell_{\bar\alpha}\bar\theta + \sum_{j=1}^{r} \alpha_{\dot\omega^{-1}j}\bar\alpha^{(j)^\vee} . \tag{6.15}$$

We can use this result to derive a relation which we will need in Subsect. 6.6; since for untwisted affine Lie algebras we have $a^\vee_0 = a^\vee_{\dot\omega 0} = 1$, so that $(\bar\Lambda_{(\dot\omega 0)}, \bar\theta) = a^\vee_{\dot\omega 0} = 1$, the relation (6.15) implies

$$(\bar\Lambda_{(\dot\omega 0)}, \bar\omega^\star\bar\alpha) = -\ell_{\bar\alpha} + \alpha_0 = -\ell_{\bar\alpha} . \tag{6.16}$$

6.3. The derivation. Let us now describe how ω acts on L_0. The derivation L_0 is defined as the unique element of the Cartan subalgebra of $\mathfrak g$ which has the commutation relations

$$[L_0, E^i_\pm] = \mp\delta_{i,0}\, E^0_\pm \tag{6.17}$$

and satisfies $(L_0 \mid L_0) = 0$. Then the automorphism property of ω demands that $[\omega(L_0), E_\pm^i] = \mp\delta_{i,\dot\omega 0} E_\pm^{\dot\omega 0}$; this fixes $\omega(L_0)$ up to a term proportional to the central element K, $\omega(L_0) = L_0 - (\bar\Lambda_{(\dot\omega 0)}, H) + cK$. Indeed, using $(\bar\Lambda_{(\dot\omega 0)}, \bar\alpha^{(i)}) = \delta_{\dot\omega 0,i}$ for $i = 1, 2, \dots, r$, and $(\Lambda_{(\dot\omega 0)}, -\bar\theta) = -a^\vee_{\dot\omega 0} = -1$, we find that $[\omega(L_0), E_\pm^i] = \mp\delta_{\dot\omega 0,i} E_\pm^i$. The constant of proportionality c can be obtained from the requirement that the invariant bilinear form is ω-invariant, which implies that $(\omega(L_0) \,|\, \omega(L_0))$ vanishes,

$$\begin{aligned} 0 &= (\omega(L_0) \mid \omega(L_0)) = (L_0 - (\bar\Lambda_{(\dot\omega 0)}, H) + cK \mid L_0 - (\bar\Lambda_{(\dot\omega 0)}, H) + cK) \\ &= -2c + (\Lambda_{(\dot\omega 0)}, \Lambda_{(\dot\omega 0)}) . \end{aligned} \tag{6.18}$$

Hence we obtain

$$\omega(L_0) = L_0 - (\bar\Lambda_{(\dot\omega 0)}, H) + \tfrac{1}{2} (\bar\Lambda_{(\dot\omega 0)}, \bar\Lambda_{(\dot\omega 0)}) \, K . \tag{6.19}$$

With this result we can now make the relation between the derivation D that we used in the general setting and the derivation L_0 explicit. To this end recall that D is uniquely characterized by the relations (3.15) for $(D \,|\, \cdot\,)$. To make the definition concrete, we choose a basis of eigenvectors of ω for the Cartan subalgebra $\hat{\mathfrak{g}}_\circ$ of the derived algebra. Thus we introduce vectors

$$h^i_m := \frac{1}{N_i} \sum_{l=1}^{N_i} \zeta_i^{-ml} H^{\dot\omega^l i} , \tag{6.20}$$

where i takes values in $\breve I$, $m = 1, 2, \dots, N_i$, and $\zeta_i = \zeta^{N/N_i}$, with ζ as defined in (3.10), is a primitive N_i^{th} root of unity. As a basis of $\hat{\mathfrak{g}}_\circ$ we now choose all h^i_m, except for $i = 0 = m$, together with the central element K. Rewriting the conditions (3.15) in terms of the generators H^i, we then find that D is characterized by

$$(D \,|\, D) = 0 , \qquad (D \,|\, H^i) = 0 \;\text{ for } i \neq \dot\omega^l 0 , \qquad (D \,|\, H^{\dot\omega^l 0}) = \frac{1}{N} \;\text{ for all } l . \tag{6.21}$$

This fixes D to

$$D = -L_0 + \frac{1}{N} \sum_{l=1}^{N-1} (\bar\Lambda_{(\dot\omega^l 0)}, H) - \frac{\varGamma_{00}}{2N^2} K \tag{6.22}$$

with

$$\varGamma_{00} := \sum_{l,l'=1}^{N-1} (\bar\Lambda_{(\dot\omega^l 0)}, \bar\Lambda_{(\dot\omega^{l'} 0)}) \equiv \sum_{l,l'=1}^{N-1} \bar G_{\dot\omega^l 0, \dot\omega^{l'} 0} \,. \tag{6.23}$$

Let us also determine the relation between the derivation $\breve D = \mathrm{P}_\omega(D)$ defined by (3.30) and (3.31) and the element $\breve L_0$ of the orbit Lie algebra $\breve{\mathfrak{g}}$. From $\mathrm{P}_\omega(h^i_0) = \breve H^i$ we learn that $\breve D$ is characterized by

$$(\breve D \,|\, \breve D) = 0 , \qquad (\breve D \,|\, \breve H^i) = 0 \;\text{ for } i \in \breve I \setminus \{0\} , \qquad (\breve D \,|\, \breve K) = N . \tag{6.24}$$

This fixes $\breve D$ to $\breve D = -N \breve L_0$. Together with $\breve D = \mathrm{P}_\omega(D)$ and (6.22), this relation shows that

$$\mathrm{P}_\omega(L_0) = N \breve L_0 + \frac{1}{N} \mathrm{P}_\omega(\sum_{l=1}^{N-1} \bar\Lambda_{(\dot\omega^l 0)}, H) - \frac{\varGamma_{00}}{2N^2} \breve K . \tag{6.25}$$

6.4. The action of $\omega^\star$. From the action of ω we can also derive the action of its dual map $\omega^\star$. First,

$$(\omega^\star(\alpha^{(i)}))(H^j) = \alpha^{(i)}(H^{\dot\omega^{-1}j}) = A^{i,\dot\omega^{-1}j} = A^{\dot\omega^i,j} = \alpha^{(\dot\omega i)}(H^j)\,. \tag{6.26}$$

It follows that $\omega^\star(\alpha^{(i)}) = \alpha^{(\dot\omega i)} + \xi_i\delta$, where ξ_i is some number and δ is the specific element

$$\delta := \sum_{i=1}^{r} a_i\alpha^{(i)}\,, \tag{6.27}$$

with the a_i the Coxeter labels of g, of the weight space $\mathsf{g}^\star$. (Note that the imaginary roots of g are precisely the integral multiples of δ.) Also, applying ω to the relation $[D, E^i_+] = \alpha^{(i)}(D)E^i_+$, we obtain

$$\alpha^{(i)}(D)E^{\dot\omega i}_+ = \omega([D, E^i_+]) = [D, E^{\dot\omega i}_+] = \alpha^{(\dot\omega i)}(D)E^{\dot\omega i}_+\,, \tag{6.28}$$

which shows that $\alpha^{(i)}(D) = \alpha^{(\dot\omega i)}(D)$. To determine the constants ξ_i, we now apply $\omega^\star(\alpha^{(i)})$ to D. By the results just obtained, this yields

$$(\omega^\star(\alpha^{(i)}))(D) = \alpha^{(\dot\omega i)}(D) + \xi_i\,\delta(D)\,; \tag{6.29}$$

on the other hand, from the definition of $\omega^\star$ we obtain

$$(\omega^\star(\alpha^{(i)}))(D) = \alpha^{(i)}(\omega^{-1}D) = \alpha^{(i)}(D)\,. \tag{6.30}$$

Thus $\xi_i\delta(D) = 0$, which because of $\delta(D) \neq 0$ means that $\xi_i = 0$. Hence we have

$$\omega^\star(\alpha^{(i)}) = \alpha^{(\dot\omega i)} \tag{6.31}$$

for all $i \in I$. This implies in particular that

$$\omega^\star(\delta) = \sum_i a_i\,\omega^\star(\alpha^{(i)}) = \delta\,. \tag{6.32}$$

Analogously, one derives how the fundamental weights $\Lambda_{(i)} \in \mathsf{g}^\star_\circ$, defined by

$$\Lambda_{(i)}(H^j) = \delta_i^j \qquad \text{and} \qquad \Lambda_{(i)}(L_0) = 0\,, \tag{6.33}$$

transform under $\omega^\star$. We find

$$\omega^\star(\Lambda_{(i)}) = \Lambda_{(\dot\omega i)} + [\bar G_{\dot\omega^{-1}0,i} - \tfrac12 a_i^\vee\,(\bar\Lambda_{(\dot\omega^{-1}0)}, \bar\Lambda_{(\dot\omega^{-1}0)})]\,\delta\,. \tag{6.34}$$

Together with the element δ (6.27), the fundamental weights $\Lambda_{(i)}$ form a basis of the weight space $\mathsf{g}^\star$. Another basis of $\mathsf{g}^\star$ is given by δ, $\kappa := \Lambda_{(0)}$ and by the horizontal fundamental weights

$$\bar\Lambda_{(i)} = \Lambda_{(i)} - a_i^\vee\Lambda_{(0)} \tag{6.35}$$

with $i = 1, 2, \ldots, r$. The relation between the components of a weight λ in the two bases is

$$\sum_{i=0}^{r}\lambda^i\Lambda_{(i)} + n_\lambda\delta = \lambda = \sum_{i=1}^{r}\bar\lambda^i\bar\Lambda_{(i)} + k_\lambda^\vee\kappa + n_\lambda\delta\,, \tag{6.36}$$

i.e. $\bar\lambda^i = \lambda^i$ for $i = 1, 2, \ldots, r$ and $k_\lambda^\vee = \sum_{i=0}^r a_i^\vee\lambda^i$. We also set $\bar\Lambda_{(0)} \equiv 0$, which will be convenient in some calculations.

The horizontal fundamental weights (6.35) transform under $\omega^\star$ as

$$\begin{aligned}\omega^\star(\bar\Lambda_{(i)}) &= \Lambda_{(\dot\omega i)} - a_i^\vee \Lambda_{(0)} + a_i^\vee(\Lambda_{(0)} - \Lambda_{(\dot\omega 0)}) + \bar G_{\dot\omega^{-1}0,i}\,\delta \\ &= \bar\Lambda_{(\dot\omega i)} - a_i^\vee \bar\Lambda_{(\dot\omega 0)} + \bar G_{\dot\omega^{-1}0,i}\,\delta\,.\end{aligned} \tag{6.37}$$

Using the relations

$$(\delta \mid \bar\Lambda_{(i)}) = \bar\Lambda_{(i)}(K) = 0\,, \qquad (\delta \mid \delta) = \delta(K) = 0\,, \tag{6.38}$$

and the fact that along with ω also $\omega^\star$ is an isometry, we then find that the metric $\bar G$ on the horizontal weight space obeys

$$\begin{aligned}\bar G_{i,j} &\equiv (\bar\Lambda_{(i)} \mid \bar\Lambda_{(j)}) = (\omega^\star\bar\Lambda_{(i)} \mid \omega^\star\bar\Lambda_{(j)}) \\ &= \bar G_{\dot\omega i,\dot\omega j} - a_i^\vee \bar G_{\dot\omega 0,\dot\omega j} - a_j^\vee \bar G_{\dot\omega 0,\dot\omega i} + a_i^\vee a_j^\vee \bar G_{\dot\omega 0,\dot\omega 0}\,.\end{aligned} \tag{6.39}$$

Applying the analogous relation for the automorphism ω^m, we see that

$$\begin{aligned}\textstyle\sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^l i,\dot\omega^{l'} j} = &\textstyle\sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^{l+m} i,\dot\omega^{l'+m} j} - N a_i^\vee \sum_{l=0}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^{l+m} j} \\ &\textstyle -N a_j^\vee \sum_{l=0}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^{l+m} i} + N^2 a_i^\vee a_j^\vee \bar G_{\dot\omega^m 0,\dot\omega^m 0}\,,\end{aligned} \tag{6.40}$$

and hence

$$N a_i^\vee a_j^\vee \bar G_{\dot\omega^m 0,\dot\omega^m 0} = a_i^\vee \sum_{l=0}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^l j} + a_j^\vee \sum_{l=0}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^l i}\,. \tag{6.41}$$

Define now $X_i := \sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^l 0,\dot\omega^{l'} i}$. Summing Eqn. (6.41) over m, we obtain the system

$$a_i^\vee X_j + a_j^\vee X_i = N \sum_{m=1}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^m 0}\, a_i^\vee a_j^\vee =: \xi\, a_i^\vee a_j^\vee \tag{6.42}$$

of linear equations for the X_i. These equations have the unique solution $X_i = \frac{1}{2}\,\xi\, a_i^\vee$. With $a_0^\vee = 1$, it then follows that

$$\Gamma_{00} \equiv \sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^l 0,\dot\omega^{l'} 0} = X_0 = \frac{N}{2} \sum_{m=1}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^m 0} \tag{6.43}$$

and

$$X_i = \sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^l 0,\dot\omega^{l'} i} = a_i^\vee X_0 = a_i^\vee \Gamma_{00} \tag{6.44}$$

with Γ_{00} as in (6.23).

6.5. The action of $\mathrm{P}_\omega^\star$. It is straightforward to determine the action of $\mathrm{P}_\omega^\star$ on $\breve{\mathfrak g}_\circ^\star$ from the action of P_ω on $\mathfrak g_\circ$. We only need to observe that the invariant bilinear form identifies $\breve{\mathfrak g}_\circ$ with the weight space $\breve{\mathfrak g}_\circ^\star$ in such a way that $\breve\alpha^{(i)}$ corresponds to $\breve H^i/\breve d_i$, and that since ω leaves the bilinear form invariant and $(\mathrm{P}_\omega h \mid \mathrm{P}_\omega h') = N(h \mid h')$, the identification of $\breve{\mathfrak g}_\circ$ with $\breve{\mathfrak g}_\circ^\star$ corresponds to identifying P_ω and $\mathrm{P}_\omega^{\star-1}$ up to a rescaling by N. Then in particular for the simple roots we have

$$\mathrm{P}^\star_\omega(\breve\alpha^{(i)}) = s_i \cdot \sum_{l=0}^{N_i-1} \alpha^{(\dot\omega^l i)} \,, \tag{6.45}$$

as already deduced for the general case in (5.12).

For untwisted affine Lie algebras the general relation (2.18) between the Coxeter labels and dual Coxeter labels of g and of the orbit algebra $\breve{\mathsf{g}}$ can be made more concrete: because of the normalizations $\breve a_0 = a_0 = 1$ and $\breve a_0^\vee = (N_0/N) a_0^\vee = a_0^\vee = 1$, the relations (2.20) and (2.19) tell us that the numbers defined by (2.18) are precisely the conventionally normalized Coxeter and dual Coxeter labels of $\breve{\mathsf{g}}$, respectively (in particular for all untwisted affine algebras they are integral, which for $\breve a_i^\vee$ is not manifest in (2.18)). For the generator $\breve\delta := \sum_{i\in\breve I} \breve a_i \breve\alpha^{(i)}$ whose integral multiples are the imaginary $\breve{\mathsf{g}}$-roots, this implies

$$\mathrm{P}^\star_\omega(\breve\delta) = \delta \,. \tag{6.46}$$

We can now also determine how the fundamental weights $\breve\Lambda_{(i)} \in \mathsf{g}^\star_\circ$ are mapped by $\mathrm{P}^\star_\omega$. As can be checked by considering the action on L_0 and on the h^i_0, we have

$$\mathrm{P}^\star_\omega(\breve\Lambda_{(i)}) = \sum_{l=0}^{N_i-1} \Lambda_{(\dot\omega^l i)} + \xi a_i^\vee N_i \,\delta \,, \tag{6.47}$$

where $\xi := (1-2N)\Gamma_{00}/2N^3$ is a constant that depends only on ω.

Using the relation $a_i^\vee = \frac{N}{N_i} \breve a_i^\vee$ we compute the action of $\mathrm{P}^\star_\omega$ on the horizontal fundamental weights:

$$\begin{aligned} \mathrm{P}^\star_\omega(\bar{\breve\Lambda}_{(i)}) &= \sum_{l=0}^{N_i-1} \Lambda_{(\dot\omega^l i)} + \xi a_i^\vee N_i \,\delta - \breve a_i^\vee \big(\sum_{l=0}^{N-1} \Lambda_{(\dot\omega^l 0)} + \xi a_0^\vee N \delta\big) \\ &= \sum_{l=0}^{N_i-1} \bar\Lambda_{(\dot\omega^l i)} - \frac{N_i}{N} a_i^\vee \sum_{l=0}^{N-1} \Lambda_{(\dot\omega^l 0)} = \frac{N_i}{N} \big(\sum_{l=0}^{N-1} \Lambda_{(\dot\omega^l i)} - a_i^\vee \Lambda_{(\dot\omega^l 0)}\big) \,. \end{aligned} \tag{6.48}$$

With the definition (6.23) of Γ_{00} and the identity (6.44), this yields the relation

$$\bar{\breve G}_{ij} = (\bar{\breve\Lambda}_{(i)} \,|\, \bar{\breve\Lambda}_{(j)}) = \frac{1}{N} (\mathrm{P}^\star_\omega \bar{\breve\Lambda}_{(i)} \,|\, \mathrm{P}^\star_\omega \bar{\breve\Lambda}_{(j)}) = \frac{N_i N_j}{N^3} \big[\sum_{l,l'=0}^{N-1} \bar G_{\dot\omega^l i, \dot\omega^{l'} j} - a_i^\vee a_j^\vee \Gamma_{00} \big] \tag{6.49}$$

between the metrics of the horizontal part of the weight spaces of g and $\breve{\mathsf{g}}$.

6.6. The Virasoro algebra. It is natural to consider the extension of the affine algebra g to the semi-direct sum of g and the Virasoro algebra $\mathcal{V}ir$. The Lie algebra $\mathcal{V}ir$ is spanned by generators C and L_n, $n \in \mathbf{Z}$; C is a central element, and L_0 is the derivation of g described in Subsect. 6.3. The Virasoro algebra has Lie brackets

$$[L_m, L_n] = (m-n)\, L_{m+n} + \tfrac{1}{12}\, (m^3 - m)\, C \,, \tag{6.50}$$

and its semi-direct sum with g is defined by

$$[L_m, H^i_n] = -n\, H^i_{m+n} \,, \qquad [L_m, E^{\bar\alpha}_n] = -n\, E^{\bar\alpha}_{m+n} \tag{6.51}$$

and $[C,\cdot\,] = 0 = [K,\cdot\,]$. It is in fact possible to extend ω to this semi-direct sum, namely via

$$\omega(L_m) = L_m - (\bar\Lambda_{(\dot\omega 0)}, H_m) + \tfrac{1}{2}\,(\bar\Lambda_{(\dot\omega 0)}, \bar\Lambda_{(\dot\omega 0)})\,\delta_{m,0}\,K \tag{6.52}$$

and $\omega(C) = C$. It is readily checked (using in particular the identity (6.16)) that ω defined by (6.7) and (6.52) is an automorphism of the Virasoro algebra and of its semi-direct sum with g. Note that, just as the extension from the derived algebra $\hat{\mathsf{g}}$ to all of g, the extension of ω to the semi-direct sum $\mathsf{g} \oplus \mathscr{V}ir$ is unique.

A symmetric weight satisfies by definition $\omega^\star\lambda = \lambda$, which because of (6.34) implies in particular that

$$\lambda^i = \lambda^{\dot\omega^l i} \tag{6.53}$$

for all $i = 0, 1, \ldots, r$ and all l. This identity is certainly a necessary condition for $\omega^\star\lambda = \lambda$, but in fact it is also sufficient. Namely, for any g-weight $\lambda = \sum_{i=0}^r \lambda^i \Lambda_{(i)} + n_\lambda\delta$ with $\lambda^i = \lambda^{\dot\omega i}$ one has

$$k^\vee_\lambda \equiv \sum_{i=0}^r a^\vee_i \lambda^i = \sum_{i\in\breve I} a^\vee_i N_i \lambda^i\,. \tag{6.54}$$

Furthermore, by an argument analogous to the derivation of (6.44) from (6.42), it can be deduced from the set of Eqns. (6.41) that the metric on the weight space of $\bar{\mathsf{g}}$ satisfies the identities

$$\sum_{m=0}^{N-1} \bar G_{\dot\omega 0,\dot\omega^m i} = \tfrac{1}{2}\,N\,a^\vee_i\,\bar G_{\dot\omega 0,\dot\omega 0} \tag{6.55}$$

for all $i \in I$ which are not of the form $i = \dot\omega^n 0$ for some n, and

$$\sum_{m=2}^{N-1} \bar G_{\dot\omega 0,\dot\omega^m 0} = \tfrac{1}{2}\,(N-2)\,\bar G_{\dot\omega 0,\dot\omega 0} \tag{6.56}$$

(which is of course non-trivial only for $N > 2$).

Combining these identities with (6.54), one finds that for any symmetric g-weight one has $\sum_{i=1}^r \bar G_{\dot\omega 0,i}\lambda^i = \frac{1}{2}\,k^\vee_\lambda \bar G_{\dot\omega 0,\dot\omega 0}$, or what is the same,

$$(\bar\Lambda_{(\dot\omega 0)}, \bar\lambda) = \tfrac{1}{2}\,k^\vee_\lambda\,(\bar\Lambda_{(\dot\omega 0)}, \bar\Lambda_{(\dot\omega 0)})\,. \tag{6.57}$$

Now according to (6.32) and (6.34) the weight $\lambda = \sum_{i=0}^r \lambda^i\Lambda_{(i)} + n_\lambda\delta$ is mapped by $\omega^\star$ to

$$\begin{aligned}\omega^\star(\lambda) &= \sum_{i=0}^r \lambda^i \Lambda_{(\dot\omega i)} + (n_\lambda + \sum_{i=0}^r [\bar G_{\dot\omega^{-1}0,i} - \frac{1}{2} a^\vee_i (\bar\Lambda_{(\dot\omega^{-1}0)}, \bar\Lambda_{(\dot\omega^{-1}0)})]\lambda^i)\,\delta \\ &= \sum_{i=0}^r \lambda^i \Lambda_{(\dot\omega i)} + n_\lambda\delta + (\sum_{i=0}^r \bar G_{\dot\omega^{-1}0,i}\lambda^i - \tfrac{1}{2}(\bar\Lambda_{(\dot\omega^{-1}0)}, \bar\Lambda_{(\dot\omega^{-1}0)})k^\vee_\lambda)\,\delta\,.\end{aligned} \tag{6.58}$$

The relation (6.57) thus shows that $\omega^\star\lambda = \sum_{i=0}^r \lambda^i\Lambda_{(\dot\omega i)} + n_\lambda\delta = \lambda$ if (6.53) holds. Thus (6.53) is a sufficient condition for λ being a symmetric weight.

It follows in particular that the pre-image $\mathrm{P}_\omega^{\star-1}(\lambda)$ of a symmetric weight λ is the unique weight of $\breve{\mathsf{g}}$ that is obtained by restricting to components λ^i with i in the index set $\breve I$, and with $\breve n_{\breve\lambda} = n_\lambda$, i.e.

$$\mathrm{P}_\omega^{\star-1}:\quad \lambda \mapsto \breve\lambda = \mathrm{P}_\omega^{\star-1}(\lambda)\,,\quad \breve\lambda^i = \lambda^i \ \text{ for } i \in \breve I,\ \breve n_{\breve\lambda} = n_\lambda\,. \tag{6.59}$$

7. The Order N Automorphism of $A^{(1)}_{N-1}$

We would like to be able to treat all diagram automorphisms of all affine Lie algebras. Except for the automorphism of order N of $\mathsf{g} = A^{(1)}_{N-1}$ which rotates the Dynkin diagram, all of these are already covered by Theorem 1. The remaining exceptional case is the subject of Theorem 2, which we prove in the present section.

For the automorphism ω of order N of $A^{(1)}_{N-1}$, the symmetric weights Λ obey Λ^i = const = $1/k^\vee_\Lambda$ for $i = 0, 1, \dots, N-1$, so that the level $k^\vee_\Lambda$ of any dominant integral symmetric weight is divisible by N. Furthermore, the subspace $\mathsf{g}^{(0)}_\circ$ of $\mathsf{g}_\circ$ that stays fixed under ω is two-dimensional; it is spanned by the two elements $K = h_0 = \sum_{i=1}^{N-1} H^i$ and $D = -\frac{1}{N}\sum_{l=0}^{N-1} \omega^l(L_0) + \xi K$, with ξ some number which can be deduced from (6.22). Now only symmetric weights $\lambda \in \mathsf{g}^{\star(0)}_\circ$ contribute to the twining character; for these we have

$$\lambda(D) = -\frac{1}{N}\sum_{l=0}^{N-1}((\omega^\star)^l\lambda)(L_0) + \lambda(\xi K) = \lambda(-L_0 + \xi K). \tag{7.1}$$

This implies that the twining character of the Verma module obeys

$$\mathscr{V}^{(\omega)}_\Lambda(tK + \tau L_0) = \mathscr{V}^{(\omega)}_\Lambda((t+\xi\tau)K - \tau D), \tag{7.2}$$

and an analogous formula holds for the irreducible twining character $\chi^{(\omega)}_\Lambda$. As K acts as a constant k_Λ on any highest weight module, the dependence of the twining character on the variable t is only via an exponential factor,

$$\mathscr{V}^{(\omega)}_\Lambda(t,\tau) \equiv \mathscr{V}^{(\omega)}_\Lambda(tK + \tau L_0) = \mathrm{e}^{2\pi\mathrm{i}tk_\Lambda} \cdot \mathrm{tr}_{V_\Lambda}\tau_\omega \mathrm{e}^{2\pi\mathrm{i}\tau R(L_0)}. \tag{7.3}$$

In the rest of this section we will show that the only non-vanishing contribution to the trace in (7.3) comes from the highest weight vector, thereby proving Theorem 2. This vector is never a null vector, so that this statement holds both for the Verma module and for the irreducible module. Thus we have

$$\chi^{(\omega)}_\Lambda(t,\tau) = \mathscr{V}^{(\omega)}_\Lambda(t,\tau) = \mathrm{e}^{2\pi\mathrm{i}tk_\Lambda}\mathrm{e}^{2\pi\mathrm{i}\tau\Delta_\Lambda}, \tag{7.4}$$

where Δ_Λ denotes the eigenvalue of L_0 on the highest weight vector.

To show that only the highest weight vector contributes to the twining character of the Verma module, we label the positive real roots of $A^{(1)}_{N-1}$ in the following way. The positive roots of A_{N-1} are $\bar\alpha^{(i,j)} := \bar\alpha^{(i)} + \bar\alpha^{(i+1)} + \ldots + \bar\alpha^{(j-1)}$ with $1 \le i < j \le N$. Then all positive real roots of $A^{(1)}_{N-1}$ are covered by

$$\alpha^{(i,j)}_n := \begin{cases} (\bar\alpha^{(i,j)}, 0, n) & \text{for} \quad i < j\,, \\ (-\bar\alpha^{(j,i)}, 0, n+1) & \text{for} \quad i > j \end{cases} \tag{7.5}$$

with $1 \le i \ne j \le N$ and $n \in \mathbf{Z}_{\ge 0}$.

The outer automorphism acts on the positive roots as

$$\omega^\star(\alpha^{(i,j)}_n) = \alpha^{(i+1,j+1)}_n; \tag{7.6}$$

here (as well as in some formulæ below) for convenience the upper indices are considered as being defined only mod$N$. Hence for any fixed n there are exactly $N-1$ orbits of length N, which can be written as

$$\{\alpha_n^{(i,j)} \mid i - j = \mathrm{const}\}\,, \tag{7.7}$$

where const $\in \{1, 2, \ldots, N-1\}$. It follows directly from the definition (7.5) that

$$\sum_{k=0}^{N-1} \alpha_n^{(i+k,i+k+l)} = (0, 0, Nn + l) \tag{7.8}$$

for $1 \leq l \leq N-1$. Thus the horizontal projection of the sum of the roots of each orbit vanishes, and the grade of $\sum_{l=0}^{N-1} \omega^{\star l}(\alpha_n^{(i,j)})$ is $nN + j - i$.

On the step operators H_n^i ($n \geq 1$) associated to lightlike roots, ω acts as $H_n^i \mapsto H_n^{i+1}$ for $1 \leq i \leq N-2$, while it sends H_n^{N-1} to $-\sum_{k=1}^{N-1} H_n^k$ (compare (6.5) and (6.7)). Thus the linear combinations

$$h_n^p := \sum_{j=1}^{N-1} (\zeta^{-pj} - 1) H_n^j\,, \tag{7.9}$$

with $n \geq 1$ and $p = 1, 2, \ldots, N-1$, and with $\zeta = \exp(2\pi\mathrm{i}/N)$ a primitive N^{th} root of unity, obey

$$\begin{aligned} \omega(h_n^p) &= \sum_{j=1}^{N-2} (\zeta^{-pj} - 1)\, H_n^{j+1} - (\zeta^{-p(N-1)} - 1) \sum_{j=1}^{N-1} H_n^j \\ &= \sum_{j=1}^{N-1} (\zeta^{-p(j-1)} - 1 - \zeta^p + 1) H_n^j = \zeta^p h_n^p\,, \end{aligned} \tag{7.10}$$

i.e. they are eigenvectors of ω to the eigenvalue ζ^p.

According to the Poincaré-Birkhoff-Witt theorem a basis for $\mathsf{U}(\mathsf{g}_-)$ (the subalgebra of the universal enveloping algebra $\mathsf{U}(\mathsf{g})$ that is generated by the step operators corresponding to negative roots of $\mathsf{g} = A_{N-1}^{(1)}$) can be described as follows. Consider an arbitrary, but definite ordering of the generators of g_-, starting, say, with the step operators corresponding to lightlike roots. Then for any sequences $\mathbf{n} \equiv (n(m,j))$ and $\mathbf{n}' \equiv (n'(m,j,l))$ which take values in the non-negative integers and for which only finitely many elements are different from zero, we denote by $[\mathbf{n}, \mathbf{n}']$ the element

$$[\mathbf{n}, \mathbf{n}'] := \prod_{m=1}^{\infty} \Big[\prod_{j=1}^{N-1} (h_{-m}^j)^{n(m,j)} \prod_{l=0}^{N-1} (E^{-\alpha_m^{(l,l+j)}})^{n'(m,j,l)} \Big] \tag{7.11}$$

of $\mathsf{U}(\mathsf{g}_-)$. Here it is to be understood that the products are ordered according to the chosen ordering of the basis of g_-. The Poincaré-Birkhoff-Witt theorem asserts that the set

$$\{\, [\mathbf{n}, \mathbf{n}'] \mid \mathbf{n},\, \mathbf{n}' \,\} \tag{7.12}$$

is in fact a basis of $\mathsf{U}(\mathsf{g}_-)$.

To compute the contribution of the state $v = [\mathbf{n}, \mathbf{n}'] \cdot v_\Lambda$ to the twining character, we consider the standard filtration of $\mathsf{U}(\mathsf{g}_-)$; thus we denote by U_p the subspace of $\mathsf{U}(\mathsf{g}_-)$ that is spanned by all elements of $\mathsf{U}(\mathsf{g}_-)$ which can be written as the product of p or less elements of g_-. Now under ω, the generator $[\mathbf{n}, \mathbf{n}']$ is mapped to

$$\omega([\mathbf{n},\mathbf{n}']) = \prod_{m=1}^{\infty} [\prod_{j=1}^{N-1} (\zeta^j h^j_{-m})^{n(m,j)} \prod_{l=0}^{N-1} (E^{-\alpha_m^{(l+1,l+j+1)}})^{n'(m,j,l)}] , \tag{7.13}$$

and hence ω maps the subspace U_p bijectively to itself.

Moreover, for any p elements x_i of g_- and any permutation π of $\{1, 2, \ldots, p\}$ we have

$$x_1 x_2 \ldots x_p - x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(p)} \in \mathsf{U}_{p-1} \tag{7.14}$$

(it is sufficient to check this statement only for π a transposition, in which case it follows from the properties of the commutator). Now both $[\mathbf{n},\mathbf{n}']$ (7.11) and $\omega([\mathbf{n},\mathbf{n}'])$ (7.13) are elements of U_p with

$$p = \sum_{m,j} n(m,j) + \sum_{m,j,l} n'(m,j,l) , \tag{7.15}$$

but not of U_{p-1}. In computing the trace of τ_ω we are therefore allowed to reorder the factors in $\omega([\mathbf{n},\mathbf{n}'])$ without changing the value of the trace, since reordering only introduces terms in U_{p-1}. This shows that a state $[\mathbf{n},\mathbf{n}'] \cdot v_\Lambda$ with $[\mathbf{n},\mathbf{n}']$ of the form (7.11) can only contribute to the twining character if the number $n'(m,j,l)$ is constant on any orbit, or in other words, if it does not depend on l at all. Correspondingly, we will write $n'(m,j)$ from now on. To proceed, it is convenient to drop the trivial dependence of the twining character of the Verma module on the central element and shift $\mathscr{V}_\Lambda^{(\omega)}$ by $\mathrm{e}^{-\tau\Delta_\Lambda}$; thus we define

$$\mathscr{V}^{(\omega)}(\tau) := \mathrm{e}^{-\tau\Delta_\Lambda} \mathscr{V}_\Lambda^{(\omega)}(0,\tau) . \tag{7.16}$$

We will show that

$$\mathscr{V}^{(\omega)}(\tau) \equiv 1 . \tag{7.17}$$

To see this, first note that a vector $[\mathbf{n},\mathbf{n}'] \cdot v_\Lambda$ in the Verma module V_Λ which fulfills the conditions formulated in the preceding paragraphs gives a contribution of ηq^n to $\mathscr{V}^{(\omega)}(\tau)$, where $q \equiv \exp(2\pi \mathrm{i}\tau)$,

$$\eta := \prod_{m,j} (\zeta^j)^{n(m,j)} \qquad \text{and} \qquad n := \sum_{m,j} n(m,j) \cdot m + \sum_{m,j} n'(m,j) \cdot (Nm+j) . \tag{7.18}$$

The function $\mathscr{V}^{(\omega)}(\tau)$ just keeps track of the contributions (7.18) from all states in V_Λ. It is convenient to combine the contributions from all powers of any fixed generator of g_-; thus any h^j_m yields a contribution of a factor of

$$1 + \zeta^j q^m + (\zeta^j q^m)^2 + \ldots = (1 - \zeta^j q^m)^{-1} \tag{7.19}$$

to $\mathscr{V}^{(\omega)}(\tau)$, while any orbit characterized by j and m contributes a factor of

$$1 + q^{Nm+j} + (q^{Nm+j})^2 + \ldots = (1 - q^{Nm+j})^{-1} . \tag{7.20}$$

Thus

$$(\mathscr{V}^{(\omega)}(\tau))^{-1} = \prod_{m=0}^{\infty} \prod_{j=1}^{N-1} [(1 - \zeta^j q^m)(1 - q^{Nm+j})] . \tag{7.21}$$

By arranging the terms in the first product differently this can be rewritten as

$$(\mathcal{Y}^{(\omega)}(\tau))^{-1} = \prod_{m=0}^{\infty} \left\{ \prod_{j=1}^{N-1} [(1-q^{Nm+j}) \cdot \prod_{j'=1}^{N-1}(1-\zeta^{j'}q^{Nm+j})] \cdot \prod_{j=1}^{N-1}(1-\zeta^{j}q^{Nm}) \right\} . \tag{7.22}$$

For any fixed m and j the term in the square brackets evaluates to

$$\begin{aligned} \prod_{j'=0}^{N-1}(1-\zeta^{j'}q^{Nm+j}) &= q^{N(Nm+j)} \prod_{j'=0}^{N-1}(q^{-Nm-j}-\zeta^{j'}) \\ &= q^{N(Nm+j)}(q^{-N^2m-jN}-1) = 1-q^{N(Nm+j)} . \end{aligned} \tag{7.23}$$

Inserting this identity into (7.22), we find

$$(\mathcal{Y}^{(\omega)}(q))^{-1} = \prod_{m=0}^{\infty}\prod_{j=1}^{N-1} [(1-\zeta^{j}q^{Nm})(1-q^{N(Nm+j)})] = (\mathcal{Y}^{(\omega)}(q^N))^{-1} . \tag{7.24}$$

This functional equation for $\mathcal{Y}_\Lambda^{(\omega)}$ implies that $\mathcal{Y}^{(\omega)}(q)$ is constant. Evaluating the function for $q=0$ we thus find $\mathcal{Y}^{(\omega)}(q) \equiv 1$, as was claimed in (7.17).

According to the definition (7.16) of $\mathcal{Y}^{(\omega)}(\tau)$, it then follows immediately that the twining character of the Verma module is given by (7.4), and hence the proof of Theorem 2 is completed.

8. Modular Transformations

One important property of the untwisted affine Lie algebras (and of the twisted affine Lie algebras $A_1^{(2)}$ and $\tilde{B}_n^{(2)}$) is that at any fixed value $k^\vee$ of the level, the set of irreducible highest weight modules with dominant integral highest weights carries a unitary representation of the twofold covering SL(2,**Z**) of the modular group of the torus. To be precise, this representation does not act on the characters χ as we used them in the previous sections, but rather on the so-called *modified* characters $\tilde{\chi}$. From Table (2.24) we read off that if ω is a simple current automorphism, which is the case we are considering, then also the characters of the orbit Lie algebra $\breve{\mathfrak{g}}$ – and hence the twining characters as well – give rise to a unitary representation of $SL(2,\mathbf{Z})$.

The modified characters are defined as

$$\tilde{\chi}_\Lambda := \mathrm{e}^{-s_\Lambda \delta}\chi_\Lambda , \tag{8.1}$$

where $\delta = \sum_{i=1}^{r} a_i \alpha^{(i)}$ is the element of the weight space that was defined in (6.27), and where for any integrable highest weight Λ of $\mathfrak{g}$ the number s_Λ is the so-called modular anomaly

$$s_\Lambda := \frac{1}{(\bar\theta,\bar\theta)}\left(\frac{(\bar\Lambda+\bar\rho,\bar\Lambda+\bar\rho)}{k^\vee+g^\vee} - \frac{(\bar\rho,\bar\rho)}{g^\vee}\right) . \tag{8.2}$$

For later reference, we also remark that using the strange formula $(\bar\rho,\bar\rho)/(\bar\theta,\bar\theta) = g^\vee \dim \bar{\mathsf g}/24$ and the eigenvalues

$$\bar C_2(\Lambda) = (\bar\Lambda + 2\bar\rho, \bar\Lambda) = (\bar\Lambda+\bar\rho, \bar\Lambda+\bar\rho) - (\bar\rho,\bar\rho) \tag{8.3}$$

of the second order Casimir operator $\overline{\mathscr{C}}_2$ of $\bar{\mathsf g}$, one can rewrite the modular anomaly as

$$s_\Lambda = \frac{\bar C_2(\Lambda)}{(\bar\theta,\bar\theta)(k^\vee+g^\vee)} - \frac{k^\vee \dim\bar{\mathsf g}}{24\,(k^\vee+g^\vee)} = \Delta_{\bar\Lambda} - \frac{c}{24}\,. \tag{8.4}$$

Here in the last step we implemented our convention that $(\bar\theta,\bar\theta) = 2$, and introduced the central charge $c := k^\vee \dim\bar{\mathsf g}/(k^\vee+g^\vee)$ of the Virasoro algebra and the conformal weight $\Delta_\Lambda := \bar C_2(\Lambda)/2(k^\vee+g^\vee)$ of the highest weight Λ.

In the present section we treat the case where also the orbit Lie algebra $\breve{\mathsf g}$ is an untwisted affine Lie algebra; the alternative case that $\breve{\mathsf g} = \tilde B_n^{(2)}$ will be described in Sect. 9. The modular anomaly of the twining character of $\mathsf g$ is *not* the one of the ordinary character of $\mathsf g$, i.e. $\exp(-s_\Lambda\delta)$, but rather the pull back of the modular anomaly of the $\breve{\mathsf g}$-character. Defining $\hat\delta := \mathrm{P}^\star_\omega(\breve\delta)$ and

$$\hat s_\Lambda := \breve s_{\mathrm{P}^{\star-1}_\omega \Lambda} \equiv \breve s_{\breve\Lambda} = \frac{\bar C_2(\breve\Lambda)}{2(\breve k^\vee+\breve g^\vee)} - \frac{\breve c}{24}\,, \tag{8.5}$$

we can introduce *modified twining characters* by

$$\tilde\chi_\Lambda^{(\omega)} := \mathrm{e}^{-\hat s_\Lambda \hat\delta}\chi_\Lambda^{(\omega)}\,. \tag{8.6}$$

They are related to the modified characters of the orbit Lie algebra $\breve{\mathsf g}$ by a relation analogous to (4.17):

$$\begin{aligned}\tilde\chi_\Lambda^{(\omega)}(h) &= \mathrm{e}^{-\hat s_\Lambda\hat\delta}\chi_\Lambda^{(\omega)} = \exp\,[\breve s_{\mathrm{P}^{\star-1}_\omega\Lambda}(\mathrm{P}^\star_\omega(\breve\delta))(h)]\,\breve\chi_{\breve\Lambda}(\mathrm{P}_\omega h)\\ &= \exp\,[\breve s_{\mathrm{P}^{\star-1}_\omega\Lambda}\,\breve\delta(\mathrm{P}_\omega h)]\,\breve\chi_{\breve\Lambda}(\mathrm{P}_\omega h) = \tilde{\breve\chi}_{\breve\Lambda}(\mathrm{P}_\omega h)\,.\end{aligned} \tag{8.7}$$

We will now show that the difference between the modular anomaly of the twining character and the one of the usual character is not as big as one might expect. In fact, they differ by a constant which only depends on the level of the weight Λ. First, the relation $\mathrm{P}^\star_\omega(\breve\delta) = \delta$ (6.46) shows that the two modifications differ only in the value of the modular anomaly; closer inspection shows that the difference $\hat s_\Lambda - s_\Lambda$ is only a function of the level of the weight Λ. Since when analysing the modular transformation behavior one has to restrict oneself to weights at a fixed level, this shows that the modular anomalies differ only by a constant. This constant is precisely the "shift" of the conformal weights that was observed in [4].

More precisely, the relation between $\hat s_\Lambda$ and s_Λ is as follows. For affine $\mathsf g$ the relation (2.18) between the dual Coxeter labels of $\mathsf g$ and $\breve{\mathsf g}$ implies that the dual Coxeter numbers $\breve g^\vee \equiv \sum_{i\in\breve I}\breve a_i^\vee$ of $\breve{\mathsf g}$ and $g^\vee \equiv \sum_{i=0}^r a_i^\vee$ of $\mathsf g$ are related by

$$N\,\breve g^\vee = g^\vee\,. \tag{8.8}$$

Further, for any symmetric g-weight λ of level $k^\vee_\lambda$, the level of the weight $\breve\lambda = \mathrm{P}^{\star-1}_\omega(\lambda)$ is given by

$$\breve k^\vee_\lambda \equiv \sum_{i\in\breve I} \breve a^\vee_i \breve\lambda^i = \frac{1}{N}\sum_{i\in I} a^\vee_i \lambda^i = \frac{1}{N}\, k^\vee_\lambda\,. \tag{8.9}$$

To compare $\hat s_\Lambda$ and s_Λ, we also need a relation between the second order Casimir operators of the horizontal projection of the weights Λ and $\breve\Lambda$. Thus consider two symmetric weights $\lambda, \mu \in \mathrm{g}^{\star(0)}_\circ$. The scalar product $(\bar\lambda, \bar\mu) = \sum_{i,j=1}^r \bar G_{i,j}\lambda^i\mu^j$ of their horizontal components can be written as

$$\begin{aligned}(\bar\lambda,\bar\mu) = & \sum_{i,j\in\breve I\setminus\{0\}} \breve\lambda^i\breve\mu^j \sum_{m,n=0}^{N-1} \frac{N_iN_j}{N^2}\,\bar G_{\dot\omega^m i,\dot\omega^n j} + \lambda^0 \sum_{j\in\breve I\setminus\{0\}} \breve\mu^j \sum_{m=1}^{N-1}\sum_{n=0}^{N-1} \frac{N_j}{N}\,\bar G_{\dot\omega^m 0,\dot\omega^n j} \\ & + \mu^0 \sum_{i\in\breve I\setminus\{0\}} \breve\lambda^i \sum_{m=0}^{N-1}\sum_{n=1}^{N-1} \frac{N_i}{N}\,\bar G_{\dot\omega^m i,\dot\omega^n 0} + \lambda^0\mu^0 \sum_{m,n=1}^{N-1} \bar G_{\dot\omega^m 0,\dot\omega^n 0}\,.\end{aligned} \tag{8.10}$$

Further, we have

$$\lambda^0 = k^\vee_\lambda - \sum_{i=1}^r a^\vee_i\lambda^i = k^\vee_\lambda - \sum_{i\in\breve I\setminus\{0\}} N_i\, a^\vee_i \lambda^i - (N-1)\,\lambda^0\,, \tag{8.11}$$

which owing to the relation (2.18) between the dual Coxeter labels of g and $\breve{\mathrm g}$ can be rewritten as

$$\lambda^0 = \breve k^\vee_\lambda - \sum_{i\in\breve I\setminus\{0\}} \breve a^\vee_i \breve\lambda^i = \breve\lambda^0\,. \tag{8.12}$$

Inserting this identity into the right-hand side of (8.10) and using the formula (6.49) for $\bar{\breve G}$, we can express the scalar product (8.10) entirely in terms of the horizontal subalgebra of the orbit Lie algebra. We obtain (compare [4])

$$(\bar\lambda,\bar\mu) = N\,(\bar{\breve\lambda},\bar{\breve\mu}) + \Gamma_{00}\,\breve k^\vee_\lambda \breve k^\vee_\mu\,, \tag{8.13}$$

where $\breve k^\vee_\lambda$ and $\breve k^\vee_\mu$ are the levels of the $\breve{\mathrm g}$-weights $\breve\lambda$ and $\breve\mu$, respectively.

Then in particular, the quadratic Casimir eigenvalue of a symmetric highest g-weight Λ at level $k^\vee$ can be written as

$$(\bar\Lambda,\bar\Lambda+2\bar\rho) = N\,(\bar{\breve\Lambda},\bar{\breve\Lambda}+2\bar{\breve\rho}) + \Gamma_{00}\,\breve k^\vee(\breve k^\vee + 2\breve g^\vee)\,. \tag{8.14}$$

(In [4], this formula was obtained in a different guise, which is obtained from the present one by (8.8) and the identity

$$\Gamma_{00}\,\breve g^\vee = \frac{N}{12}\,(d-\breve d)\,, \tag{8.15}$$

where d and $\breve d$ are the dimensions of the simple Lie algebras $\bar{\mathrm g}$ and $\bar{\breve{\mathrm g}}$, respectively.) Dividing (8.14) by $2(k^\vee + g^\vee)$ and using (8.8) and (8.9), we then obtain a simple relation between the conformal dimensions of primary fields of the g and $\breve{\mathrm g}$ WZW conformal field theories (at levels $k^\vee$ and $\breve k^\vee$, respectively), namely

$$\Delta_\Lambda = \breve{\Delta}_{\breve{\Lambda}} + \frac{1}{2N^2}\,\Gamma_{00}k^\vee\,(1+\frac{g^\vee}{k^\vee+g^\vee})\,, \tag{8.16}$$

or, equivalently, using (8.15)

$$\Delta_\Lambda = \breve{\Delta}_{\breve{\Lambda}} + \frac{1}{24}\,[\,\frac{k^\vee}{g^\vee}\,(d-\breve{d})+c-\breve{c}\,]\,, \tag{8.17}$$

This shows that the two modular anomalies in fact only differ by a (level-dependent) constant:

$$s_\Lambda = \Delta_\Lambda - \frac{c}{24} = \breve{\Delta}_{\breve{\Lambda}} - \frac{\breve{c}}{24} + \frac{1}{24}\,\frac{k^\vee}{g^\vee}\,(d-\breve{d}) = \breve{s}_\Lambda + \frac{1}{24}\,\frac{k^\vee}{g^\vee}\,(d-\breve{d})\,. \tag{8.18}$$

The analysis above reproduces in particular the results concerning the fixed point conformal field theories that have been obtained in [4]. Note that in [4] the fixed point theory has been found by looking for those affine Lie algebras $\mathring{\mathfrak{g}}$ for which a relation of the form (8.18) between the conformal dimensions of the symmetric weights of $\mathfrak{g}$ and the weights of $\mathring{\mathfrak{g}}$ exists. Equation (8.18) shows that the orbit Lie algebra $\breve{\mathfrak{g}}$, which was defined by a folding procedure of the Cartan matrix, fulfills precisely these requirements. Also note that from the explicit formulæ (2.7) and (6.49) it is by no means manifest that the symmetrized Cartan matrix of the horizontal orbit Lie algebra is the inverse of the quadratic form matrix $\overline{\breve{G}}$ as defined in (6.49), which coincides with the result obtained in [4] for the quadratic form matrix of $\bar{\mathring{\mathfrak{g}}}$; that this is nevertheless true can thus be seen as a non-trivial check of the identification of $\breve{\mathfrak{g}}$ with $\mathring{\mathfrak{g}}$.

9. Twisted Orbit Lie Algebras

When comparing the list of orbit Lie algebras in (2.24) with the list of "fixed point conformal field theories" as presented in [4], for the cases involving the simple current automorphisms of order two of $\mathfrak{g} = C^{(1)}_{2n}$ or $B^{(1)}_{n+1}$ some additional explanations are in order. In these cases the orbit Lie algebra is $\breve{\mathfrak{g}} = \tilde{B}^{(2)}_n$, while in [4] the fixed point theory was conjectured to be the $C^{(1)}_n$ WZW theory at level ℓ if the level of $\mathfrak{g}$ is $k^\vee = 2\ell+1$. For even level the spectrum could not be matched with any known conformal field theory apart from a few special cases. Based on a level-rank duality of $N=2$ superconformal coset models, an S-matrix for the spectra at even levels was conjectured in [6].

In this section we explain how these observations fit together. For odd levels $k^\vee = 2\ell+1$ of $\mathfrak{g}$, we show that the S-matrix of $\breve{\mathfrak{g}} = \tilde{B}^{(2)}_n$ at level $\breve{k}^\vee = k^\vee$ coincides (up to sign factors which are related to certain shifts appearing in the application to fixed point resolution) with the S-matrix of $\mathring{\mathfrak{g}} := C^{(1)}_n$ at level $\ell = (k^\vee-1)/2$. For even levels $k^\vee = 2\ell$ we prove the conjecture of [6] related to level-rank duality. Note that the level of $\tilde{B}^{(2)}_n$ is defined with a conventional factor of two [1],

$$\breve{k}^\vee_{\breve{\lambda}} = 2\sum_{i\in\breve{I}} \breve{a}^\vee_i \breve{\lambda}^i\,, \tag{9.1}$$

as compared to the formula (6.6) of the untwisted case; this cancels the factor of $1/2$ that according to (8.9) is present in the relation between $\breve{k}^\vee$ and $k^\vee$.

The modular S-matrix of $\breve{\mathfrak{g}} = \tilde{B}_n^{(2)}$ at level $\breve{k}^\vee$ is given by the Kac–Peterson formula

$$S_{\breve{\Lambda},\breve{\Lambda}'} = \mathrm{i}^{|\bar{\Delta}_+|} \left| \frac{M^\star}{(\breve{k}^\vee + \breve{g}^\vee) M} \right|^{-1/2} \sum_{\bar{w} \in \overline{W}} \epsilon(\bar{w}) \exp\Big[-2\pi\mathrm{i}\, \frac{(\bar{\breve{\Lambda}} + \bar{\breve{\rho}}, \bar{w}(\bar{\breve{\Lambda}}' + \bar{\breve{\rho}}))}{\breve{k}^\vee + \breve{g}^\vee} \Big]. \tag{9.2}$$

Such a formula holds for all untwisted affine Lie algebras, while among the twisted algebras it is valid only for $\tilde{B}_n^{(2)}$ (here and below we employ the convention that the twisted algebra $A_1^{(2)}$ is denoted by $\tilde{B}_1^{(2)}$ and hence is included in the $\tilde{B}_n^{(2)}$ series; also recall that in [1] these twisted algebras are denoted by $A_{2n}^{(2)}$). The notation used in (9.2) is as follows. For $\breve{\mathfrak{g}} = \tilde{B}_n^{(2)}$, $\bar{\breve{\mathfrak{g}}}$ is the unique diagram subalgebra isomorphic to C_n, while for untwisted affine Lie algebras it is the horizontal subalgebra generated by the zero modes. The summation is over the Weyl group $\overline{W}$ of $\bar{\breve{\mathfrak{g}}}$, and $\bar{\breve{\lambda}}$ is the projection of the $\breve{\mathfrak{g}}$-weight $\breve{\lambda}$ to $\bar{\breve{\mathfrak{g}}}$. In the prefactor of the sum, $\bar{\Delta}_+$ is the set of positive roots of $\bar{\breve{\mathfrak{g}}}$, M is the translation subgroup of the Weyl group of $\breve{\mathfrak{g}}$, and $M^\star$ its dual lattice. We also note that the dual Coxeter number of $\tilde{B}_n^{(2)}$ is $\breve{g}^\vee = 2n+1$, and that the translation lattice M of the Weyl group of $\tilde{B}_n^{(2)}$ is isomorphic to the root lattice of the simple Lie algebra B_n. Moreover [1, Corollary 6.4.], if we normalize the invariant bilinear form of $\mathfrak{g}$ such that the longest roots have length 2, the restriction of this invariant bilinear form to $\tilde{B}_n^{(2)}$ is twice the normalized form of C_n.

For concreteness, from now on we consider $\breve{\mathfrak{g}} = \tilde{B}_n^{(2)}$ as the orbit Lie algebra of $\mathfrak{g} = B_{n+1}^{(1)}$. The case $\mathfrak{g} = C_{2n}^{(1)}$ is very similar.

9.1. Odd level. Let us first treat the case of $B_{n+1}^{(1)}$ at odd level $k^\vee = 2\ell+1$. We start by showing that the prefactors in the Kac–Peterson formula for $\mathring{\mathfrak{g}} = C_n^{(1)}$ at level ℓ and $\breve{\mathfrak{g}} = \tilde{B}_n^{(2)}$ at level $2p+1$ coincide. The powers of i are identical because $\bar{\mathring{\mathfrak{g}}} = C_n = \bar{\breve{\mathfrak{g}}}$, i.e. the horizontal algebras coincide. Further, for $\tilde{B}_n^{(2)}$, M is the root lattice L of B_n, while for $C_n^{(1)}$ it is the coroot lattice $L^\vee$ of C_n; these lattices are proportional because B_n and C_n are dual Lie algebras. To determine the relative normalization, we notice that the simple coroots of C_n are $\gamma^{(i)^\vee} = 2\gamma^{(i)}$ with length squared 4 for $i = 1, \dots, n-1$, and $\gamma^{(n)^\vee} = \gamma^{(n)}$ with length squared 2, while the simple roots of B_n are $\beta^{(i)}$ with length squared 2 for $i = 1, \dots, n-1$, and $\beta^{(n)}$ with length 1. Hence

$$\begin{aligned} &M(\tilde{B}_n^{(2)}) = L(B_n) = \tfrac{1}{\sqrt{2}}\, L^\vee(C_n) = \tfrac{1}{\sqrt{2}}\, M(C_n), \\ &M^\star(\tilde{B}_n^{(2)}) = \sqrt{2}\, M^\star(C_n). \end{aligned} \tag{9.3}$$

Finally for $C_n^{(1)}$ we have $\mathring{k}^\vee + \mathring{g}^\vee = \ell + (n+1) = \ell + n + 1$, while for $\tilde{B}_n^{(2)}$, $\breve{k}^\vee + \breve{g}^\vee = (2\ell+1)+(2n+1) = 2(\ell+n+1)$. Taking these results together, we find that the prefactors coincide as claimed.

Furthermore, the terms in the exponent coincide as well. As already seen, the denominators differ by a factor of two; this is cancelled by a factor of two in the numerator from the different normalization of the invariant bilinear form. Finally, the Weyl groups of B_n and C_n are isomorphic so that also the summation is the same for both cases. Thus we conclude that the formulæ for the S-matrices coincide.

However, we still have to determine the precise relation between the weights in the two descriptions. Now clearly, the mappings of the symmetric integrable highest weights of $\mathfrak{g}$ to those of $\breve{\mathfrak{g}}$ and the mapping to weights of $\mathring{\mathfrak{g}}$ that was considered in [4] are different. But even the restrictions of these maps to the isomorphic subalgebras $\bar{\breve{\mathfrak{g}}} = C_n$ and $\bar{\mathring{\mathfrak{g}}} = C_n$ do not coincide; rather, the two mappings are related as follows.

A weight $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{n+1})$ of $\mathfrak{g} = B^{(1)}_{n+1}$ is symmetric if $\lambda^0 = \lambda^1$. The mapping of symmetric $\mathfrak{g}$-weights to weights of $\breve{\mathfrak{g}} = \tilde{B}^{(2)}_n$ reads

$$\lambda \mapsto \breve{\lambda} := (\lambda^1, \lambda^2, \dots, \lambda^{n+1}), \tag{9.4}$$

or in other words, $\breve{\lambda}^i := \lambda^{i+1}$ for $i = 0, 1, \dots, n$. The restriction of this mapping to the diagram subalgebra $\bar{\breve{\mathfrak{g}}} = C_n$ of $\tilde{B}^{(2)}_n$ is then given (in the conventional labelling of the C_n Dynkin diagram, i.e. with the n^{th} node corresponding to the long simple root) by

$$\lambda \mapsto \bar{\breve{\lambda}} := (\lambda^n, \lambda^{n-1}, \dots, \lambda^1), \tag{9.5}$$

i.e. by $\bar{\breve{\lambda}}^i := \lambda^{n-i+1}$ for $i = 1, 2, \dots, n$. On the other hand, the mapping to weights of $\bar{\mathring{\mathfrak{g}}} = C_n$ described in [4] is

$$\lambda \mapsto \bar{\mathring{\lambda}} := (\lambda^2, \lambda^3, \dots, \lambda^n, \tfrac{1}{2}(\lambda^{n+1} - 1)), \tag{9.6}$$

i.e. $\bar{\mathring{\lambda}}^i := \lambda^{i+1}$ for $i = 1, 2, \dots, n-1$ and $\bar{\mathring{\lambda}}^n := \frac{1}{2}(\lambda^{n+1} - 1)$. Extending this map to the affinization $\mathring{\mathfrak{g}} = C^{(1)}_n$ of $\bar{\mathring{\mathfrak{g}}}$ at level p one has $\mathring{\lambda}^i = \bar{\mathring{\lambda}}^i$, with $\bar{\mathring{\lambda}}$ as defined in (9.6), for $i = 1, 2, \dots, n$, supplemented by

$$\mathring{\lambda}^0 = \ell - \sum_{i=1}^{n} \mathring{\lambda}^i = \tfrac{1}{2}((k^\vee - 1) - 2\sum_{i=2}^{n} \lambda^i - (\lambda^{n+1} - 1)) = \tfrac{1}{2}(\lambda^0 + \lambda^1) = \lambda^1 . \tag{9.7}$$

These relations, as well as the analogous mapping from $\mathfrak{g} = C^{(1)}_{2n}$ to $\tilde{B}^{(2)}_n$, are displayed in Fig. 1.

Finally, we can also extend the map (9.5) to the affinization $C^{(1)}_n$ of $\bar{\breve{\mathfrak{g}}}$; this yields a weight $\mathring{\mu}$ of $C^{(1)}_n$ with $\mathring{\mu}^i = \bar{\breve{\lambda}}^i$ for $i = 1, 2, \dots, n$ and zeroth component

$$\mathring{\mu}^0 = \ell - \sum_{i=1}^{n} \mathring{\mu}^i = \tfrac{1}{2}((k^\vee - 1) - 2\sum_{i=1}^{n} \lambda^i = \tfrac{1}{2}(\lambda^{n+1} - 1) . \tag{9.8}$$

Combining these formulæ, we learn that the $C^{(1)}_n$-weights $\mathring{\lambda}$ and $\mathring{\mu}$ are related by

$$\mathring{\mu}^i = \mathring{\lambda}^{n-i} \tag{9.9}$$

for $i = 1, 2, \dots, n$. This means that $\mathring{\lambda}$ and $\mathring{\mu}$ are mapped to each other by the non-trivial (simple current) diagram automorphism of $C^{(1)}_n$.

We can now use the relation between S-matrix elements involving fields transforming into each other under a simple current automorphism that was found in [2] to relate the S-matrix $\mathring{S}$ proposed in [4] to the S-matrix $\breve{S}$ of the orbit Lie algebra $\breve{\mathfrak{g}}$. We obtain

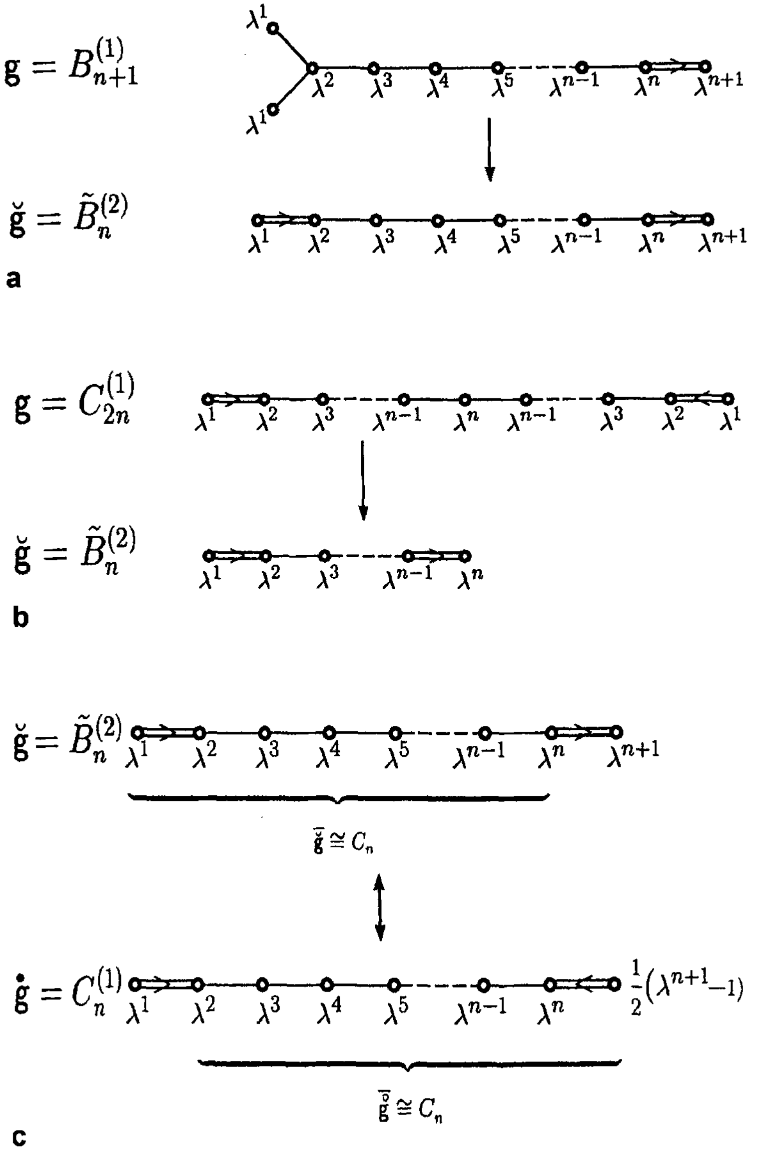

Fig. 1a–c. **a** Relation between symmetric weights ("fixed points") of $B^{(1)}_{n+1}$ and weights of the orbit Lie algebra $\tilde{B}^{(2)}_n$. **b** Relation between symmetric weights of $C^{(1)}_{2n}$ and weights of the orbit Lie algebra $\tilde{B}^{(2)}_n$. **c** Map between weights of $\tilde{B}^{(2)}_n$ and weights of $C^{(1)}_n$ at odd levels

$$\breve{S}_{\breve{\Lambda},\breve{\Lambda}'} = (-1)^{Q(\breve{\Lambda})+Q(\breve{\Lambda}')+Q(\breve{\Lambda}_J)} \cdot \mathring{S}_{\mathring{\Lambda},\mathring{\Lambda}'} , \tag{9.10}$$

where $\breve{\Lambda}_J = \ell\breve{\Lambda}_{(n)}$ is the weight of the simple current of $C_n^{(1)}$ and $Q(\breve{\Lambda}) := \sum_{j=1}^n j\, \breve{\Lambda}^j$ is the so-called monodromy charge of $\breve{\Lambda}$ with respect to the simple current, which coincides modulo 2 with the conjugacy class of the C_n-weight $\bar{\breve{\Lambda}}$. Thus the two modular S-matrices coincide up to sign factors, as claimed. These sign factors factorize into a global sign and signs associated to each row and column of the S-matrix.

To compare this result with the description of the fixed point theory $\mathring{\mathfrak{g}}$ in [4], we note that there the S-matrix $\mathring{S}$ was only defined up to a one-dimensional representation of the modular group; this allows for a global sign between $\mathring{S}$ and the S-matrix of $\breve{\mathfrak{g}}$. Further, the second type of sign factors which depends on the representations can be compensated in the process of fixed point resolution by interchanging the role of the two fields into which the fixed point is resolved, so that they cannot be noticed in the fixed point resolution procedure at the level of representations of the modular group either.

9.2. Even level. Consider now $B_{n+1}^{(1)}$ at even level $k^\vee = 2\ell$. It will be convenient to describe the symmetric weights of $B_{n+1}^{(1)}$ in terms of an orthogonal basis of the weight space of B_{n+1}. Thus for the weight λ with Dynkin components λ^i we introduce the numbers

$$l_i \equiv l_i(\lambda) := \sum_{j=i}^{n} \lambda^j + n + 2 - i + \tfrac{1}{2}\lambda^{n+1} \tag{9.11}$$

for $i = 1, 2, \ldots, n+1$, which are the components of $\lambda + \rho$ in the orthogonal basis. We have $l_1 > l_2 > \ldots > l_{n+1} \geq 1$. Furthermore, that the weight λ is symmetric means that $\lambda^0 = \lambda^1$, and hence the level can be written as

$$2\ell \equiv \lambda^0 + \lambda^1 + 2\sum_{j=2}^{n} \lambda^j + \lambda^{n+1} = 2\sum_{j=1}^{n} \lambda^j + \lambda^{n+1} . \tag{9.12}$$

This relation shows that for symmetric weights λ^{n+1} must be even at even level, so that all the numbers l_i are integers, and it also implies that for a symmetric weight the number $l_1 = \sum_{j=1}^n \lambda^j + \frac{1}{2}\lambda^{n+1} + n + 1 = \ell + n + 1$ is independent of λ. A symmetric weight can therefore be characterized by a subset $M_{B_{n+1}}(\lambda)$ of n numbers out of the set $M_{\ell+n} := \{1, 2, \ldots, \ell + n\}$.

Let us now compute the weight with respect to the subalgebra $\bar{\breve{\mathfrak{g}}} = C_n$ of the orbit Lie algebra $\breve{\mathfrak{g}}$. For a C_n-weight $\bar{\breve{\mu}} = (\breve{\mu}^i)$ the components of $\breve{\mu} + \breve{\rho}$ in the orthogonal basis read

$$m_i \equiv m_i(\breve{\mu}) = \sum_{j=i}^{n} \bar{\breve{\mu}}^j + n + 1 - i \tag{9.13}$$

for $i = 1, 2, \ldots, n$. As $\breve{\mu}$ has level ℓ, we have $n + \ell \geq m_1 > m_2 > \ldots > m_n \geq 1$. Thus these weights are again characterized by a subset of n elements of $M_{\ell+n}$, which we denote by $M_{C_n}(\breve{\mu})$. The relations (9.4) and (9.5) between a symmetric $B_{n+1}^{(1)}$-weight λ and the associated weights $\breve{\lambda}$ of $\tilde{B}_n^{(2)}$ and $\bar{\breve{\lambda}}$ of C_n then imply that $m_i(\breve{\lambda}) = \sum_{j=1}^{n+1-i} \lambda^j + n + 1 - i$ for $i = 1, 2, \ldots, n$, and hence

$$m_i(\breve{\lambda}) + l_{n-i+2}(\lambda) = \sum_{j=1}^{n} \lambda^j + \tfrac{1}{2}\lambda^{n+1} + n + 1 = \ell + n + 1\,. \tag{9.14}$$

This means that the sets M_{C_n} and $M_{B_{n+1}}$ obtained from a weight λ are related by

$$M_{C_n}(\breve{\lambda}) = \{\ell + n + 1 - i \mid i \in M_{B_{n+1}}(\lambda)\}\,. \tag{9.15}$$

With this information, we can express the S-matrix of $\breve{\mathfrak{g}}$ as follows. In the prefactor $\mathcal{N}$ in (9.2), we now have $\breve{k}^\vee + \breve{g}^\vee = 2\ell + (2n+1) = 2(\ell + n + \frac{1}{2})$. Comparing this with the corresponding number $\ell + (n+1)$ in the prefactor for the S-matrix of $C_n^{(1)}$ at level ℓ, we see that

$$\mathcal{N}(\tilde{B}_n^{(2)}) = \left(\tfrac{\ell+n+1}{\ell+n+1/2}\right)^{-n/2} \mathcal{N}(C_n^{(1)})\,. \tag{9.16}$$

With the known S-matrix of $C_n^{(1)}$ at level ℓ, we then find

$$\begin{aligned} S_{\breve{\mu},\breve{\mu}'}(\tilde{B}_n^{(2)}) &= \left(\tfrac{\ell+n+1/2}{\ell+n+1}\right)^{n/2} \cdot S_{\breve{\mu},\breve{\mu}'}(C_n^{(1)}) \\ &= (-1)^{n(n-1)/2} 2^{n/2} (\ell + n + \tfrac{1}{2})^{-n/2} \cdot \det{}_{\substack{\ell \in M_{C_n}(\breve{\mu}),\\ q \in M_{C_n}(\breve{\mu}')}} \mathcal{M}_{pq}\,, \end{aligned} \tag{9.17}$$

where

$$\mathcal{M}_{pq} := \sin\left(\tfrac{\pi p q}{\ell+n+1/2}\right) \tag{9.18}$$

for all $p, q \in M_{\ell+n}$.

This result will now be compared with the conjecture for the S-matrix obtained from level-rank duality. By level-rank duality, symmetric weights of $B_{n+1}^{(1)}$ at level 2ℓ are mapped to a pair of so-called "spinor non-symmetric simple current orbits" of weights of $D_\ell^{(1)}$ at level $2n+3$ [6]; the latter are simple current orbits which contain a $D_\ell^{(1)}$-weight ν with $\nu^\ell \neq \nu^{\ell-1}$. Again we characterize the weights ν by the components of $\nu + \rho$ in the orthogonal basis, i.e. by

$$\begin{aligned} &n_i(\nu) := \textstyle\sum_{j=i}^{\ell-2} \nu^j + \frac{1}{2}(\nu^{\ell-1} + \nu^\ell) + \ell - i \quad \text{for } i = 1, 2, \ldots, \ell - 2\,, \\ &n_{\ell-1}(\nu) = \tfrac{1}{2}(\nu^{\ell-1} + \nu^\ell) + 1\,, \qquad n_\ell(\nu) = \tfrac{1}{2}(-\nu^{\ell-1} + \nu^\ell)\,. \end{aligned} \tag{9.19}$$

To characterize pairs of simple current orbits, we choose the unique representative of each pair of orbits which has $\nu^\ell - \nu^{\ell-1} \in 2\mathbf{Z}_{\geq 0}$ and $\nu^0 \geq \nu^1$; for this representative, all n_i are positive integers and $n_1 > n_2 > \ldots > n_\ell \geq 0$. In fact, $n_\ell \geq 1$ because of $\nu^\ell \neq \nu^{\ell-1}$. Moreover, as ν is spinor non-symmetric and at level $2n+3$, the integer n_1 obeys

$$\begin{aligned} n_1 &= \sum_{j=1}^{\ell-2} \nu^j + \tfrac{1}{2}(\nu^{\ell-1} + \nu^\ell) + \ell - 1 \\ &= \tfrac{1}{2}(2n+3) + \tfrac{1}{2}(\nu^1 - \nu^0) + \ell - 1 \leq n + \ell + \tfrac{1}{2}\,, \end{aligned} \tag{9.20}$$

implying that $n_1 \leq n + \ell$. It follows that we can characterize each pair of spinor non-symmetric orbits by a subset $M_{D_\ell}(\nu)$ of ℓ elements of $M_{\ell+n}$.

In terms of this subset, the level-rank duality between symmetric weights λ of $B_{n+1}^{(1)}$ at level 2ℓ and these orbits of $D_\ell^{(1)}$-weights reads [6]

$$M_{D_\ell}(\nu) = \{\ell + n + 1 - j \mid j \notin M_{B_{n+1}}(\lambda)\}\,. \tag{9.21}$$

Combining the results (9.15) and (9.21), we find that the weights with respect to $C_n^{(1)}$ and to $D_\ell^{(1)}$ that are associated to a symmetric weight of $B_{n+1}^{(1)}$ are related by

$$M_{D_\ell}(\nu) \,\dot{\cup}\, M_{C_n}(\breve{\lambda}) = M_{\ell+n}\,, \tag{9.22}$$

where $\dot{\cup}$ denotes the disjoint union.

In [6] it was conjectured that, up to a phase, the S-matrix for the fixed point resolution for $B_{n+1}^{(1)}$ is

$$\mathring{S}_{\Lambda,\Lambda'} = 2^{\ell/2-2}(\ell+r-\tfrac{1}{2})^{-\ell/2} \cdot \det_{\substack{p\in M_{D_\ell}(\Lambda),\\ q\in M_{D_\ell}(\Lambda')}} \mathscr{M}_{pq}\,, \tag{9.23}$$

with $\mathscr{M}_{pq}$ as defined in (9.18). It is not difficult to verify that this matrix indeed coincides, up to sign factors, with the S-matrix (9.17) of $\breve{\mathfrak{g}}$; to see this one has to employ Jacobi's theorem on determinants of submatrices of an invertible square matrix, together with the identities

$$\det\mathscr{M} = (-1)^{(\ell+n)(\ell+n-1)/2}(\tfrac{2\ell+2n+1}{4})^{(\ell+n)/2}\,, \qquad \mathscr{M}^2 = \tfrac{2\ell+2n+1}{4}\,\mathbb{1}\,, \tag{9.24}$$

and the fact that $\mathscr{M}$ is symmetric (compare [6]).

We have thus completed the proof of the conjecture for the S-matrix that was derived using level-rank duality. The conjecture for the S-matrix given in [6] was based on a resolution of fixed points at the level of representations of the modular group. As we have remarked at the end of the previous subsection, any such conjecture is not sensitive to both a global sign of the S-matrix and to multiplying corresponding rows and columns of the S-matrix with the same sign. When comparing $\mathring{S}$ and $\breve{S}$, we therefore did not pay attention to such sign factors.

Appendix: $\hat{W}$ as a Subgroup of W

In Subsect. 5.2 we have already seen that $\breve{m}_{ij}$ is a divisor of $\hat{m}_{ij}$. In this appendix we show that $\hat{m}_{ij}$ is a divisor of $\breve{m}_{ij}$, or in other words, that

$$(\hat{w}_i\hat{w}_j)^{\breve{m}_{ij}} = id\,, \tag{A.1}$$

in the cases where $\breve{m}_{ij} \in \{2,3,4,6\}$, i.e. when $\breve{A}^{i,j}\breve{A}^{j,i} \in \{0,1,2,3\}$. Together, it then follows that $\hat{m}_{ij} = \breve{m}_{ij}$ also in the cases; this completes the proof that the Weyl group $\breve{W}$ of $\breve{\mathfrak{g}}$ is isomorphic to the subgroup $\hat{W}$ of W.

Recall that the generators $\hat{w}_i$ are defined by (5.2) and (5.6) for $s_i = 1$ and $s_i = 2$, respectively. We will deal with the various cases separately; the corresponding restriction of the Dynkin diagram of $\mathfrak{g}$ to the orbits of i and j is depicted in Fig. 2.

Consider first the case $\breve{m}_{ij} = 2$. Then we have

$$0 = \breve{A}^{i,j}\breve{A}^{j,i} = s_i s_j\,\frac{N_i N_j}{N^2}\sum_{l,l'=0}^{N-1} A^{\dot{\omega}^l i,j}A^{\dot{\omega}^{l'} j,i}\,. \tag{A.2}$$

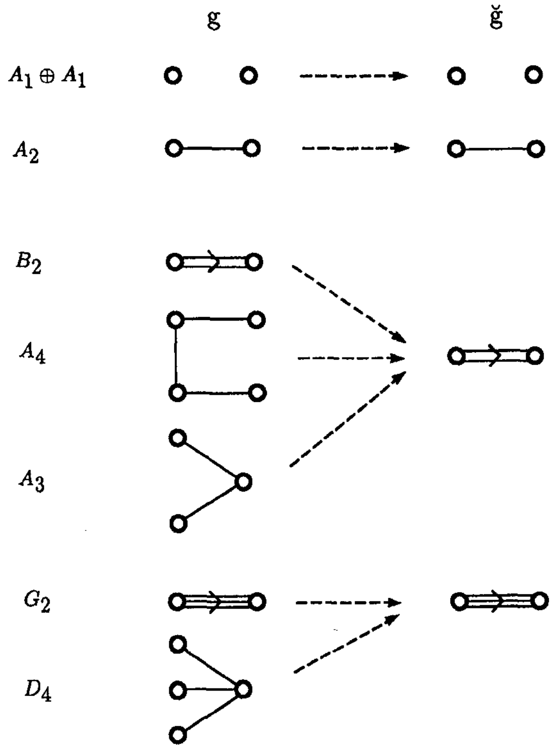

Fig. 2. The foldings of Dynkin diagrams with $\breve{A}^{i,j}\breve{A}^{j,i} \leq 3$.

As $i \neq \dot\omega^l j$ for all l, all terms in the sum are non-negative; this implies that $A^{\dot\omega^l i,j}A^{\dot\omega^{l'} j,i} = 0$ for all l, l'. Assume now that there is a value of l such that $A^{\dot\omega^l i,j}$ is different from zero (i.e. negative). Then the fact that $\dot\omega$ is an automorphism of A implies that also $A^{i,\dot\omega^{-l} j} < 0$; since A is a Cartan matrix, it follows that also $A^{\dot\omega^{-l} j,i} < 0$. This in turn implies that the term $A^{\dot\omega^l i,j}A^{\dot\omega^{-l} j,i}$ gives a positive contribution to $\breve{A}^{i,j}\breve{A}^{j,i}$, which is in contradiction with (A.2). Thus we learn that $A^{\dot\omega^l i,j}$ has to vanish for all l. Using again the fact that $\dot\omega$ is an automorphism of the Cartan matrix, we then find that also $A^{\dot\omega^l i,\dot\omega^{l'} j}$ vanishes for all values of l and l', which implies that $w_{\dot\omega^l i}w_{\dot\omega^{l'} j} = w_{\dot\omega^{l'} j}w_{\dot\omega^l i}$. This relation implies that also $\hat{w}_i\hat{w}_j = \hat{w}_j\hat{w}_i$, which shows that $\hat{m}_{ij} = 2 = \breve{m}_{ij}$ in this case. (We also see that for $N_i = N_j$ the restriction of the Dynkin diagram of A to the orbits of i and j consists of N_i copies of the situation $A_1 \oplus A_1$ of Fig. 2, and analogously for $N_i \neq N_j$.)

In the remaining cases $\breve{A}^{i,j}\breve{A}^{j,i} \in \{1,2,3\}$, we need either $\breve{A}^{i,j} = -1$ or $\breve{A}^{j,i} = -1$. Since the labels i and j appear symmetrically in the definition of $\breve{m}_{ij}$, we can assume without loss of generality that $\breve{A}^{i,j} = -1$. The relation $\breve{A}^{i,j} = s_i \sum_{l=0}^{N_i-1} A^{\dot\omega^l i,j}$ for $\breve{A}$ then implies that $s_i = 1$. Moreover, for any representative $\dot\omega^m j$ of the orbit of j, the Cartan matrix element $A^{\dot\omega^l i,\dot\omega^m j}$ is different from zero only for a single value of l, for which it is equal to -1.

Let us first deal with the case $s_j = 2$. Then the product

$$\breve{A}^{i,j}\breve{A}^{j,i} = 2\sum_{l,l'} A^{\dot\omega^l i,j} A^{\dot\omega^{l'} j,i} \tag{A.3}$$

is even, so that the only case we have to analyze is when it is equal to two, and hence $\breve{m}_{ij} = 4$. As in Subsect. 5.2 we can assume that $N = 2$; then $s_j = 2$ implies that $N_j = 2$. Assume now first that $N_i = 1$, i.e. $\dot\omega i = i$; this implies that $\breve{A}^{j,i} = 2(A^{j,i} + A^{\dot\omega j,i}) = 2(A^{j,i} + A^{j,\dot\omega i}) = 4A^{j,i}$. Thus if $\breve{A}^{j,i}$ is non-zero, it is in fact ≤ -4, which implies that $\breve{A}^{i,j}\breve{A}^{j,i} \geq 4$. This does not belong to the cases we are investigating here, and hence we can assume that $N_i = 2$.

Now $s_j = 2$ means that $A^{j,\dot\omega j} = A^{\dot\omega j,j} = -1$, while $s_i = 1$ tells us that $A^{i,\dot\omega i} = A^{\dot\omega i,i} = 0$. Further, because of $\breve{A}^{i,j} = A^{i,j} + A^{\dot\omega i,j} = -1$, we can assume without loss of generality that $A^{\dot\omega i,j} = 0$ and $A^{i,j} = -1$. The automorphism property of $\dot\omega$ then implies that $A^{i,\dot\omega j} = 0$ and $A^{\dot\omega i,\dot\omega j} = -1$. As A is a Cartan matrix, we then also have $A^{j,\dot\omega i} = 0 = A^{\dot\omega j,i}$. To determine the matrix elements $A^{j,i} = A^{\dot\omega j,\dot\omega i}$ (which because of $A^{i,j} \neq 0$ are non-zero), we observe that $\breve{A}^{j,i} = 2(A^{j,i} + A^{\dot\omega j,i}) = 2A^{j,i}$ must be ≥ -3 in order to yield $\breve{A}^{i,j}\breve{A}^{j,i} \leq 3$; thus $A^{j,i} = A^{\dot\omega j,\dot\omega i} = -1$, and we are in situation A_4 of Fig. 2.

Having found these Cartan matrix elements, we know that

$$\begin{array}{ll} w_i w_j w_i = w_j w_i w_j\,, & w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} = w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j}\,, \\ w_j w_{\dot\omega j} w_j = w_{\dot\omega j} w_j w_{\dot\omega j} & \end{array} \tag{A.4}$$

and

$$w_i w_{\dot\omega j} = w_{\dot\omega j} w_i\,, \qquad w_j w_{\dot\omega i} = w_{\dot\omega i} w_j\,, \qquad w_i w_{\dot\omega i} = w_{\dot\omega i} w_i\,. \tag{A.5}$$

Applying these relations repeatedly, we obtain

$$\begin{aligned} (w_i w_{\dot\omega i} w_j w_{\dot\omega j} w_j)^2 &= w_{\dot\omega i} w_i w_{\dot\omega j} w_j w_{\dot\omega j}\; w_i w_{\dot\omega i} w_j w_{\dot\omega j} w_j \\ &= w_{\dot\omega i} w_{\dot\omega j} w_i w_j w_i\; w_{\dot\omega j} w_{\dot\omega i} w_j w_{\dot\omega j} w_j \\ &= w_{\dot\omega i} w_{\dot\omega j} w_j w_i w_j\; w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j} w_j w_{\dot\omega j} \\ &= w_{\dot\omega i} w_{\dot\omega j} w_j w_i w_j\; w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} w_j w_{\dot\omega j} \\ &= w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} w_j w_i\; w_j w_{\dot\omega j} w_{\dot\omega i} w_j w_{\dot\omega j} \\ &= w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j} w_j w_i\; w_j w_{\dot\omega j} w_j w_{\dot\omega i} w_{\dot\omega j} \\ &= w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j} w_j w_i\; w_{\dot\omega j} w_j w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j} \\ &= w_{\dot\omega j} w_{\dot\omega i} w_{\dot\omega j} w_j w_{\dot\omega j}\; w_i w_j w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} \\ &= w_{\dot\omega j} w_{\dot\omega i} w_j w_{\dot\omega j} w_j\; w_i w_j w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} \\ &= w_{\dot\omega j} w_j w_{\dot\omega i} w_{\dot\omega j} w_i\; w_j w_i w_{\dot\omega i} w_{\dot\omega j} w_{\dot\omega i} \\ &= w_{\dot\omega j} w_j w_i w_{\dot\omega i} w_{\dot\omega j}\; w_j w_{\dot\omega i} w_{\dot\omega j} w_i w_{\dot\omega i} \\ &= w_{\dot\omega j} w_j w_i w_{\dot\omega i} w_{\dot\omega j}\; w_{\dot\omega i} w_j w_{\dot\omega j} w_i w_{\dot\omega i} \\ &= w_{\dot\omega j} w_j w_i w_{\dot\omega j} w_{\dot\omega i}\; w_{\dot\omega j} w_j w_{\dot\omega j} w_i w_{\dot\omega i} \\ &= w_{\dot\omega j} w_j w_{\dot\omega j} w_i w_{\dot\omega i}\; w_{\dot\omega j} w_j w_{\dot\omega j} w_i w_{\dot\omega i} \\ &= (w_j w_{\dot\omega j} w_j w_i w_{\dot\omega i})^2\,. \end{aligned} \tag{A.6}$$

Thus the generators $\hat{w}_i := w_i w_{\dot\omega i}$ and $\hat{w}_j := w_j w_{\dot\omega j} w_j$ of $\hat{W}$ satisfy $(\hat{w}_i\hat{w}_j)^2 = (\hat{w}_j\hat{w}_i)^2$, or what is the same,

$$(\hat{w}_i\hat{w}_j)^4 = id\,, \tag{A.7}$$

which is the relation we need, since $\breve{A}^{i,j}\breve{A}^{j,i} = 2$.

Let us now turn to the case $s_j = 1$. As the number t of those values of l for which $A^{\dot\omega^l j, \dot\omega^m i}$ is non-zero is the same for any representative $\dot\omega^m i$ of the orbit of i, we then find that

$$\breve{A}^{j,i} = \sum_{l=0}^{N-1} A^{\dot\omega^l j, i} \le -t\,, \tag{A.8}$$

which shows that t can only have the values 1, 2 or 3. Note that we still have $\breve{A}^{i,j} = -1$, so that $N_i = N_j/t$ for each of these values of t. We can now classify the possible situations through the restriction of the Dynkin diagram of $\mathfrak{g}$ to the orbits of i and j. For $t = 1$, we have $N_i = N_j$, and the restriction of the Dynkin diagram to the two relevant orbits consists of N_i disconnected copies of the Dynkin diagram of either A_2, $B_2 \equiv C_2$, or G_2, according to whether $\breve{A}^{i,j}\breve{A}^{j,i}$ is 1, 2 or 3. These algebras have also been used to denote the corresponding folding in Fig. 2. For $t = 2$, there is only one possibility which satisfies all required constraints, namely that one has N_i disconnected copies of the Dynkin diagram of A_3, such that the middle node lies on the orbit of i while the two extremal nodes lie on the orbit of j. Finally, for $t = 3$ there are N_i disconnected copies of the Dynkin diagram of D_4, with the middle node lying on the orbit of i and the three extremal nodes on the orbit of j; this corresponds to the last case in Fig. 2.

We will deal with the different values of t consecutively. All cases with $t = 1$ can be treated simultaneously. In these cases the orbits of i and j have the same length. We can therefore label the simple reflections in W associated to the elements of these orbits as follows. We define $r_l := w_{\dot\omega^l i}$ for $l = 1, \ldots, N_i$, and then set $r'_l := w_{\dot\omega^{l'} j}$, with l' chosen such that r'_l commutes with all r_l for $l \neq l'$. With this notation we have

$$r_l r'_m = r'_m r_l \quad \text{for } l \neq m\,, \qquad (r_l r'_l)^{\tilde{m}_{ij}} = id\,. \tag{A.9}$$

Moreover, it follows from $s_i = s_j = 1$ that for all l, m the reflections of the same "type" commute, $r_l r_m = r_m r_l$ and $r'_l r'_m = r'_m r'_l$. Using these relations, we find that

$$(\hat{w}_i\hat{w}_j)^{\tilde{m}_{ij}} = (\textstyle\prod_{l=0}^{N_i-1} r_l \cdot \prod_{m=0}^{N_j-1} r'_m)^{\tilde{m}_{ij}} = \displaystyle\prod_{l=0}^{N_j-1} (r_l r'_l)^{\tilde{m}_{ij}} = id \tag{A.10}$$

as required.

Next consider the case $t = 2$. Then we have N_i commuting copies of A_3. By similar arguments as in the $t = 1$ case, we can restrict our attention to just one of these copies, and we can assume that the labelling is such that the relevant representatives of the orbits of i and j are i and j themselves together with $\dot\omega j$. Then we have the relations

$$w_j w_{\dot\omega j} = w_{\dot\omega j} w_j\,, \qquad w_i w_j w_i = w_j w_i w_j\,, \qquad w_i w_{\dot\omega j} w_i = w_{\dot\omega j} w_i w_{\dot\omega j}\,, \tag{A.11}$$

which imply in particular

$$\begin{aligned}(w_i w_j w_{\dot\omega j})^2 &= w_i w_j w_{\dot\omega j} w_i w_{\dot\omega j} w_j = w_i w_j w_i w_{\dot\omega j} w_i w_j = w_j w_i w_j w_{\dot\omega j} w_i w_j \\ &= w_j w_i w_{\dot\omega j} w_j w_i w_j = w_j w_i w_{\dot\omega j} w_i w_j w_i = w_j w_{\dot\omega j} w_i w_{\dot\omega j} w_j w_i \\ &= (w_j w_{\dot\omega j} w_i)^2\,.\end{aligned} \tag{A.12}$$

Thus $\hat w_i := w_i$ and $\hat w_j := w_j w_{\dot\omega j}$ satisfy $(\hat w_i \hat w_j)^2 = (\hat w_j \hat w_i)^2$, i.e. $(\hat w_i \hat w_j)^4 = id$ as required by $\breve A^{i,j} \breve A^{j,i} = t = 2$.

Finally, for $t = 3$ the calculation is similar, though somewhat lengthier. There are N_i commuting copies of D_4, and we can restrict ourselves to one of these copies, with the labels of its nodes being i for the middle node and j, $\dot\omega j$ and $\dot\omega^2 j$ for the others. The associated simple reflections of W satisfy $w_i w_{\dot\omega^m j} w_i = w_{\dot\omega^m j} w_i w_{\dot\omega^m j}$ for $m = 0,1,2$, and $w_{\dot\omega^l j} w_{\dot\omega^m j} = w_{\dot\omega^m j} w_{\dot\omega^l j}$ for $l, m = 0,1,2$; repeated use of these relations yields

$$\begin{aligned}(w_i w_j w_{\dot\omega j} w_{\dot\omega^2 j})^3 &= w_i w_j w_{\dot\omega j} w_i w_{\dot\omega^2 j} w_i \; w_j w_{\dot\omega j} w_i w_{\dot\omega j} w_j w_{\dot\omega^2 j} \\ &= w_i w_j w_{\dot\omega j} w_i w_{\dot\omega^2 j} w_i \; w_j w_i w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} \\ &= w_i w_j w_{\dot\omega j} w_i w_{\dot\omega^2 j} w_j \; w_i w_j w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} \\ &= w_i w_{\dot\omega j} w_i w_j w_i w_{\dot\omega^2 j} \; w_i w_j w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} \\ &= w_{\dot\omega j} w_i w_{\dot\omega j} w_j w_i w_{\dot\omega^2 j} \; w_i w_{\dot\omega j} w_i w_j w_i w_{\dot\omega^2 j} \\ &= w_{\dot\omega j} w_i w_j w_{\dot\omega j} w_i w_{\dot\omega j} \; w_{\dot\omega^2 j} w_i w_{\dot\omega j} w_j w_i w_{\dot\omega^2 j} \\ &= w_{\dot\omega j} w_i w_j w_i w_{\dot\omega j} w_i \; w_{\dot\omega^2 j} w_i w_{\dot\omega j} w_j w_i w_{\dot\omega^2 j} \\ &= w_{\dot\omega j} w_j w_i w_j w_{\dot\omega j} w_{\dot\omega^2 j} \; w_i w_{\dot\omega^2 j} w_{\dot\omega j} w_j w_i w_{\dot\omega^2 j} \\ &= w_j w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} w_i \; w_{\dot\omega j} w_i w_j w_i w_{\dot\omega^2 j} w_i \\ &= w_j w_{\dot\omega j} w_i w_{\dot\omega^2 j} w_j w_i \; w_j w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} w_i \\ &= w_j w_{\dot\omega j} w_i w_{\dot\omega^2 j} w_i w_j \; w_i w_{\dot\omega j} w_i w_j w_{\dot\omega^2 j} w_i \\ &= (w_j w_{\dot\omega j} w_{\dot\omega^2 j} w_i)^3\,.\end{aligned} \tag{A.13}$$

Thus $\hat w_i := w_i$ and $\hat w_j := w_j w_{\dot\omega j} w_{\dot\omega^2 j}$ satisfy $(\hat w_i \hat w_j)^3 = (\hat w_j \hat w_i)^3$, i.e. $(\hat w_i \hat w_j)^6 = id$, which is again the required Coxeter relation for $\breve A^{i,j} \breve A^{j,i} = t = 3$.

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