

# From individual behaviors to an evaluation of the collective evolution of crowds along footbridges

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# From individual behaviors to an evaluation of the collective evolution of crowds along footbridges

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## Abstract

In the present work a mathematical model aimed at evaluating the dynamics of crowds is derived and discussed. In particular, on the basis of some phenomenological considerations focused on pedestrians' individual behavior, a model dealing with the collective evolution of the crowd, formalized in terms of a conservation law for a *crowding measure*, is obtained. To suit engineering needs, modeling efforts are made toward the simulation of real world crowd events specifically happening along pedestrian walkways. The response of the model as well as its tunability are studied in some computational domains inspired by existing footbridges.

**Keywords:** pedestrian dynamics, individual behavior, collective evolution, Civil Engineering applications.

**Mathematics Subject Classification:** 35L65, 35Q70, 90B20, 97M50

## 1 Introduction

The study of the dynamics of crowds defines a wide research sector that, in recent times, is knowing an increasing expansion fostered by different interests and disciplines. In the authors' opinion, throughout the years, the multiple facets of the field have been approached almost independently by different scientific communities, such as those of Applied Mathematics, Physics, Biomechanics, Transportation and Civil Engineering [29]. The present study develops from a collaborative multidisciplinary approach between applied mathematicians and structural engineers and aims at crossing and linking such facets. First, a mathematical model to simulate the evolution of crowds is introduced and inquired into; second, modeling efforts are made toward the simulation of real world crowd events specifically happening along footbridges.

In the proposed model, the crowd evolution is primarily approached considering the phenomenological perspective of an individual pedestrian. In particular, an evolution equation for agents' positions is derived. Such an equation takes into account the *active* attitude of pedestrians, namely the fact that their motion is not passively driven by force fields, as it happens with

inert particles. Rather, pedestrian positions evolve according to personal desires and external stimuli (see e.g., [10, 12, 16]; the reader can refer to [2, 14] for a general picture on crowd modeling). However, since macroscopic or statistical information is more significant in the engineering design and evaluation of the facility performances, a formal procedure is introduced to derive an evolution equation for pedestrian collectivity. Such a procedure, in the spirit of Reynolds Transport Theorem, generates an evolution equation for a mesoscopic measure of the pedestrian crowding, in the form of a conservation law. Moreover, in order to meet engineering codified approaches, a statistical interpretation of the model output is further proposed.

To deduce the model, *completeness* as well as *minimality* are assumed as conceptual guidelines. In particular, two kinds of interactions are retained for determining the dynamics: the interactions among pedestrians and those between pedestrians and domain walls. Doing so allows one to obtain simple equations depending on a limited amount of tunable free parameters. Furthermore, looking from the engineering practice perspective, ways of imposing boundary conditions, handling articulated domain geometries, and discretizing the equations of motion are proposed to complement the mathematical model. Finally, the response of the model as well as its tunability are studied in some computational domains inspired by existing footbridges.

The paper is organized in four more sections that follow the conceptual approach described above. In particular, in Sect. 2 the phenomenological model is proposed and the mathematical model is therefrom deduced; in Sect. 3 further elements required to simulate real world crowd events along footbridges are discussed; in Sect. 4 results from simulations of crowd events in four computational domains inspired by real world as well as the tunability of the model are discussed. Finally, in Sect. 5 conclusions are outlined.

## 2 Crowd modeling

The aim of this section is to introduce and discuss the crowd model considered throughout the paper. A phenomenological model describing the dynamics of pedestrians in a moving crowd is firstly obtained (Sect. 2.1). On such a basis, a mathematical model suitable to provide data on the collective evolution of the crowd is derived (Sect. 2.2). Finally, a statistical interpretation of the latter is discussed (Sect. 2.3). This interpretation is aimed at providing statistical data on crowds, in the direction of the current engineering practice, which, more and more frequently, require probabilistic characterization of events (see e.g., [1] in Civil Engineering).

In the modeling approach we propose, *individual* pedestrians *actively* determine their velocity on the basis of *interactions* they have with the *collectivity* around them, being the latter observed at a *mesoscopic*-like scale. Indeed, in this approach, the individual and the collective points of view coexist and interact, determining the overall system dynamics. For the sake of simplicity, the pedestrian-collectivity interaction mechanism is treated considering two phenomenological aspects: the *perception* a walker has of her close neighbors and her consequent *reaction*. From the modeling point of view, the coupling of these two aspects is done via a velocity term called *interaction velocity*. Such a term is calculated on the basis of a *spatial measure*, namely the mesoscopic variable involved, which models what one can perceive. In the model, the interaction velocity plays the role of an additive perturbation of a second velocity term, the *desired velocity*, which, instead, models the velocity one would keep in the absence of other nearby agents. The sum of desired velocity and interaction velocity gives rise to pedestrian *total velocity*.

### 2.1 Phenomenological modeling of pedestrians dynamics

Let us imagine a crowd walking in a domain  $D \subseteq \mathbb{R}^d$ . Usually  $d = 2$ , but the approach we present is sufficiently general to allow one to model possibly different systems of active interacting agents in different space dimensions. For instance, vehicular traffic with  $d = 1$  or swarms with  $d = 3$ . Therefore, we keep the dimension  $d$  generic. Throughout the current section we set  $D = \mathbb{R}^d$ ; nonetheless, in Sect. 3 this requirement will be relaxed allowing bounded domains, closer to engineering needs, to be considered.

At time  $t > 0$ , the point of view of individuals is treated in terms of the spatial position  $X_t \in D$  of a generic pedestrian, henceforth referred to as *test pedestrian*. On the other hand, the point of view of collectivity is expressed in terms of a time evolving *positive Radon measure*  $\nu_t$ . Such a measure, supported on  $D$ , quantifies the pedestrian mass in any Borel measurable set  $E \in \mathcal{B}(D)$ , being  $\mathcal{B}(D)$  the Borel  $\sigma$ -algebra of  $D$ , i.e.

$$\begin{aligned} \nu_t & : \mathcal{B}(D) \rightarrow \mathbb{R}_+ \\ \nu_t(E) & = \text{measure of the } \textit{crowding} \text{ of } E \text{ at time } t. \end{aligned} \quad (1)$$

From now on, we will simply write  $E \subset D$  to denote at once a subset of  $D$  which is also measurable, i.e., belonging to  $\mathcal{B}(D)$ .

Owing to Lebesgue Decomposition Theorem, the measure  $\nu_t$  admits a unique decomposition as a sum of an absolutely continuous part and a singular part with respect to a given measure, which, in our case, is the  $d$ -dimensional Lebesgue measure  $\mathcal{L}^d$  in  $\mathbb{R}^d$ . Moreover, by Radon-Nykodim theorem, the absolutely continuous part can be expressed by means of a non negative density function  $\rho_t \in L^1(D)$ . On the other hand, the singular component can be further written as an atomic part (i.e., a sum of Dirac masses denoted by  $\nu_t^a$ ) plus a Cantor part (see e.g., [22]). In view of modeling human perception, we are essentially interested in the absolutely continuous and atomic parts, hence we set:

$$d\nu_t = \rho_t dx + d\nu_t^a.$$

In particular, the density  $\rho_t$  is intended to model a gross, coarse grained, perception of individuals, while the atoms constituting  $\nu_t^a$  are aimed at reproducing a more granular and sharper perception, which is precise and spatially localized.

As previously mentioned, pedestrians are assumed to have a *desired* walk velocity, which is kept in absence of other nearby people. This velocity is expressed through a vector field

$$v_d : D \rightarrow \mathbb{R}^d,$$

which is evaluated at the agent's position. On the other hand, to obtain the *interaction velocity*, a mapping

$$k : D \times D \rightarrow \mathbb{R}^d$$

is considered, so that  $k(X_t, y)$  models the *reaction* that the test pedestrian in  $X_t$  has to a pedestrian perceived in  $y$ . As a pedestrian generally interacts just with nearby agents, on the basis of what she *perceives*, the interaction velocity is recovered first by restricting  $k$  to a region  $S(X_t) \in \mathcal{B}(D)$  around  $X_t$ , and then by weighting the pairwise interaction expressed by  $k$  with the perceived mass  $\nu_t$ , i.e.

$$v_i[\nu_t](X_t) = \int_{S(X_t)} k(X_t, y) d\nu_t(y) = \int_D \underbrace{\mathbf{1}_{S(X_t)}(y)k(X_t, y)}_{:=K(X_t, y)} d\nu_t(y), \quad (2)$$

where  $\mathbf{1}_{S(X_t)}$  denotes the characteristic function of the set  $S(X_t)$ . The latter is intended to model to the so-called *sensory region* of the test pedestrian (see Fig. 1 and, e.g., [10]). Coherently with human physiology, such a region is expected to be bounded and possibly anisotropic with respect to the pedestrian direction. Since the sensory region is bounded, agents cannot voluntarily produce the collective trends that one can observe by looking at their global distribution  $\nu_t$ , because they have a limited perception of the group they are part of.

In order to obtain the agent's total velocity, the interaction velocity is *added* to the desired velocity, giving

$$\frac{dX_t}{dt} = v_d(X_t) + v_i[\nu_t](X_t) = v_d(X_t) + \int_D K(X_t, y) d\nu_t(y), \quad (3)$$

which details how  $v_i$  generates deviations from the trajectory indicated by  $v_d$ .

The model deduced thus far features two state variables,  $X_t$  and  $\nu_t$ , although just one evolution equation (3) has been provided. A second equation can be obtained via the following argument. In

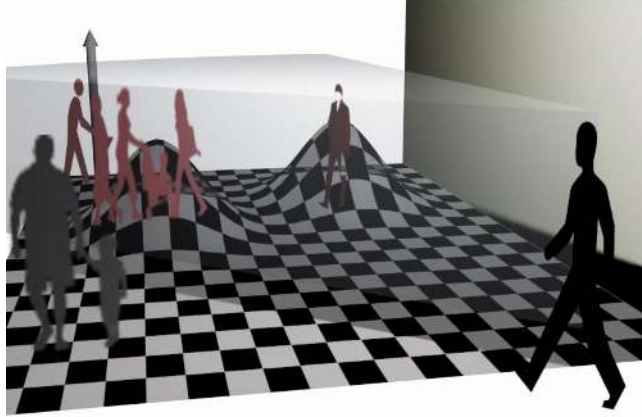


Figure 1: Sensory region of a given pedestrian - pedestrians perceived in terms of their distributed (checker texture) or localized (arrow) mass

order to be consistent with its definition of pedestrian mass,  $\nu_t$  must be a *material quantity* with respect to pedestrians and their movement. In other words, the initial perceived mass distribution  $\nu_0$  has to be transported by pedestrians themselves. This fact is expressed by the equation

$$\nu_t = X_t \# \nu_0, \quad (4)$$

which states that  $\nu_t$  is the *push forward* through  $X_t$  of the initial perceived pedestrian mass distribution  $\nu_0$ . Equation (4) means that for every measurable subset  $E$  of  $D$ , the relation

$$\nu_t(E) = \nu_0(X_t^{-1}(E))$$

holds.

## 2.2 Mathematical model

In this section, we derive a self consistent mathematical model from the coupled evolution equations (3) and (4). To this end, we compute a formal expression of the time derivative of the measure  $\nu_t$ , thereby obtaining a conservation law which expresses its evolution in time and space.

Let us consider  $\nu_t$  as a distribution, i.e., for every smooth test function  $\psi \in C_c^\infty(D)$  compactly supported in  $D$  we consider the action

$$\psi \mapsto \langle \psi, \nu_t \rangle = \int_D \psi d\nu_t.$$

In view of Eq. (4), the duality  $\langle \psi, \nu_t \rangle$  satisfies

$$\langle \psi, \nu_t \rangle = \int_D \psi d(X_t \# \nu_0) = \int_D \psi(X_t) d\nu_0, \quad (5)$$

hence, taking a formal time derivative (very much in the spirit of Reynolds Transport Theorem), we recover

$$\frac{d}{dt} \langle \psi, \nu_t \rangle = \int_D \frac{d}{dt} \psi(X_t) d\nu_0 = \int_D \nabla \psi(X_t) \cdot \frac{dX_t}{dt} d\nu_0.$$

From Eq. (3) we further have

$$\int_D \nabla \psi(X_t) \cdot \frac{dX_t}{dt} d\nu_0 = \int_D \left( \nabla \psi(X_t) \cdot \left( v_d(X_t) + \int_D K(X_t, y) d\nu_t(y) \right) \right) d\nu_0,$$

thus

$$\frac{d}{dt} \langle \psi, \nu_t \rangle = \int_D \left( \nabla \psi \cdot \left( v_d + \int_D K(\cdot, y) d\nu_t(y) \right) \right) d\nu_t. \quad (6)$$

We say that the measure  $\nu_t$  satisfies in the weak (i.e. distributional) sense the conservation law<sup>1</sup>

$$\frac{\partial \nu_t}{\partial t} + \nabla \cdot (\nu_t (v_d + v_i[\nu_t])) = 0 \quad (7)$$

provided Eq. (6) holds for all test functions  $\psi \in C_c^\infty(D)$  and for almost every (a.e.) time  $t$  in a given time interval of interest.

Under reasonable assumptions on the velocity field, one can have existence of solutions to the Cauchy problem obtained by complementing Eq. (7) with an initial condition

$$\nu_0 = \bar{\nu}. \quad (8)$$

The interested reader is referred to [26] for technical details.

The great generality of the concept of measure has been transferred into Eq. (7). For instance, if  $\nu_t$  is the counting measure, the obtained system is formally equivalent to a discrete dynamical system. On the other hand,  $\nu_t$  can be absolutely continuous with respect to the Lebesgue measure on  $D$  and thus (7) becomes a usual conservation law for a density. As already mentioned,  $\nu_t$  can also be hybrid in nature, having a discrete and a continuous component at the same time. This allows one to embed the dynamical effects of the microscopic granularity typical of crowds also in a continuous description given in terms of a density [8].

### 2.3 Statistical interpretation of (7)-(8)

The model deduced thus far is self-consistent and can be considered on its own. However, a further statistical reading can be *attached* to it. Such a reading has a twofold aim. Firstly, it allows one to better inquire into the real scale at which we are observing the crowd, i.e., the scale which the measure  $\nu_t$  refers to. Secondly, statistic-like information is required in the current engineering practice as previously mentioned.

Let us suppose that the considered crowd is constituted by  $N$  indistinguishable agents. Given an abstract sample space  $(\Omega, \mathcal{F}, \mathbb{P})$ , we consider the  $N$  random variables  $X_t^1, X_t^2, \dots, X_t^N$  referring to their position, i.e.,

$$\begin{aligned} X_t^j &: (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow D, \quad \forall j \in \{1, 2, \dots, N\} \\ X_t^j &= \text{position of agent } j \text{ at time } t. \end{aligned}$$

Of course, the  $X_t^j$ 's, being positions of pedestrians, evolve according to (3).

Let us further assume that, at time  $t = 0$ , the positions  $X_0^j$  identically distribute in  $D$  according to the *normalized* initial perceived mass measure, i.e.,

$$X_0^j \# \mathbb{P} = \underbrace{\frac{1}{N} \bar{\nu}}_{:= \bar{\mu}} \quad (9)$$

that is

$$\mathbb{P}(X_0^j \in E) = \bar{\mu}(E), \quad \forall E \subset D.$$

Equation (9) can be motivated, at least intuitively, on the basis of the idea that the probability of finding a pedestrian is higher where more pedestrians are present.

Because  $X_t^j$  solves Eq. (3), the following identity between measurable mappings from  $\Omega$  into  $D$  holds:

$$X_t^j(\omega) = (X_t \circ X_0^j)(\omega), \quad \text{for a.e. } \omega \in \Omega, \quad (10)$$

(see also Fig. 2). This equation details the real nature of the stochasticity of the  $X_t^j$ 's, which is just a legacy of the stochasticity in the initial positions  $X_0^j$ . In fact, the evolution of the uncertainty

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<sup>1</sup>The term  $v_i$  has been defined in Eq. (2)

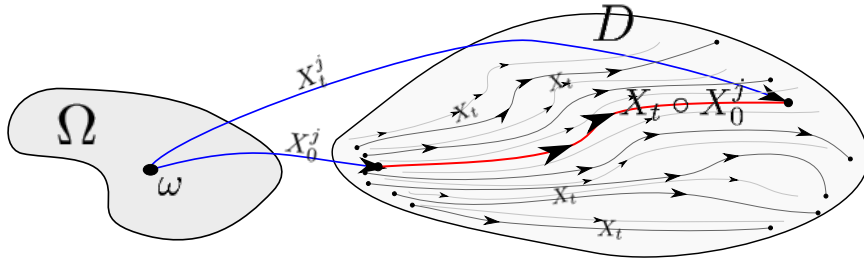


Figure 2: Conceptual representation of the measurable maps  $X_0^j$  and  $X_t^j$

on the  $X_t^j$ 's at any time is deterministically operated by the flow map  $X_t$ . In particular, the probability measure  $X_t^j \# \mathbb{P}$  satisfies

$$X_t^j \# \mathbb{P} = (X_t \circ X_0^j) \# \mathbb{P} = X_t \# (X_0^j \# \mathbb{P}) = X_t \# \bar{\mu} = \frac{1}{N} X_t \# \bar{\nu} = \frac{1}{N} \nu_t.$$

Hence, the measure  $\mu_t := \frac{1}{N} \nu_t$  is exactly the probability law referring to the position of the  $j$ -th pedestrian at time  $t$ , i.e.

$$\mathbb{P}(X_t^j \in E) = \mu_t(E), \quad \forall E \subset D.$$

In other words, the  $X_t^j$ 's, “dragged” by  $X_t$ , transport the uncertainty associated to their initial condition.

Accepting this statistical interpretation, one can count the total amount of pedestrians in a given set  $E \subset D$  at time  $t$  through the variable

$$Y_{E,t} := \sum_{j=1}^N \mathbf{1}_E(X_t^j)$$

which is random because it is a function of the random variables  $X_t^j$ . Its expected value can be easily computed:

$$\mathbb{E}[Y_{E,t}] := \sum_{j=1}^N \mathbb{E}[\mathbf{1}_E(X_t^j)] = N \mu_t(E) = \nu_t(E). \quad (11)$$

The quantity  $\nu_t$ , initially introduced as a mesoscopic quantity referring to collectivity, can clearly have a macroscopic interpretation as an expression of the average pedestrian mass. On the other hand, a linear relation exists between  $\nu_t$  and  $\mu_t$ , and the latter embeds a genuinely mesoscopic-like information. Actually, this scaling is *linear* because we are considering first order models, i.e., models which do not invoke explicitly any concept of acceleration. The link just found can have some flavor of “equipartition of energy”, namely all pedestrians contribute the same way to the macroscopic mass as they are indistinguishable.

### 3 Toward the simulation of real world crowd events

The aim of this section is to obtain a tool to analyze crowd events in real domains on the basis of the modeling framework previously deduced. From now on, two working assumptions are made: the measures involved are assumed to be absolutely continuous with respect to the Lebesgue measure, hence any granular effect is neglected, and the dimension of the working domain is fixed to  $d = 2$ .

First, to obtain a self-consistent tool a functional expression must be given to  $K$ , the interaction kernel defined in Eq. (2). In order to meet engineering needs, such an expression is expected to be synthetic, although obtained on a phenomenological basis.

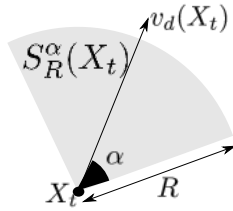


Figure 3: Sensory region  $S_R^\alpha(X_t)$  of the test pedestrian in  $X_t$

Second, the current study considers problems concerning crowd flows in built environment. Such flows usually develop in very large phenomenological environments, in the sense that pedestrian dynamics are affected by phenomena which originate far from the spatial region of engineering interest<sup>2</sup>. Thus, on the one hand, the mathematical domain is a restriction of the phenomenological environment and inflow conditions need to be prescribed on some conventional boundaries. On the other hand, the environment is bounded by built perimeters having articulated geometries in plan (e.g., building facades, obstacles, parapets). Hence, pedestrian behavior must be sensitized to walls and the discretization procedure must be able to handle such possibly complicated geometries. In the following, our work will be mostly inspired by problems arising when considering footbridges. In particular, the computational domains considered own to the class of the *elongated* domains and the considered inflow conditions may mimic queue processes happening on footbridges [11].

In summary, modeling solutions have to be given so as to address the following aspects: i) interactions among individuals; ii) limited spatial domains  $D \subsetneq \mathbb{R}^2$ ; iii) pedestrian behavior at walls; iv) numerical implementation in complicated domain geometries; v) crowd inflow. The current section details the modeling solutions for these aspects.

### 3.1 Modeling pedestrian interactions

The interaction velocity  $v_i$  introduced in Sect. (2) is the modeling artifact to account for the interaction among pedestrians. The kernel  $K = K(X_t, y)$ , in particular, is in charge of modeling its intangible psychological aspects. It is worth remarking that, in order to make model tuning affordable,  $K$  should depend on a limited amount of free parameters.

In the current section, an expression of  $K$  based on three positive parameters is proposed. In Sect. 4, the parameters involved will be tuned and used to perform simulations.

Commonly, in normal crowd avoidance, pedestrian interactions are repulsive and limited to a frontal region (being, thus, *anisotropic*). Therefore a minimal expression of  $K$  can be:

$$K(X_t, y) = -c \frac{\mathbf{e}_r}{r} \mathbf{1}_{S_R^\alpha(X_t)}(y), \quad (12)$$

where  $c$  is a positive constant (which determines the repulsiveness of the interaction) and  $r = |y - X_t|$  is the inter-pedestrian distance whose direction is  $\mathbf{e}_r = \frac{y - X_t}{|y - X_t|}$ .

The interaction described in Eq. (12) is confined to the sensory region  $S(X_t)$ , which we model as a circular sector of radius  $R$ , center  $X_t$ , and angular semi-amplitude  $0 < \alpha < \pi/2$  around  $v_d(X_t)$ . Formally, the set  $S(X_t)$  can be written as

$$S(X_t) = S_R^\alpha(X_t) = \{y \in D : r < R, v_d(X_t) \cdot \mathbf{e}_r > |v_d(X_t)| \cos \alpha\},$$

see Fig. 3.

<sup>2</sup>For instance, one can consider spectators leaving a stadium at the end of a sport event and then crossing a footbridge which connects the stadium with the car park [11]. A structural/transportation engineer may be interested in simulating the pedestrian flow over the footbridge, omitting a detailed description of the grandstand evacuation process.



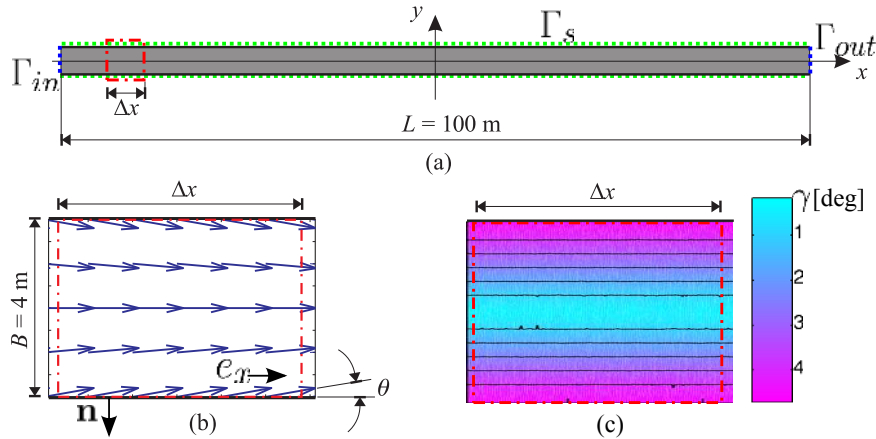


Figure 4: (a) Rectangular domain  $D$  - (b) Detail of  $D$  with the desired velocity vector field. (c) Iso-lines of angle  $\gamma$

## 3.2 Modeling the effects of walls

The model considered thus far operates on the whole unbounded physical space  $\mathbb{R}^2$ . However, domains relevant in applications commonly feature built perimeters, in general *walls*, which agents cannot cross. Therefore, proper behavioral rules describing the agents' reaction to walls should be introduced.

The rules proposed in the current section are obtained on the basis of a pure physical intuition, supported nonetheless by the evidence that both desired and total velocities should not allow for wall penetration. In particular, constructing suitable desired velocity fields in presence of walls for generic bounded domains is a nontrivial task, thus in the next section *elongated* domains will be specifically considered in view of the application to footbridges.

### 3.2.1 Modeling the desired velocity in presence of walls

In the current modeling framework, the agent desired velocity field  $v_d$ , which ideally drives pedestrian motion in the domain, is supposed to be known *a priori*. Therefore, it should be constructed out of the knowledge of the geometry of the domain.

Let us consider a generic wall bounding the domain. We denote by  $\mathbf{n}$  the outward normal unit vector (see Fig. 4(b)). Because of the impenetrableness constraint, the following *compatibility* condition, to be understood as a basic design guideline for  $v_d$ , can be introduced:

$$v_d \cdot \mathbf{n} \leq 0. \quad (13)$$

When general domains are considered, constructing a field  $v_d$  which is phenomenologically acceptable and satisfies Eq. (13) is a nontrivial task, which has been considered both in crowd modeling literature [17, 19] and, more generally, when *path planning* is concerned (e.g., robot motion planning [21]). In the following, a method to build such a field in case of elongated domains is proposed. These domains are characterized by a longitudinal dimension crossed by pedestrians (*length*) much larger than the transversal one (*chord*), the latter possibly slowly varying along the former. They may model, for instance, pedestrian footbridges, sidewalks, platforms and so on. Moreover, as in [17, 19], we wish to consider velocity fields that are (normalized) potential fields, i.e., such that  $v_d = -\nabla u / |\nabla u|$  for some potential function  $u$  that has to be determined.

Let us initially consider the simplest domain geometry, where the problem can be easily set and modeling considerations appear more intuitive. Let  $D$  be a rectangular domain of length  $L$ , chord  $B$ , and aspect ratio  $\tilde{B} = B/L = \frac{1}{25} \ll 1$  (Fig. 4(a)). To find  $u$ , we assume:

- that pedestrians flow from left to right in  $D$ . Therefore, the field  $-\nabla u$  has to be rightward. Moreover, a *unit* potential difference across  $L$  is assumed;

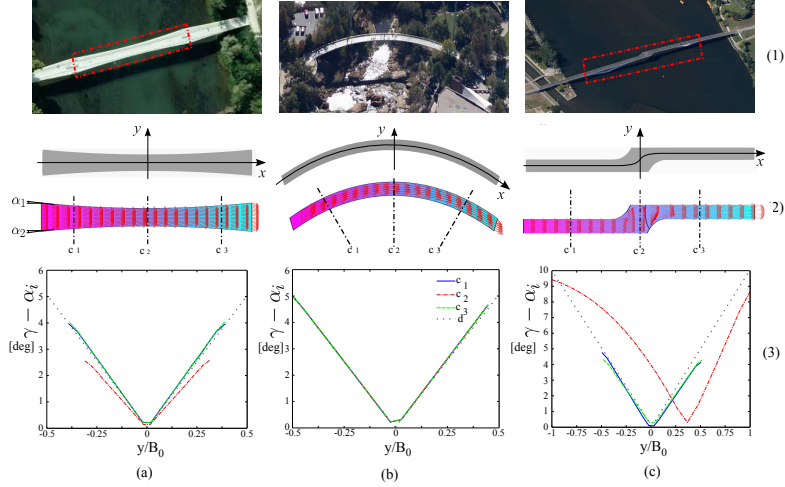


Figure 5: (1) Pictures of real footbridges - (2) Computational domains and desired velocity vector fields - (3) Distribution of  $\gamma - \alpha_i$  across given chords

- that pedestrians aim at not scraping against lateral boundaries. Hence,  $-\nabla u$  must be directed inwards at the ends of the chord and longitudinally at mid-chord. In other words, being  $e_x$  the longitudinal direction, the angle  $\gamma = \cos^{-1}((v_d \cdot e_x)/|v_d|)$  is expected to decrease monotonically to zero when approaching the mid-chord. We parametrize the convergence rate of  $v_d$  in terms of its slope at walls:

$$\tan \theta = \tan \gamma|_{\Gamma_s}, \quad (14)$$

see Fig. 4(b). We stress that, since inter-pedestrian interactions are repulsive, if pedestrian-wall repulsion is not taken into account then agent agglomerations are likely to appear in the proximity of the lateral boundaries  $\Gamma_s$ .

A minimal potential complying with the assumptions above is:

$$u(x, y) = -x + qy^2, \quad (15)$$

where the coordinates  $x$  and  $y$  are scaled with respect to the span  $L$  and moreover  $q = \tan \theta / \tilde{B}$ . Hence,  $u$  defines the potential field  $-\nabla u = (1, -2qy)$ , which, up to normalization, generates the desired velocity field  $v_d = (1, -2qy)/\sqrt{1 + 4q^2y^2}$ .

In order to deal with more general low aspect ratio domains, we primarily observe that Eq. (15) solves the following Poisson problem:

$$\begin{cases} -\nabla \cdot \nabla u = -2q & \text{in } D \\ \partial_{\mathbf{n}} u = \tan \theta \frac{\tilde{b}}{\tilde{B}} & \text{on } \Gamma_s \\ u = -x_i + qy^2 & \text{on } \Gamma_i \ (i \in \{in, out\}), \end{cases} \quad (16)$$

which can be imposed on more general domains than the rectangle. The term  $\theta$  is intended to parametrize the convergence rate of  $v_d$  toward the longitudinal direction at mid-chord. Therefore, the factor  $\tilde{b}/\tilde{B}$ , where  $\tilde{b} = \tilde{b}(x)$  is the (spatially variable) chord amplitude, is introduced. Factually, in rectangular domains, where  $\tilde{b} \equiv \tilde{B}$ , the Neumann boundary condition on  $\Gamma_s$  turns out to be exactly condition (14).

The potential defined in Eq. (16) is subharmonic. In the literature, analogous velocity fields have been usually built by using harmonic functions [7, 19, 21], relying on the so called *min-max principle*. Such a principle ensures that any local maximum or minimum of  $u$ , i.e., a stationary point of the gradient field, lies on the boundary of  $D$  [9]. Therefore, provided proper Dirichlet

conditions are imposed (e.g.,  $u = 0$  on exits,  $u = 1$  anywhere else on the boundary), velocity fields driving pedestrians to a given exit can be obtained. On the other hand, no control on the behavior of  $-\nabla u$  at walls is allowed, hence phenomenologically unsatisfactory behaviors can arise. Nevertheless, subharmonic functions do not satisfy a minimum principle, hence the absence of internal local minima cannot be guaranteed anymore. Aside from this, their use through Eq. (16) allows phenomenologically consistent fields [15] to be obtained.

The model in Eq. (16) with  $\theta = 5^\circ$  is applied to some real world footbridges having different walkway shapes (Fig. 5): bottleneck shape (Chiaves footbridge, [23]), curved shape (Liberty footbridge, [3]), shifted shape (Coimbra footbridge, [6]). As far as the longitudinal direction is concerned (Fig. 5(2)), the obtained  $v_d$  fields are constant and a unit potential difference across  $L$  is achieved. For what concerns the chord-wise direction, the desired walking angle  $\gamma$  is sensitized to the local sidewall geometrical inclination, for the sake of clarity. To do so, we introduce the angle  $\beta := \gamma - \alpha_i$ , where  $\alpha_i$  extends the geometric angle between the boundary and  $e_x$ . In particular, it is expressed in terms of the function  $\alpha_i(y) = |\alpha_1(y - y_2)/(y_1 - y_2) - \alpha_2(y - y_1)/(y_2 - y_1)|$ , where  $\alpha_j$  ( $j = 1, 2$ ) are defined in Fig. 5(2a). In Fig. 5(3) the angle  $\beta$  is plotted along the sections c1 ( $x = -0.35L$ ), c2 ( $x = 0$ , mid-span), c3 ( $x = 0.35L$ ) (Fig. 5(2)). It is worth pointing out that the obtained  $\beta$  chord-wise profiles are close to the expected trend (gray dotted lines, Fig. 5(3)) when the walkway section is almost constant; nonetheless, the  $\beta$  profile coherently departs from this trend when significant geometry variations take place (e.g. 5(a)-c2, 5(c)-c2).

### 3.2.2 Effects of walls on the total velocity

The constraint of not penetrating walls reflects also on the conditions to be enforced on the total pedestrian velocity at the boundaries. The following conditions should then be satisfied:

$$\frac{dX_t^j}{dt} \cdot \mathbf{n} = (v_d + v_i) \cdot \mathbf{n} \leq 0. \quad (17)$$

Two complying conditions, indeed referring to antipodal behaviors, can be found:

1. Pedestrians may scrape against walls, i.e.,

$$\left. \frac{dX_t^j}{dt} \right|_{\text{walls}} = (v_d + v_i) - (\mathbf{n} \cdot (v_d + v_i) \mathbf{n}) [(v_d + v_i) \cdot \mathbf{n} > 0];$$

2. Pedestrians must stop at walls, i.e.,

$$\left. \frac{dX_t^j}{dt} \right|_{\text{walls}} = (v_d + v_i) [(v_d + v_i) \cdot \mathbf{n} < 0]. \quad (18)$$

Equation (18) can be thought as an equivalent of the *no-slip* condition in fluid mechanics. In the simulations presented in Sect. 4 condition 1 is used.

### 3.3 Simulation of crowd events in articulated domains

The current section deals with the numerical discretization of Eq. (7) for producing numerical simulations of crowd events in real, possibly articulated, domains. Spatial discretizations, done via unstructured triangular meshes, are considered.

Problem (7)-(8) can be put in discrete form by using the *ad hoc* scheme introduced in [19]. Operatively, discretized equations are obtained after a twofold discretization in time and space. First, a first order explicit-in-time discretization is needed. Let  $[0, T]$  be a time interval of interest, which we evenly partition by the instants  $t_n = n\Delta t$ , so that  $n$  ranges from 0 to a value  $M$  such that  $T = M\Delta t$ . In this setting, we recursively generate a countable family of measures  $\{\tilde{\nu}_n\}_{n=0}^M$  approximating the  $\nu_{t_n}$ 's. To this end, let us introduce the discrete-in-time flow map:

$$\tilde{X}_n(x) = x + \tilde{w}_n(x)\Delta t, \quad (19)$$

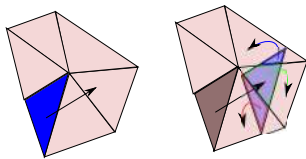


Figure 6: Conceptual sketch of the numerical scheme in action

where  $\tilde{w}_n(x) := v_d(x) + v_i[\tilde{\nu}_n](x)$ . By using Eq. (19), one can define the recursion

$$\begin{cases} \tilde{\nu}_{n+1} &= \tilde{X}_n \# \tilde{\nu}_n, & n \geq 1 \\ \tilde{\nu}_0 &= \bar{\nu} \end{cases}$$

which produces the desired time approximation:

$$\tilde{\nu}_n \approx \nu_n \Delta t, \quad n = 1, 2, \dots, M.$$

As  $\bar{\nu}$  is absolutely continuous, if the discrete flow map meets suitable regularity conditions then  $\tilde{\nu}_n$  is absolutely continuous as well for all  $n$  (for a detailed discussion on the requirements the reader can refer to [19]).

After the time discretization has been established, we consider a finite-volume-type partition of  $D$ . Let  $\{E_k\}_{k=1}^Q$  be a grid of  $Q$  measurable elements of centroids  $x^k$  and let  $\tilde{\rho}_n$  be the density of  $\tilde{\nu}_n$ , i.e.,  $d\tilde{\nu}_n = \tilde{\rho}_n dx$ . We approximate  $\tilde{\rho}_n$  by means of a piecewise constant function  $\hat{\rho}_n$  given by:

$$\hat{\rho}_n(x) = \sum_{k=1}^Q \rho_n^k \mathbf{1}_{E_k}(x),$$

where  $\rho_n^k$  is a characteristic value of  $\tilde{\rho}_n$  when restricted to the element  $E_k$  (e.g.,  $\rho_n^k = \tilde{\rho}_n(x^k)$ ). We finally call  $\hat{\nu}_n$  the measure whose density is  $\hat{\rho}_n$ , i.e.,  $d\hat{\nu}_n = \hat{\rho}_n dx$ .

A piecewise constant space approximation of the flow map (19) can be further considered:

$$\hat{X}_n(x) = x + \hat{w}_n(x) \Delta t,$$

where the piecewise version of the velocity  $\hat{w}_n(x)$  is used, i.e.,

$$\hat{w}_n(x) = \sum_{k=1}^Q (v_d(x^k) + v_i[\hat{\nu}_n](x^k)) \mathbf{1}_{E_k}(x).$$

As the mesh is fixed, i.e., it does not get deformed in time, the recursive relation

$$\hat{\nu}_{n+1} = \hat{X}_n \# \hat{\nu}_n, \quad n \geq 1$$

is further approximated by

$$\hat{\rho}_{n+1}^q = \sum_{k=1}^Q \hat{\rho}_n^k \frac{|E_q \cap \hat{X}_n^{-1}(E_k)|}{|E_q|}, \quad q = 1, 2, \dots, Q, \quad (20)$$

where, for the sake of conciseness, we have denoted by  $|\cdot|$  the Lebesgue measure  $\mathcal{L}^2$ . If the spatiotemporal grid is properly refined (viz. under a suitable relationship between the characteristic size of the elements and the time step  $\Delta t$ ), the scheme converges to the weak solution of Problem (7)-(8) (for technical details see [26]).

This numerical scheme can be ideally used with any type of grid, independently of the element shape. From the strict implementation point of view, one needs to compute rigid movements of the elements  $E_k$  and evaluate their intersections. For instance, in [8, 19] orthogonal grids formed by

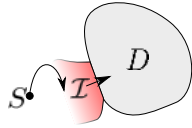


Figure 7: Pedestrian entering region  $\mathcal{I}$

square-shaped elements have been used. Nonetheless, when articulated domains are considered, triangular meshes are often used. Pairs of triangles exhibit a wide selection of (topologically) different ways of intersecting<sup>3</sup> (see Fig. 6 and e.g., [24]), which makes the evaluation of (20) expensive. It is worth pointing out that the problem of intersecting triangles (and convex polygon in general) is typical in computational geometry and computer graphics and can be solved very efficiently. Algorithms having linear complexity with respect to the number of edges of the polygons have been conceived (see e.g., [18, 27]), which allow computational times to be contained. Domains considered in simulations in Sect. 4 (see Fig. 5) have been discretized by using triangular meshes.

### 3.4 Handling pedestrian entering

Often, in applications, the spatial domain considered is just a subset of a larger phenomenological domain in which the whole crowd event develops. Therefore, assuming that at the initial time all involved pedestrians are already inside the domain can be over-restrictive. This is especially true when footbridges are considered. On the contrary, this is not the case when dealing, for instance, with evacuation problems [25]. In the current section, an approach allowing for pedestrian entrance, conceptually consistent with the model previously developed, is proposed. Specifically, arrival processes mimicking queues will be considered.

As in Fig. 7, let us consider a region  $\mathcal{I} \subset \mathbb{R}^2$  adjacent to  $D$  which models a *buffer* through which pedestrians coming from an external (i.e., not explicitly modeled) part of the phenomenological domain enter  $D$ . In other words,  $\mathcal{I}$  can be thought of as a *lobby* or a *waiting room*. Consistently, we assume that it has a non-zero Lebesgue measure, i.e.  $|\mathcal{I}| > 0$ . From now on,  $\mathcal{I}$  will be referred to as the *entering region*.

For our purposes, let us imagine the perceived mass measure of  $\mathcal{I}$  as a bulk, *super-macroscopic*, variable of the system, which we henceforth denote by  $I_t := \nu_t(\mathcal{I})$ . At time  $t$ , we denote by  $S_t$  the number of pedestrians involved in the crowd event who are waiting to *appear* in  $\mathcal{I}$ . They are ideally stored in a 0-dimensional *reservoir*  $\mathcal{S}$ . Once they are in  $\mathcal{I}$ , system dynamics (cf. Eq. (7)) drive them out to  $D$ . Again from a super-macroscopic perspective, we call  $\Phi_t = d\nu_t(D)/dt$  the outflow of pedestrian from  $\mathcal{I}$  to  $D$ . On the whole, the following conservation principle holds:

$$\frac{dI_t}{dt} + \frac{dS_t}{dt} + \Phi_t = 0. \quad (21)$$

We indicate the rate of emptying of  $\mathcal{S}$ , i.e., the rate at which pedestrians appear in  $\mathcal{I}$  (neglecting their spatial distribution), by

$$\frac{dS_t}{dt} = f(S_t, I_t; t).$$

From the phenomenological point of view, such a rate can depend on several factors such as, for instance, the number of pedestrians in  $\mathcal{I}$ , the amount of agents who still have to enter (i.e.,  $S_t$ ), or even the time of the day.

Coherently, the function  $f$  should be designed in order to mimic dynamical arrival processes. In the following, an expression for  $f$ , suitable to represent smoothly ending queue processes is constructed. In particular, we assume:

- $f \propto \sigma(S_t)$  for some function  $\sigma : [0, 1] \rightarrow \mathbb{R}^+$  satisfying  $\sigma(0) = 0$  (i.e., the rate is zero if all agents have appeared in  $\mathcal{I}$ );

<sup>3</sup>Notice that intersections of rectangles taken from orthogonal grids, instead, give rise just to rectangular intersections.

- that  $\mathcal{I}$  cannot be filled up indiscriminately. A certain capacity  $C > 0$  must not be overcome. Hence, a logistic factor is considered:  $f \propto (1 - \frac{I_t}{C})$ . The constant  $C$  may satisfy  $C = \rho_C |\mathcal{I}|$ ,  $\rho_C > 0$  being a capacity density threshold of  $\mathcal{I}$ . In this setting, when  $I_t < C$  a mass inflow in  $\mathcal{I}$  from  $\mathcal{S}$  is allowed, otherwise a reverse flux takes place.

On the whole, we obtain the expression:

$$f(S_t, I_t; t) = \sigma(S_t) \left(1 - \frac{I_t}{C}\right). \quad (22)$$

As a matter of fact,  $\sigma(S_t)$  represents the ideal outflow of mass from  $\mathcal{S}$  to  $\mathcal{I}$  when  $I_t = 0$  (i.e.,  $\mathcal{I}$  is empty). If the queue features a constant arrival flux which smoothly ends as  $N$  pedestrians enter the domain, the function  $\sigma$  can be chosen as:

$$\sigma(S_t) = F \left[ \mathbf{1}_{[0, p)} \left( \frac{S_t}{N} \right) \frac{S_t}{Np} + \mathbf{1}_{[p, 1]} \left( \frac{S_t}{N} \right) \right], \quad (23)$$

$F$  being a positive constant and  $0 < p < 1$  the ratio at which the appearance rate begins to fade out.

### 3.4.1 Numerical treatment of the entrance region

In this section the numerical scheme in Eq. (20) is extended so that pedestrian entering regions introduced in Sect. 3.4 can be handled.

An *extended* version of domain  $D \cup \mathcal{I}$ , including entrance regions, is considered. Therein, a finite volume-like partition  $\{E_j\}_{j=1}^Q$  consistent with<sup>4</sup> both subsets  $D$  and  $\mathcal{I}$  is introduced. Coherently, the discrete density  $\hat{\rho}_n^j$  is defined on such a partition and thus it is defined also on  $\mathcal{I}$ .

The dynamic arrival process described by Eqs. (21) and (22) is taken into account in its discrete-in-time version, that is:

$$\begin{cases} \hat{S}_{n+1} &= \hat{S}_n - \Delta t f(\hat{S}_n, \hat{I}_{n+1}; t_n) \\ \hat{I}_{n+1} &= \hat{I}_{n+1} + \Delta t f(\hat{S}_n, \hat{I}_{n+1}; t_n) - (\hat{\Phi}_{n+1} - \hat{\Phi}_n), \end{cases} \quad (24)$$

where  $\hat{I}_{n+1}$  and  $\hat{\Phi}_{n+1}$  satisfy respectively

$$\hat{I}_{n+1} = \sum_{j: E_j \subset \mathcal{I}} \hat{\rho}_{n+1}^j |E_j| \quad (25)$$

$$\hat{\Phi}_{n+1} = \sum_{j: E_j \subset D} \hat{\rho}_{n+1}^j |E_j|. \quad (26)$$

On the whole, the evolution of the crowd through  $\mathcal{S}$ ,  $\mathcal{I}$ , and  $D$  is computed using the following algorithm:

```

for  $n \leftarrow 0$  to  $M$  do
  Evolvea  $\hat{\rho}_n^j$  on  $D \cup \mathcal{I}$ ;
  Evaluate  $\hat{\Phi}_{n+1}$  and  $\hat{I}_{n+1}$  (using Eqs. (26) and (25));
  Evaluate  $\hat{S}_{n+1}$  and  $\hat{I}_{n+1}$  (using Eq. (24));
  Assignb
    
$$\hat{\rho}_{n+1}^j \leftarrow \hat{I}_{n+1} \frac{|E_j|}{|\mathcal{I}|}, \quad \forall j : E_j \subset \mathcal{I};$$

end

```

**Algorithm 1:** Procedure to handle pedestrian entering

<sup>a</sup>i.e. evolve the discrete densities  $\hat{\rho}_n^j$  according to algorithm in Eq. (20)

<sup>b</sup>i.e. distribute uniformly the mass  $\hat{I}_{n+1}$  on  $\mathcal{I}$  - thus *overwrite* the discrete densities  $\hat{\rho}_n^j$  in  $\mathcal{I}$ .

<sup>4</sup>I.e.  $\{E_j \in \{E_j\}_{j=1}^Q : E_j \cap D \neq \emptyset \wedge E_j \cap \mathcal{I} \neq \emptyset\} = \emptyset$ .

Table 1: Parameters used for crowd event simulations

$L$	$B$	$\rho_C$	$N$	$V$
100 m	4 m	1.3 ped/m <sup>2</sup>	1500 ped	1.18 m/s

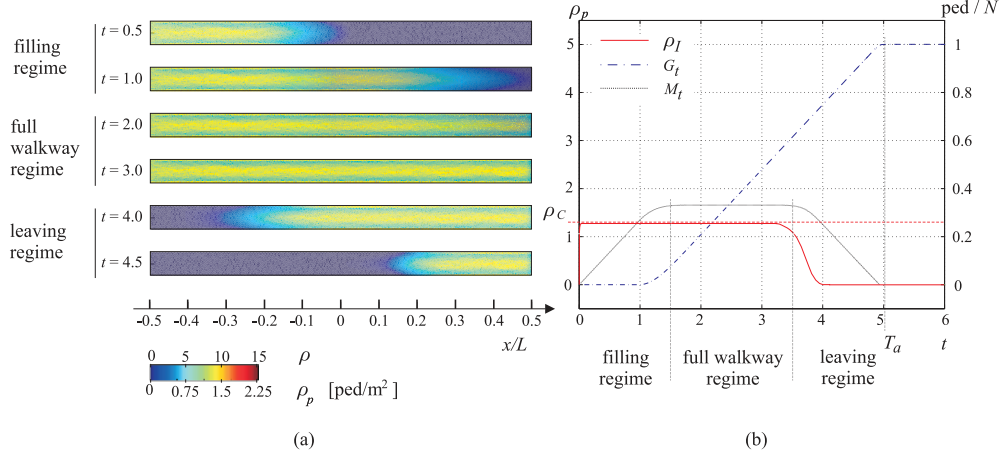


Figure 8: (a) Instantaneous density fields and recognized regimes - (b) Time history of some crowd bulk parameters

## 4 Application

This section is devoted to provide some real world applications of the proposed model. In particular, model sensitivity and tunability are discussed in the reference configuration in Fig. 4. Moreover, simulation results are discussed for the geometrical the configurations in Fig. 5.

### 4.1 Setup overview and some simulated phenomena

In this section, a prototypical crowd event happening in the straight rectangular walkway  $D$  depicted in Fig. 4 is considered. The application is defined by assuming that a hypothetical client is able to provide some data of the problem setup, particularly  $L$ ,  $B$ , the expected total number  $N$  of incoming pedestrians, capacity density  $\rho_C$  in Eq. (22), the pedestrian desired speed  $V = |v_d|$ . In the present case values have been selected on the basis of data available in Transportation and Civil Engineering literature [5, 11, 28] and they are summarized in table 4.1. The length  $L$  and the desired speed  $V$  are reference quantities, thus the time scale  $T = L/V$  can be defined as the time required to an undisturbed pedestrian to cross the whole facility. In the following, we denote by  $\rho_p$  the dimensional pedestrian mass density [ $ped/m^2$ ] which is related to the density  $\rho$  through the scaling  $\rho_p = \rho N/L^2$ .

The quantities  $N$  and  $\rho_C$  are required to characterize the time history of the incoming crowd (Sect. 3.4), while the parameters  $F = 2$  and  $p = 0.9$  governing the entrance queue (Eq. 23) have been arbitrarily set to simulate a sudden start of the entrance and a smoother end queue, already observed in some real world crowd events (e.g. in [4, 11]). The resulting time history of crowd density in the region  $\mathcal{I}$  is reported in Fig. 8(b).

In order to outline the main features of the simulated crowd event, Fig. 8(a) samples some instantaneous density fields and groups them in three recognized regimes: during the *filling regime* pedestrians advance on the partially empty walkway, which is homogeneously filled in the *full walkway regime*. The crowd event gradually ends during the *leaving regime*. In Fig. 8(b), the time history of two integral parameters of the simulated crowd event is also plotted: the number  $M_t$  of pedestrians along the walkway and the cumulative number  $G_t$  of pedestrians walked out, both scaled with respect to  $N$ . A further bulk parameter can be easily recognized, i.e. the *total time*

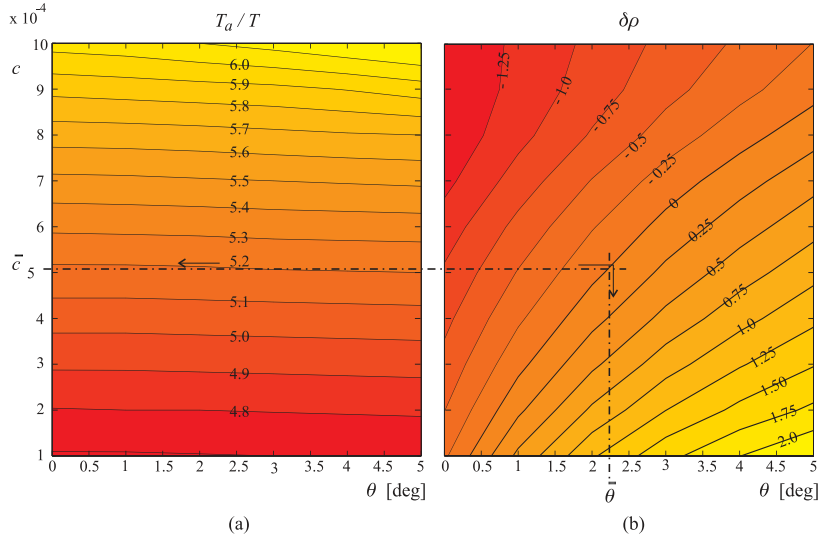


Figure 9: (a) Contours of  $T_a = T_a(c, \theta)$  - (b)  $\delta\rho = \delta\rho(c, \theta)$

of the crowd event  $T_a$  defined as

$$T_a = \inf \left\{ t : \frac{G_t}{N} = 1 \right\}. \quad (27)$$

#### 4.1.1 Sensitivity to free model parameters

Four free model parameters remains and determine the evolution of the solution, namely the constants  $c$ ,  $\theta$ ,  $R$ , and  $\alpha$ . On the one hand,  $R = 2$  m and  $\alpha = 45^\circ$ , depending upon the geometry of the sensory region, are regarded as case-independent and recoverable from existing literature (see e.g., [10, 28]). On the other hand, the sensitivity of the model to the repulsion constant  $c$  and to the angle  $\theta$  is inquired. The analysis is based on the observable variables  $T_a$ , which has been previously defined in Eq. 27, and  $\delta\rho$ :

$$\delta\rho = \frac{\rho_m - \rho_s}{\rho_C}, \quad (28)$$

where  $\rho_s$  and  $\rho_m$  are the crowd density at the walkway side and at the mid-line, respectively, evaluated at the mid-span  $x = L/2$  during the full walkway regime. In other words, such a variable is a measure of the chord-wise uniformity of the crowd density, being the span-wise uniformity assured during the full walkway regime and the chord-wise symmetry of the solution assured by the selected setup.

In Fig. 9, such variables are plotted versus the dimensionless parameters  $2.5 \cdot 10^{-4} \leq c \leq 12.5 \cdot 10^{-4}$ ,  $0^\circ \leq \theta \leq 5^\circ$ .

The crowd event time  $T_a$  (Fig. 9(a)) is mainly sensitive to the pedestrian-pedestrian repulsion, i.e., to  $c$ . For the selected incoming pedestrian density  $\rho_C$ ,  $T_a$  approximatively varies from three to four times the undisturbed pedestrian crossing time, depending on the value of  $c$ . On the contrary,  $\delta\rho$  (Fig. 9(b)) depends upon both parameters and shows both positive values (higher density at the walkway sides) and negative ones (high density along the mid-line) in the selected range of the model parameters. In other words, for any considered value of  $T_a$ , there exist a value  $\theta^*$  of the angle allowing a nearly homogeneous chord-wise crowd density ( $\delta\rho = 0$ ), while larger or smaller values produce nonuniform distributions. In order to detail the chord-wise trend of the crowd density and to discuss its phenomenological features, the  $\rho_p$  profiles at mid-length are plotted in Fig. 10 for different values of  $\theta$  and fixed value of  $c = 5 \cdot 10^{-4}$ .

For  $0 \leq \theta < \theta^*$ , where  $\theta^* \approx 2^\circ$  with  $c = 5 \cdot 10^{-4}$ , the pedestrian-wall repulsion dictated by the desired velocity field is lower than the pedestrian-pedestrian repulsion, the chord-wise component



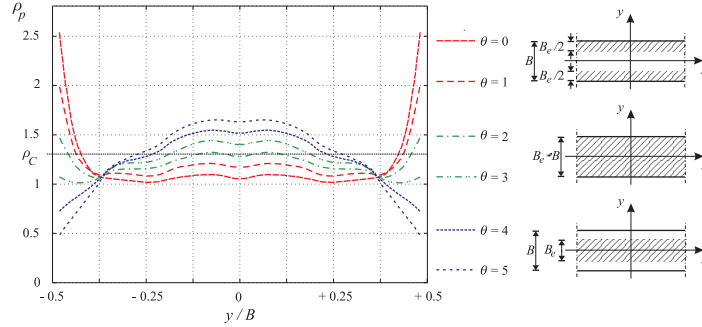


Figure 10: Chord-wise crowd density profiles at mid-length for different  $\theta$  ( $c = 5 \cdot 10^{-4}$ )

of the total velocity field is directed toward the lateral walls and crowd density results larger at the walkway sides than at mid-chord (Fig. 10, red profiles). Conversely, the pedestrian-wall repulsion predominates over the pedestrian-pedestrian one for  $\theta^* < \theta \leq 5^\circ$ , and the crowd density at mid-chord is larger than at walkway sides (Fig. 10, blue profiles). The crowd density profile is almost flat around the value  $\theta^*$  (Fig. 10, green profiles). In a general modeling perspective: i) the parameter  $\theta$  allows one to account for different degrees of repulsion of either both or a single walkway side, e.g., a panoramic point on one side only; ii) the parameter  $c$  allows one to account for different attitudes of pedestrians in accepting the proximity of other walkers, e.g., dictated by different travel purposes [5, 28]. From a transportation engineering perspective, the parameter  $\theta$  allows one to model the shy distance of pedestrians from the wall [20] and to evaluate the *effective width* of the walkway (e.g., [13]): for  $\theta = \theta^*$  the effective width of the walkway matches the geometric width, while shorter effective widths are obtained otherwise (Fig. 10, conceptual sketches).

#### 4.1.2 Tunability of the free parameters

On the basis of the previous sensitivity analysis a tuning strategy can be suggested. On the one hand, the model is compact, in the sense that only two parameters are introduced to characterize the two velocity components and to account for distinct repulsion phenomena. On the other hand, bulk experimental data directly obtained from real world crowd events are preferable to pointwise measurements obtained in laboratory tests, in the perspective of a tuning procedure to be easily applied to the engineering practice.

Hence, we assume the hypothetical client is able to complement the setup data with measurements obtained on the same walkway of interest or on analogous facilities. In particular, the crowd event time  $T_a$  is easy to be measured, hence it is usually provided (e.g. in [11]) or estimated. Moreover, information about the degree of repulsion of the lateral sides to be adopted is required, either qualitatively identified by referring to the prototypical blue-red-green profile shapes in Fig. 10 or quantitatively expressed by  $\delta\rho$ .

Being such data available, say  $\check{T}_a = 5.2$ ,  $\check{\delta\rho} = 0$ , the graphs of Fig. 9 can be used as tuning charts: first, the value of the constant  $c = \check{c}$  is obtained by setting the value of  $T_a/T = \check{T}_a$  in Fig. 9(a), and then the value of  $\theta = \check{\theta}$  is recovered by setting the value of  $\delta\rho = \check{\delta\rho}$  in Fig. 9(b). In the following, the values  $\check{c} \approx 5 \cdot 10^{-4}$  and  $\check{\theta} = \theta^* \approx 2^\circ$  are retained.

## 4.2 Crowd flow along real world footbridges

The crowd flow along the same real world footbridges described in Sect. 3.2.1, Fig. 5, is now simulated in order to show the ability of the proposed approach to face actual engineering problems. The simulations adopt the same setup (Tab. 4.1) and the values of the model parameters set in the previous subsection, so that the comparison to the reference test-case (rectangular walkway) allows to qualitatively point out the effects of the domain geometry.

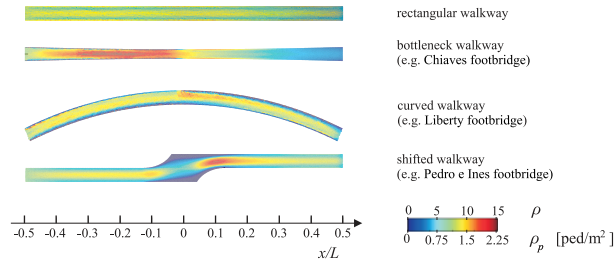


Figure 11: Comparison among real world footbridges: density field during the stationary full walking regime

For instance, Fig. 11(a) collects the crowd density instantaneous fields during the stationary full walking regime for the considered domains.

The fields differ significantly both qualitatively and quantitatively: asymmetric patterns arise with respect to the chord-wise axis at midspan (bottleneck walkway), to the longitudinal axis (curved walkway) or to both (shifted walkway); the maximum crowd density approximatively doubles in the bottleneck and shifted walkway respect to the rectangular one.

Finally, it is worth pointing out that the output of all considered simulations can be equivalently interpreted in the statistic way proposed in Sect. 2.3. In particular, both Figs. 8 and 11 report alongside the pedestrian density  $\rho_p$  the probability  $\rho$  density associated to the random variables  $X_t^j$ .

## 5 Conclusions

In this paper, a model aimed at evaluating the collective evolution of a crowd has been derived and analyzed. A model describing the motion of a generic test individual has been firstly obtained on a phenomenological basis, incorporating the effect of the perception she has of the collectivity in her surroundings. It is worth pointing out that the collective point of view has been treated in the general terms of Radon measures, which allow one to distinguish two contributions in the perception: an atomic localized one and a continuous distributed one. A procedure has then been provided to obtain, out of the equation governing the motion of single individuals, equations describing the collective evolution of the crowd. Next, the specific structural class of pedestrian footbridges has been considered and the elements required to simulate crowd events therein have been given. In particular, a working strategy to construct suitable desired velocity fields, satisfying some minimal phenomenological requirements, has been proposed. Issues concerning the equation discretization as well as the handling of boundary conditions have been addressed. The reduced amount of free parameters (four) present in the final equations, made possible the definition of a tuning procedure based on a free parameter sensitivity analysis. To show the ability of the proposed approach to face actual engineering problems, crowd events have been simulated in different computational domains inspired by real footbridges.

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