FROM: Large–scale Earth Structure from Analyses of Free Oscillation Splitting and Coupling

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In: E. Boschi, G. Ekström and A. Morelli (eds) ”Problems in Geophysics for the New Millennium”, Istituto Nazionale di Geofisica e Vulcanologia, Editrice Compositori, Bologna, Italy, 2000

APPENDIX A – A mode tutorial

This appendix describes some basic theoretical results – a more complete treatment can be found in Lapwood and Usami (1981) and Dahlen and Tromp (1998).

Since departures from spherical symmetry are small (particularly in the deep Earth), it is useful to consider an approximate Earth model which is spherically symmetric, non-rotating, and elastically isotropic. Departures from this state (anelasticity, anisotropy, rotation and three-dimensional structure) are supposed sufficiently small that they can be treated by perturbation theory. The model is assumed to be initially quiescent and in a state of hydrostatic equilibrium. The equations governing the small oscillations of such a body are given by:

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \nabla \cdot \mathbf{T} - \nabla(s_r \rho_0 g_0) - \rho_0 \nabla \phi_1 + \hat{\mathbf{r}} g_0 \nabla \cdot (\rho_0 \mathbf{s}) + \mathbf{f} \qquad (A1)$$

and

$$\nabla^2 \phi_1 = -4\pi G \nabla \cdot (\rho_0 \mathbf{s}) \qquad (A2)$$

Equation (A1) is the linearized equation for conservation of momentum. ρ_0 is the unperturbed density and $\mathbf{s}$ is the displacement field. The acceleration due to gravity is given by $\mathbf{g}(r) = -\hat{\mathbf{r}} g_0(r)$, $\mathbf{T}$ is the elastic stress tensor and $\mathbf{f}$ is a body force density used to represent the earthquake source ($\mathbf{f}$ = 0 when the Earth is in free oscillation). Since the motion of the Earth causes a disturbance of the gravitational potential which in turn affects the motion, we must also solve Poisson's equation (equation (A2)) where ϕ_1 is the perturbation in the gravitational potential associated with the motion. Considering the linearized momentum equation (A1), the force densities contributing to the acceleration of material at a fixed point in space are respectively: 1) elastic forces caused by the deformation of the body, 2) motion of material in the initial stress field, 3) forces due to the change in gravitational potential, and 4) forces due to motion within the initial gravitational field.

We seek solutions to equations (A1) and (A2) which satisfy certain boundary conditions on the displacement field and the tractions acting on interfaces (the traction acting on a surface with normal $\hat{\mathbf{n}}$ is $\mathbf{T} \cdot \hat{\mathbf{n}}$). In particular, the displacement field must be continuous everywhere except at a fluid-solid boundary where slip is allowed. The tractions on horizontal surfaces must be continuous at all interfaces and must vanish at the free surface. The usual way to proceed is to use separation of variables and expand the displacement field in vector spherical harmonics. We recognize that there may be more than one solution to equations (A1) and (A2) and designate the k'th solution by $\mathbf{s}_k$ where:

$$\mathbf{s}_k = \hat{\mathbf{r}}\,{}_kU + \nabla_1\,{}_kV - \hat{\mathbf{r}} \times (\nabla_1\,{}_kW) \qquad (A3)$$

$\nabla_1 = \hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\,\mathrm{cosec}\,\theta\partial_\phi$ is the horizontal gradient operator and ${}_kU$, ${}_kV$ and ${}_kW$ are scalar functions of position. We now expand ${}_kU$, ${}_kV$ and ${}_kW$ in ordinary spherical harmonics as well as ϕ_{1k}. Each of these functions have an expansion of the form

$$\left.\begin{aligned} {}_kU &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} {}_kU_l^m(r)Y_l^m(\theta,\phi) \\ {}_kV &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} {}_kV_l^m(r)Y_l^m(\theta,\phi), \quad \text{etc.} \end{aligned}\right\} \qquad (A4)$$

where the $Y_l^m(\theta,\phi)$ are fully normalized spherical harmonics (Edmonds, 1960):

$$Y_l^m(\theta,\phi) = (-1)^m \left[\frac{2l+1}{4\pi} \cdot \frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} P_l^m(\cos\theta)e^{im\phi} \qquad (A5)$$

and the P_l^m are associated Legendre functions. Figure 17 illustrates some of the Y_l^m's geographical shapes in the Hammer–Aitoff projection we also use to illustrate splitting functions.

To proceed, we adopt a constitutive relation for the material to relate $\mathbf{T}$ to the deformation of the body. In the simplest case, a perfectly elastic relation is adopted (the weak attenuation being amenable to treatment by perturbation theory) and $\mathbf{T} = \mathbf{C} : \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon}$ is the strain tensor and $\mathbf{C}$ is a fourth-order tensor of elastic moduli. In component form, the constitutive relation is

$$T_{ij} = C_{ijkl}\epsilon_{kl},$$

where summation over repeated indices is implied. The most general elastic Earth exhibiting spherical symmetry is "transversely isotropic" (i.e. seismic velocities in the radial direction are different from velocities in the tangent plane) and is described by five elastic moduli which, in Love notation, are: $\mathrm{A}, \mathrm{C}, \mathrm{L}, \mathrm{N}$, and F. An elastically isotropic body is described by two elastic moduli: λ and μ where μ is known as the shear modulus or rigidity. The results for an isotropic body can be recovered from those for a transversely isotropic body using the substitutions: A = C = $\lambda + 2\mu$, F = λ, and L = N = μ.

Substitution of these forms into equations (A1) and (A2) results in four coupled second-order ordinary differential equations governing the radial dependence of the scalars in equation (A4). (For clarity, we drop the subscripts on U, V, W, and Φ so, in the next equation $U \equiv {}_kU_l^m$ etc.)

$$\left.\begin{aligned}
\frac{1}{r^2}\left(\frac{d}{dr}\; r^2\frac{d\Phi_1}{dr}\right) - l(l+1)\frac{\Phi_1}{r^2} &= \\
-4\pi G\left[\frac{d}{dr}(\rho_0 U) + \rho_0 F\right]& \\
-\rho_0\omega_k^2 U = \frac{d}{dr}\left(\mathrm{C}U' + \mathrm{F}F\right)& \\
-\frac{1}{r}\left[2(\mathrm{F}-\mathrm{C})U' + 2(\mathrm{A}-\mathrm{N}-\mathrm{F})F + l(l+1)\mathrm{L}X\right]& \\
-\rho_0\Phi_1' + g_0\left((\rho_0 U)' + \rho_0 F\right) - (\rho_0 g_0 U)'& \\
-\rho_0\omega_k^2 V = \frac{d}{dr}(\mathrm{L}X)& \\
+\frac{1}{r}\left[(\mathrm{A}-\mathrm{N})F + \mathrm{F}U' + 3\mathrm{L}X - \frac{\mathrm{N}V}{r}(l+2)(l-1)\right]& \\
-\frac{\rho_0\Phi_1}{r} - \frac{\rho_0 g_0 U}{r}&
\end{aligned}\right\} \qquad (A6)$$

$$-\rho_0\omega_k^2 W = \frac{d}{dr}(\mathrm{L}Z) + \frac{1}{r}\left[3\mathrm{L}Z - \frac{\mathrm{N}W}{r}(l+2)(l-1)\right] \qquad (A7)$$

where

$$X = V' + \frac{U-V}{r}, \qquad Z = W' - \frac{W}{r}, \qquad F = \frac{1}{r}(2U - l(l+1)V)$$

and a prime denotes radial derivative. Note that the equations are dependent upon harmonic degree (ℓ) but are independent of azimuthal order number m. Consider equation (A7). For a chosen harmonic degree and frequency, the solution to these equations will not necessarily match the boundary conditions, notably vanishing of traction at the free surface. There are however, discrete frequencies for each ℓ when solutions $W(r)$ can be found which match all boundary conditions. Such frequencies are the frequencies of free toroidal oscillation of the Earth. For a particular ℓ, the mode with the lowest frequency of free oscillation is labelled ${}_0T_\ell$, the next highest is ${}_1T_\ell$ and so on. The displacement field of the n'th mode ${}_nT_\ell$ which has frequency ${}_n\omega_\ell$ say is proportional to

$${}_n\mathbf{s}_l^m = \left[\hat{\boldsymbol{\theta}}\operatorname{cosec}\theta\; {}_nW_l(r)\frac{\partial Y_l^m}{\partial\phi} - \hat{\boldsymbol{\phi}}\; {}_nW_l(r)\frac{\partial Y_l^m}{\partial\theta}\right] e^{i\,{}_n\omega_l t} \qquad (A8)$$

for $-l \leq m \leq l$. This kind of motion is called toroidal because it consists of twisting on concentric shells. Toroidal motion can be sustained only in a solid so toroidal modes are confined to the mantle (another class is confined to the inner core but cannot be observed at the surface). Note that there is no radial component of motion and no compression or dilation so there is no perturbation to the gravitational field. This is not true for solutions to the other three coupled ODEs (equation (A6)). Again there are discrete frequencies for a fixed harmonic degree where solutions can be found which match all boundary conditions. These are the frequencies of free spheroidal motion (sometimes called poloidal motion). For a particular ℓ, the mode with the lowest frequency of free oscillation is labelled ${}_0S_\ell$, the next highest is ${}_1S_\ell$ and so on. The displacement field of the n'th mode ${}_nS_\ell$ which has frequency ${}_n\omega_\ell$ say is proportional to

$${}_n\mathbf{s}_l^m = \left[\hat{\mathbf{r}}\,{}_nU_l(r)Y_l^m(\theta,\phi) + \hat{\boldsymbol{\theta}}\,{}_nV_l(r)\frac{\partial Y_l^m}{\partial\theta}(\theta,\phi) + \operatorname{cosec}\theta\hat{\boldsymbol{\phi}}\,{}_nV_l(r)\frac{\partial Y_l^m}{\partial\phi}(\theta,\phi)\right]e^{i_n\omega_l t} \qquad (A9)$$

Note that in both equations (A8) and (A9), for each n and ℓ there are $2\ell + 1$ modes of oscillation with exactly the same frequency since the governing equations do not depend upon m. This is the phenomenon of degeneracy which is a consequence of the assumed symmetry of the Earth model. This group of $2\ell + 1$ modes is called a "multiplet" while the individual members of the multiplet are called "singlets". Departures of the Earth from spherical symmetry remove the degeneracy and, in general, each singlet within a multiplet will have a slightly different frequency.

We can also use the results given above to compute the elastic energy density of a mode. The total elastic energy of a mode is:

$$E = \int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon}\, dV \qquad (A10)$$

where the double dots indicate tensor contraction. This can be written in terms of the mode scalars as:

$$E = \int \left[l(l+1)(l-1)(l+2)\frac{\mathrm{N}}{r^2}W^2 + l(l+1)\mathrm{L}Z^2\right] r^2\, dr \qquad (A11)$$

for toroidal modes, and

$$E = \int \left[l(l+1)(l-1)(l+2)\frac{\mathrm{N}}{r^2}V^2 + l(l+1)\mathrm{L}X^2 + 2\mathrm{F}U'F + (\mathrm{A}-\mathrm{N})F^2 + \mathrm{C}U'^2\right] r^2\, dr \qquad (A12)$$

for spheroidal modes. For an elastically isotropic material, we can divide the elastic energy into its shear and compressional components by substituting the bulk modulus K_s for λ using the relationship $K_s = \lambda + 2/3\mu$ yielding

$$E = \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2)W^2 + \mu l(l+1)Z^2 \right] r^2\, dr \tag{A13}$$

for toroidal modes, and

$$E = \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2)V^2 + \mu l(l+1)X^2 + \frac{\mu}{3}(2U' - F)^2 + K_s(U' + F)^2 \right] r^2\, dr \tag{A14}$$

for spheroidal modes. The integrands are plotted in Figure 18 for some representative modes of oscillation and show how a particular mode samples the Earth.

Returning now to equations (A8) and (A9). For a spherical Earth model, all singlets within a multiplet have the same frequency so it is easy to sum all singlets to get an expression for a synthetic seismogram for a multiplet:

$$u(\mathbf{r}, \mathbf{r}_0) = \sum_{m=-\ell}^{\ell} {}_n s_l^m(\mathbf{r}) {}_n a_l^m(\mathbf{r}_0) e^{i\,{}_n\omega_l t} \tag{A15}$$

where the real part is understood. For simplicity, we are considering a single component of recording (e.g. the $\hat{\mathbf{r}}$ component) with a receiver at $\mathbf{r}$ and the source at $\mathbf{r}_0$. The a_l^m are excitation factors for each singlet which can be computed using a moment tensor model of the source (Gilbert and Dziewonski, 1975; Dahlen and Tromp, 1998). This equation is equivalent to equation (4) in the text when no 3D structure is present. For a particular source, we use the j index to indicate a specific recording at location $\mathbf{r}$ and the k index to indicate one of the $2\ell + 1$ singlets. Letting $\bar{\omega} \equiv {}_n\omega_\ell$ and $R_{jk} \equiv {}_n s_l^m(\mathbf{r})$ then

$$u_j = \sum_k R_{jk} a_k e^{i\bar{\omega}t} \tag{A16}$$

This equation is modified when 3D structure is present and an application of perturbation theory results in equation (4) where the vector $\mathbf{a} = a_k, k = 1, ..., 2\ell + 1$ is now weakly time dependent with the time dependence given by equation (5):

$$\mathbf{a}(t) = \exp(i\mathbf{H}t) \cdot \mathbf{a}(0) \tag{A17}$$

and $\mathbf{H}$ is the all-important *splitting matrix* where all of the effects of 3D structure are encoded. As noted in the text, the elements of $\mathbf{H}$ are linearly related to *structure coefficients* (equations 11 and 12) which is the usual way we describe the effects of 3D structure on a mode. For 3D elastic perturbations, $\mathbf{H} \equiv \mathbf{E}$ and

$$E_{mm'} = \sum_s \gamma_s^{mm'} c_s^{m-m'}; \qquad (A18)$$

The c_s^t are the elastic structure coefficients and are linear functionals of the spherical harmonic expansion coefficients for 3D structure (equation 2). The γs are integrals over three spherical harmonics:

$$\gamma_s^{mm'} = \int_\Omega Y_\ell^{m*} Y_s^{m-m'} Y_\ell^{m'} \, d\Omega \qquad (A19)$$

which can be easily computed. Iterative spectral fitting assumes an initial guess for a set of c_s^ts and computes a synthetic seismogram, $\mathbf{u}_0$. The derivative of a seismogram with respect to a c_s^t is computed by differentiating $\mathbf{a}(t)$ with respect to a c_s^t. Let

$$\mathbf{e}(t) = \frac{\partial \mathbf{a}(t)}{\partial c_s^t} \quad \text{so} \quad \frac{\partial \mathbf{u}(t)}{\partial c_s^t} = \mathbf{R} \cdot \mathbf{e}(t) e^{i\bar{\omega}t} \qquad (A20)$$

Iterative spectral fitting proceeds by iteratively solving:

$$\mathbf{u}(t) - \mathbf{u}_0(t) = \frac{\partial \mathbf{u}(t)}{\partial c_s^t} \cdot \delta c_s^t \qquad (A21)$$

for perturbations to the structure coefficients, δc_s^t. The fitting is typically done in the frequency domain in a small frequency band around a mode of interest. The method can be highly non-linear and requires a model of the source to compute **a** and **e**. The matrix AR technique described in the body of the text circumvents these problems.

APPENDIX B: Coriolis Coupling and Splitting parameters

We shall make reference to Appendix D of Dahlen and Tromp (1998). We write U, V and W for the displacement scalers u,v and w in their notation. Let us write the elastic splitting matrix in the form

$$\mathbf{E} = \mathbf{E}^{3D} + \mathbf{W} + (\mathbf{V}^{ell+cen} - \omega_0^2 \mathbf{T}^{ell})/2\omega_0 \qquad (B1)$$

where ω_0 (sometimes written $\bar{\omega}$) is the reference or fiducial frequency for a set of coupled multiplets. The effects of $\mathbf{V}^{ell+cen} - \omega_0^2 \mathbf{T}^{ell}$ cannot be distinguished from those of aspherical structure (c_2^0) and it is customary to calculate $\mathbf{V}^{ell+cen} - \omega_0^2 \mathbf{T}^{ell}$ for a 1D model and subtract it from equation (B1) so that

$$\mathbf{E} = \mathbf{E}^{3D} + \mathbf{W} \tag{B2}$$

In equation (B2) $\mathbf{E}^{3D}$ is the elastic splitting matrix due to 3D structure and $\mathbf{W}$ is due to the Coriolis force.

For a split spheroidal multiplet (a self–coupled block of $\mathbf{E}$) the diagonal of $\mathbf{E}^{3D}$ is symmetric in m because it represents axisymmetric structure. It is well known that $\mathbf{W}$ in this case is diagonal and linear in m. Let

$$\phi_m = 3m/[(2\ell + 1)\ell(\ell + 1)] \tag{B3}$$

Then

$$\sum_m \phi_m E^{3D}_{mm} = 0 \tag{B4}$$

and

$$\sum_m \phi_m W_{mm} = \sum_m E_{mm} = \chi \cdot \Omega \tag{B5}$$

where χ is the Coriolis splitting parameter

$$\chi = \int_0^a \rho(r) \left[V^2 + 2UV\right] r^2 \, dr \tag{B6}$$

We see that χ can be determined from the splitting matrix. It is a linear integral constraint on the 1D density.

For a Coriolis–coupled spheroidal–toroidal pair of multiplets ${}_n\mathrm{S}_l$ and ${}_{n'}\mathrm{T}_{l'}$ we must have $\ell' = \ell \pm 1$. In this case the diagonal of the coupling block of $\mathbf{E}$ for which $t = 0 (m' = m)$ is odd in m. Let

$$\phi_m = 3(\delta_{\ell\ell'+1} S_{\ell m}/\ell + \delta_{\ell\ell'-1} S_{\ell' m}/\ell') \tag{B7}$$

where $S_{\ell m} = [(\ell + m)(\ell - m)/(2\ell + 1)(2\ell - 1)]^{\frac{1}{2}}$ then equation (B4) is again true and

$$\sum_m \phi_m E_{mm} = -i\zeta\Omega \tag{B8}$$

where ζ is the Coriolis coupling parameter

$$\zeta = \int_0^a \rho(r) W^A(r) r^2 \, dr \qquad (B9)$$

and $W^A(r)$ is given in equation (D.71) of Dahlen and Tromp (1998). We see that ζ, like χ, is a linear constraint on the 1D density. Both can be recovered from **E**.

Every spheroidal multiplet ($\ell > 0$) is a potential source for a value of χ and every Coriolis–coupled spheroidal–toroidal pair is a potential source for a value of ζ. There are hundreds of the former and many tens of the latter below 5 mHz. Consequently, it is reasonable to expect to be able to resolve the 1D density with great accuracy in the not too distant future.

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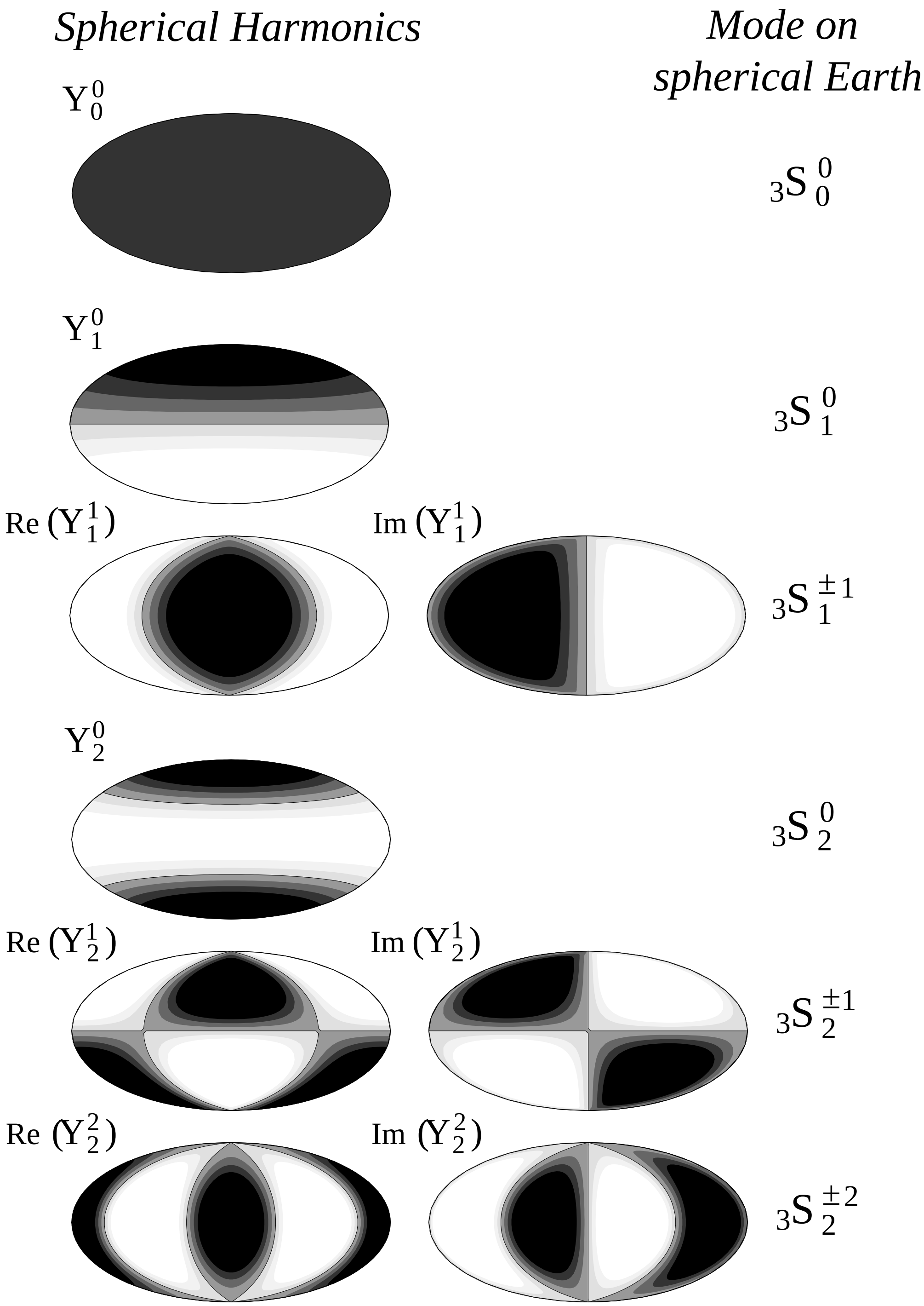

Figure 17. Some low–order spherical harmonics plotted in Hammer–Aitoff projection. Note that the singlets of a spheroidal mode have these shapes on the $\hat{\mathbf{r}}$ component of recordings on a spherical Earth (equation A3).

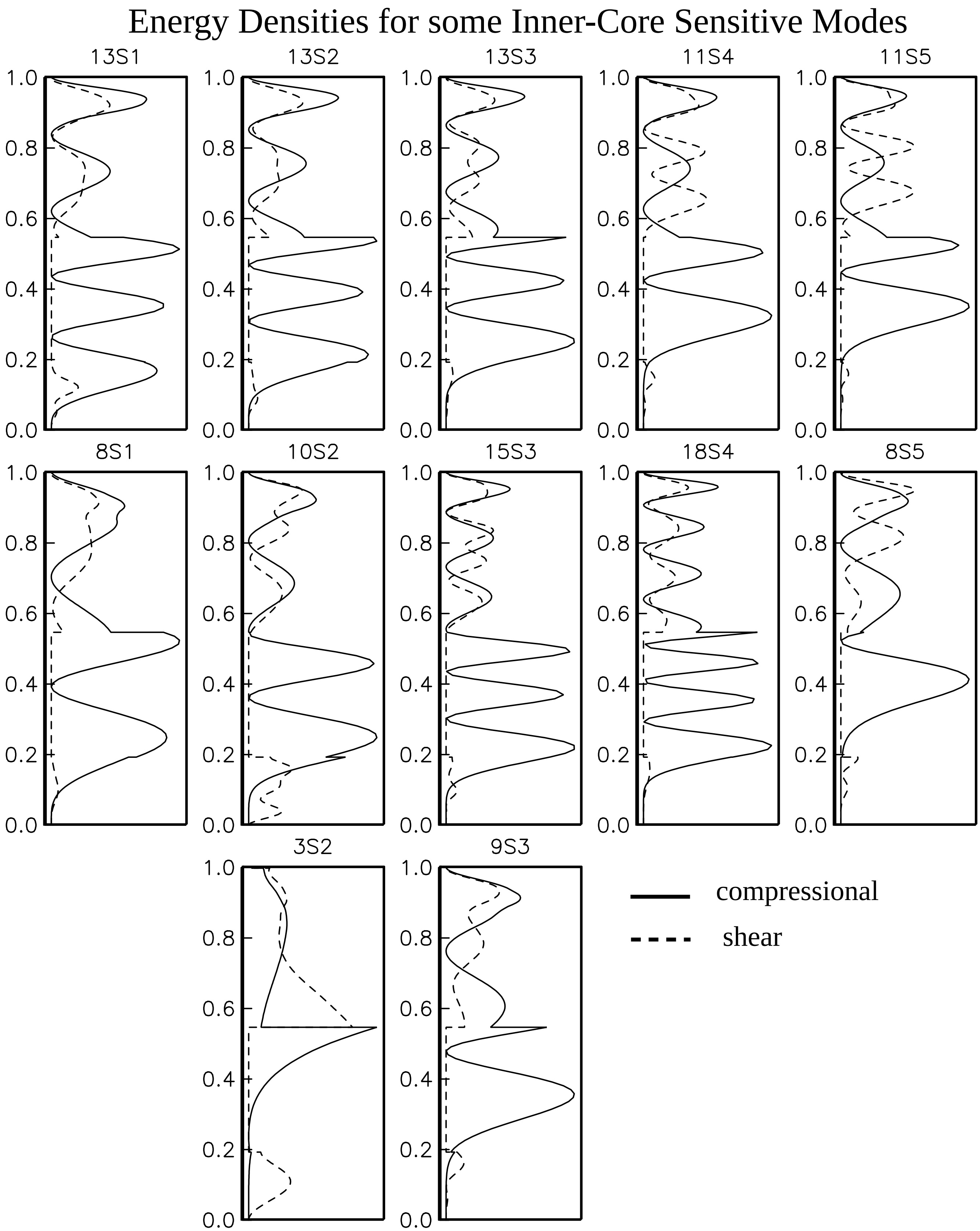

Figure 18. Energy densities for compression and shear as a function of radius for a selection of inner-core sensitive modes.