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From Lipschitzian to non-Lipschitzian characteristics: continuity of behaviors

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Abstract

Linear complementarity systems are used to model discontinuous dynamical systems such as networks with ideal diodes and mechanical systems with unilateral constraints. In these systems mode changes are modeled by a relation between nonnegative, complementarity variables. We consider approximating systems obtained by replacing this non-Lipschitzian relation with a Lipschitzian function and investigate the convergence of the solutions of the approximating system to those of the ideal system as the Lipschitzian characteristic approaches to the (non-Lipschitzian) complementarity relation. It is shown that this kind of convergence holds for linear passive complementarity systems for which solutions are known to exist and to be unique. Moreover, this result is extended to systems that can be made passive by pole shifting.

1 Introduction

The well-posedness (in the sense of existence and uniqueness of solutions) of a class of hybrid systems, namely complementarity systems, has been the main theme of our previous work (see [1, 2, 5, 7, 10, 11] and also [6, 8] for related work). Having networks with ideal diodes as the most typical examples (see for other examples [4]), the complementarity systems are of the form

$$\dot{x} = f(x, u) \tag{1a}$$
$$y = g(x, u) \tag{1b}$$
$$0 \le u \perp y \ge 0 \tag{1c}$$

where the inequalities are understood componentwise, which implies together with the orthogonality relation that $u_i = 0$ or $y_i = 0$ for all i. Of course, one has to be precise about what a solution of such a system means. In [1, 2, 5, 7, 10, 11], solution concepts for several families of systems (1a)-(1b) (e.g. linear, Hamiltonian etc.) are developed and sufficient conditions for well-posedness are presented. Notice that the so-called complementarity conditions (1c) as depicted in Figure 1 do not define a function between u and y. However, a slight perturbation of the piece

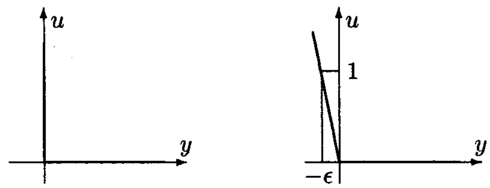

Figure 1: Complementarity characteristic and one of its possible approximations

with infinite slope allows to express u as a piecewise-linear (and hence Lipschitz continuous) function of y. Naturally, one might expect/desire that this approximated characteristic generates trajectories 'close' to ones of complementarity system (1). However, it is not hard to find examples for which this property does not hold whenever the complementarity system is ill-posed. The main objective of the present paper is to prove the convergence of the trajectories generated by the Lipschitzian characteristics to those generated by the (non-Lipschitzian) complementarity characteristic for a class of well-posed complementarity systems including linear passive ones. We will mainly focus on the linear complementarity systems given by

$$\dot{x} = Ax + Bu \tag{2a}$$
$$y = Cx + Du \tag{2b}$$
$$0 \le u \perp y \ge 0. \tag{2c}$$

It can be verified that the linear system (2a)-(2b) with the approximated characteristic of Figure 1 is equivalent to the complementarity system given by

$$\dot{x}^\epsilon = A_\epsilon x^\epsilon + B_\epsilon u^\epsilon \tag{3a}$$
$$y^\epsilon = C_\epsilon x^\epsilon + D_\epsilon u^\epsilon \tag{3b}$$
$$0 \le u^\epsilon \perp y^\epsilon \ge 0 \tag{3c}$$

with $(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon) = (A, B, C, D + \epsilon I)$ in the sense that there is a one-to-one correspondence between the trajectories of the two systems. Keeping this equivalence in mind, we will investigate the convergence of the trajectories of general linear complementarity systems instead of some specific approximation schemes. Later on, several approximation schemes including the one depicted in Figure 1 will

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be treated as special cases of our general setting. Continuity of linear dynamical systems are addressed for instance in [3, 13]. While continuity is defined via pointwise convergence of trajectories in [13], [3] considers continuity in the graph topology. What we understand as continuity is quite close to the notion used in [13]. Our treatment heavily relies on the concept of passivity. In particular, the infinite zero structure imposed by passivity.

The organization of the paper is as follows. In the next section, we recall several facts such as Carathéodory solution of a differential equation, and the definition and characterization of the notion of passivity in order to be reasonably self-contained. Section 3 is devoted to linear complementarity systems. After recalling the solution concept developed previously for such systems, we will present known well-posedness results under the passivity assumption. In section 4, these results will be extended to the class of systems that we call passifiable by pole shifting. This will be followed by results on convergence in section 5. In section 6 some examples will be treated as special cases of the general framework of the previous section. By means of an example, it will be illustrated that the trajectories of the approximating systems are not convergent for the *irregular* initial states. The paper will be closed by conclusions in section 8 and an appendix containing the proofs.

2 Preliminaries

Consider the continuous-time, linear and time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4a}$$
$$y(t) = Cx(t) + Du(t) \tag{4b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A, B, C, and D are matrices with appropriate sizes. We denote (4) by $\Sigma(A, B, C, D)$.

A triple $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ is said to be an $\mathcal{L}_2$-*solution* of $\Sigma(A, B, C, D)$ if it satisfies (4a) in the sense of Carathéodory, i.e., for almost all $t \in [t_0, t_1]$, (4b) holds and

$$x(t) = x(t_0) + \int_{t_0}^{t} [Ax(s) + Bu(s)]ds. \tag{5}$$

Next, we recall the definition of the passivity notion.

Definition 2.1 [12] The system $\Sigma(A, B, C, D)$ given by (4) is said to be *passive (dissipative with respect to the supply rate $u^\top y$)* if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (a *storage function*), such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geq V(x(t_1)) \tag{6}$$

holds for all t_0 and t_1 with $t_1 \geq t_0$, and all $\mathcal{L}_2$-solutions $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ of $\Sigma(A, B, C, D)$.

We state a well-known result on passive systems which characterizes passivity in terms of linear matrix inequalities.

Lemma 2.2 [12] *Assume that (A, B, C) is minimal. Then $\Sigma(A, B, C, D)$ is passive if and only if the matrix inequalities*

$$K = K^\top > 0 \text{ and } \begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leq 0$$

have a solution. Moreover, $V(x) = \frac{1}{2}x^\top Kx$ is a quadratic storage function if and only if K is a solution of the above matrix inequalities.

In what follows, we introduce the following notation.

Notation 2.3 For a given matrix quadruple (A, B, C, D) and K, $\mathcal{K}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ denotes the matrix

$$\begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix}.$$

3 Linear complementarity systems

The main objects of study in the present paper are linear complementarity systems, that is to say, linear systems with complementarity conditions given by

$$\dot{x} = Ax + Bu \tag{7a}$$
$$y = Cx + Du \tag{7b}$$
$$0 \leq u \perp y \geq 0. \tag{7c}$$

We denote the linear complementarity system (7) by LCS(A, B, C, D). Next, we shall define what is meant by a solution of a linear complementarity system by clarifying the meaning of the complementarity conditions in (7c).

Definition 3.1 The triple $(u, x, y) \in \mathcal{L}_2^{m+n+m}(0, T)$ is a *solution* of LCS(A, B, C, D) on $[0, T]$ with initial state x_0 if the following conditions hold.

1. (u, x, y) is a $\mathcal{L}_2$-solution of $\Sigma(A, B, C, D)$ on $[0, T]$.
2. For almost all $t \in [0, T]$, $0 \leq u(t) \perp y(t) \geq 0$.

The initial state is said to be *regular* if there exists a solution with this initial state and *irregular* otherwise.

Throughout the paper, we will frequently use the following assumption.

Assumption 3.2 (A, B, C) is a minimal representation and B is of full column rank.

The passivity of the system $\Sigma(A,B,C,D)$, together with Assumption 3.2, guarantees the existence and uniqueness of solutions (in the sense of Definition 3.1) to LCS(A,B,C,D)as will be presented in the next theorem. Before stating these results, we recall the notion of a dual cone. For a given nonempty set $\mathcal{S}$, we say that the set $\{v \mid v^\top w \geq 0 \text{ for all } w \in \mathcal{S}\}$ is the *dual cone* of $\mathcal{S}$. It is denoted by $\mathcal{S}^*$. In particular, the dual cone of the set $\mathcal{S}_D = \{v \mid v \geq 0,\ Dv \geq 0, \text{ and } v^\top Dv = 0\}$ plays an important role in the above mentioned characterization.

Theorem 3.3 *Consider a matrix quadruple (A,B,C,D) satisfying Assumption 3.2. Suppose that $\Sigma(A,B,C,D)$ is passive. Let $T > 0$ be given. Then, there exists a unique solution of LCS(A,B,C,D) on $[0,T]$ with initial state x_0 if and only if $Cx_0 \in \mathcal{S}_D^*$.*

The proof can be found in [1,5].

4 Passifiability by pole shifting

In this section, we will extend the well-posedness results presented in Theorem 3.3 to a class of nonpassive systems. To do so, note that if (u,x,y) is a solution of LCS(A,B,C,D) with some initial state then $e^{\rho\cdot}(u,x,y)$ is a solution of LCS$(A+\rho I,B,C,D)$ with the same initial state and vice versa. Clearly, it may be possible to find ρ such that the *pole-shifted* system $\Sigma(A+\rho I,B,C,D)$ is passive even $\Sigma(A,B,C,D)$ is not passive. Hence, above mentioned correspondence ensures us to apply Theorem 3.3 to systems which can be made passive by pole shifting. In what follows, this class of systems will be introduced.

Definition 4.1 The quadruple (A,B,C,D) is said to be *passifiable by pole shifting* if there exists $\rho \in \mathbb{R}$ such that $\Sigma(A+\rho I,B,C,D)$ is passive.

Next, we give necessary and sufficient conditions for passifiability by pole shifting in the following theorem.

Theorem 4.2 *Consider a matrix quadruple (A,B,C,D) satisfying Assumption 3.2. Let E be such that* $\ker E = \{0\}$ *and* $\operatorname{im} E = \ker\,(D+D^\top)$. *Then (A,B,C,D) is passifiable by pole shifting if and only if D is nonnegative definite and $E^\top CBE$ is symmetric positive definite.*

In the light of the discussion preceding Definition 4.1, we can extend the well-posedness results presented in Theorem 3.3 to the class of passifiable systems.

Corollary 4.3 *Theorem 3.3 still holds if $\Sigma(A,B,C,D)$ is passifiable by pole shifting rather than passive.*

5 Continuity of behaviors

In this section, we will present some continuity results for linear complementarity systems. In this respect, only spesific approximations are admissible.

Definition 5.1 The sequence $\{(A_\epsilon,B_\epsilon,C_\epsilon,D_\epsilon)\}$ is said to be *an admissible approximation of* (A,B,C,D) if the following conditions hold.

1. D_ϵ is positive definite for all sufficiently small ϵ.
2. $\{(A_\epsilon,B_\epsilon,C_\epsilon,D_\epsilon)\}$ converges to (A,B,C,D) as ϵ tends to zero.

Note that the positive definiteness of D_ϵ implies passifiability by pole shifting. Also note that approximating linear complementarity systems have unique solution for all initial states.

Now we can present the main result of this section.

Theorem 5.2 *Consider a matrix quadruple (A,B,C,D) satisfying Assumption 3.2. Suppose that $\Sigma(A,B,C,D)$ is passifiable by pole shifting. Let $T > 0$ and a regular initial state of LCS(A,B,C,D) x_0 be given. Also let $\{(A_\epsilon,B_\epsilon,C_\epsilon,D_\epsilon)\}$ be an admissible approximation of (A,B,C,D) and let $(u^\epsilon,x^\epsilon,y^\epsilon)$ be the unique solution of LCS$(A_\epsilon,B_\epsilon,C_\epsilon,D_\epsilon)$ on $[0,T]$ with the initial state x_0. If $\{u^\epsilon\}$ is bounded then $\{x^\epsilon\}$ converges (strongly) to x and $\{(u^\epsilon,y^\epsilon)\}$ converges weakly to (u,y) in $\mathcal{L}_2$-sense as ϵ tends to zero.*

As illustrated in the following example, not all admissible approximations produce bounded u-trajectories.

Example 5.3 Consider the linear complementarity system LCS(A,B,C,D)given by

$$\begin{aligned}
\dot{x}_1 &= u_1\\
\dot{x}_2 &= u_2\\
y_1 &= x_1\\
y_2 &= x_2\\
0 \leq u &\perp y \geq 0
\end{aligned}$$

and the approximating systems LCS$(A_\epsilon,B_\epsilon,C_\epsilon,D_\epsilon)$given by

$$\begin{aligned}
\dot{x}_1^\epsilon &= u_1^\epsilon\\
\dot{x}_2^\epsilon &= u_2^\epsilon\\
y_1^\epsilon &= x_1^\epsilon - \epsilon x_2^\epsilon + \epsilon^k u_1^\epsilon\\
y_2^\epsilon &= -\epsilon x_1^\epsilon + x_2^\epsilon + \epsilon^k u_2^\epsilon\\
0 \leq u^\epsilon &\perp y^\epsilon \geq 0.
\end{aligned}$$

It is easy to see that the above approximations are admissible. The unique solution $(u^\epsilon,x^\epsilon,y^\epsilon)$ of

LCS$(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ with the initial state $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ can be computed as

$$(u^\epsilon, x^\epsilon) = \left(\begin{bmatrix} \epsilon^{-k_1} e^{-\epsilon^{-k}t} \\ 0 \end{bmatrix}, \begin{bmatrix} -\epsilon e^{-\epsilon^{-k}t} + \epsilon \\ 1 \end{bmatrix} \right)$$

$$y^\epsilon = \begin{pmatrix} 0 \\ 1 - \epsilon^2 + \epsilon^2 e^{-\epsilon^{-k}t} \end{pmatrix}$$

One can check that

$$\|u_1^\epsilon\|^2 = \frac{\epsilon^{-k+2}}{2}(1 - e^{-2\epsilon^{-k}T})$$

on a given interval $[0, T]$. Consequently, $\{u^\epsilon\}$ is not bounded unless $k \leq 2$.

6 Examples

We consider two types of approximation schemes in this section. It will be shown that these two schemes are addmissible approximations. Consider a quadruple (A, B, C, D) satisfying Assumption 3.2, and suppose that $\Sigma(A, B, C, D)$ is passive.

For the first scheme in Figure 6, it can be verified that the overall system can be written as LCS$(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ where $(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon) = (A, B, C, D + \epsilon I)$. Since $\Sigma(A, B, C, D)$ is passive, $D_\epsilon > 0$ for all $\epsilon > 0$. Besides, $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ converges to (A, B, C, D) as ϵ tends to zero. Therefore, $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ is an admissible approximation. For the second scheme, one can check that

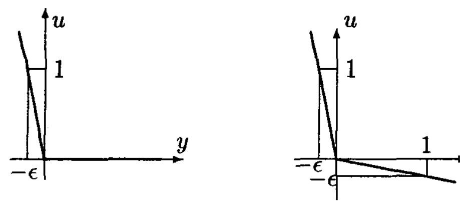

Figure 2: Examples of characteristics that give admissible approximations

the overall system can be rewritten as LCS$(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ where $A_\epsilon = A - \epsilon B(I + \epsilon D)^{-1}C$, $B_\epsilon = B(I + \epsilon D)^{-1}$, $C_\epsilon = (1-\epsilon^2)(I+\epsilon D)^{-1}C$, and $D_\epsilon = (\epsilon I + D)(I + \epsilon D)^{-1}$. It can be verified that $D_\epsilon > 0$ for all $\epsilon > 0$. Since $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ converges to (A, B, C, D) as ϵ tends to zero, it follows that $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ is an admissible approximation of (A, B, C, D).

Then, Theorem 5.2 imply that the trajectories of these approximating systems converge to those of LCS(A, B, C, D) in both cases provided that u-trajectories are bounded.

7 Irregular initial states

So far, what has been done is to investigate the convergence of the solutions, only those with a regular initial state of the limit system, of approximating systems. Although the limit system does not have solutions with the irregular initial states, the admissible approximations have. Then, it is natural to raise the question if and in what sense the approximating solutions with irregular initial states converge. By means of the following example, we will illustrate that different approximations may yield different limits in this case.

Example 7.1 Consider the LCS(A, B, C, D) given by

$$\begin{aligned} \dot{x}_1 &= u_1 + 2u_2 \\ \dot{x}_2 &= 2u_1 + u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \\ 0 \leq u &\perp y \geq 0, \end{aligned}$$

the approximating systems LCS$(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ given by

$$\begin{aligned} \dot{x}_1^\epsilon &= u_1^\epsilon + 2u_2^\epsilon \\ \dot{x}_2^\epsilon &= 2u_1^\epsilon + u_2^\epsilon \\ y_1^\epsilon &= x_1^\epsilon + \epsilon u_1^\epsilon \\ y_2^\epsilon &= x_2^\epsilon + \epsilon u_2^\epsilon \\ 0 \leq u^\epsilon &\perp y^\epsilon \geq 0 \end{aligned}$$

and LCS$(A_\mu, B_\mu, C_\mu, D_\mu)$ given by

$$\begin{aligned} \dot{x}_1^\mu &= u_1^\mu + 2u_2^\mu \\ \dot{x}_2^\mu &= 2u_1^\mu + u_2^\mu \\ y_1^\mu &= x_1^\mu + \mu u_1^\mu + 2\mu u_2^\mu \\ y_2^\mu &= x_2^\mu + 2\mu u_1^\mu + \mu u_2^\mu \\ 0 \leq u^\mu &\perp y^\mu \geq 0. \end{aligned}$$

Evidently, both $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ and $\{(A_\mu, B_\mu, C_\mu, D_\mu)\}$ qualify as admissible approximations of (A, B, C, D). Let $(u^\epsilon, x^\epsilon, y^\epsilon)$ and (u^μ, x^μ, y^μ) denote the solutions of LCS$(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ and LCS$(A_\mu, B_\mu, C_\mu, D_\mu)$ with the initial state $x_0 = \begin{bmatrix} -5 & -1 \end{bmatrix}^\top$. It can be checked that both $\{u^\epsilon\}$ and $\{u^\mu\}$ are convergent in the distributional sense. Indeed, they converge to $\begin{bmatrix} 3 - \frac{4\sqrt{2}}{3\sqrt{3}} & \frac{4\sqrt{2}}{3\sqrt{3}} - 1 \end{bmatrix}^\top \delta$ and $\begin{bmatrix} \frac{5}{2} & 0 \end{bmatrix}^\top \delta$, respectively. The fact that these approximations converge to different limits naturally weakens the power of ideal modeling in this context. In fact, it shows that the ideal model cannot capture the *fast dynamics* of the actual system.

8 Conclusions

We have considered linear complementarity systems described by linear time invariant systems coupled to ideal diode type complementarity characteristics. It is known that these systems possess unique solutions if the underlying linear system is passive. For these systems, it has been shown that the solutions of the system obtained by approximating the complementarity characteristic by a smoother Lipschitzian characteristic converge to the solution of the complementarity system as the approximating characteristics get closer to the complementarity one.

Motivated by the relation between the solutions of a linear complementarity system and its pole-shifted version, we have introduced the notion of passifiability by pole shifting. After establishing necessary and sufficient conditions under which a given linear system can be made passive by pole shifting, the same convergence result has been proved for such systems.

Appendix: Proofs

To prove Theorem 4.2, we need the following technical lemma.

Lemma 8.1 *Let $A,\ B \in \mathbb{R}^{m\times n}$ and let A be of full row rank. Then, there exists a symmetric positive definite matrix X such that $AX = B$ if and only if $BA^\top$ is symmetric positive definite.*

Proof *only if:* Postmultiplying $AX = B$ by $A^\top$, we get $AXA^\top = BA^\top$. Since $X = X^\top > 0$, $BA^\top = AB^\top > 0$.

if: Note that A can be written as $A = \begin{bmatrix} I & 0 \end{bmatrix} V$ for some nonsingular $V \in \mathbb{R}^{n\times n}$. Postmultiplying both sides of $AX = B$ by $V^\top$ and defining $Y := VXV^\top$, we get $\begin{bmatrix} I & 0 \end{bmatrix} Y = BV^\top$. Clearly, finding a solution to the latter equation with $Y = Y^\top > 0$ is equivalent to finding a solution to $AX = B$ with $X = X^\top > 0$. Let Y and $BV^\top$ be partitioned as follows:

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \qquad BV^\top = \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

To satisfy $\begin{bmatrix} I & 0 \end{bmatrix} Y = BV^\top$, we can take $Y_{12} = B_2$ and $Y_{11} = B_1 = BV^\top \begin{bmatrix} I & 0 \end{bmatrix}^\top = BA^\top$. Hence, by the hypothesis $Y_{11} = Y_{11}^\top > 0$. It remains to determine Y_{21} and Y_{22} in such a way that $Y = Y^\top > 0$. Choose $Y_{21} = Y_{12}^\top$ and $Y_{22} = I + Y_{12}^\top Y_{11}^{-1} Y_{12}$. Then, it follows from

$$Y = \begin{bmatrix} I & 0 \\ Y_{12}^\top Y_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} - Y_{12}^\top Y_{11}^{-1} Y_{12} \end{bmatrix} \times \begin{bmatrix} I & Y_{11}^{-1} Y_{12} \\ 0 & I \end{bmatrix}$$

that $Y = Y^\top > 0$. ■

Proof of Theorem 4.2 *if:* Since both E and B are of full column rank, the equation $E^\top C = E^\top B^\top K$ has a symmetric positive definite solution K according to Lemma 8.1. Define $\mu = \lambda_{max}(K)$. Let F be such that $\ker F = \{0\}$ and $\mathrm{im}\ F = (\mathrm{im}\ E)^\perp$. Note that $\mathrm{im}\ E \oplus \mathrm{im}\ F = \mathbb{R}^m$. Clearly, $F^\top DF$ is nonnegative definite. Suppose that $v^\top F^\top DFv = 0$, i.e., $(D + D^\top)Fv = 0$. This means that $Fv \in \mathrm{im}\ E$. It is easy to see that $v = 0$. Hence, we can conclude that $F^\top DF$ is positive definite. Define $\alpha = \frac{1}{2\mu}\lambda_{max}(A^\top K + KA)$, $\beta = \frac{1}{2\mu}\|KBF - C^\top F\|$ and $\gamma = -\frac{1}{2\mu}\lambda_{min}(F^\top(D + D^\top)F)$. Note that $\gamma < 0$. Take $\rho \le \frac{\beta^2}{\gamma} - \alpha$ and note that $\begin{bmatrix} \alpha+\rho & \beta \\ \beta & \gamma \end{bmatrix}$ is nonpositive definite. It can be verified that $(A + \rho I, B, C, D)$ is passive with the storage function $V(x) = x^\top Kx$. Indeed,

$$\begin{aligned} \begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{K}\left(\begin{smallmatrix} A+\rho I & B \\ C & D \end{smallmatrix}\right) \begin{bmatrix} x \\ u \end{bmatrix} &= x^\top(A^\top K + KA)x + 2\rho x^\top Kx \\ &\quad + 2x^\top(KB - C^\top)u - u^\top(D + D^\top)u \\ &= x^\top(A^\top K + KA)x + 2\rho x^\top Kx \\ &\quad + 2x^\top(KB - C^\top)Fu_f - u_f^\top F^\top(D + D^\top)Fu_f \end{aligned}$$

where $u = Eu_e + Fu_f$. From the Rayleigh-Ritz (see e.g. [9, Theorem 5.2.2.2]) and Cauchy-Schwarz inequalities, we get

$$\begin{aligned} \begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{K}\left(\begin{smallmatrix} A+\rho I & B \\ C & D \end{smallmatrix}\right) \begin{bmatrix} x \\ u \end{bmatrix} &\le \lambda_{max}(A^\top K + KA)\|x\|^2 \\ &\quad + 2\rho\lambda_{max}(K)\|x\|^2 + 2\|KBF - C^\top F\|\|u_f\|\|x\| \\ &\quad - \lambda_{min}(F^\top(D + D^\top)F)\|u_f\|^2 \\ &\le 2\mu \begin{bmatrix} \|x\| \\ \|u_f\| \end{bmatrix}^\top \begin{bmatrix} \alpha+\rho & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \|x\| \\ \|u_f\| \end{bmatrix} \le 0. \end{aligned}$$

Since K is positive definite and minimality of (A, B, C) implies that $(A + \rho I, B, C)$ is also minimal, we can conclude that $(A + \rho I, B, C, D)$ is passive due to Lemma 2.2.

only if: If (A, B, C, D) is passifiable by pole shifting then there exist a $\rho \in \mathbb{R}$ and $K = K^\top > 0$ such that $\mathcal{K}\left(\begin{smallmatrix} A+\rho I & B \\ C & D \end{smallmatrix}\right)$ is nonpositive definite. It follows that D is nonnegative definite and $(KB - C^\top)E = 0$. The latter together with the hypothesis that B is of full column rank implies that $E^\top CBE$ is symmetric positive definite because $E^\top CBE = E^\top B^\top KBE$. ■

Proof of Theorem 5.2 Without loss of generality, we can assume that $\Sigma(A, B, C, D)$ is passive due to the discussion preceeding Definition 4.1. Since u^ϵ is bounded for all sufficiently small ϵ, it has a weakly convergent subsequence, say $\{u^{\epsilon_k}\}$. Let u be the weak limit of this subsequence. Define the operators

- $(\mathcal{T}v)(t) = \int_0^t e^{A(t-s)}Bv(s)\,ds$,,
- $(\mathcal{T}_\epsilon v)(t) = \int_0^t e^{A_\epsilon(t-s)}B_\epsilon v(s)\,ds$,
- $(\mathcal{S}_\epsilon v)(t) = D_\epsilon v(t)$.

It can be verified that

- $\mathcal{S}_\epsilon$ is nonnegative definite for all sufficiently small ϵ,

- $\mathcal{T}$ is a compact operator,
- $\{S_{\epsilon_k} u^{\epsilon_k}\}$ converges to Du,
- $\{C_{\epsilon_k} \mathcal{T}_{\epsilon_k} u^{\epsilon_k} - C\mathcal{T} u^{\epsilon_k}\}$ converges to zero,
- $\{e^{A_\epsilon \cdot} x_0\}$ converges to $e^{A\cdot} x_0$.

Therefore, [2, Theorem 6.9] implies that

- $\{x^{\epsilon_k}\}$ converges (strongly) to x where $x = \mathcal{T}u$,
- $\{y^{\epsilon_k}\}$ converges weakly to $y := Cx + Du$,
- (u, x, y) is a solution of LCS(A, B, C, D) on $[0, T]$ with the initial state x_0.

We already know from Theorem 3.3 that this solution is unique. Then, it follows from [2, Lemma 6.1 item 2] that not only a subsequence of $\{u^\epsilon\}$ but $\{u^\epsilon\}$ itself converges weakly to u as ϵ tends to zero.■

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