From Low-distortion Norm Embeddings to Explicit Uncertainty Relations and Efficient Information Locking

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Encryption of a classical message

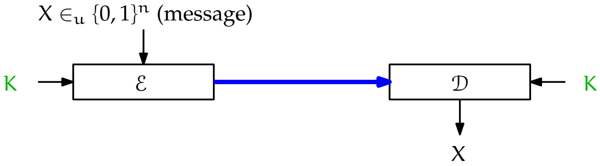
Resources

Shared secret key $\mathsf{K} \in_u \{0,1\}^s$

Public communication channel
classical or quantum

Task

Transmit X to Bob



- Bob: K known $\rightarrow$ Decode $\mathcal{E}(X, K)$ using K to get X

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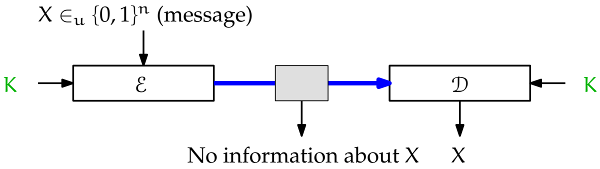

- Bob: K known $\rightarrow$ Decode $\mathcal{E}(X, K)$ using K to get X
- Eve: K unknown $\rightarrow$ $\mathcal{E}(X, K)$ gives no information about X

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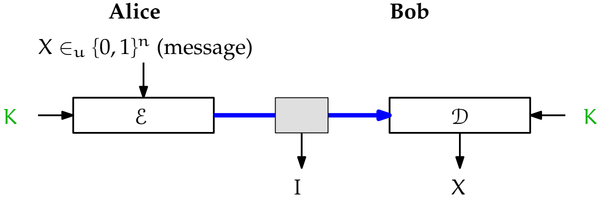

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Encryption of a classical message

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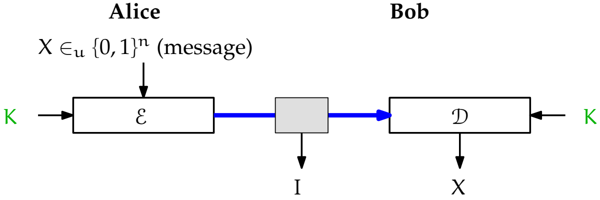

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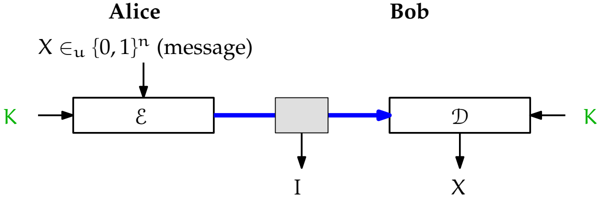

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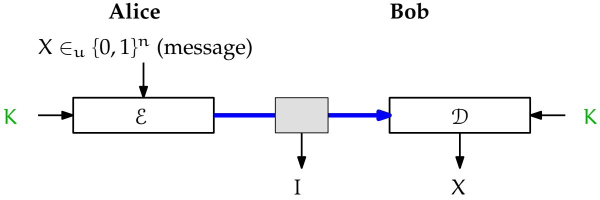

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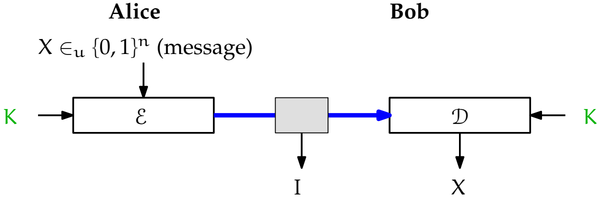

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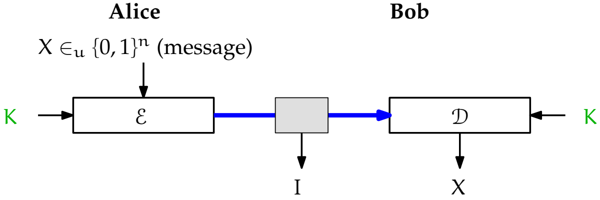

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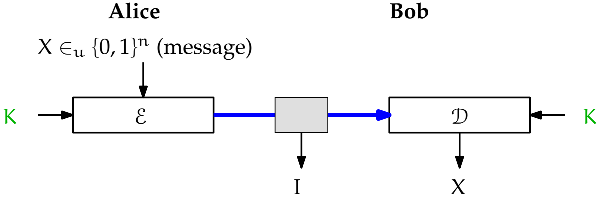

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There exists $\mathcal{E}, \mathcal{D}$ with $s = 3\log(1/\epsilon)$

There exists $\mathcal{E}, \mathcal{D}$ efficient quantum circuits with $s = O(\log(n/\epsilon))$

Outline

1. Metric uncertainty relations: definition and applications
    - Definition
    - Application: Encryption
    - Application: Quantum equality testing
2. Metric uncertainty relations: constructions
    - Known constructions
    - Metric interpretation
    - Efficient metric uncertainty relation

Outline

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Uncertainty relations

Property of:

- A set of measurements $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{t-1}\}$ (bases here)
- **Notational convenience:** $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{t-1}\} \leftrightarrow \{U_0, U_1, \ldots, U_{t-1}\}$ where $U_k : \mathcal{B}_k \mapsto \{|x\rangle\}_{x\in\{0,1\}^n}$ fixed computational basis

$$\text{Measure } \mathcal{B}_k \quad \Longleftrightarrow \quad \text{apply } U_k \text{ and measure } \{|x\rangle\}_{x\in\{0,1\}^n}$$

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Expresses:

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Expresses:

- Uncertainty of outcome distributions $\left\{p_{U_0|\psi\rangle}, \ldots, p_{U_{t-1}|\psi\rangle}\right\}$ $\forall|\psi\rangle$
- Measurements “incompatible”

Example: $\{+, \times\} \;\leftrightarrow\; \{I, H\}$

$$p_{I|\psi\rangle} = \left[|\langle 0|I|\psi\rangle|^2, |\langle 1|I|\psi\rangle|^2\right] = \left[|\alpha|^2, |\beta|^2\right]$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$p_{H|\psi\rangle} = \left[|\langle 0|H|\psi\rangle|^2, |\langle 1|H|\psi\rangle|^2\right] = \left[\tfrac{|\alpha+\beta|^2}{2}, \tfrac{|\alpha-\beta|^2}{2}\right]$$

Incompatibility of $+$ and $\times$:

$$\text{For all } |\psi\rangle, \quad \text{uncertainty}(p_{I|\psi\rangle}) + \text{uncertainty}(p_{H|\psi\rangle}) \geqslant \text{large}$$

Quantifying uncertainty

$$\text{For all } |\psi\rangle, \quad \sum_{k=0}^{t-1} \text{uncertainty}(p_{U_k|\psi\rangle}) \geqslant \text{large}$$

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$$\text{For all } |\psi\rangle, \quad \sum_{k=0}^{t-1} \mathbf{H}(p_{U_k|\psi\rangle}) \geqslant \text{large}$$

Uncertainty:

- Entropy $\mathbf{H}(\cdot)$

Quantifying uncertainty

$$\text{For all } |\psi\rangle, \quad \sum_{k=0}^{t-1} \Delta(p_{U_k|\psi\rangle}, \text{unif}) \leqslant \text{small}$$

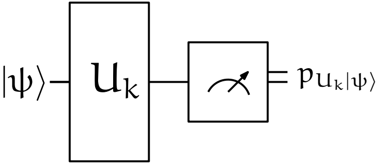
Uncertainty:

- Entropy $\mathbf{H}(\cdot)$
- Closeness to uniform $\Delta(\cdot, \text{unif})$
  (the smaller, the more uncertain)
  $\Delta(p,q) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{x\in\mathcal{X}} |p(x) - q(x)|$ total variation distance

Metric uncertainty relations

Recap of definitions:

$p_{U_k|\psi\rangle}(x) \stackrel{\text{def}}{=} |\langle x|U_k|\psi\rangle|^2$



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Definition (Metric uncertainty relation)

$\{U_0,\ldots,U_{t-1}\}$ acting on $(\mathbb{C}^2)^{\otimes n}$

For all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ $$\frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p_{U_k|\psi\rangle}, \mathrm{unif}(\{0,1\}^n)\right) \leqslant \epsilon$$

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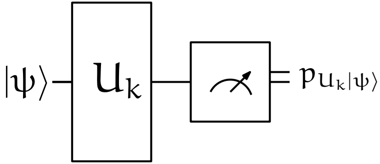

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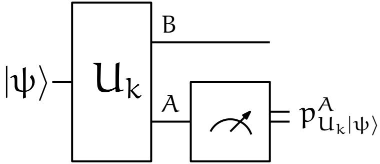
Intuition: $\forall|\psi\rangle$, for most values of k, $\Delta\left(p_{U_k|\psi\rangle}, \mathrm{unif}(\{0,1\}^n)\right) \lessapprox \epsilon$

Objectives: t, ϵ small

Metric uncertainty relations

Recap of definitions:

$$p^A_{U_k|\psi\rangle}(a) \stackrel{\text{def}}{=} \sum_{b\in\{0,1\}^{n_B}} |\langle a|^A \langle b|^B U_k|\psi\rangle|^2$$



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Definition (Metric uncertainty relation)

$\{U_0,\dots,U_{t-1}\}$ acting on $(\mathbb{C}^2)^{\otimes n} = A \otimes B$ with $A = (\mathbb{C}^2)^{\otimes n_A}$ and $B = (\mathbb{C}^2)^{\otimes n_B}$

For all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ $\quad \displaystyle\frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathrm{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$

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Objectives: t, ϵ small and n_A large

Metric and entropic uncertainty relations

Entropic uncertainty relations
Use (Shannon) entropy [Bialynicki-Birula, Mycielski, 1975; Deutsch, 1983]

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$$\mathbf{H}(p_{U_k|\psi\rangle}) \geqslant \mathbf{H}(p^A_{U_k|\psi\rangle}) \qquad \text{recall } p^A_{U_k|\psi\rangle}(a) = \sum_b p_{U_k|\psi\rangle}(a,b)$$

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Proposition (Metric UR ⇒ Entropic UR)

$U_0, \ldots, U_{t-1}$ define an ϵ-metric UR, then

$$\textit{For all } |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \qquad \frac{1}{t}\sum_{k=0}^{t-1} \mathbf{H}(p_{U_k|\psi\rangle}) \geqslant (1-2\epsilon)n_A - \eta(\epsilon)$$

Proof: Fannes' inequality □

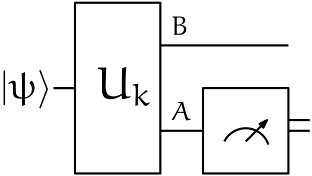
Metric uncertainty relations: parameters

Theorem (Metric uncertainty relations)

$\exists\, U_0, \dots, U_{t-1}$ *acting on* $(\mathbb{C}^2)^{\otimes n} = A \otimes B$ *with*

	$\log t$	n_A
Non-explicit	$3 \log(1/\epsilon)$	$n - 2\log(1/\epsilon)$
Efficient	$O(\log(n/\epsilon))$	$0.99n$
Efficient	$O(\log^2(n/\epsilon))$	$n - O(\log(n/\epsilon))$

for all $|\psi\rangle$ $$\frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathit{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon.$$



$p^A_{U_k|\psi\rangle} \approx_\epsilon \mathrm{unif}(\{0,1\}^{n_A})$
for most values of k

Encryption of classical messages

Definition (Locking scheme)

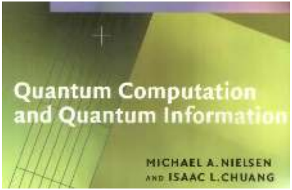
Message $X \in_u \{0,1\}^n$, key $K \in_u \{0,1\}^s$ (think $s \ll n$)

$\mathcal{E}$ is ϵ-locking scheme if:

Knowing K, can determine X using $\mathcal{E}(X,K)$

Not knowing K, for any measurement whose outcome is I: $\Delta(p_{XI}, p_X \times p_I) \leqslant \epsilon$

Composability



A QKD protocol is defined as being *secure* if, for any security parameters $s > 0$ and $\ell > 0$ chosen by Alice and Bob, and for any eavesdropping strategy, either the scheme aborts, or it succeeds with probability at least $1 - O(2^{-s})$, and guarantees that Eve's mutual information with the final key is less than $2^{-\ell}$. The key string must also be essentially random.

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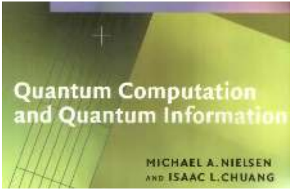

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Security of Quantum Key Distribution

A dissertation submitted to

SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH

for the degree of
tor of Natural Sciences

presented by

Renato Renner
Dipl. Phys. ETH

2.2.1 Standard security definitions are not universal

Unfortunately, many security definitions that are commonly used in quantum cryptography are not universal. For instance, the security of the key S generated by a QKD scheme is typically defined in terms of the mutual information $I(S;W)$ between S and the classical outcome W of a measurement of the adversary's system (see, e.g., [LC99, SP00, NC00, GL03, LCA05] and also the discussion in [BOHL+05] and [RK05]). Formally, S is said to be secure if, for some small ε,

$$\max_W I(S;W) \leq \varepsilon\,, \qquad (2.5)$$

where the maximum ranges over all measurements on the adversary's system with output W. Such a definition—although it looks reasonable—does, however, not guarantee that the key S can safely be used in applications. Roughly speaking, the reason for this flaw is that criterion (2.5) does not account for the fact that an adversary might wait with the measurement of her system until she learns parts of the key. (We also refer to [RK05]

Not necessarily composable!

[Ben-Or, Horodecki, Leung, Mayers, Oppenheim, 2005; Konig, Renner, Bariska, Maurer, 2007]

Information locking: History

[DiVincenzo, Horodecki, Leung, Smolin, Terhal, 2004]

- $X \in_u \{0,1\}^n$ (message) and $K \in_u \{0,1\}$ (key)
- If $K = 0$, $\mathcal{E}(x,0) = |x\rangle$
- If $K = 1$, $\mathcal{E}(x,1) = H^{\otimes n}|x\rangle$

Knowing K, can determine X

Without knowing K, for any measurement whose outcome is I: $\mathbf{I}(X;I) \leqslant n/2$

One bit of information (K) can unlock $\frac{n}{2}$ bits about X hidden in the quantum system $\mathcal{E}(X,K)$

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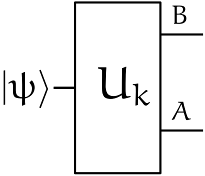
Encoding in random bases

- [Hayden, Leung, Shor, Winter, 2004] $\mathbf{I}(X; I) \leqslant 3$ with $K \in \{0,1\}^{4 \log n}$
- [Dupuis, Florjanczyk, Hayden, Leung, 2010] $\mathbf{I}(X; I) \leqslant \epsilon$ with $K \in \{0,1\}^{O(\log(n/\epsilon))}$ and stronger definition

Locking scheme from a metric uncertainty relation

$\{U_k\}$ satisfies metric uncertainty relation



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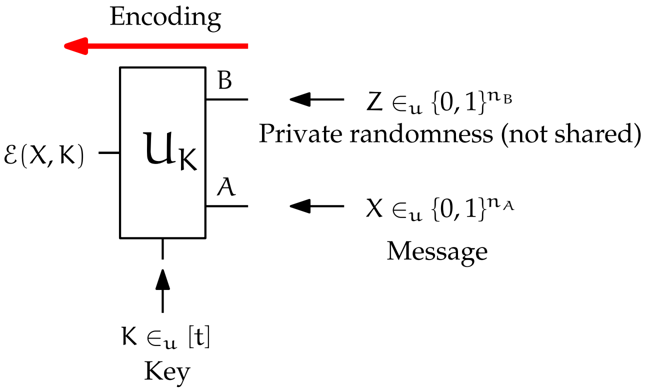

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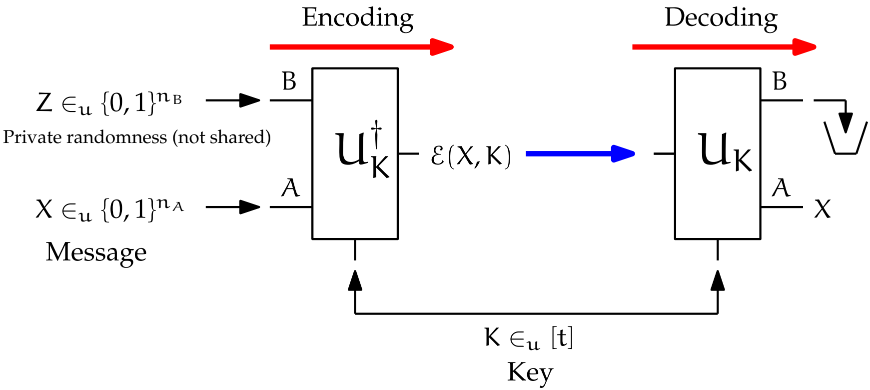
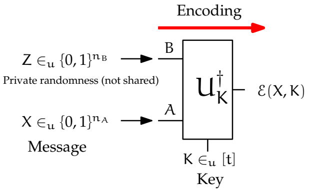


Locking scheme from a metric UR: proof

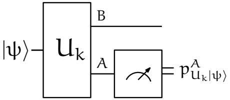
For $a \in \{0,1\}^{n_A}$ and $k \in [t]$

$$\mathcal{E}(a,k) = U_k^\dagger \left(|a\rangle\langle a|^A \otimes \frac{\mathbb{I}^B}{2^{n_B}} \right) U_k$$



- Can assume measurement $\{\xi_i |e_i\rangle\langle e_i|\}_i$
- Outcome I
- Unknown K:

$$\mathbf{P}\{X = a | I = i\} = \frac{1}{t} \sum_{k=0}^{t-1} p^A_{U_k|e_i\rangle}(a)$$



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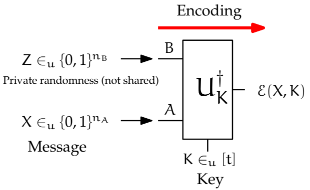

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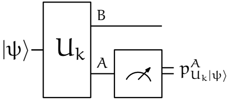

By definition of metric UR: $\Delta\left(\frac{1}{t}\sum_{k=0}^{t-1} p^A_{U_k|e_i\rangle}, \mathrm{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$

$\Rightarrow \quad \Delta\left(p_{X|I=i}, \mathrm{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$ for any i □

Parameters of locking scheme

Theorem

There exists ϵ-locking schemes

	Bits of key	*Qubits of $\mathcal{E}(x,k)$*
Non-explicit	$5\log(1/\epsilon)$	n
Efficient	$O(\log(n/\epsilon))$	$1.01n$
Efficient	$O(\log^2(n/\epsilon))$	n

	Inf. leakage	Key	Ciphertext	Efficient ?
[DHLST04]	$n/2$	1	n	yes
[HLSW04]	3	$4\log(n)$	n	no
[DFHL10]	ϵn	$2\log(n/\epsilon^2)$	n	no
I	ϵn	$5\log(1/\epsilon)$	n	no
II	ϵn	$O(\log(n/\epsilon))$	$1.01n$	yes
III	ϵn	$O(\log^2(n/\epsilon))$	n	yes

Note: Can take $\epsilon = \eta/n$

Another application: Quantum equality testing

Quantum identification or **approximate measurement simulation**

	Alice	**Bob**
Inputs	$\lvert\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$	description of $\lvert\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$
Ouput		YES with prob $\lvert\langle\psi\lvert\phi\rangle\rvert^2 \pm \epsilon$
		NO with prob $1 - \lvert\langle\psi\lvert\phi\rangle\rvert^2 \pm \epsilon$
Objective	Minimize quantum communication	

Relaxation of quantum info transmission [Winter, 2004]

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Classical equality testing or identification

	Alice	Bob
Inputs	$x \in \{0,1\}^n$	$y \in \{0,1\}^n$
Ouput		YES with prob $\mathbf{1}_{x=y} \pm \epsilon$
		NO with prob $\mathbf{1}_{x\neq y} \pm \epsilon$
Objective	Minimize classical communication	

Communication complexity EQUALITY

Remark: Communication is one way

Quantum equality testing

	Alice	**Bob**
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Using free classical communication

- *There exists a protocol using $O(\log(1/\epsilon))$ qubits communication*
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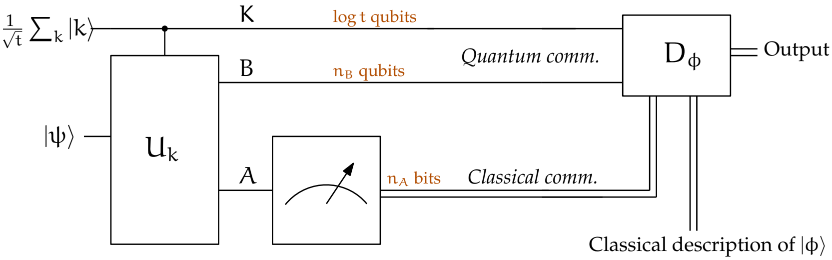
Classical equality testing:

- With free shared randomness: $O(\log(1/\epsilon))$ bits communication
- Public-coin randomized comm. complexity of EQUALITY is $O(\log(1/\epsilon))$

From metric UR to quantum equality testing



Quantum communication: $\log t + n_B$ qubits
Classical communication: n_A bits

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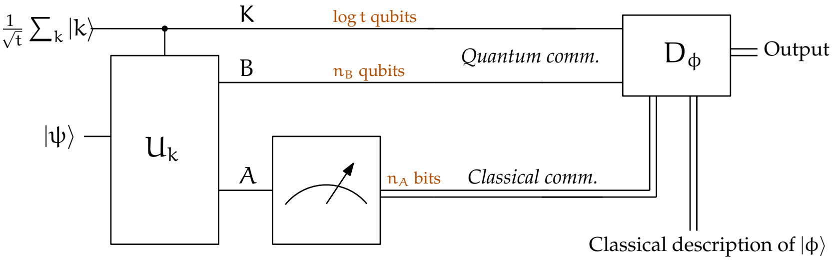

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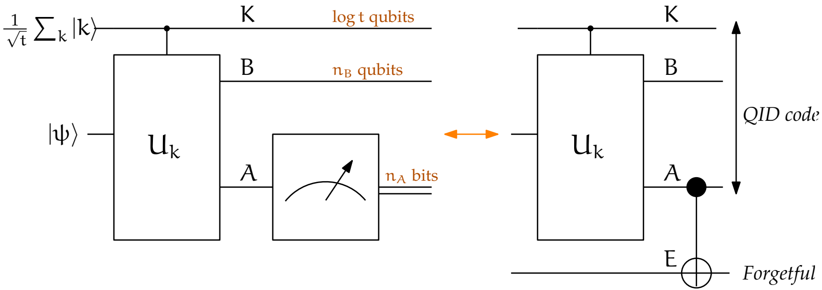

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Outline

Entropic URs with $t = 2$ measurements

Rectilinear and diagonal basis

- $I, H^{\otimes n}$

$$\frac{1}{2}\left(\mathbf{H}(p_{|\psi\rangle}) + \mathbf{H}(p_{H^{\otimes n}|\psi\rangle})\right) \geqslant \frac{1}{2}n$$

- U_0, U_1 mutually unbiased: $\forall x, y \in \{0,1\}^n \; |\langle x|U_0 U_1^\dagger|y\rangle|^2 = \frac{1}{2^n}$

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[Maassen, Uffink, 1989]

Recall: $p_{|\psi\rangle}(x) = |\langle x|\psi\rangle|^2$

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To increase the lower bound, need $t > 2$ measurements

Entropic URs with $t > 2$ measurements

Want: $$\frac{1}{t}\sum_{k=0}^{t-1} \mathbf{H}(p_{U_k|\psi\rangle}) \geqslant h(t) \qquad \text{for all } |\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$$

with $h(t) > n/2$ large

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- For $t = 2^n + 1$ (full set of MUBs):
  $h(t) \geqslant \log(2^n + 1) - 1 \geqslant n - 1$ [Sanchez, 1993; Ivanovic, 1994]

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- For $t = 2^n + 1$ (full set of MUBs):
  $h(t) \geqslant \log(2^n + 1) - 1 \geqslant n - 1$ [Sanchez, 1993; Ivanovic, 1994]
- For $t < 2^{n/2}$, general MUBs do not work well:
  $\exists t$ MUBs with $h(t) \approx n/2$ [Ballester and Wehner, 2007; Ambainis, 2009]

Entropic URs with $t > 2$ measurements

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with $h(t) > n/2$ large

Other candidate: random bases [Hayden, Leung, Shor, Winter, 2004]
For $t = n^4$, there exists $U_0, \ldots, U_{t-1}$

$$\frac{1}{t}\sum_{k=0}^{t-1} \mathbf{H}(p_{U_k|\psi\rangle}) \geqslant n - 3$$

Remark: Not explicit

Metric URs: metric interpretation

Definition (Metric uncertainty relation)

For all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ $\quad \frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathrm{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$

In terms of fidelity

$1 - \epsilon \leqslant \frac{1}{t}\sum_k F\left(p^A_{U_k|\psi\rangle}, \mathrm{unif}(\{0,1\}^{n_A})\right)$

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Define $\qquad V : |\psi\rangle \mapsto \frac{1}{\sqrt{t}}\sum_k |k\rangle \otimes U_k|\psi\rangle$

For all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, $\qquad \| V|\psi\rangle \|_1 \geqslant (1-\epsilon)\sqrt{t2^n} \| |\psi\rangle \|_2$

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$$\text{For all } |\psi\rangle \in (\mathbb{C}^2)^{\otimes n}, \quad \sqrt{t2^n}\||\psi\rangle\|_2 \geqslant \|V|\psi\rangle\|_1 \geqslant (1-\epsilon)\sqrt{t2^n}\||\psi\rangle\|_2$$

V is a low-distortion embedding $(\mathbb{C}^{2^n}, \ell_2) \hookrightarrow (\mathbb{C}^{t2^n}, \ell_1)$

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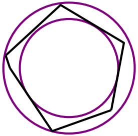
For $|\psi\rangle \in A \otimes B$, $\quad \||\psi\rangle\|_{\ell_1^A(\ell_2^B)} = \sum_{a\in\{0,1\}^{n_A}} \|\langle a|\psi\rangle\|_2$

$\ell_2 \hookrightarrow \ell_1$ embeddings

Dvoretzky's theorem:
For any normed space $(\mathbb{R}^d, \|\cdot\|)$, there is a large subspace $\|\cdot\| \approx_\epsilon \|\cdot\|_2$
[Dvoretzky, 1961; Milman, 1971; Milman and Schechtman, 1986;...]



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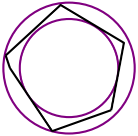

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For ℓ_1 norm

- Explicit constructions [Indyk, 2007; Guruswami, Lee, Razborov, 2009;...]
- Applications: high-dimensional nearest neighbour search and compressed sensing

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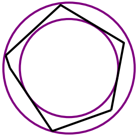

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- Explicit constructions [Indyk, 2007; Guruswami, Lee, Razborov, 2009;...]
- Applications: high-dimensional nearest neighbour search and compressed sensing

For Schatten p-norms [Aubrun, Szarek, Werner, 2010]

- Counterexample additivity minimum output entropy [Hayden and Winter 2008; Hastings, 2009]

Metric uncertainty relations: existence

Theorem (Metric uncertainty relations)

$\exists\, U_0, \dots, U_{t-1}$ acting on $(\mathbb{C}^2)^{\otimes n} = A \otimes B$ with

$$\log t = 3 \log(1/\epsilon) \qquad \textit{and} \qquad n_A = n - 2\log(1/\epsilon)$$

$$\textit{for all } |\psi\rangle \qquad \frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathit{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon.$$

Proof: Probabilistic argument, $U_0, \dots, U_{t-1}$ at random [Milman, 1971]

Efficient metric UR: Structure of the construction

Use ideas of explicit ℓ_2 into ℓ_1 embedding of [Indyk, 2007]

Two ingredients:

1. Min-entropy uncertainty relation (mutually unbiased bases)
2. Permutation extractors

Min-entropy uncertainty relation

Lemma (MUBs define min-entropy uncertainty relations)

$V_0, \ldots, V_{r-1}$ define MUBs with $r = 1/\epsilon^2$, for all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$

$$\frac{1}{r}\sum_{j=0}^{r-1} \mathbf{H}_{\min}^{\epsilon}(p_{V_j|\psi\rangle}) \gtrapprox (1-\epsilon)n/2$$

$$\mathbf{H}_{\min}(p) = -\log \max_{x \in \mathcal{X}} p(x)$$

$$\mathbf{H}_{\min}^{\epsilon}(p) = \max_{q : \Delta(p,q) \leqslant \epsilon} \mathbf{H}_{\min}(q)$$

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Remarks

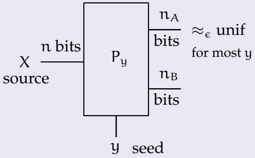
- Interpret as: for most values of j, $\mathbf{H}_{\min}^{\epsilon}(p_{V_j|\psi\rangle}) \gtrapprox (1-\epsilon)n/2$
- Min-entropy UR of [Damgaard, Fehr, Renner, Salvail, Schaffner, 2007] uses $r = 2^n$ bases
- Rate $1/2$ is best possible

Permutation extractors

Definition (Strong permutation extractor)

$P_0, \ldots, P_{s-1}$ permutations of $\{0,1\}^n$

$\mathbf{H}_{\min}(X) \geqslant \ell$



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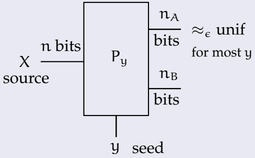
$$\mathbf{H}_{\min}(X) \geqslant \ell \Rightarrow \frac{1}{s}\sum_{y=0}^{s-1} \Delta\left(P_y^A(X), \mathrm{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$$



$P^A(x)$: first n_A bits of $P(x)$

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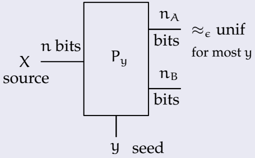

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Remarks:

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- Want efficient P_y and P_y^{-1}

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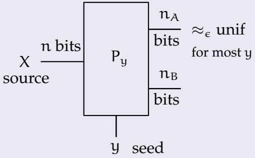

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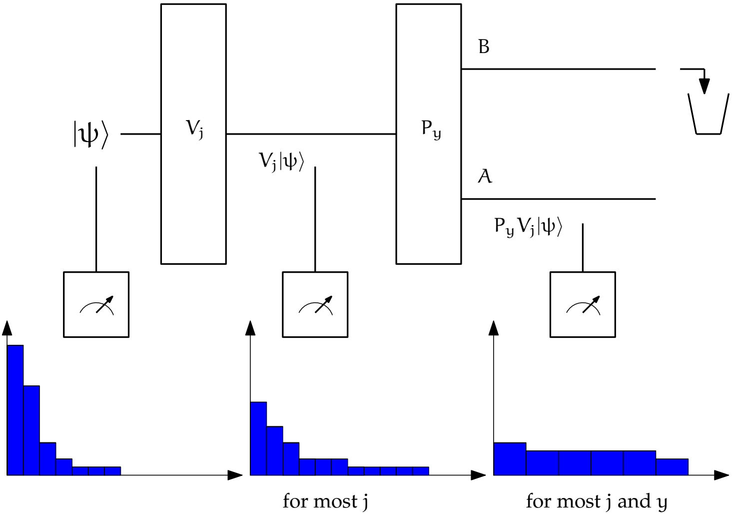
Adapting [Guruswami, Umans, Vadhan, 2009]

Theorem

$\exists$ *efficient strong perm. extractor with* $\log s = O(\log(n/\epsilon))$ *and* $n_A = (1-\delta)\ell$

Putting things together



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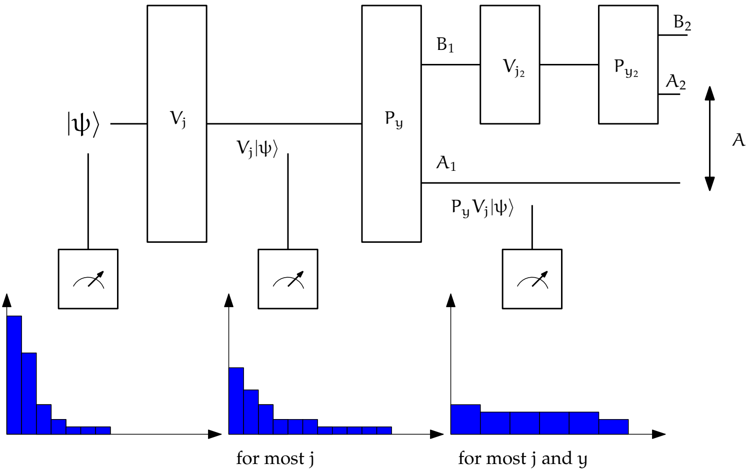

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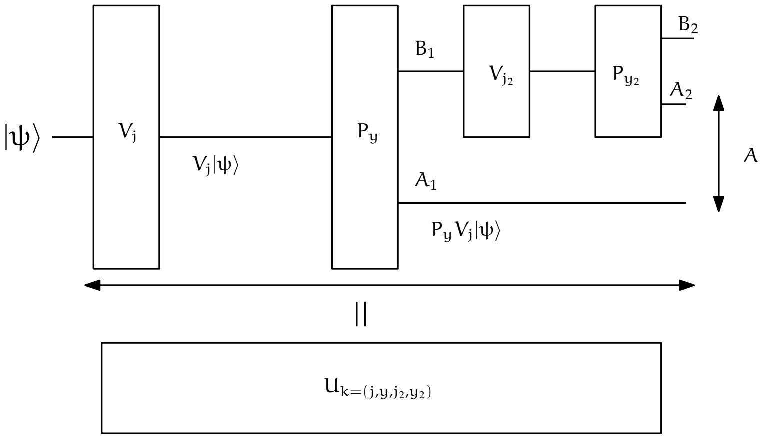

Parameters of the metric uncertainty relation

Theorem (Efficient MURs: key optimized)

$\exists U_0, \ldots, U_{t-1}$ *with* $\log t = c_\delta \log(n/\epsilon)$ *and* $n_A = (1-\delta)n$

$$\textit{For all } |\psi\rangle, \qquad \frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathit{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$$

$U_0, \ldots, U_{t-1}$ *have quantum circuits of size* $O(n \operatorname{polylog}(n/\epsilon))$

Theorem (Efficient MURs: A system maximized)

$\exists U_0, \ldots, U_{t-1}$ *with* $\log t = c \log^2(n/\epsilon)$ *and* $n_A = n - O(\log(1/\epsilon) + \log\log n)$

$$\textit{For all } |\psi\rangle, \qquad \frac{1}{t}\sum_{k=0}^{t-1} \Delta\left(p^A_{U_k|\psi\rangle}, \mathit{unif}(\{0,1\}^{n_A})\right) \leqslant \epsilon$$

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Summary

Inspired by definitions and results in asymptotic geometric analysis:

- Define metric uncertainty relations
- Prove random bases satisfy URs with better params
- Construct efficient metric URs
- First efficient strong information locking schemes
  - One of the schemes uses only Hadamard gates and classical computation
- Quantum equality testing
- Other results in paper:
  - Quantum hiding fingerprint [Gavinsky, Ito, 2010]
  - String commitment protocol [Buhrman, Christandl, Hayden, Lo, Wehner, 2006]

Open questions

- Other cryptographic applications? Bounded/noisy storage model?
- Explicit constructions of UR matching probabilistic argument?
- Existence results of UR matching lower bounds? Are there $U_0, \dots, U_{t-1}$

$$\frac{1}{t}\sum_{k=0}^{t-1} \mathbf{H}(p_{U_k|\psi\rangle}) \geqslant \left(1 - \frac{1}{t}\right) n \quad \text{for } t > 2?$$

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Thank you!

`arXiv:1010.3007`

See also `arXiv:1011.1612` [Dupuis, Florjancyk, Hayden, Leung, 2010]

Many thanks to Ivan Savov for comments on the presentation

Extra: Proof of min-entropy uncertainty relation

Lemma (MUBs define min-entropy uncertainty relations)

For "most" values of j, there exists q_j s.t. $\Delta\left(p_{V_j|\psi\rangle}, q_j\right) \leqslant \epsilon$ *and* $q_j(x) \lessapprox 2^{-n/2}$

Proof:

$$\vec{v} = \begin{bmatrix} V_0 \\ \vdots \\ V_{r-1} \end{bmatrix} |\psi\rangle \in \mathbb{C}^{r2^n} \qquad \vec{v}_{j,x} = \langle x|V_j|\psi\rangle \qquad V = \begin{bmatrix} V_0 \\ \vdots \\ V_{r-1} \end{bmatrix} \in \mathbb{C}^{r2^n \times 2^n}$$

1. $\vec{v}$ is spread: for any $|S| \leqslant 2^{n/2}$, $\|\vec{v}_S\|_2^2 \leqslant \frac{2}{r}\|\vec{v}\|_2^2$
   - $\vec{v}_S = V_S|\psi\rangle$
   - $\|\vec{v}_S\|_2^2 = |\langle\psi|V_S^\dagger V_S|\psi\rangle| \leqslant \max$ eigenvalue of $V_S^\dagger V_S$

$$V_S^\dagger V_S = \begin{bmatrix} 1 & \langle y|V_{j'}^\dagger V_j|x\rangle & \dots \\ \langle x|V_j^\dagger V_{j'}|y\rangle & \ddots & \vdots \\ \vdots & \dots & 1 \end{bmatrix}$$

   - $\max$ eigenvalue of $V_S^\dagger V_S \leqslant 1 + |S|2^{-n/2}$ ← use MUB here

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3. Define $q_j(x) = |w_{j,x}|^2$ (recall $p_{V_j|\psi\rangle}(x) = |\vec{v}_{j,x}|^2$)
4. For "most" values of j, $q_j \approx_\epsilon$ distribution
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Extra: Min-entropy uncertainty relation (generalized)

Approximate MUB: $\forall x, y \; |\langle x|V_j V_{j'}^{\dagger}|y\rangle| \leqslant \frac{1}{2^{\gamma n/2}}$ $\quad \gamma \in [0,1]$

Lemma (Min-entropy uncertainty relations)

$V_0, \ldots, V_{r-1}$ define γ-MUBs with $r = 1/\epsilon^2$, for all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$

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Lemma (1/2-MUBs with single qubit unitaries)

There exist $V_j \in \left\{H^{u_1} \otimes H^{u_2} \otimes \cdots \otimes H^{u_n} : u_i \in \{0,1\}\right\}$ for $j \in [t]$ that define $1/2$-MUBs

H: transforms $+$ to $\times$