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From Newton's method to exotic basins
Part II: Bifurcation of the Mandelbrot-like sets

by

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Abstract. This is a continuation of the work [Ba] dealing with the family of all cubic rational maps with two supersinks. We prove the existence of the following parabolic bifurcation of Mandelbrot-like sets in the parameter space of this family. Starting from a Mandelbrot-like set in cubic Newton maps and changing parameters in a continuous way, we construct a path of Mandelbrot-like sets ending in the family of parabolic maps with a fixed point of multiplier 1. Then it bifurcates into two paths of Mandelbrot-like sets, contained respectively in the set of maps with exotic or non-exotic basins. The non-exotic path ends at a Mandelbrot-like set in cubic polynomials.

1. Introduction. This paper extends the work started in [Ba]. We study the family $\mathcal{F} = \{f_{a,b}\}$, where

$$f_{a,b}(z) = az^2 \frac{bz + 1 - 2b}{(2-b)z - 1}, \qquad a \in \mathbb{C} \setminus \{0\},\ b \in \mathbb{C} \setminus \{0, 1\}. \tag{1}$$

This family consists of cubic rational maps with two supersinks: 0 and ∞ and a critical point at 1. As one can easily check,

$$\begin{gathered} \text{the critical points of } f_{a,b} \text{ are } 0, \infty, 1, u = u_{a,b} = \frac{2b-1}{b(2-b)}; \\ f_{a,b}(0) = 0,\ f_{a,b}(\infty) = \infty,\ f_{a,b}(1) = a,\ f_{a,b}(u) = \frac{a(2b-1)^3}{b(2-b)^3}. \end{gathered} \tag{2}$$

The paper [Ba] contains a general description of $\mathcal{F}$ and its moduli space together with a detailed study of the subfamily $\mathcal{F}_2 \subset \mathcal{F}$ consisting of maps for which the critical point 1 is periodic of period 2. In particular, a parabolic bifurcation in $\mathcal{F}_2$ from a Newton map to maps with exotic and non-exotic basins is described ([Ba], Theorem 4.20). This theorem says that there exists

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a curve in the parameter plane of $\mathcal{F}_2$ joining a Newton map $N \in \mathcal{F}_2$ via maps with an attracting fixed point to a parabolic map $P \in \mathcal{F}_2$ with a fixed point of multiplier 1. Then this curve bifurcates to two curves starting at P, contained respectively in the set of maps with exotic or non-exotic basins. Recall that an exotic basin (a notion introduced by F. Przytycki [P1]) is a non-simply connected completely invariant basin of an attracting fixed point, containing k critical points counted with multiplicity, with k less than the degree of the map. (For more information we refer to [Ba].)

This paper generalizes this result, proving the existence of a parabolic bifurcation of Mandelbrot-like sets in $\mathcal{F}$ (Theorem 4.1). It is known (see [CGS], [DH2]) that in the parameter plane of cubic Newton maps $\{N_\sigma\}_{\sigma\in\mathbb{C}}$ there exist Mandelbrot-like sets quasiconformally homeomorphic to the standard Mandelbrot set. These sets correspond to some regions of parameters σ for which N_σ^k is quadratic-like on some topological disc U_σ. Given such a Mandelbrot-like set M_N, we prove the existence of a continuous path of Mandelbrot-like sets joining M_N via M_t, $t \in (0,1)$, contained in the family of maps with an attracting fixed point of multiplier t to a parabolic M_1 contained in the family of maps with a fixed point of multiplier 1. Then this path bifurcates into two paths of Mandelbrot-like sets, contained in the set of maps with exotic or non-exotic basins (see Figs. 1 and 2–5). Moreover, the non-exotic path can be extended in such a way that it terminates with a Mandelbrot-like set in the family of maps conformally conjugate to cubic polynomials with a supersink. Theorem 4.1 answers a question asked by F. Przytycki [P1] (see [Ba], Section 1).

The plan of the paper is as follows. In Section 2 we prove some technical lemmas which are used in the proof of the main theorem. They concern the limit behaviour of some invariant curves under the action of Riemann mappings. Section 3 contains definitions and properties of Mandelbrot-like families taken from [DH2]. The formulation and the proof of the main theorem (Theorem 4.1) are contained in Section 4. The proof, quite long and technical, is divided into several parts. In Subsection 4.1 we modify the initial Mandelbrot-like family in Newton maps to an equivalent family which has a nice combinatorial description. Then in Subsection 4.2 we construct the sets M_t for $t \in (0,1)$ using quasiconformal surgery. The most delicate part is to prove that M_t tends to a parabolic M_1 for $t \to 1^-$. This is done in Subsection 4.3. (The problem of obtaining a parabolic map as a limit of hyperbolic maps was studied in [Ma]. However, his theory is not applicable to our case.) In Subsection 4.4 we describe the bifurcation of M_1 into two paths by the use of the "tour de valse" method from [DH1]. In Subsection 4.5 we determine which path is exotic and in Subsection 4.6 we extend the non-exotic path in such a way that it ends in maps with a double supersink (i.e. conformally conjugate to cubic polynomials).

NOTATION. The symbols cl, ∂ and int denote respectively the closure, boundary and interior of a set, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, $\mathbb{D}_r$ is the open disc centred at 0 of radius r and $\mathbb{D}_r(x)$ is the open disc centred at x of radius r. Define also $\mathbb{D}^+ = \{z \in \mathbb{D} : \mathrm{Im}(z) > 0\}$, $\mathbb{D}^- = \{z \in \mathbb{D} : \mathrm{Im}(z) < 0\}$. By a simple arc we mean a curve homeomorphic to a line segment, and a topological disc is a set homeomorphic to $\mathbb{D}$. If $\gamma : [0, \infty) \to \widehat{\mathbb{C}}$ is a curve such that $\gamma(s) \to z$ as $s \to \infty$, then we say that γ begins at $\gamma(0)$ and lands at z. If ζ is an attracting (resp. parabolic) point for a map f, then $B(\zeta)$ denotes the immediate basin of attraction to ζ (resp. invariant parabolic basin of ζ).

REMARK. We often refer to the notions and results from [Ba], so it is advisable to read the two papers together. We also adopt some notational conventions from the first part of the work. Referring to [Ba], we write Theorem I.N.M for Theorem N.M of [Ba] etc.

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2. Convergence of Riemann mappings and invariant curves

DEFINITION 2.1. Denote by Sing the set of singular parameters $(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ for the family $\mathcal{F}$, i.e.

$$\mathrm{Sing} = \{(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : a \in \{0, \infty\} \text{ or } b \in \{0, 1, \infty\}\}.$$

For $\varrho \in \mathbb{C}$ let $\mathrm{Fix}(\varrho)$ be the set of parameters $(a, b) \in (\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}) \setminus \mathrm{Sing}$ for which $f_{a,b} \in \mathcal{F}$ has a fixed point $\xi \neq 0, \infty$ of multiplier ϱ.

REMARK. There are four fixed points of $f_{a,b}$ (counted with multiplicity): 0, ∞ and two others; denote them by ξ_1, ξ_2. Let $\varrho_i = f'_{a,b}(\xi_i)$. By the holomorphic fixed point formula (see e.g. [Mi]), if $\varrho_i \neq 1$, then

$$\frac{1}{1 - \varrho_1} + \frac{1}{1 - \varrho_2} = -1. \tag{3}$$

In other words, the value of one multiplier determines the other one and

$$\mathrm{Fix}(\varrho) = \mathrm{Fix}\left(2 + \frac{1}{\varrho - 2}\right).$$

In this section we prove two technical lemmas, which will be used in the proof of the main theorem. The first one describes some standard facts on convergence of maps from $\mathcal{F}$ and suitable Riemann mappings. Recall first the definition of the convergence in the Carathéodory topology (see e.g. [McM]).

Definition 2.2. Let (U_n, z_n), $z_n \in U_n \subset \widehat{\mathbb{C}}$, be a sequence of pointed topological discs. We say that (U_n, z_n) tends to a pointed topological disc (U, z), $z \in U \subset \widehat{\mathbb{C}}$, in the *Carathéodory topology* if $z_n \to z$, for every compact set $K \subset U$ we have $K \subset U_n$ for sufficiently large n and for every open connected set V containing z, if $V \subset U_n$ for infinitely many n, then $V \subset U$.

Lemma 2.3. *Let $f_n = f_{a_n,b_n} \in \mathcal{F}$ such that $(a_n, b_n) \to (a, b)$ as $n \to \infty$ for some $a, b \in \widehat{\mathbb{C}}$, $a \neq 0, \infty$. Suppose that for every n the map f_n has a fixed point $\xi_n \neq 0, \infty$, either attracting with the immediate basin $B(\xi_n)$ or parabolic with an invariant parabolic basin $B(\xi_n)$. Let ζ_n be one of the points $0, \infty, \xi_n$ (the same choice for all n) and let c_n be an f_n-critical point such that $c_n \in B(\zeta_n)$, $c_n \to c \in \widehat{\mathbb{C}}$. Assume also that there exists $r > 0$ such that for all n the basin $B(\zeta_n)$ contains a disc of spherical radius r centred at c_n and $\widehat{\mathbb{C}} \setminus B(\zeta_n)$ contains a non-empty open set independent of n.*

Then $B(\zeta_n)$ is simply connected for every n, the map $f = f_{a,b}$ has a fixed point $\zeta \in \widehat{\mathbb{C}}$, either attracting with the immediate basin $B(\zeta)$ or parabolic of multiplier 1 with exactly one invariant parabolic basin $B(\zeta)$ such that in both cases $B(\zeta)$ is simply connected, $c \in B(\zeta)$ and $(B(\zeta_n), c_n)$ tends to $(B(\zeta), c)$ in the Carathéodory topology. The sequence ζ_n does not always converge to ζ, but if $\zeta_n \not\to \zeta$, then ζ is parabolic and $\zeta_n \in \{0, \infty\}$ for infinitely many n.

Moreover, if $\psi_n : \mathbb{D} \to B(\zeta_n)$ is the unique Riemann mapping such that $\psi_n(0) = c_n$ and $\psi_n'(0) > 0$, then $\psi_n \to \psi$ almost uniformly on $\mathbb{D}$, where $\psi : \mathbb{D} \to B(\zeta)$ is the Riemann mapping such that $\psi(0) = c$ and $\psi'(0) > 0$. Furthermore, the Blaschke product $h_n = \psi_n^{-1} \circ f_n \circ \psi_n$ tends to $h = \psi^{-1} \circ f \circ \psi$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}}$ outside at most one point $z_0 \in \partial\mathbb{D}$.

Remark. We allow here (a, b) to be a singular parameter, which means that $f_{a,b}$ can be a map of degree two.

Proof. Observe first that by the Flower Theorem, if there exists an invariant parabolic basin, then the multiplier of the fixed point is equal to 1. Moreover, if f_n has an attracting fixed point, then by (3), the fourth fixed point is repelling and if f_n has a parabolic fixed point of multiplier 1, then there are only three fixed points. According to [Sh], if for a rational map only one fixed point is repelling or parabolic of multiplier 1, then each Fatou component is simply connected. Hence, $B(\zeta_n)$ is simply connected.

By assumption, $a \notin \{0, \infty\}$, so if $b \notin \{0, 1, \infty\}$, then $(a, b) \notin \mathrm{Sing}$ and $f_n \to f$ uniformly in the spherical metric on $\widehat{\mathbb{C}}$. Moreover, by Lemma I.2.2, if $b \in \{0, 1, \infty\}$, then $f_n \to f$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{1/b\}$ (with the convention $1/0 = \infty$, $1/\infty = 0$). Hence, we get:

(4) If $\quad b \notin \{0, 1, \infty\}$ or $z \in \widehat{\mathbb{C}} \setminus \{1/b\}, \quad$ then $\quad f_n \to f$ uniformly on V_z

for some neighbourhood $V_z \subset \widehat{\mathbb{C}}$ of z. Moreover, using Lemma I.2.2, it is easy to check that

$$\text{(5)} \qquad \text{If} \quad b \in \{0, 1, \infty\}, \quad \text{then} \quad \operatorname{dist}(1/b, J(f_n)) \to 0,$$

where $J(f_n)$ is the Julia set of f_n and dist denotes the spherical distance.

By assumption, $\widehat{\mathbb{C}} \setminus B(\zeta_n)$ contains a non-empty open set independent of n, so the family $\{\psi_n\}_{n>0}$ is normal and $\psi_{n_k} \to \psi$ as $k \to \infty$ almost uniformly on $\mathbb{D}$ for some holomorphic map $\psi : \mathbb{D} \to \mathbb{C}$. Moreover, $\psi_{n_k}(\mathbb{D})$ contains the disc of spherical radius $r/2$ centred at c for large k. Hence, ψ is univalent.

Note that h_n is a quadratic or cubic Blaschke product with an attracting or parabolic fixed point and a critical point at 0. Passing to a subsequence, we can assume that $h_{n_k} \to h$ almost uniformly on $\mathbb{D}$ for some holomorphic map h, which is a Blaschke product of degree at most three or a constant. By definition, for $z \in \mathbb{D}$ we have

$$\text{(6)} \qquad f_{n_k}(\psi_{n_k}(z)) = \psi_{n_k}(h_{n_k}(z)).$$

Fix $z \in \mathbb{D}$. Then $\psi_{n_k}(h_{n_k}(z)) \to \psi(h(z))$. Moreover, $f_{n_k}(\psi_{n_k}(z)) \to f(\psi(z))$, provided $f_{n_k} \to f$ uniformly in a neighbourhood of $\psi(z)$. By (4), this holds if $b \notin \{0, 1, \infty\}$ or $\psi(z) = 1/b$. Suppose now $b \in \{0, 1, \infty\}$, $\psi(z) = 1/b$. Then $1/b \in \psi(\mathbb{D})$, so by the almost uniform convergence $\psi_{n_k} \to \psi$, there exists a neighbourhood of $1/b$ contained in $B(\zeta_{n_k})$ for sufficiently large k. This contradicts (5), which shows that this case is impossible. We conclude that we can pass to the limit in (6), obtaining

$$\text{(7)} \qquad f \circ \psi = \psi \circ h,$$

which means that f on $\psi(\mathbb{D})$ is conformally conjugate to h on $\mathbb{D}$. Note that since ψ is univalent, h is not constant and has a critical point at 0, so it is a quadratic or cubic Blaschke product. Moreover, $f(\psi(\mathbb{D})) \subset \psi(\mathbb{D})$, so $\psi(\mathbb{D})$ is contained in an invariant Fatou component F. It cannot be a rotation domain, since it contains a critical point. Hence, either F is the immediate basin $B(\zeta)$ of an attracting fixed point ζ or F is an invariant parabolic basin $B(\zeta)$ for a parabolic fixed point ζ (by the Flower Theorem, in this case the multiplier must be equal to 1). To prove that in the latter case $B(\zeta)$ is the unique invariant parabolic basin of ζ, we show that for all maps $f_{a,b}$ for $a, b \in \widehat{\mathbb{C}}$, if there exists a fixed point of multiplier 1, then it has only one invariant basin. This is obvious for $(a, b) \in \text{Sing}$. To check the case $(a, b) \notin \text{Sing}$, change conformally the coordinates fixing $0, \infty$ and sending the parabolic point to 1. It is easy to check that such a map in these coordinates has the form

$$\widetilde{f}(z) = z^2 \frac{z + \beta - 2}{\beta z - 1}, \qquad \beta \in \mathbb{C} \setminus \{1\},$$

and $\widetilde{f}(1) = \widetilde{f}'(1) = 1$, $\widetilde{f}''(1) = 2/(b-1) \neq 0$. By the Flower Theorem, the fixed point of multiplier 1 has only one petal. Hence, $B(\zeta)$ is unique.

Now we return to the general case. By (7) and the dynamics of f, it is easy to check that no point $z \in \partial\psi(\mathbb{D})$ can be in $B(\zeta)$, so in fact we have $\psi(\mathbb{D}) = B(\zeta)$. Hence, $B(\zeta)$ is simply connected and contains the critical point $c = \psi(0)$. It is clear that ψ is the unique Riemann mapping from $\mathbb{D}$ onto $B(\zeta)$ such that $\psi(0) = c$ and $\psi'(0) > 0$.

By normality, for any subsequence n_j we can choose a subsubsequence n_{j_k} and repeat the above arguments, showing that $\psi_{n_{j_k}} \to \psi$ and $h_{n_{j_k}} \to h$ as $k \to \infty$ for the same ψ, h. It follows that in fact ψ_n converges to ψ and h_n converges to h almost uniformly on $\mathbb{D}$. Then the sequence of pointed topological discs $(\psi_n(\mathbb{D}), \psi_n(0)) = (B(\zeta_n), c_n)$ tends to $(\psi(\mathbb{D}), \psi(0)) = (B(\zeta), c)$ in the Carathéodory topology (see e.g. [McM]).

Now we check when $\zeta_n \to \zeta$. Suppose first that ζ is attracting. Then by the Carathéodory convergence of $(B(\zeta_n), c_n)$ to $(B(\zeta), c)$ and by (5), we have $b \notin \{0, 1, \infty\}$ or $\zeta \neq 1/b$. Hence, by (4), $f_n \to f$ uniformly in some neighbourhood of ζ, which easily implies that for large n there exists an attracting f_n-fixed point ζ_n' such that $\zeta_n' \to \zeta$. But by (3), there can be at most one attracting fixed point for f_n, so $\zeta_n = \zeta_n' \to \zeta$.

Assume now that ζ is parabolic. As previously, if $b \notin \{0, 1, \infty\}$ or $\zeta \neq 1/b$, then $f_n \to f$ uniformly in a neighbourhood of ζ. Hence, in this case there exist f_n-fixed points ζ_n', ζ_n'' such that $\zeta_n', \zeta_n'' \to \zeta$ and $\zeta_n' = \zeta_n''$ if and only if they have multiplier 1. But since f_n has only four (resp. three) fixed points in the attracting (resp. parabolic case), it follows that $\zeta_n \in \{\zeta_n', \zeta_n'', 0, \infty\}$. Hence, if $\zeta_n \not\to \zeta$, then $\zeta_n = 0$ or ∞ for infinitely many n. We are left with the case $b \in \{0, 1, \infty\}$, $\zeta = 1/b$. Suppose $\zeta_n \not\to \zeta$ and take a convergent subsequence $\zeta_{n_j} \to \widetilde{\zeta} \neq \zeta$. By (4), $f_n \to f$ uniformly in some neighbourhood of $\widetilde{\zeta}$. Therefore, $\widetilde{\zeta}$ is an f-fixed point, so it must be 0 or ∞. Moreover, by the uniform convergence $f_n \to f$ near $\widetilde{\zeta}$, we have $\zeta_{n_j} \in B(\widetilde{\zeta})$ for the map f_{n_j} with large j, so $\zeta_{n_j} = \widetilde{\zeta}$ for large j. Hence, $\zeta_n = 0$ or ∞ for infinitely many n.

It is clear that since $h_n \to h$ and h_n, h are quadratic or cubic Blaschke products, the convergence is uniform in the spherical metric on $\widehat{\mathbb{C}}$ unless $\deg h_n = 3$ and $\deg h = 2$. In the latter case,

$$h_n(z) = e^{i\theta_n} \frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \cdot \frac{z - \beta_n}{1 - \overline{\beta}_n z} \cdot \frac{z - \gamma_n}{1 - \overline{\gamma}_n z}$$

with $\alpha_n \to z_0 \in \partial\mathbb{D}$, $\beta_n \to z_1 \in \mathbb{D}$, $\gamma_n \to z_2 \in \mathbb{D}$ and it is easy to check that

$$\frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \to -z_0$$

almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{z_0\}$. This easily implies that $h_n \to h$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{z_0\}$. ■

The second lemma asserts that in the situation of Lemma 2.3, a suitable converging sequence of f_n-invariant curves converges in the Hausdorff metric to a suitable f-invariant curve. We consider two cases. In the case (a) we have a sequence of backward f_n-invariant curves landing at a repelling or parabolic point and in the case (b) we consider a sequence of forward f_n-invariant curves containing an attracting point, landing at a repelling point and converging to a forward f-invariant curve landing at a parabolic point. The advantage of the lemma is that we do not assume that the sequence f_n is uniformly convergent in a neighbourhood of the fixed point of f.

LEMMA 2.4. *Under the assumptions of the previous lemma, assume additionally that for every n the basin $B(\xi_n)$ contains a disc centred at an f_n-critical point of radius independent of n. Let γ_n, γ be simple arcs in $\mathbb{D}$ parameterized by $s \in [0,\infty)$ such that $\gamma_n(s) \to \gamma(s)$ as $n \to \infty$ almost uniformly on $[0,\infty)$. Suppose one of the following two possibilities is satisfied*:

(a) $h_n(\gamma_n(s)) = \gamma_n(s-1)$ *for* $s \geq 1$ *and* γ_n *(resp.* γ*) is disjoint from the closure of the postcritical set for* h_n *(resp.* h*).*

(b) $h_n(\gamma_n(s)) = \gamma_n(s+(n-s)/n)$ *for* $n > 1$, $s \geq 0$, $\gamma_n((n,\infty))$ *is disjoint from the closure of the* h_n*-postcritical set and* $\operatorname{diam}\gamma_n([n,\infty)) \to 0$.

Then γ_n (resp. γ) lands at some h_n-fixed point q_n (resp. h-fixed point q) in $\partial\mathbb{D}$ and $\psi_n \circ \gamma_n$ (resp. $\psi \circ \gamma$) lands at some f_n-fixed point p_n (resp. f-fixed point p) in $\widehat{\mathbb{C}}$. In the case (a) *the points q_n, p_n, q, p are either repelling or parabolic of multiplier* 1 *and in the case* (b) *the points q_n, p_n are repelling and q, p are parabolic of multiplier* 1.

Moreover, in both cases, for any sequences n_k, s_k,

$$\text{if} \quad n_k, s_k \underset{k\to\infty}{\longrightarrow} \infty, \quad \text{then} \quad \gamma_{n_k}(s_k) \underset{k\to\infty}{\longrightarrow} q \quad \text{and} \quad \psi_{n_k}(\gamma_{n_k}(s_k)) \underset{k\to\infty}{\longrightarrow} p.$$

This implies that $q_n \to q$, $p_n \to p$ and $\gamma_n \cup \{q_n\}$ (resp. $\psi_n(\gamma_n) \cup \{p_n\}$) tends to $\gamma \cup \{q\}$ (resp. $\psi(\gamma) \cup \{p\}$) in the Hausdorff metric as compact subsets of the Riemann sphere.

Proof. The case (a). It is a general fact (the Landing Theorem) that under the assumptions of the lemma, γ_n (resp. $\psi_n \circ \gamma_n$) lands at an h_n-fixed point $q_n \in \partial\mathbb{D}$ (resp. f_n-fixed point $p_n \in \widehat{\mathbb{C}}$), repelling or parabolic of multiplier 1 (see [TY]). Since $\gamma_n(s) \to \gamma(s)$ almost uniformly on $[0,\infty)$, passing to the limit we obtain

$$h(\gamma(s)) = \gamma(s-1), \qquad f(\psi(\gamma(s))) = \psi(h(\gamma(s))) = \psi(\gamma(s-1)),$$

so the Landing Theorem implies that γ lands at an h-fixed point $q \in \partial\mathbb{D}$ and $\psi(\gamma)$ lands at an f-fixed point $p \in \widehat{\mathbb{C}}$.

Proof of $\gamma_{n_k}(s_k) \to q$. Let $n_k, s_k \to \infty$ and suppose $\gamma_{n_k}(s_k) \not\to q$. Passing to a subsequence, we can assume $\gamma_{n_k}(s_k) \notin D$, where $D = \mathbb{D}_\varepsilon(q)$ for a small $\varepsilon > 0$. By Lemma 2.3, $h_n \to h$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}}$ outside at most one point in $\partial\mathbb{D}$. Hence, changing ε if necessary, we can assume $h_n \to h$ uniformly on ∂D. Since γ lands at q, we have $\gamma(s) \in \mathbb{D}_{\varepsilon/2}(q)$ for large s, so by the almost uniform convergence $\gamma_n \to \gamma$, we can find $s_0 > 1$ such that for sufficiently large k,

$$\gamma_{n_k}([s_0 - 1, s_0]) \subset D. \tag{8}$$

Let

$$s'_k = \inf\{s \in (s_0, s_k) : \gamma_{n_k}(s) \in \partial D\}. \tag{9}$$

Then (8) and (9) give

$$h_{n_k}(\gamma_{n_k}(s'_k)) = \gamma_{n_k}(s'_k - 1) \in D. \tag{10}$$

If q is repelling for h, then $h(\partial D) \cap \operatorname{cl} D = \emptyset$ (if we take ε small enough), so by the uniform convergence $h_n \to h$ on ∂D, we have $h_{n_k}(\partial D) \cap \operatorname{cl} D = \emptyset$ for large k, which contradicts (10).

We are left with the case when q is parabolic for h of multiplier 1. Then by the local study of a holomorphic map near such a point, the uniform convergence $h_n \to h$ on ∂D and (10) we obtain

$$\gamma_{n_k}(s'_k) \in K \subset \mathbb{D}, \tag{11}$$

where K is a compact set independent of k. Note that in the case when q is parabolic we have $\zeta_n = \xi_n$, because if $\zeta_n \in \{0, \infty\}$, then h_n (and hence h) would have a supersink at 0, which is impossible, because h is a Blaschke product and $\mathbb{D} \subset B(q)$.

By (11), $\gamma_{n_k}(s'_k) \in \mathbb{D}_{r_0}$ for some $r_0 < 1$ independent of k. Since the segment $[0, q]$ forms the attracting axis of the parabolic point q, by the almost uniform convergence $h_n \to h$ on $\mathbb{D}$ we can find a positive integer j and a small $\varepsilon_1 > 0$ such that for $s''_k = s'_k - j > 1$ we have

$$|\gamma_{n_k}(s''_k) - q| < \varepsilon_1 \quad \text{and} \quad \operatorname{dist}(\gamma_{n_k}(s''_k), [0, q]) < |\gamma_{n_k}(s''_k) - q|/10 \tag{12}$$

for large k.

Now we prove

$$\psi_n^{-1}(\xi_n) \to q. \tag{13}$$

To do it, suppose (13) does not hold and (passing to a subsequence) assume $\psi_n^{-1}(\xi_n) \to \widetilde{q} \neq q$. Recall that $h_n \to h$ almost uniformly in the spherical metric outside at most one point $z_0 \in \partial\mathbb{D}$. If $\widetilde{q} \neq z_0$, then by the uniform convergence in a neighbourhood of $\widetilde{q}$, the map h has two fixed non-repelling points $\widetilde{q} \neq q$ in $\operatorname{cl}\mathbb{D}$, which is clearly impossible for a quadratic or cubic Blaschke product. If $\widetilde{q} = z_0$, then by the uniform convergence in a neighbourhood of q, it is easy to check that for large n we have two repelling h_n-fixed points $q', q'' \in \partial\mathbb{D}$ in a small neighbourhood of q. But by uniform convergence, the small arc (q', q'') in $\partial\mathbb{D}$ is h_n-invariant, so there exists another h_n-fixed point in this arc, which is impossible. This proves (13).

Due to (13), we can assume

$$|\psi_{n_k}^{-1}(\xi_{n_k}) - q| < \varepsilon_1/10. \tag{14}$$

Let $D_k \subset \mathbb{D}$ be the open hyperbolic disc (i.e. the disc with respect to the hyperbolic metric in $\mathbb{D}$) centred at $\psi_{n_k}^{-1}(\xi_{n_k})$ such that $\gamma_{n_k}(s_k'') \in \partial D_k$ (in the case when $\psi_{n_k}^{-1}(\xi_{n_k})$ is attracting) or the horodisc tangent to $\partial\mathbb{D}$ at $\psi_{n_k}^{-1}(\xi_{n_k})$ such that $\gamma_{n_k}(s_k'') \in \partial D_k$ (in the case when $\psi_{n_k}^{-1}(\xi_{n_k})$ is parabolic). Then in both cases we have

$$\gamma_{n_k}(s_k'' - 1) \in D_k \quad \text{and} \quad h_{n_k}(D_k) \subset D_k$$

(see e.g. [St]), so D_k contains a point from $\gamma_{n_k}([0,1])$. By the almost uniform convergence $\gamma_n \to \gamma$, this point lies in some fixed compact set in $\mathbb{D}$. On the other hand, it is easy to check by elementary geometry that (12) and (14) imply the Euclidean diameter of D_k is less than $2\varepsilon_1$, so $D_k \subset \mathbb{D}_{3\varepsilon_1}(q)$. For sufficiently small ε_1 we get a contradiction.

Proof of $\psi_{n_k}(\gamma_{n_k}(s_k)) \to p$. Define ε, D, s_k, s_0, s_k' as previously, replacing h_{n_k}, h, q, γ_{n_k} respectively by f_{n_k}, f, p, $\psi_{n_k} \circ \gamma_{n_k}$. If p is repelling, then the proof is the same. Consider the case when p is parabolic. Then repeating the proof, instead of (11) we obtain

$$\psi_{n_k}(\gamma_{n_k}(s_k')) \in K, \tag{15}$$

where K is a compact set independent of k contained in the unique f-invariant parabolic basin $B(p)$ of p (recall that by Lemma 2.3, p has only one invariant basin). By assumption, $B(\xi_n)$ contains a disc of a fixed radius centred at some f_n-critical point $\widetilde{c}_n$. It is easy to see that $B(0) \cup B(\infty)$ also contains a disc of a spherical radius independent of n. Hence, the assumptions of Lemma 2.3 are fulfilled for $B(\xi_n)$, so $(B(\xi_n), \widetilde{c}_n)$ tends in the Carathéodory topology to $(\widetilde{B}(\zeta), \lim_n \widetilde{c}_n)$ for an f-invariant parabolic basin $\widetilde{B}(\zeta)$ of ζ. It is clear that $\zeta = p$ and $\widetilde{B}(\zeta) = B(p)$. By the Carathéodory convergence, $K \subset B(\xi_{n_k})$ for large k. Hence, $\zeta_{n_k} = \xi_{n_k}$ and $\psi(\mathbb{D}) = B(p)$, so

$$K \subset \psi(\mathbb{D}).$$

Thus, $\psi^{-1}(K) \subset \mathbb{D}_{r_1}$ for some $r_1 < 1$. By the uniform convergence $\psi_{n_k} \to \psi$ on $\mathbb{D}_{r_1}$, we have $K \subset \psi_{n_k}(\mathbb{D}_{r_1})$ for large k. But by (15), $\psi_{n_k}^{-1}(K) \subset \mathbb{D}_{r_1}$ contains $\gamma_{n_k}(s_k')$, which contradicts $\gamma_{n_k}(s_k') \to q \in \partial\mathbb{D}$.

The case (b). Note first that the point $\gamma_n(n)$ is an attracting h_n-fixed point for every n. Hence, $\psi_n(\gamma_n(n)) = \xi_n = \zeta_n$ is an attracting f_n-fixed point. Moreover, the curve $\gamma_n|_{[n+1,\infty)}$ fulfills (after rescaling) the conditions of the case (a). Therefore, γ_n lands at an h_n-fixed point $q_n \in \partial\mathbb{D}$ and $\psi_n(\gamma_n)$

lands at an f_n-fixed point $p_n \in \widehat{\mathbb{C}}$. The points q_n, p_n are repelling, since h_n and f_n have an attracting fixed point. By the almost uniform convergence $\gamma_n(s) \to \gamma(s)$, after passing to the limit we get

$$h(\gamma(s)) = \gamma(s+1), \qquad f(\psi(\gamma(s))) = \psi(\gamma(s+1)).$$

By assumption, $\operatorname{diam} \gamma_n([n,\infty)) \to 0$, so $\gamma_n(n) \to \partial\mathbb{D}$, which implies that h (and hence f) must be parabolic. Thus, $\gamma(n) = h^n(\gamma(0))$ converges to a parabolic h-fixed point $q \in \partial\mathbb{D}$ of multiplier 1 and $\psi(\gamma(n))$ converges to a parabolic f-fixed point $p \in \widehat{\mathbb{C}}$ of multiplier 1. Since the hyperbolic distance between any point from γ and the set $\{\gamma(n)\}_{n>0}$ (resp. $\psi(\gamma)$ and $\{\psi(\gamma(n))\}_{n>0}$) is bounded by a fixed constant, we conclude that γ lands at q and $\psi(\gamma)$ lands at p.

Proof of $\gamma_{n_k}(s_k) \to q$. Note that since q is parabolic, by (13) we have $\gamma_{n_k}(n_k) = \psi_{n_k}^{-1}(\xi_{n_k}) \to q$. Hence, using the assumption $\operatorname{diam} \gamma_n([n,\infty)) \to 0$, we can assume $s_k < n_k$. Take a small $\varepsilon_2 > 0$. As $[0,q]$ forms the attracting axis of the parabolic point q, by the almost uniform convergence $h_n \to h$ we can find s_0 such that for every $s \in [s_0, s_0+1]$,

$$|\gamma_{n_k}(s) - q| < \varepsilon_2 \quad \text{and} \quad \operatorname{dist}(\gamma_{n_k}(s), [0,q]) < |\gamma_{n_k}(s) - q|/10$$

for k larger than some $k(s)$ (cf. (12)). As in the proof of the case (a), we find an open disc $D_{s,k}$ of Euclidean diameter less than $2\varepsilon_2$ such that $h_{n_k}(D_{s,k}) \subset D_{s,k}$ and $\xi_{n_k}, \gamma_{n_k}(s) \in D_{s,k}$ for $k > k(s)$. By the compactness of $[s_0, s_0+1]$, there exists k_0 such that for every $k > k_0$ and every $s \ge s_0$ we have $\xi_{n_k}, \gamma_{n_k}(s) \in D_{s,k}$. Hence, $|\gamma_{n_k}(s) - \xi_{n_k}| < 2\varepsilon_2$, so $|\gamma_{n_k}(s) - q| < 3\varepsilon_2$ for every $s \ge s_0$. This proves $\gamma_{n_k}(s_k) \to q$.

Proof of $\psi_{n_k}(\gamma_{n_k}(s_k)) \to p$. Suppose this is not true and define ε and D as in the proof of the case (a). Note that $\psi_n(\gamma_n(n)) = \xi_n \to p$. (To see this, it is sufficient to notice that by Lemma I.2.2, $f_n \to f$ almost uniformly on $\widehat{\mathbb{C}} \setminus \{p\}$ and use similar arguments to those for (13).) Hence, $\psi_{n_k}(\gamma_{n_k}(n_k)) \in D$ for large k. Let

$$s_k' = \begin{cases} \sup\{s \in (s_k, n_k) : \psi_{n_k}(\gamma_{n_k}(s)) \in \partial D\} & \text{if } s_k < n_k, \\ \inf\{s \in (n_k, s_k) : \psi_{n_k}(\gamma_{n_k}(s)) \in \partial D\} & \text{if } s_k > n_k. \end{cases}$$

Then $f_{n_k}(\psi_{n_k}(\gamma_{n_k}(s_k'))) \in D$. In the same way as in the case (a) we show that there exists a compact set K and $r_1 < 1$ such that

$$\psi_{n_k}(\gamma_{n_k}(s_k')) \in K \subset \psi_{n_k}(\mathbb{D}_{r_1})$$

for sufficiently large k. This contradicts $\gamma_{n_k}(s_k') \to q \in \partial\mathbb{D}$. Hence, $\psi_{n_k}(\gamma_{n_k}(s_k)) \to p$.

Note that the facts proven above easily imply $q_n \to q$, $p_n \to p$. Moreover, it is easy to see that they show (together with the almost uniform convergence $\gamma_n \to \gamma$, $\psi_n \circ \gamma_n \to \psi \circ \gamma$) that $\gamma_n \cup \{q_n\}$ tends to $\gamma \cup \{q\}$ and $\psi_n(\gamma_n) \cup \{p_n\}$ tends to $\psi(\gamma) \cup \{p\}$ in the Hausdorff metric. ∎

3. Mandelbrot-like families. It is known (see [CGS], [DH2]) that in the parameter plane of cubic Newton maps N there exist Mandelbrot-like sets (quasiconformally homeomorphic to the standard Mandelbrot set) corresponding to certain sets of parameters for which $N^k|_U$ is quadratic-like on some topological disc U with some $k \geq 2$. More precisely, consider the family $\mathcal{N}_{(0,u,\infty)} \subset \mathcal{F}$ of Newton maps with supersinks $0, u, \infty$. This family consists of maps $f_{a,b}$ with $a = \left(\frac{b-2}{2b-1}\right)^2$ (see Section I.3). To avoid confusion in notation, we parameterize these maps by a new parameter $\sigma \in \Sigma = \mathbb{C} \setminus \{0, 1/2, 1, 2\}$ setting

$$N_\sigma = f_{a,b} \quad \text{for} \quad a = \left(\frac{\sigma - 2}{2\sigma - 1}\right)^2, \ b = \sigma.$$

Take σ_0 such that for N_{σ_0} the critical point 1 is periodic with some period $k \geq 2$. According to Lemma I.4.19, $N_{\sigma_0}^k$ is quadratic-like on some topological disc U_{σ_0} containing the critical point 1. Perturbing σ within some open set $\mathcal{U} \subset \Sigma$ we obtain a family $\{N_\sigma\}_{\sigma \in \mathcal{U}}$ of maps such that N_σ^k is quadratic-like on some topological disc U_σ (see Subsection 4.1).

The general theory of such analytic families of polynomial-like maps was developed in [DH2]. For convenience, we recall here the definitions and some results from this work, which will be used in what follows.

Definition 3.1. A *polynomial-like map* of degree d is a proper holomorphic mapping of degree d from a topological disc $U \subset \widehat{\mathbb{C}}$ onto a topological disc $V \subset \widehat{\mathbb{C}}$ such that $\operatorname{cl} U \subset V$. The *filled-in Julia set* K_f of f is the set of points $z \in U$ such that the entire forward trajectory of z is contained in U. A polynomial-like map of degree two is called a *quadratic-like map*.

Definition 3.2. Let $\{f_\sigma\}_{\sigma \in \mathcal{U}}$ for some complex manifold $\mathcal{U}$ be a family of quadratic-like maps $f_\sigma : U_\sigma \to f_\sigma(U_\sigma)$. Let $\mathcal{V} = \{(\sigma, z) : z \in U_\sigma\}$, $\mathcal{W} = \{(\sigma, z) : z \in f_\sigma(U_\sigma)\}$, $f(\sigma, z) = (\sigma, f_\sigma(z))$. Then this family is an *analytic family of quadratic-like maps* if:

(i) $\mathcal{V}$ and $\mathcal{W}$ are homeomorphic over $\mathcal{U}$ to $\mathcal{U} \times \mathbb{D}$,
(ii) the projection $\operatorname{cl}_{\mathcal{W}} \mathcal{V} \to \mathcal{U}$ is a proper map,
(iii) $f : \mathcal{V} \to \mathcal{W}$ is a proper holomorphic map.

Remark. The condition (ii) of the above definition is fulfilled if $\operatorname{cl} U_\sigma$ depends continuously on σ in the Hausdorff metric. Indeed, in this case for every $\sigma_n \in \mathcal{U}$ and $z_n \in \operatorname{cl} U_{\sigma_n}$, if $z_n \to z \in \widehat{\mathbb{C}}$ and $\sigma_n \to \sigma \in \mathcal{U}$, then $z \in U_\sigma$, which easily implies the condition (ii).

The Straightening Theorem says that every quadratic-like map f is hybrid equivalent to some polynomial z^2+c, i.e. f is conjugate to z^2+c by a quasiconformal homeomorphism h which is conformal almost everywhere on the filled-in Julia set K_f of f. Moreover, if K_f is connected, then c is uniquely determined. Hence, for a family $\{f_\sigma\}_{\sigma\in\mathcal{U}}$ we can define a function χ setting $\chi(\sigma)=c$ for σ such that K_{f_σ} is connected. Let M be the standard Mandelbrot set for the family z^2+c. The following facts were proved in [DH2]:

Theorem 3.3. *For any analytic family $\{f_\sigma\}_{\sigma\in\mathcal{U}}$ of quadratic-like mappings, if $\mathcal{U}$ is a contractible complex manifold, then the mapping χ can be extended continuously to $\mathcal{U}$ and χ is holomorphic on* $\operatorname{int}\chi^{-1}(M)$. *Moreover, for every $c\in M$ the set $\chi^{-1}(c)$ is analytic. However, χ need not to be holomorphic at the boundary of $\chi^{-1}(M)$.* ■

Theorem 3.4. *For any analytic family $\mathbf{f}=\{f_\sigma\}_{\sigma\in\mathcal{U}}$ of quadratic-like mappings with $\mathcal{U}$ connected of complex dimension one, if χ is not constant, then it is topologically holomorphic over M, i.e. for all $\sigma\in\chi^{-1}(M)$, σ is isolated in its fibre and the index of χ at σ is strictly positive.* ■

Proposition 3.5. *Suppose that, in addition, $\mathcal{U}$ is a topological disc and $\chi^{-1}(M)$ is compact. Let ω_σ be the critical point of f_σ and let A be a closed topological disc in $\mathcal{U}$ such that $\chi^{-1}(M)\in\operatorname{int}A$. Then χ has degree d on $\chi^{-1}(M)$, where d is the number of times $f_\sigma(\omega_\sigma)-\omega_\sigma$ turns around* 0 *as σ describes ∂A. In particular, if $d=1$, then χ is a homeomorphism on $\chi^{-1}(M)$ and we call the family $\{f_\sigma\}_{\sigma\in\mathcal{U}}$ a Mandelbrot-like family and $M_{\mathbf{f}}=\chi^{-1}(M)$ a Mandelbrot-like set.* ■

Definition 3.6. Let $\mathcal{U}$ be a Riemann surface homeomorphic to a disc and let

$$\mathbf{f}=\{f_{\sigma,t}:U_{\sigma,t}\to f_{\sigma,t}(U_{\sigma,t})\}_{(\sigma,t)\in\mathcal{U}\times[0,1]}$$

be a family of quadratic-like maps. Suppose that the conditions (i) and (ii) from Definition 3.2 hold for $\mathcal{U}$ replaced by $\mathcal{U}\times[0,1]$ and the mapping $(\sigma,t,z)\mapsto(\sigma,t,f_{\sigma,t}(z))$ is continuous and proper in (σ,t,z) and holomorphic in (σ,z). Assume also that for every $t\in[0,1]$ the family $\mathbf{f}_t=\{f_{\sigma,t}\}_{\sigma\in\mathcal{U}}$ is Mandelbrot-like and all sets $M_{\mathbf{f}_t}$ are contained in a common compact set $A\subset\mathcal{U}$. Then we say that $\mathbf{f}_0$ and $\mathbf{f}_1$ can be connected by a *continuous path of Mandelbrot-like families.*

Remark. As previously, the condition (ii) is satisfied if $\operatorname{cl}U_{\sigma,t}$ depends continuously on σ,t in the Hausdorff metric.

Let $\mathcal{U}$ and $\mathcal{U}_t$ for $t\in[0,1]$ be disc-equivalent Riemann surfaces such that there exists a homeomorphism $\mathcal{H}:\mathcal{U}\times[0,1]\to\bigcup_{t\in[0,1]}\mathcal{U}_t$ which maps holomorphically $\mathcal{U}\times\{t\}$ onto $\mathcal{U}_t$ for every t. If $\{f_{\mathcal{H}(\sigma,t)}\}_{(\sigma,t)\in\mathcal{U}\times[0,1]}$ is a continuous path of Mandelbrot-like families, we will also say that $\{f_\tau\}_{\tau\in\mathcal{U}_t,\, t\in[0,1]}$ is a continuous path of Mandelbrot-like families.

PROPOSITION 3.7. *Let* $\mathbf{f}=\{f_\sigma\}_{\sigma\in\mathcal{U}}$ *and* $\mathbf{g}=\{g_\sigma\}_{\sigma\in\mathcal{U}}$ *be two Mandelbrot-like families parameterized by the same Riemann surface* $\mathcal{U}$. *If* $\mathbf{f}$ *and* $\mathbf{g}$ *can be connected by a continuous path of Mandelbrot-like families*, *then the homeomorphism* $\chi_{\mathbf{g}}^{-1}\circ\chi_{\mathbf{f}}: M_{\mathbf{f}}\to M_{\mathbf{g}}$ *is quasiconformal.* ■

Moreover, [DH2] gives examples of topological discs $\mathcal{U}$ in the parameter plane of cubic Newton maps such that (in our parameterization) $\mathcal{U}\subset\Sigma$, the family $\{N_\sigma^k|_{U_\sigma}\}_{\sigma\in\mathcal{U}}$ is Mandelbrot-like for some topological discs U_σ containing the critical point 1 and χ is quasiconformal on $\chi^{-1}(M)$.

DEFINITION 3.8. We will say that two Mandelbrot-like families

$$\mathbf{f}=\{f_\sigma: U_\sigma\to f_\sigma(U_\sigma)\}_{\sigma\in\mathcal{U}_{\mathbf{f}}}\quad\text{and}\quad \mathbf{g}=\{g_\sigma: V_\sigma\to g_\sigma(V_\sigma)\}_{\sigma\in\mathcal{U}_{\mathbf{g}}}$$

with $\mathcal{U}_{\mathbf{f}},\mathcal{U}_{\mathbf{g}}\subset\mathcal{U}$ for some Riemann surface $\mathcal{U}$ are *equivalent* if

(i) $M_{\mathbf{f}}=M_{\mathbf{g}}\subset\mathcal{U}_{\mathbf{f}}\cap\mathcal{U}_{\mathbf{g}}$,
(ii) $K_{f_\sigma}=K_{g_\sigma}\subset U_\sigma\cap V_\sigma$ and $f_\sigma|_{U_\sigma\cap V_\sigma}=g_\sigma|_{U_\sigma\cap V_\sigma}$ for every $\sigma\in\mathcal{U}_{\mathbf{f}}\cap\mathcal{U}_{\mathbf{g}}$.

4. The bifurcation theorem.

In this section we formulate and prove the main theorem.

THEOREM 4.1 (The bifurcation theorem). *Let* $\mathbf{N}=\{N_\sigma^k|_{U_\sigma}\}_{\sigma\in\mathcal{U}}$ *for a topological disc* $\mathcal{U}\subset\Sigma$ *be a Mandelbrot-like family in* $\mathcal{N}_{(0,u,\infty)}$ *and let* $M_N=\chi^{-1}(M)$ *be the Mandelbrot-like set in* $\mathcal{U}$. *Set* $\sigma_0=\chi^{-1}(0)$ *and assume that for* N_{σ_0}, *Head's angle* α *is periodic* mod 1 *with period* k. *Then there exist continuous paths of Mandelbrot-like families* $\{\mathbf{f}_t\}_{t\in[0,1]}$, $\{\mathbf{f}_t^0\}_{t\in[1,1+\varepsilon]}$, $\{\mathbf{f}_t^\infty\}_{t\in[1,1+\varepsilon]}$ *for some* $0<\varepsilon<1$, *with Mandelbrot-like sets* M_t, M_t^0, M_t^∞ *respectively*, *such that*:

- $\mathbf{f}_0=\{N_\sigma^k|_{U'_\sigma}\}_{\sigma\in\mathcal{U}'}$ *for some topological disc* $\mathcal{U}'\subset\Sigma$ *and* $\mathbf{f}_0$ *is equivalent to* $\mathbf{N}$, *in particular* $M_0=M_N$.
- *For every* $t\in(0,1)$ *we have* $\mathbf{f}_t=\{f_{a,b}^k|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}_t}$ *for some topological disc* $\mathcal{U}_t\subset\mathrm{Fix}(t)$ *and for every* $(a,b)\in\mathcal{U}_t$ *the* $f_{a,b}$*-critical point* $u_{a,b}$ *is in the immediate basin of an attracting* $f_{a,b}$*-fixed point of multiplier* t.
- $\mathbf{f}_1=\mathbf{f}_1^0=\mathbf{f}_1^\infty=\{f_{a,b}^k|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}_1}$ *for some topological disc* $\mathcal{U}_1\subset\mathrm{Fix}(1)$ *and for every* $(a,b)\in\mathcal{U}_1$ *the* $f_{a,b}$*-critical point* $u_{a,b}$ *is in the unique invariant parabolic basin of an* $f_{a,b}$*-fixed point of multiplier* 1.
- *For* $\zeta\in\{0,\infty\}$ *and every* $t\in(1,1+\varepsilon]$ *we have* $\mathbf{f}_t^\zeta=\{f_{a,b}^k|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}_t^\zeta}$ *for some topological disc* $\mathcal{U}_t^\zeta\subset\widehat{\mathbb{C}}\times\widehat{\mathbb{C}}\setminus\mathrm{Sing}$ *such that* $u_{a,b}\in B(\zeta)$ *for every* $(a,b)\in\mathcal{U}_t^\zeta$ *and* $\Phi_{a,b}^\zeta(f_{a,b}(u_{a,b}))=2-t$ *for every* $(a,b)\in M_t^\zeta$, *where* $\Phi_{a,b}^\zeta$ *denotes the Böttcher coordinates defined on the maximal subset of* $B(\zeta)$ *for the map* $f_{a,b}$.

• *For* $\zeta \in \{0,\infty\}$, *if the set* $\partial B(u) \cap \partial B(\zeta)$ *for the map* N_{σ_0} *in the initial family is infinite*, *then for* $t \in (1, 1+\varepsilon]$ *the basin* $B(\zeta)$ *is not simply connected for every* $(a,b) \in \mathcal{U}_t^\zeta$ *and is exotic for every* $(a,b) \in M_t^\zeta$; *otherwise* $B(\zeta)$ *is simply connected for every* $(a,b) \in M_t^\zeta$ *and is not exotic for every* $(a,b) \in \mathcal{U}_t^\zeta$.

• *For* $\zeta \in \{0,\infty\}$, *if* $B(\zeta)$ *is not exotic for* $(a,b) \in \mathcal{U}_t^\zeta$, $t \in (1, 1+\varepsilon]$, *then the path* $\{\mathbf{f}_t^\zeta\}_{t\in[1,1+\varepsilon]}$ *can be extended to a continuous path* $\{\mathbf{f}_t^\zeta\}_{t\in[1,2]}$ *for* $\mathbf{f}_t^\zeta = \{f_{a,b}^k|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}_t^\zeta}$ *such that for every* $(a,b) \in \mathcal{U}_t^\zeta$ *we have* $u_{a,b} \in B(\zeta)$ *for* $t \in (1,2]$ *and* $u_{a,b} = \zeta$ *for* $t = 2$. *In particular*, *if* $B(\infty)$ *is not exotic for* $(a,b) \in \mathcal{U}_t^\infty$, *then the maps* $f_{a,b}$ *for* $(a,b) \in \mathcal{U}_2^\infty$ *are cubic polynomials with a supersink at* 0.

Moreover, *if* $\chi : M_N \to M$ *is quasiconformal*, *then all the Mandelbrot-like sets* M_t, M_t^0, M_t^∞ *are quasiconformally homeomorphic to* M.

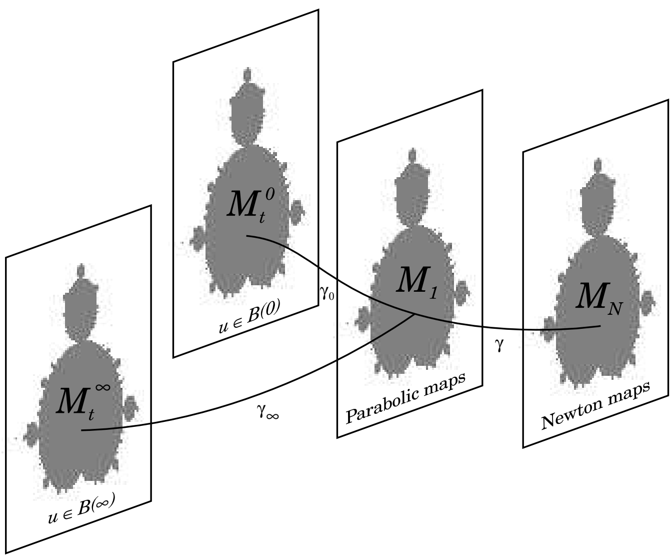

Fig. 1. Bifurcation of the Mandelbrot-like sets

Remark. It is not possible to extend the exotic path $\{\mathbf{f}_t^\zeta\}_{t\in[1,1+\varepsilon]}$ to $\{\mathbf{f}_t^\zeta\}_{t\in[1,2]}$ as in the non-exotic case, because for $t = 2$ the map $f_{a,b}$ can degenerate (see Section I.2).

The parabolic bifurcation is demonstrated in Fig. 1, and Figs. 2–5 present computer pictures of the Mandelbrot-like sets. In all of them white (resp. dark grey) colour corresponds to parameters for which the critical point 1 is in the basin of ∞ (resp. 0). Note the remarkable similarity between the parameter space and the dynamical space in Fig. I.7.

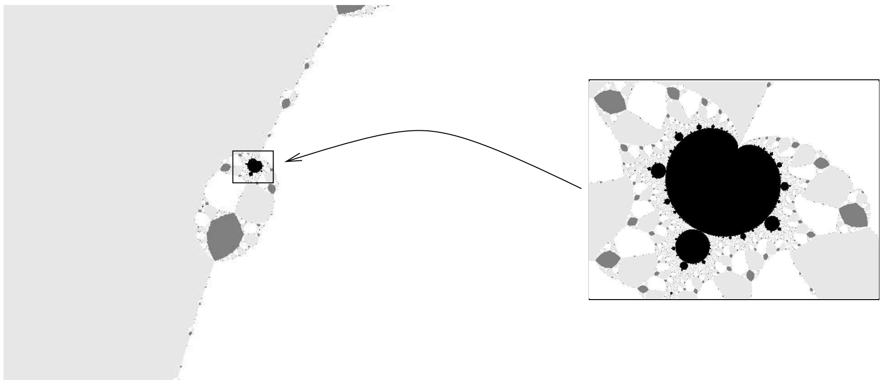

Fig. 2. A fragment of the parameter space of Newton maps with supersinks $0, \infty, u$. Light grey colour corresponds to parameters for which the critical point 1 is in the basin of u. The enlargement shows the Mandelbrot-like set M_N.

Fig. 3. A fragment of the parameter space of parabolic maps with supersinks $0, \infty$, and a fixed point of multiplier 1. Light grey colour corresponds to parameters for which the critical point 1 is in the parabolic basin. The enlargement shows the Mandelbrot-like set M_1.

Let us remark that compared to the general case, it is much easier to prove the existence of the above bifurcation for the "centre" of the main cardioid in the Mandelbrot-like sets, i.e. the bifurcation on the surface

$$\mathrm{Per}(k) = \{(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \setminus \mathrm{Sing} : 1 \text{ is periodic with period } k \text{ for } f_{a,b}\}.$$

(For $k = 2$, this was done in Theorem I.4.20.) One can show that for some open set $\mathcal{V} \subset \mathbb{C} \times \mathbb{C}$ containing $\bigcup_{\varrho \in \mathbb{D} \cup \{1\}} \mathrm{Fix}(\varrho)$, the set $\mathrm{Per}(k) \cap \mathcal{V}$ is a one-dimensional complex manifold, transversal to the set of Newton maps with supersinks $0, u, \infty$. (The proof is similar to the proof of Theorem I.4.14.) Moreover, one can show there are no singular parameters in $\mathrm{cl}\,\mathrm{Per}(k) \cap \mathcal{V}$. Let $\xi_{a,b} \neq 0, \infty$ be the $f_{a,b}$-attracting fixed point for $(a, b) \in \mathrm{Per}(k) \cap \bigcup_{\varrho \in \mathbb{D}} \mathrm{Fix}(\varrho)$. Then the function $(a, b) \mapsto \xi_{a,b}$ is holomor-

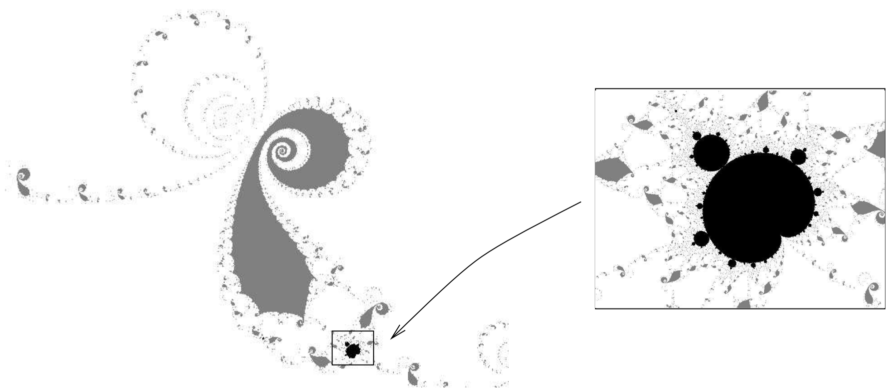

Fig. 4. A fragment of the parameter space of maps with supersinks 0, ∞ such that u is in the exotic basin of ∞. The enlargement shows the Mandelbrot-like set M_t^∞.

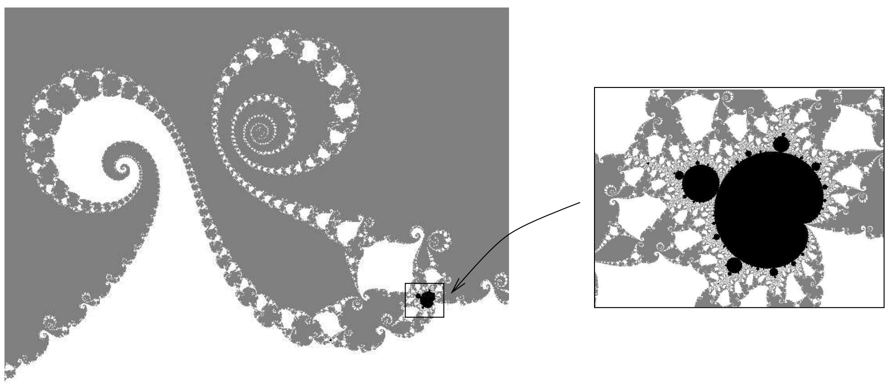

Fig. 5. A fragment of the parameter space of maps with supersinks 0, ∞ such that u is in the non-exotic basin of 0. The enlargement shows the Mandelbrot-like set M_t^0.

phic and the boundary of $\mathrm{Per}(k) \cap \mathcal{V} \cap \bigcup_{\varrho \in \mathbb{D}} \mathrm{Fix}(\varrho)$ is a piecewise analytic curve in $\bigcup_{\varrho \in \partial\mathbb{D}} \mathrm{Fix}(\varrho)$. Hence, one can connect the map N_{σ_0} to a parabolic map in $\mathrm{Per}(k) \cap \mathrm{Fix}(1)$ by a suitable curve γ in $\mathrm{Per}(k) \cap \bigcup_{\varrho \in \mathbb{D}} \mathrm{Fix}(\varrho)$. Then using the Douady–Sentenac theorem on the parabolic bifurcation (Theorem I.4.21) one can show the existence of two curves γ_0, γ_∞ leading to maps for which the critical point u is respectively in the (exotic or not) basin of 0 or ∞.

The proof of Theorem 4.1 in the general case is longer and consists of several parts. First, in Subsection 4.1 we show that we can modify the initial Mandelbrot-like family $\mathbf{N}$ to an equivalent family $\mathbf{f}_0 = \{N_\sigma^k|_{U'_\sigma}\}_{\sigma \in \mathcal{U}'}$ such that the topological discs U'_σ have a nice combinatorial description. In Subsection 4.2 we construct the family $\mathbf{f}_t$ for $t \in (0,1)$ using quasiconformal surgery. In Subsection 4.3 we define the family $\mathbf{f}_1$ and prove that $\{\mathbf{f}_t\}_{t \in [0,1]}$ is a continuous path of Mandelbrot-like families. This is the most technical part of the proof, which makes use of Lemmas 2.3 and 2.4. Subsection 4.4 contains the proof of the existence of the families $\mathbf{f}_t^0$ and $\mathbf{f}_t^\infty$, similar to the proof of Theorem I.4.20. In Subsection 4.5 we determine which path is exotic. Finally, in Subsection 4.6 we extend the non-exotic path so that it ends in the family of maps with a double critical point (conformally conjugate to cubic polynomials with a supersink), using quasiconformal surgery once more.

Consider the initial family $\mathbf{N}$ and the map N_{σ_0}. Then the boundaries of two of the three immediate basins $B(0), B(u), B(\infty)$ have an infinite intersection and the boundary of the third one has a one-point intersection with the others (see Subsection I.4.4). Hence, we have three possibilities: $\partial B(u) \cap \partial B(\infty)$ is infinite, $\partial B(u) \cap \partial B(0)$ is infinite or $\partial B(0) \cap \partial B(\infty)$ is infinite. We will consider only the first case (then, according to Theorem 4.1, the path $\{\mathbf{f}_t^\infty\}$ is exotic and $\{\mathbf{f}_t^0\}$ is non-exotic). The second case is symmetric to the first (it differs only by the exchange of 0 and ∞) and the third can be proved in the same way (in fact, it is technically much easier).

4.1. *Modifying the sets U_σ.* Recall that for $\sigma \in \mathcal{U}$ the set U_σ is a topological disc and N_σ^k is quadratic-like on U_σ. It is obvious that U_σ contains the critical point 1. Let K_σ be the filled-in Julia set of $N_\sigma^k|_{U_\sigma}$. We construct a topological disc $\mathcal{U}' \subset \mathcal{U}$ containing M_N and topological discs U'_σ for $\sigma \in \mathcal{U}'$ such that U'_σ contains the critical point 1, $K_\sigma \subset U'_\sigma$ and $N_\sigma^k|_{U'_\sigma}$ is quadratic-like. The boundary of the sets U'_σ will consist of suitable parts of some equipotential curves and preperiodic internal rays in the basins of $0, u, \infty$.

Recall that for any Newton map N_σ the three immediate basins $B_\sigma(\zeta) = B(\zeta)$ for $\zeta \in \{0, u, \infty\}$ are simply connected and N_σ has a unique repelling fixed point $p_\sigma = \partial B_\sigma(0) \cap \partial B_\sigma(u) \cap \partial B_\sigma(\infty)$ (see [P2]).

Definition 4.2. For $\zeta \in \{0, u, \infty\}$ and $\sigma \in \Sigma$ let ψ_σ^ζ be the unique Riemann mapping from $\mathbb{D}$ onto $B_\sigma(\zeta)$ such that $\psi_\sigma^\zeta(0) = \zeta$ and $(\psi_\sigma^\zeta)'(0) > 0$.

It is clear that ψ_σ^ζ conjugates N_σ to

$$\widetilde{h}_\sigma^\zeta(z) = \begin{cases} e^{2\pi i \theta_\sigma^\zeta} z^2 & \text{if } 1 \notin B_\sigma(\zeta), \\ e^{2\pi i \theta_\sigma^\zeta} z^2 \dfrac{z - \widetilde{\varrho}_\sigma^\zeta}{1 - \overline{\widetilde{\varrho}_\sigma^\zeta} z} & \text{if } 1 \in B_\sigma(\zeta), \end{cases} \tag{16}$$

for some $\theta_\sigma^\zeta \in \mathbb{R}/\mathbb{Z}$, $\widetilde{\varrho}_\sigma^\zeta \in \mathbb{D}$.

Note that by [Ro], the boundaries of the immediate basins are locally connected. Hence, ψ_σ^ζ extends continuously to $\operatorname{cl}\mathbb{D}$. Moreover, by [Ta], if $1 \notin B_\sigma(\zeta)$, then $\partial B_\sigma(\zeta)$ is a Jordan curve, so by the Carathéodory theorem, the extension is a homeomorphism. If $1 \in B_\sigma(\zeta)$, then there are two $\widetilde{h}_\sigma^\zeta$-fixed points in $\partial\mathbb{D}$ and only one N_σ-fixed point $p_\sigma \in \partial B_\sigma(\zeta)$, so the extension is not a homeomorphism.

It is easy to see that for $\zeta \in \{0, u, \infty\}$ and every sequence $\sigma_n \to \sigma$, $\sigma_n, \sigma \in \Sigma$ the maps N_{σ_n} satisfy the assumptions of Lemma 2.3 with $\zeta_n = \zeta$. Hence, the maps ψ_σ^ζ and $\widetilde{h}_\sigma^\zeta$ depend continuously on $\sigma \in \Sigma$ in the almost uniform convergence topology, so $\theta_\sigma^\zeta, \widetilde{\varrho}_\sigma^\zeta$ are continuous functions of $\sigma \in \Sigma$.

Definition 4.3. For $\zeta \in \{0, u, \infty\}$ and $\sigma \in \Sigma$ let φ_σ^ζ be a Riemann mapping from $\mathbb{D}$ onto $B_\sigma(\zeta)$ such that $\varphi_\sigma^\zeta(0) = \zeta$ and $\varphi_\sigma^\zeta(1) = p_\sigma$.

The map φ_σ^ζ is a composition of ψ_σ^ζ with a suitable rotation in $\mathbb{D}$. If $1 \notin B_\sigma(\zeta)$, then φ_σ^ζ is uniquely determined and coincides with the inverse of the Böttcher coordinates (for the definition of the Böttcher coordinates see Subsection I.4.3). If $1 \in B_\sigma(\zeta)$, then there are two $\widetilde{h}_\sigma^\zeta$-fixed points in $\partial\mathbb{D}$, so there are two maps φ_σ^ζ satisfying Definition 4.3. Note that φ_σ^ζ conjugates N_σ to

$$h_\sigma^\zeta(z) = \begin{cases} z^2 & \text{if } 1 \notin B_\sigma(\zeta), \\ \dfrac{1-\overline{\varrho_\sigma^\zeta}}{1-\varrho_\sigma^\zeta} z^2 \dfrac{z-\varrho_\sigma^\zeta}{1-\overline{\varrho_\sigma^\zeta} z} & \text{if } 1 \in B_\sigma(\zeta), \end{cases} \tag{17}$$

for some $\varrho_\sigma^\zeta \in \mathbb{D}$.

Definition 4.4. For $\zeta \in \{0, u, \infty\}$ and $\sigma \in \Sigma$ such that $1 \notin B_\sigma(\zeta)$ let $R_\zeta(\theta) = \varphi_\sigma^\zeta(\{re^{2\pi i\theta} : r \in [0,1)\})$ be the internal ray in $B_\sigma(\zeta)$ of angle $\theta \in \mathbb{R}/\mathbb{Z}$ and let $l_\zeta(\theta)$ be its landing point. Set also $\widetilde{B}_\sigma(\zeta) = N_\sigma^{-1}(B_\sigma(\zeta)) \setminus B_\sigma(\zeta)$, $\widetilde{R}_\zeta(\theta) = N_\sigma^{-1}(R_\zeta(\theta)) \cap \widetilde{B}_\sigma(\zeta)$ and let $\widetilde{l}_\zeta(\theta)$ be the landing point of $\widetilde{R}_\zeta(\theta)$ (cf. Subsection I.4.4).

The case $\sigma \in M_N$. Consider first the map N_{σ_0}. Then $1 \notin B_{\sigma_0}(0) \cup B_{\sigma_0}(u) \cup B_{\sigma_0}(\infty)$, so for $\zeta \in \{0, u, \infty\}$ the map φ_σ^ζ conjugates N_σ on $B_\sigma(\zeta)$ to $z \mapsto z^2$. As was remarked, we assume that the set $\partial B_{\sigma_0}(u) \cap \partial B_{\sigma_0}(\infty)$ is infinite. The positions of the three basins are presented in Fig. I.6 and Fig. 7.

By assumption, Head's angle α (see Subsection I.4.4 for the definition) is periodic with period k mod 1, so $\alpha = m/(2^k - 1)$ for some m. The angle β defined in Subsection I.4.4 is equal to $m/2^k$. Define also another angle $\delta = m/2^k + 1/2^{k+1}$. Since $\alpha < 1/2$, we have $\beta < \alpha < \delta$. Using the definitions of α, β, δ and the combinatorics of Newton maps described in Subsection I.4.4, it is easy to show that $l_u(2\beta) = l_\infty(1 - 2\beta)$, $2^{k-1}\beta = 1/2 \bmod 1$, $l_u(\delta) = l_\infty(1 - \delta)$ and $2^k\delta = 1/2 \bmod 1$. This implies that if we set $\tau(z) = 2z \bmod 1$, then

$$\tau^k([\beta, \delta]) = [0, 1/2] \supset (0, 1/2) \supset [\beta, \delta] \tag{18}$$

and τ^k is injective on $[\beta, \delta]$.

Definition 4.5. For $\sigma \in M_N$ let

$$L_\sigma = \mathrm{cl}(R_u(\delta) \cup R_\infty(1-\delta) \cup R_\infty(1-\beta) \cup \widetilde{R}_u(2\beta) \cup \widetilde{R}_u(2\delta) \cup \widetilde{R}_\infty(1-2\delta) \cup \widetilde{R}_\infty(1-2\beta) \cup R_u(\beta)).$$

(See Fig. 6, where L_σ is the dashed line.) The definitions of β and δ easily imply that L_{σ_0} is a Jordan curve.

Lemma 4.6. *The set L_σ is a Jordan curve for every $\sigma \in M_N$.*

Proof. For $\sigma \in M_N$ the critical point 1 is contained in the filled-in Julia set K_σ, so it cannot be in the basin of a supersink. Hence, the rays in $B_\sigma(u)$, $B_\sigma(\infty)$, $\widetilde{B}_\sigma(u)$ and $\widetilde{B}_\sigma(\infty)$ are defined (and have well defined landing points). We need to prove that for $\sigma \in M_N$ we have $l_u(\delta) = l_\infty(1-\delta)$, $l_\infty(1-\beta) = \widetilde{l}_u(2\beta)$, $\widetilde{l}_u(2\delta) = \widetilde{l}_\infty(1-2\delta)$ and $\widetilde{l}_\infty(1-2\beta) = l_u(\beta)$. First we show that for every $\sigma \in M_N$ the forward orbit of 1 does not hit the landing points of any of these rays. Since all these points are preimages of $p_\sigma = l_0(0) = l_u(0) = l_\infty(0)$ and $1 \in K_\sigma$, it is sufficient to prove that

$$p_\sigma \notin K_\sigma \quad \text{for all } \sigma \in \mathcal{U}. \tag{19}$$

To show (19), suppose that p_σ is a fixed point of the quadratic-like map $N_\sigma^k|_{U_\sigma}$ for some $\sigma \in \mathcal{U}$. Such a point moves continuously when we perturb σ and for $\sigma = \chi^{-1}(1/4)$ becomes a parabolic point. This is not possible, since p_σ is a repelling fixed point for all Newton maps. In this way we have shown (19), which implies that the forward orbit of 1 is disjoint from the landing points of the above rays.

Now we can prove the four equalities. We consider only the first one—others can be shown in the same way. Let

$$M' = \{\sigma \in M_N : l_u(\delta) = l_\infty(1-\delta) \text{ for } N_\sigma\}.$$

We will show that M' is open and closed in M_N. As $\sigma_0 \in M'$ and M_N is connected, this will imply $M' = M_N$.

Take $\sigma \in M'$. We have shown that the forward orbit of 1 is disjoint from $l_u(1/2)$. Hence, there exists a branch ν_σ of N_σ^{-k} defined in some open neighbourhood of $R_u(1/2) \cup l_u(1/2) \cup R_\infty(1/2)$ such that

$$\nu_\sigma(R_u(1/2) \cup l_u(1/2) \cup R_\infty(1/2)) = R_u(\delta) \cup l_u(\delta) \cup R_\infty(1-\delta). \tag{20}$$

As we perturb σ within M_N, these rays are defined and move in a continuous way (this follows e.g. from Lemma 2.4). Therefore, for a small perturbation of σ within M_N, the branch ν_σ exists and (20) holds. This shows that M' is open in M_N.

To see that M' is closed in M_N, suppose that we have a sequence $\sigma_n \in M'$ such that $\sigma_n \to \sigma \in M_N$ and $\sigma \notin M'$. Then $l_u(\delta)$ for N_σ is in the boundary of some preimage of $B_\sigma(\infty)$ different from $B_\sigma(\infty)$. Hence, there exists $0 \le j < k$ such that $l_u(2^j\delta) \in \widetilde{B}_\sigma(\infty)$, so $l_u(2^j\delta) = \widetilde{l}_\infty(1-2^{j+1}\delta)$. Therefore, there exists a branch $\widetilde{\nu}_\sigma$ of $N_\sigma^{-(k-j)}$ defined in some neighbourhood of $R_u(1/2) \cup l_u(1/2) \cup R_\infty(1/2)$ such that

$$\widetilde{\nu}_\sigma(R_u(1/2) \cup l_u(1/2) \cup R_\infty(1/2)) = R_u(2^j\delta) \cup l_u(2^j\delta) \cup \widetilde{R}_\infty(1-2^{j+1}\delta).$$

For a small perturbation of σ in M_N such a branch still exists. Thus, if we take sufficiently large n, then for the map N_{σ_n} we have

$$l_u(2^j\delta) = \widetilde{l}_\infty(1-2^{j+1}\delta).$$

But $\sigma_n \in M'$, so for N_{σ_n} we must have $l_u(2^j\delta) = l_\infty(1-2^j\delta)$. This is a contradiction. Therefore $\sigma \in M'$, so M' is closed in M_N. We conclude that $M' = M_N$. ∎

By Lemma 4.6, we can state the following definition.

Definition 4.7. For $\sigma \in M_N$ let E_σ be the component of $\widehat{\mathbb{C}} \setminus L_\sigma$ containing the critical point 1 and let

$$U'_\sigma = E_\sigma \setminus \operatorname{cl} N_\sigma^{-1}(\varphi_\sigma^u(\mathbb{D}_{\widehat{r}^{2^{-k+1}}}) \cup \varphi_\sigma^\infty(\mathbb{D}_{\widehat{r}^{2^{-k+1}}}))$$

for a fixed $\widehat{r} < 1$ close to 1.

By the definitions of β, δ and by (18), U'_σ is a topological disc containing 1 and $N_\sigma^k|_{U'_\sigma}$ is quadratic-like. Moreover,

$$\begin{aligned}
\partial N_\sigma(U'_\sigma) = \varphi_\sigma^u(&\{re^{2\pi i 2\beta} : r \in [\widehat{r}^{2^{-k+1}}, 1]\} \\
&\cup \{\widehat{r}^{2^{-k+1}} e^{2\pi i\theta} : \theta \in [2\beta, 2\delta]\} \\
&\cup \{re^{2\pi i 2\delta} : r \in [\widehat{r}^{2^{-k+1}}, 1]\}) \\
\cup\, \varphi_\sigma^\infty(&\{re^{2\pi i(1-2\delta)} : r \in [\widehat{r}^{2^{-k+1}}, 1]\} \\
&\cup \{\widehat{r}^{2^{-k+1}} e^{2\pi i\theta} : \theta \in [1-2\delta, 1-2\beta]\} \\
&\cup \{re^{2\pi i(1-2\beta)} : r \in [\widehat{r}^{2^{-k+1}}, 1]\}), \\
\partial N_\sigma^k(U'_\sigma) = \varphi_\sigma^u(&[\widehat{r}, 1] \cup \{\widehat{r}e^{2\pi i\theta} : \theta \in [0, 1/2]\} \cup [-1, -\widehat{r}]) \\
\cup\, \varphi_\sigma^\infty(&[\widehat{r}, 1] \cup \{\widehat{r}e^{2\pi i\theta} : \theta \in [1/2, 1]\} \cup [-1, -\widehat{r}]).
\end{aligned}$$

See Figs. 6 and 7.

Now we show

$$K_\sigma \subset U'_\sigma \quad \text{for all } \sigma \in M_N. \tag{21}$$

The boundary of U'_σ consists of points from the basins of attraction to u and ∞ and the landing points of some rays. It is obvious that K_σ cannot contain points from the basins of u and ∞. Since $1 \in K_\sigma$ and K_σ is connected for $\sigma \in M_N$, either $K_\sigma \subset U'_\sigma$ or K_σ contains the landing point of some ray from $\partial U'_\sigma$. But the second case is impossible by (19). Hence, (21) holds.

We have constructed a family $\{U'_\sigma\}_{\sigma \in M_N}$ of topological discs such that $N_\sigma^k|_{U'_\sigma}$ is quadratic-like and $K_\sigma \subset U'_\sigma$. This implies (see e.g. [McM]) that for $\sigma \in M_N$ the filled-in Julia sets K_σ of $N_\sigma^k|_{U_\sigma}$ and K'_σ of $N_\sigma^k|_{U'_\sigma}$ coincide.

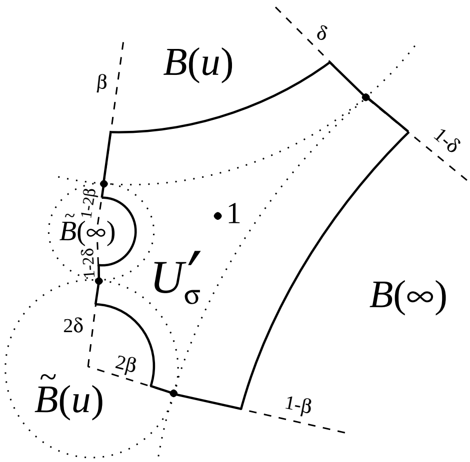

Fig. 6. The set U'_σ

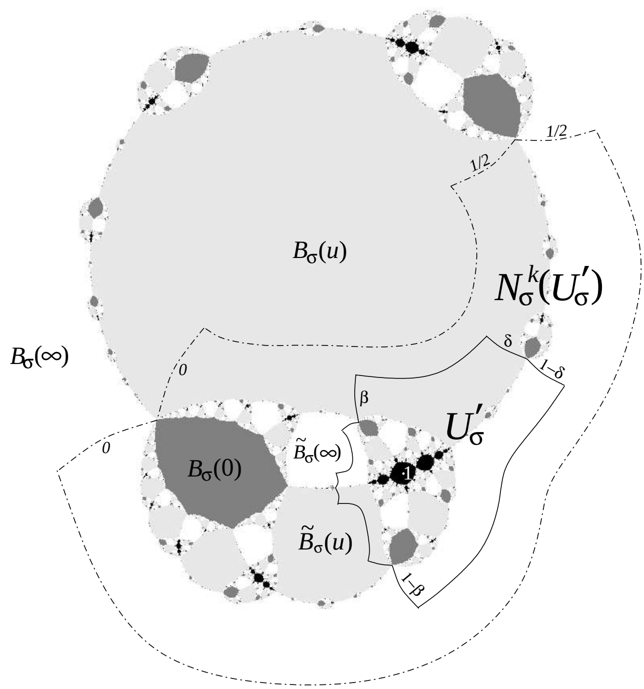

Fig. 7. The set U'_σ in the dynamical plane

The case $\sigma \notin M_N$. Now we extend the definition of U'_σ for σ in some topological disc $\mathcal{U}'$ such that $M_N \subset \mathcal{U}' \subset \mathcal{U}$. First we show that for $\zeta \in \{u, \infty\}$ and σ in some neighbourhood of M_N, if $1 \in B_\sigma(\zeta)$, then we can choose one of two Riemann maps φ_σ^ζ from Definition 4.3 such that φ_σ^ζ depends continuously on σ. To do it, we prove the following technical lemma:

LEMMA 4.8. *There exist $\varepsilon_0 > 0$ and an open set $\widetilde{\mathcal{U}} \subset \Sigma$ containing M_N such that for every $\sigma \in \widetilde{\mathcal{U}}$ and $\zeta \in \{u, \infty\}$ there exists an $\widetilde{h}_\sigma^\zeta$-fixed point $q_\sigma^\zeta \in \partial\mathbb{D}$ such that*

$$\textit{if} \quad 1 \in B_\sigma(\zeta), \quad \textit{then} \quad |\widetilde{v}_\sigma^\zeta - q_\sigma^\zeta| > \varepsilon_0,$$

where $\widetilde{h}_\sigma^\zeta$ is the map from (16) *and $\widetilde{v}_\sigma^\zeta = (\psi_\sigma^\zeta)^{-1}(1)$ is the unique non-zero $\widetilde{h}_\sigma^\zeta$-critical point in $\mathbb{D}$.*

Proof. To simplify notation, consider only the case $\zeta = u$. Suppose the lemma is not true. Then there exists a sequence $\sigma_n \in \Sigma$ such that $\sigma_n \to \sigma \in M_N$, for every n we have $1 \in B_{\sigma_n}(u)$ and

$$|\widetilde{v}_{\sigma_n}^u - q_{\sigma_n}|, |\widetilde{v}_{\sigma_n}^u - \widetilde{q}_{\sigma_n}| \to 0 \quad \text{as } n \to \infty,$$

where $q_{\sigma_n}, \widetilde{q}_{\sigma_n}$ are $\widetilde{h}_{\sigma_n}^u$-fixed points in $\partial\mathbb{D}$. For simplicity, set $h_n = \widetilde{h}_{\sigma_n}^u$, $v_n = \widetilde{v}_{\sigma_n}^u$, $q_n = q_{\sigma_n}$ and $\widetilde{q}_n = \widetilde{q}_{\sigma_n}$. By Lemma 2.3, h_n tends to $\widetilde{h}_\sigma(z) = e^{2\pi i\theta_\sigma}z^2$ and $\psi_{\sigma_n}^u$ tends to ψ_σ^u. Changing $\psi_{\sigma_n}^u, \psi_\sigma^u$ by the rotation by θ_σ, assume $\psi_\sigma^u = \varphi_\sigma^u$ so that $h_n(z) \to h(z) = z^2$. By Lemma 2.3, $h_n \to h$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{z_0\}$ for some $z_0 \in \partial D$. As 1 is the only h-fixed point in $\partial\mathbb{D}$, this easily implies that $q_n, \widetilde{q}_n \to 1$ and $z_0 = 1$. Thus, we have

$$v_n \to 1. \tag{22}$$

Let V be a small neighbourhood of the repelling N_σ-fixed point $p = p_\sigma$. Then there exists a branch ν of N_σ^{-1} defined on V such that $\nu(p) = p$ and $\operatorname{cl}\nu(V) \subset V$. Moreover, since $N_{\sigma_n} \to N_\sigma$ uniformly in the spherical metric on $\widehat{\mathbb{C}}$, for sufficiently large n there exists a branch ν_n of $N_{\sigma_n}^{-1}$ defined on V such that $\nu_n(p_{\sigma_n}) = p_{\sigma_n}$ and $\operatorname{cl}\nu_n(V) \subset V$.

Let Φ_n be the Böttcher coordinates for h_n on $\mathbb{D}$. Note that for sufficiently large n the map Φ_n^{-1} is defined on $\operatorname{cl}\mathbb{D}_{\widehat{r}2^{-k}}$ for $\widehat{r}$ from Definition 4.7 and $\Phi_n \to \mathrm{id}$ uniformly on this set. Let

$$I_n = \Phi_n^{-1}([\widehat{r}^2, \widehat{r}]).$$

By Lemma 2.3, we can assume $\psi_{\sigma_n}^u(I_n) \subset V$ for large n provided $\widehat{r}$ is fixed sufficiently close to 1. Define a curve $\gamma_n : [0, \infty) \to \mathbb{D}$ setting $\gamma_n(s) = \Phi_n^{-1}((1-s)\widehat{r}^2 + s\widehat{r})$ for $s \in [0,1]$ and $\gamma_n(s+1) = (\psi_{\sigma_n}^u)^{-1}(\nu_n(\psi_{\sigma_n}^u(\gamma_n(s))))$ for $s > 1$. Define also a curve $\gamma : [0, \infty) \to \mathbb{D}$ parameterizing the segment $[\widehat{r}^2, 1)$ such that $\gamma(0) = \widehat{r}^2$, $\gamma(1) = \widehat{r}$, γ is affine on $[0,1]$ and $\gamma(s+1) = \sqrt{\gamma(s)}$. Since $1 \notin V$, we have $v_n = (\psi_{\sigma_n}^u)^{-1}(1) \notin \gamma_n$, so there exist two branches ν_n', ν_n'' of h_n^{-1} defined in a neighbourhood of γ_n such that $\nu_n'(\gamma_n(0)) = \Phi_n^{-1}(\widehat{r})$

and $\nu_n''(\gamma_n(0)) = \Phi_n^{-1}(-\widehat{r})$. Let

$$\gamma_n' = \nu_n'(\gamma_n), \qquad \gamma_n'' = \nu_n''(\gamma_n)$$

and define

$$\begin{aligned} \Gamma_n^u &= \gamma_n' \cup \Phi_n^{-1}(\partial\mathbb{D}_{\widehat{r}} \cap \mathbb{D}^+) \cup \gamma_n'', \\ \Gamma^u &= [\widehat{r}, 1) \cup (\partial\mathbb{D}_{\widehat{r}} \cap \mathbb{D}^+) \cup (-1, -\widehat{r}], \\ \Gamma^\infty &= [\widehat{r}, 1) \cup (\partial\mathbb{D}_{\widehat{r}} \cap \mathbb{D}^-) \cup (-1, -\widehat{r}]. \end{aligned}$$

Note that since $1 \notin B_{\sigma_n}(\infty), B_\sigma(\infty)$, the maps $\varphi_{\sigma_n}^\infty, \varphi_\sigma^\infty$ are uniquely determined and coincide with the inverse of the Böttcher coordinates. Now we show that for sufficiently large n,

(23) $\operatorname{cl}(\psi_{\sigma_n}^u(\Gamma_n^u) \cup \varphi_{\sigma_n}^\infty(\Gamma^\infty))$ is a Jordan curve converging to $\partial N_\sigma^k(U_\sigma')$ in the Hausdorff metric.

Note that the curves $\psi_{\sigma_n}^u(\gamma_n')$ and $\varphi_{\sigma_n}^\infty([\widehat{r}, 1))$ land at the same point p_{σ_n} and $\varphi_{\sigma_n}^\infty((-1, -\widehat{r}])$ lands at $l_\infty(1/2)$ for the map N_{σ_n}. Hence, to prove that $\operatorname{cl}(\psi_{\sigma_n}^u(\Gamma_n^u) \cup \psi_{\sigma_n}^\infty(\Gamma^\infty))$ is a Jordan curve, we only need to show that for sufficiently large n the curve $\psi_{\sigma_n}^u(\gamma_n'')$ lands at $l_\infty(1/2)$ for the map N_{σ_n}.

By Lemma 2.3, $\gamma_n(s)$ tends to $\gamma(s)$ almost uniformly on $[0, \infty)$ and γ_n, γ satisfy the assumptions of the case (a) of Lemma 2.4. By this lemma, for arbitrarily small $\varepsilon > 0$ there exists $s_0 > 0$ such that $\psi_{\sigma_n}^u(\gamma_n([s_0, \infty))) \subset \mathbb{D}_\varepsilon(p)$ for large n. By (19), N_σ maps biholomorphically a small neighbourhood of $x = l_u(1/2) = l_\infty(1/2)$ onto a small neighbourhood of p. Moreover, there exists a small $\varepsilon' > 0$ such that N_{σ_n} for large n maps biholomorphically $\mathbb{D}_{\varepsilon'}(x)$ onto a set containing $\mathbb{D}_\varepsilon(p)$. Since x is the landing point of $\psi_\sigma^u((-1, -\widehat{r}])$ and $\varphi_\sigma^\infty((-1, -\widehat{r}])$, by almost uniform convergence, $\mathbb{D}_{\varepsilon'}(x)$ contains points from $\psi_{\sigma_n}^u(\gamma_n'')$ and $\varphi_{\sigma_n}^\infty((-1, -\widehat{r}])$ for large n. Therefore, $\psi_{\sigma_n}^u(\gamma_n'')$ and $\varphi_{\sigma_n}^\infty((-1, -\widehat{r}])$ must land at the same point near x. In this way we have proved that $\operatorname{cl}(\psi_{\sigma_n}^u(\Gamma_n) \cup \varphi_{\sigma_n}^\infty(\Gamma))$ is a Jordan curve for large n. Moreover, Lemma 2.4 implies that $\operatorname{cl}\gamma_n$ tends to $\operatorname{cl}\gamma$ and $\operatorname{cl}\psi_{\sigma_n}^u(\gamma_n)$ tends to $\operatorname{cl}\psi_\sigma^u(\gamma)$ in the Hausdorff metric. Since $\Phi_n \to \mathrm{id}$ uniformly on $\operatorname{cl}\mathbb{D}_{\widehat{r}}$, $h_n \to h$ uniformly on $\operatorname{cl}\mathbb{D} \setminus \{1\}$ and $\psi_{\sigma_n}^u \to \psi_\sigma^u$ almost uniformly on $\mathbb{D}$, we conclude that $\operatorname{cl}\Gamma_n^u$ tends to $\operatorname{cl}\Gamma^u$ in the Hausdorff metric.

It is easy to check that the case (a) of Lemma 2.4 holds for $\gamma_n^\infty = \gamma^\infty = [\widehat{r}, 1)$ with a suitable parameterization and $\varphi_{\sigma_n}^\infty, \varphi_\sigma^\infty$ instead of ψ_n, ψ. Repeating the above arguments we show that $\operatorname{cl}\varphi_{\sigma_n}^\infty(\Gamma^\infty)$ tends to $\operatorname{cl}\varphi_\sigma^\infty(\Gamma^\infty)$ in the Hausdorff metric. Hence, $\operatorname{cl}(\psi_{\sigma_n}^u(\Gamma_n^u) \cup \varphi_{\sigma_n}^\infty(\Gamma^\infty))$ tends to $\operatorname{cl}(\psi_\sigma^u(\Gamma^u) \cup \varphi_\sigma^\infty(\Gamma^\infty)) = \partial N_\sigma^k(U_\sigma')$ in the Hausdorff metric, which gives (23).

By (23), we can define A_n for large n to be the component of $\widehat{\mathbb{C}} \setminus \operatorname{cl}(\psi_{\sigma_n}^u(\Gamma_n^u) \cup \varphi_{\sigma_n}^\infty(\Gamma^\infty))$ containing $N_{\sigma_n}^k(1)$ and U_n to be the component of $N_{\sigma_n}^{-k}(A_n)$ containing 1. Then U_n is a topological disc and $\operatorname{cl}U_n$ tends to $\operatorname{cl}U_\sigma'$ in the Hausdorff metric. Moreover, $\Gamma_n^u = (\psi_{\sigma_n}^u)^{-1}(\partial N_{\sigma_n}^k(U_n) \cap B_{\sigma_n}(u))$.

It is obvious that $h_n^j(0) \notin \varGamma_n^u$ for $j = 0, \ldots, k$. Moreover, for large n we have $N_{\sigma_n}^j(1) \in U_n$ for $j = 0, \ldots, k$, so $h_n^j(v_n) \notin \varGamma_n^u$. Hence, there exists a branch $\widetilde{\nu}_n$ of h_n^{-k} defined in a neighbourhood of $\varGamma_n^u$ such that

$$\widetilde{\nu}_n(\varPhi_n^{-1}(\partial \mathbb{D}_{\widehat{r}} \cap \mathbb{D}^+)) = \varPhi_n^{-1}(\{\widehat{r}^{2^{-k}} e^{2\pi i\theta} : \theta \in (\beta, \delta)\}).$$

Then $\widetilde{\nu}_n(\varGamma_n^u)$ is a simple arc in $\mathbb{D}$ with endpoints landing at two points in $\partial\mathbb{D}$. Let Y_n be the component of $\mathbb{D} \setminus \widetilde{\nu}_n(\varGamma_n^u)$ which does not contain 0. Then

$$Y_n \supset (\psi_{\sigma_n}^u)^{-1}(U_n \cap B_{\sigma_n}(u)).$$

Let

$$Y = \{re^{2\pi i\theta} : r \in [\widehat{r}^{2^{-k}}, 1), \theta \in (\beta, \delta)\}.$$

By Definition 4.7,

$$Y = (\psi_\sigma^u)^{-1}(U_\sigma' \cap B_\sigma(u)), \quad h^k(\partial Y \cap \mathbb{D}) = \varGamma^u.$$

Recall that $\operatorname{cl} \varGamma_n^u$ tends to $\operatorname{cl} \varGamma^u$ in the Hausdorff metric, $\varPhi_n \to \mathrm{id}$ uniformly on $\operatorname{cl} \mathbb{D}_{\widehat{r}^{2^{-k}}}$ and $h_n \to h$ uniformly on $\operatorname{cl} \mathbb{D} \setminus \{1\}$. This easily implies that ∂Y_n tends to ∂Y in the Hausdorff metric. Hence, $\operatorname{dist}(\operatorname{cl} Y_n, 1) > c$ for a constant $c > 0$ independent of n. But since $1 \in U_n \cap B_{\sigma_n}(u)$, we have $v_n = (\psi_{\sigma_n}^u)^{-1}(1) \in Y_n$, so $|v_n - 1| > c$, which contradicts (22). ∎

Remark. In fact, one can show in the same way that if $\zeta \in \{u, \infty\}$ and $\sigma_n \to \sigma \in M_N$ with $1 \in B_{\sigma_n}(\zeta)$, then $\widetilde{v}_{\sigma_n}^\zeta \to e^{2\pi i(\alpha - \theta_\sigma^\zeta)}$ for Head's angle α and θ_σ^ζ from (16).

Lemma 4.8 implies that for every $\sigma \in \widetilde{\mathcal{U}}$ and $\zeta \in \{u, \infty\}$ there exists an $\widetilde{h}_\sigma^\zeta$-fixed point $q_\sigma^\zeta \in \partial\mathbb{D}$ depending continuously on σ, such that if $1 \in B_\sigma(\zeta)$, then for the other $\widetilde{h}_\sigma^\zeta$-fixed point $\widetilde{q}_\sigma^\zeta \in \partial\mathbb{D}$ we have $|q_\sigma^\zeta - \widetilde{q}_\sigma^\zeta| > \varepsilon_0$. This easily gives

Corollary 4.9. *There exists a topological disc $\mathcal{U}' \subset \mathcal{U}$ containing M_N such that for each $\sigma \in \mathcal{U}'$ and $\zeta \in \{u, \infty\}$, if $1 \in B_\sigma(\zeta)$, then we can choose a Riemann map φ_σ^ζ from Definition* 4.3 *such that φ_σ^ζ depends continuously on $\sigma \in \mathcal{U}'$ in the almost uniform convergence topology. In particular*, *Lemmas* 2.3 *and* 2.4 *hold for N_σ, $\sigma \in \mathcal{U}'$ and φ_σ^ζ instead of ψ_σ^ζ. Moreover*, *there exists $\varepsilon_0 > 0$ such that if $1 \in B_\sigma(\zeta)$, then*

$$|v_\sigma^\zeta - 1| > \varepsilon_0,$$

where v_σ^ζ is the unique non-zero h_σ^ζ-critical point in $\mathbb{D}$ for h_σ^ζ from (17). ∎

From now on, by φ_σ^ζ we will always understand the map from the above corollary.

Let $\mathcal{U}'$ be the topological disc from Corollary 4.9. Now we define U_σ' for $\sigma \in \mathcal{U}'$. For $\zeta \in \{u, \infty\}$ let $\varPhi_\sigma^\zeta$ be the Böttcher coordinates for h_σ^ζ on $\mathbb{D}$. If we take $\mathcal{U}'$ sufficiently close to M_N, then $(\Phi_\sigma^\zeta)^{-1}$ is defined on $\mathrm{cl}\,\mathbb{D}_{\widehat{r}}$ for $\widehat{r}$ from Definition 4.7. Let

$$I_\sigma^\zeta = (\Phi_\sigma^\zeta)^{-1}([\widehat{r}^2, \widehat{r}]).$$

By Lemma 2.3 and Corollary 4.9, if we take $\mathcal{U}'$ sufficiently close to M_N and choose $\widehat{r}$ sufficiently close to 1, then for every $\sigma \in \mathcal{U}'$ there exist two branches $\nu_\sigma'^\zeta$, $\nu_\sigma''^\zeta$ of $(h_\sigma^\zeta)^{-1}$ defined in a neighbourhood V of 1 such that $I_\sigma^\zeta \subset V$, $\mathrm{cl}\,\nu_\sigma'^\zeta(V) \subset V$, $\nu_\sigma'^\zeta((\Phi_\sigma^\zeta)^{-1}(\widehat{r}^2)) = (\Phi_\sigma^\zeta)^{-1}(\widehat{r})$, $\nu_\sigma''^\zeta((\Phi_\sigma^\zeta)^{-1}(\widehat{r}^2)) = (\Phi_\sigma^\zeta)^{-1}(-\widehat{r})$. Connect I_σ^ζ to 1 by the curve $\gamma_\sigma^\zeta = \bigcup_{n=0}^\infty (\nu_\sigma^\zeta)^n (I_\sigma^\zeta)$ and let

$$\gamma_\sigma'^\zeta = \nu_\sigma'^\zeta(\gamma_\sigma^\zeta), \quad \gamma_\sigma''^\zeta = \nu_\sigma''^\zeta(\gamma_\sigma^\zeta).$$

Then $\gamma_\sigma'^\zeta$ begins at $(\Phi_\sigma^\zeta)^{-1}(\widehat{r})$ and $\gamma_\sigma''^\zeta$ begins at $(\Phi_\sigma^\zeta)^{-1}(-\widehat{r})$. Define

$$\Gamma_\sigma^\zeta = \gamma_\sigma'^\zeta \cup (\Phi_\sigma^\zeta)^{-1}(\partial\mathbb{D}_{\widehat{r}} \cap \mathbb{D}^\pm) \cup \gamma_\sigma''^\zeta, \tag{24}$$

where we take $\mathbb{D}^+$ for $\zeta = u$ and $\mathbb{D}^-$ for $\zeta = \infty$. Then Γ_σ^ζ is a simple arc in $\mathbb{D}$ connecting 1 to some point of $(h_\sigma^\zeta)^{-1}(1)$ (see Fig. 8).

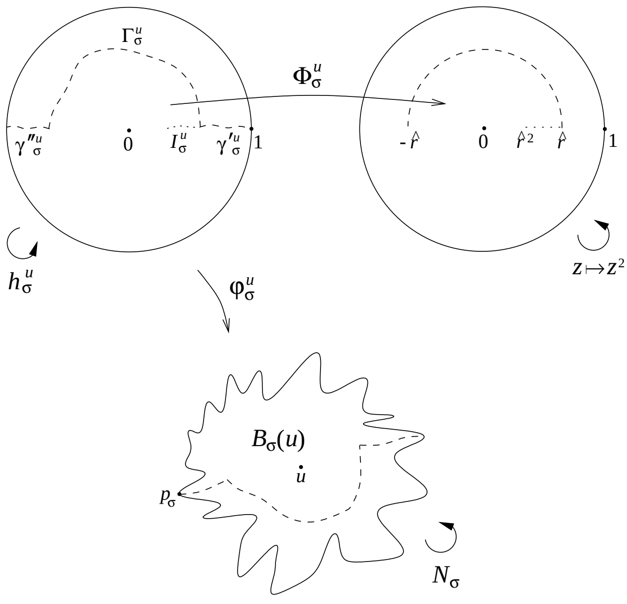

Fig. 8. The curves $\gamma_\sigma'^u$, $\gamma_\sigma''^u$ and Γ_σ^u

Now we show that if we take $\mathcal{U}'$ sufficiently close to M_N, then

(25) for every $\sigma \in \mathcal{U}'$, $\mathrm{cl}(\varphi_\sigma^u(\Gamma_\sigma^u) \cup \varphi_\sigma^\infty(\Gamma_\sigma^\infty))$ is a Jordan curve depending continuously on σ in the Hausdorff metric.

The proof of (25) is the same as for (23) and we leave it to the reader (by Corollary 4.9, we can use Lemmas 2.3 and 2.4 for φ_σ^ζ instead of ψ_σ^ζ). Note that for $\sigma \in M_N$ we have $\varphi_\sigma^\zeta(\Gamma_\sigma^\zeta) = \partial N_\sigma^k(U'_\sigma) \cap B_\sigma(\zeta)$ and $\mathrm{cl}(\varphi_\sigma^u(\Gamma_\sigma^u) \cup \varphi_\sigma^\infty(\Gamma_\sigma^\infty)) = \partial N_\sigma^k(U'_\sigma)$. By (25), we can make the following definition.

Definition 4.10. For $\sigma \in \mathcal{U}'$ let A_σ be the component of $\widehat{\mathbb{C}} \setminus \mathrm{cl}(\varphi_\sigma^u(\Gamma_\sigma^u) \cup \varphi_\sigma^\infty(\Gamma_\sigma^\infty))$ containing $N_\sigma^k(1)$ and let U'_σ be the component of $N_\sigma^{-k}(A_\sigma)$ containing 1.

Then (provided $\mathcal{U}'$ is sufficiently close to M_N) the set U'_σ is a topological disc and N_σ^k is quadratic-like on U'_σ. Moreover, $\mathrm{cl}\, U'_\sigma$ is continuous with respect to $\sigma \in \mathcal{U}'$ in the Hausdorff metric and for $\sigma \in M_N$ the definition of U'_σ coincides with the previous one.

For $\sigma \in \mathcal{U}'$ let K'_σ be the filled-in Julia set of $N_\sigma^k|_{U'_\sigma}$. Since $\mathrm{diam}\, K_\sigma$ and $\mathrm{diam}\, K'_\sigma$ are continuous functions of σ (see [McM]), we can assume that $K_\sigma \subset U'_\sigma$ for all $\sigma \in \mathcal{U}'$. As previously, this gives $K'_\sigma = K_\sigma$.

The above properties imply that $\{N_\sigma^k|_{U'_\sigma}\}_{\sigma \in \mathcal{U}'}$ is a Mandelbrot-like family equivalent to $\mathbf{N}$ (see Section 3).

4.2. *The construction of* $\mathbf{f}_t$. In this subsection we define the families $\mathbf{f}_t$ for $0 < t < 1$ using quasiconformal surgery. Consider first a quadratic Blaschke product h_t defined by

$$h_t(z) = \frac{(t+2)z^2 + t}{tz^2 + t + 2}, \qquad t \in [0,1]. \tag{26}$$

Then 0 is a h_t-critical point and $h_t(0) = t/(t+2)$. Moreover, for $t \in [0,1)$ the point

$$w_t = (1 - \sqrt{1-t^2})/t$$

is an attracting h_t-fixed point of multiplier t. Note that if $t \to 1^-$, then $h_t \to h_1$ uniformly in the spherical metric on $\widehat{\mathbb{C}}$.

For $t \in [0,1)$ let $D_t \subset \mathbb{D}$ be the open hyperbolic disc centred at w_t such that $-\widehat{r} \in \partial D_t$ for $\widehat{r}$ from Definition 4.7. Note that D_t is a Euclidean disc symmetric with respect to the real axis, containing the entire forward trajectory of the h_t-critical point 0. Let

$$W_t = h_t^{-1}(D_t) \cap \mathbb{D}.$$

By definition, W_t is a topological disc, ∂W_t is a Jordan curve and $W_t \supset \mathrm{cl}\, h_t(W_t) = \mathrm{cl}\, D_t$. Moreover, h_t has degree two on W_t, is a degree two covering on the annulus $W_t \setminus \mathrm{cl}\, D_t$ and maps the segment $(0,1)$ into itself. See Fig. 9.

Consider now the map N_σ for $\sigma \in \mathcal{U}'$. Recall that the Riemann map φ_σ^u from Corollary 4.9 conjugates N_σ on $B_\sigma(u)$ to the Blaschke product h_σ^u from (17) and Φ_σ^u are the Böttcher coordinates for h_σ^u on $\mathbb{D}$ (see Subsection 4.1). Let

$$D_\sigma = (\Phi_\sigma^u)^{-1}(\mathbb{D}_{\widehat{r}}), \qquad W_\sigma = (\Phi_\sigma^u)^{-1}(\mathbb{D}_{\sqrt{\widehat{r}}}).$$

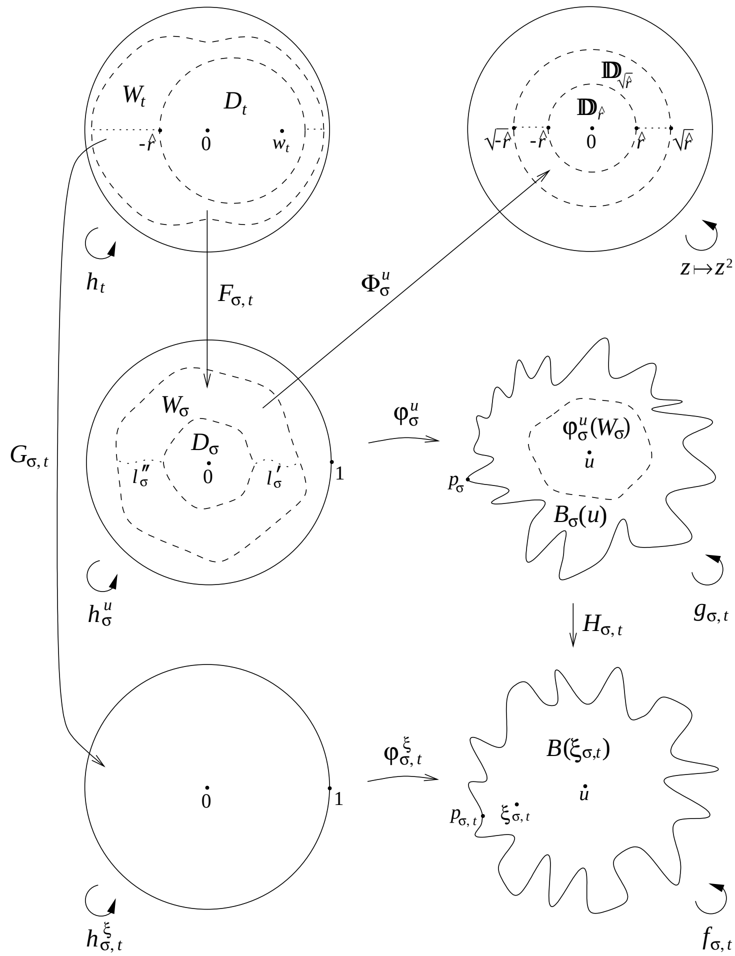

Fig. 9. The construction of $f_{\sigma,t}$

Then W_σ is a component of $(h^u_\sigma)^{-1}(D_\sigma)$ such that W_σ is a topological disc, ∂W_σ is a Jordan curve, $W_\sigma \supset \operatorname{cl} D_\sigma$, h^u_σ has degree two on W_σ and is a degree two covering on the annulus $W_\sigma \setminus \operatorname{cl} D_\sigma$. Moreover, if $\sigma \in M_N$, then $D_\sigma = \mathbb{D}_{\widehat{r}}$ and $W_\sigma = \mathbb{D}_{\sqrt{\widehat{r}}}$.

Let

$$\ell'_\sigma = (\Phi^u_\sigma)^{-1}([\widehat{r}, \sqrt{\widehat{r}}]), \qquad \ell''_\sigma = (\Phi^u_\sigma)^{-1}([-\sqrt{\widehat{r}}, -\widehat{r}])$$

(see Fig. 9). Note that $\ell'_\sigma = \gamma'^u_\sigma \cap \operatorname{cl} W_\sigma$ and $\ell''_\sigma = \gamma''^u_\sigma \cap \operatorname{cl} W_\sigma$ for $\gamma'^u_\sigma, \gamma''^u_\sigma$ from (24).

Now for $\sigma \in \mathcal{U}'$ and $t \in [0,1)$ we define a C^1-diffeomorphism

$$F_{\sigma,t} : \operatorname{cl} W_t \xrightarrow{\text{onto}} \operatorname{cl} W_\sigma$$

such that:

(a) $h^u_\sigma \circ F_{\sigma,t} = F_{\sigma,t} \circ h_t$ on ∂W_t,
(b) $F_{\sigma,t}$ maps biholomorphically D_t onto D_σ with $F_{\sigma,t}(0) = 0$,
(c) $F_{\sigma,t}((\mathrm{cl}\, W_t \setminus D_t) \cap (0,1)) = \ell'_\sigma$, $F_{\sigma,t}((\mathrm{cl}\, W_t \setminus D_t) \cap (-1,0)) = \ell''_\sigma$,
(d) $F_{\sigma,t}$ depends continuously on σ, t,
(e) $F_{\sigma,0}$ is holomorphic for $\sigma \in \mathcal{U}'$ and $F_{\sigma,0} = \mathrm{id}$ for $\sigma \in M_N$.

We define $F_{\sigma,t}$ as follows: on $\mathrm{cl}\, D_t$ let $F_{\sigma,t}$ be the Möbius map sending 0 and the two points from $\partial D_t \cap (-1,1)$ to 0, $-\widehat{r}$ and $\widehat{r}$ respectively, composed with $(\Phi^u_\sigma)^{-1}$. Then the condition (b) is obviously satisfied. On each of the two segments of $(\mathrm{cl}\, W_t \setminus D_t) \cap (-1,1)$ let $F_{\sigma,t}$ be the composition of a suitable affine map with $(\Phi^u_\sigma)^{-1}$. Since both h_t on ∂W_t and h^u_σ on ∂W_σ are degree two coverings, the conditions (a) and (c) define $F_{\sigma,t}$ uniquely on ∂W_t. Then it is easy to extend $F_{\sigma,t}$ C^1-diffeomorphically to both components of $\mathrm{cl}\, W_t \setminus (D_t \cup (-1,1))$ so that the conditions (d)–(e) are satisfied.

Let

$$g_{\sigma,t} = \begin{cases} N_\sigma & \text{on } \widehat{\mathbb{C}} \setminus \varphi^u_\sigma(W_\sigma), \\ \varphi^u_\sigma \circ F_{\sigma,t} \circ h_t \circ F^{-1}_{\sigma,t} \circ (\varphi^u_\sigma)^{-1} & \text{on } \varphi^u_\sigma(W_\sigma). \end{cases}$$

By definition, $g_{\sigma,t}$ is a C^1 cubic branched cover of $\widehat{\mathbb{C}}$, holomorphic on $\widehat{\mathbb{C}} \setminus \mathrm{cl}\, \varphi^u_\sigma(W_\sigma)$ and $g_{\sigma,t}(\varphi^u_\sigma(W_\sigma))$. Moreover, $\mathrm{cl}\, g_{\sigma,t}(\varphi^u_\sigma(W_\sigma)) \subset \varphi^u_\sigma(W_\sigma)$, the function $(\sigma,t) \mapsto g_{\sigma,t}$ is continuous, and by the condition (e) and the uniqueness of the Böttcher coordinates, we have $g_{\sigma,0} = N_\sigma$.

Now we define a continuous family of $g_{\sigma,t}$-invariant conformal structures $\mu_{\sigma,t}$ on $\widehat{\mathbb{C}}$ setting

$$\mu_{\sigma,t} = \begin{cases} \mu_0 & \text{on } g_{\sigma,t}(\varphi^u_\sigma(W_\sigma)), \\ (g^n_{\sigma,t})^*(\mu_0) & \text{on } g^{-n}_{\sigma,t}(g_{\sigma,t}(\varphi^u_\sigma(W_\sigma))), \\ \mu_0 & \text{else}, \end{cases}$$

where μ_0 is the standard structure. Note that $\mu_{\sigma,t} \in L^\infty$, because for every $z \in \widehat{\mathbb{C}}$ its forward trajectory under $g_{\sigma,t}$ hits at most twice the closed annulus $\mathrm{cl}\, \varphi^u_\sigma(W_\sigma) \setminus g_{\sigma,t}(\varphi^u_\sigma(W_\sigma))$, where the map is not holomorphic. By the measurable Riemann theorem (see [DH2]), there exists a continuous family of quasiconformal homeomorphisms $H_{\sigma,t}$ of $\widehat{\mathbb{C}}$ such that $H_{\sigma,t}(\zeta) = \zeta$ for $\zeta \in \{0, 1, \infty\}$, $H_{\sigma,t} \circ g_{\sigma,t} \circ H^{-1}_{\sigma,t}$ is rational and $H_{\sigma,0} = \mathrm{id}$. Moreover, $H_{\sigma,t}$ is holomorphic on $\widehat{\mathbb{C}} \setminus \mathrm{cl} \bigcup_{n=0}^\infty g^{-n}_{\sigma,t}(\varphi^u_\sigma(W_\sigma))$ and $g_{\sigma,t}(\varphi^u_\sigma(W_\sigma))$. It is easy to check that $H_{\sigma,t} \circ g_{\sigma,t} \circ H^{-1}_{\sigma,t} = f_{a(\sigma,t),b(\sigma,t)}$ for some continuous functions $a(\sigma,t), b(\sigma,t)$. For simplicity, write $f_{\sigma,t}$ for $f_{a(\sigma,t),b(\sigma,t)}$. Then $f_{\sigma,0} = N_\sigma$ because $H_{\sigma,0} = \mathrm{id}$. Moreover, since $g_{\sigma,t}$ has an attracting fixed point of multiplier t in $g_{\sigma,t}(\varphi^u_\sigma(W_\sigma))$, we have $(a(\sigma,t), b(\sigma,t)) \in \mathrm{Fix}(t)$. Let

$$U_{\sigma,t} = H_{\sigma,t}(U'_\sigma).$$

Note that by the definitions of U'_σ and $g_{\sigma,t}$, we have $g^k_{\sigma,t} = N^k_\sigma$ on U'_σ. Hence, $U_{\sigma,t}$ is a topological disc containing 1 and the map $f^k_{\sigma,t}|_{U_{\sigma,t}}$ is quadratic-like and hybrid equivalent to $N^k_\sigma|_{U'_\sigma}$. Moreover,

$$\partial f^k_{\sigma,t}(U_{\sigma,t}) = \mathrm{cl}(H_{\sigma,t}(\varphi^u_\sigma(\Gamma^u_\sigma)) \cup H_{\sigma,t}(\varphi^\infty_\sigma(\Gamma^\infty_\sigma))) \tag{27}$$

for $\Gamma^u_\sigma, \Gamma^\infty_\sigma$ from (24).

Let $\xi_{\sigma,t}$ be the attracting $f_{\sigma,t}$-fixed point. Denote by $B(\xi_{\sigma,t})$ its immediate basin of attraction and let $p_{\sigma,t}$ be the unique repelling $f_{\sigma,t}$-fixed point. By construction, $u \in B(\xi_{\sigma,t})$. Note that the boundary of $f^k_{\sigma,t}(U_{\sigma,t})$ in $B(\xi_{\sigma,t})$ near $p_{\sigma,t}$ forms a curve ending at $p_{\sigma,t}$. Let

$$\varphi^\xi_{\sigma,t} : \mathbb{D} \to B(\xi_{\sigma,t})$$

be the unique Riemann mapping such that $\varphi^\xi_{\sigma,t}(0) = u$ and $(\varphi^\xi_{\sigma,t})^{-1}$ maps this curve to a curve ending at 1. This map extends to $\partial\mathbb{D}$, because $\partial B(\xi_{\sigma,t}) = H_{\sigma,t}(\partial B_\sigma(u))$ is locally connected. In particular, this implies $\varphi^\xi_{\sigma,t}(1) = p_{\sigma,t}$.

The map $\varphi^\xi_{\sigma,t}$ conjugates $f_{\sigma,t}$ on $B(\xi_{\sigma,t})$ to some quadratic or cubic Blaschke product $h^\xi_{\sigma,t}$ with an attracting fixed point of multiplier t, a critical point at 0 and a fixed point at 1. If $1 \notin B(\xi_{\sigma,t})$, then $\deg h^\xi_{\sigma,t} = 2$ and it is easy to check that in this case $h^\xi_{\sigma,t} = h_t$ for h_t from (26). If $1 \in B(\xi_{\sigma,t})$, then $\deg h^\xi_{\sigma,t} = 3$.

Let

$$\varphi^\infty_{\sigma,t} = H_{\sigma,t} \circ \varphi^\infty_\sigma.$$

Since $H_{\sigma,t}$ is holomorphic on $B(\infty)$ for N_σ, the map $\varphi^\infty_{\sigma,t}$ is a Riemann mapping from $\mathbb{D}$ onto $B(\infty)$ for $f_{\sigma,t}$ such that $\varphi^\infty_{\sigma,t}(0) = \infty$ and $\varphi^\infty_{\sigma,t}(1) = p_{\sigma,t}$. Moreover, $\varphi^\infty_{\sigma,t}$ conjugates $f_{\sigma,t}$ on $B(\infty)$ to h^∞_σ from (17).

By Lemma 2.3 and Corollary 4.9, it is easy to see that for $\zeta \in \{\xi, \infty\}$ the map $\varphi^\zeta_{\sigma,t}$ depends continuously on σ, t in the almost uniform convergence topology.

Let

$$G_{\sigma,t} = (\varphi^\xi_{\sigma,t})^{-1} \circ H_{\sigma,t} \circ \varphi^u_\sigma \circ F_{\sigma,t}. \tag{28}$$

Then $G_{\sigma,t}$ is a quasiconformal homeomorphism on W_t and conjugates h_t to $h^\xi_{\sigma,t}$ (see Fig. 9). By definition, $G_{\sigma,t}$ is holomorphic on D_t and since it conjugates two holomorphic maps, it is easy to see that in fact it is holomorphic on W_t.

Now we show that if $(a(\sigma_1, t_1), b(\sigma_1, t_1)) = (a(\sigma_2, t_2), b(\sigma_2, t_2))$, then $U_{\sigma_1,t_1} = U_{\sigma_2,t_2}$. It is obvious that we must have $t_1 = t_2 = t$. Hence, $G^{-1}_{\sigma_1,t} \circ G_{\sigma_2,t}$ conjugates holomorphically h_t to h_t on W_t. Since W_t contains the infinite forward orbit of the h_t-critical point 0, it follows that

$G^{-1}_{\sigma_1,t} \circ G_{\sigma_2,t} = \mathrm{id}$ on W_t. This together with the definition of $U_{\sigma,t}$ easily implies $U_{\sigma_1,t} = U_{\sigma_2,t}$.

Note that if $1 \notin B(\xi_{\sigma,t})$ (which is equivalent to $1 \notin B_\sigma(u)$), then $G_{\sigma,t}$ conjugates h_t to h_t, so $G_{\sigma,t} = \mathrm{id}$. Moreover, $N^k_\sigma(U'_\sigma) \cap B_\sigma(u) = \varphi^u_\sigma(\mathbb{D}^+ \setminus \mathrm{cl}\, \mathbb{D}_{\widehat{r}})$ in this case. Hence, by the definition of $F_{\sigma,t}$,

$$\text{(29)} \qquad \text{If} \quad 1 \notin B(\xi_{\sigma,t}), \quad \text{then} \quad f^k_{\sigma,t}(U_{\sigma,t}) \cap B(\xi_{\sigma,t}) = \varphi^\xi_{\sigma,t}(\mathbb{D}^+ \setminus \mathrm{cl}\, D_t).$$

Let

$$Q_t(\sigma) = (a(\sigma,t), b(\sigma,t)), \qquad \sigma \in \mathcal{U}',\ t \in [0,1).$$

Lemma 4.11. *For every $t \in [0,1)$ and $(a_0,b_0) \in Q_t(\mathcal{U}')$ there exists an open neighbourhood $\mathcal{V}$ of (a_0,b_0) in $\mathbb{C} \times \mathbb{C}$ such that* $\mathrm{Fix}(t) \cap \mathcal{V}$ *is a one-dimensional complex manifold.*

Proof. Note that for (a,b) near (a_0,b_0) all the critical and fixed points for the map $f_{a,b}$ are simple. By the implicit function theorem, for (a,b) close to (a_0,b_0) there exists a holomorphic map $(a,b) \mapsto \xi(a,b) \neq 0,\infty$ such that $\xi(a,b)$ is an attracting $f_{a,b}$-fixed point and $\xi(a_0,b_0)$ has multiplier t. Then the map $z \mapsto f_{a,b}(\xi(a,b)z)/\xi(a,b)$ has supersinks at $0,\infty$ and a fixed point at 1. It is easy to check that such a map has the form

$$\widetilde{f}_{A,B}(z) = z^2 \frac{z+A+B-1}{Az+B}, \qquad A \neq 1,\ B \neq 0,\ A+B \neq 0,$$

for (A,B) depending holomorphically on (a,b). Since $\widetilde{f}_{A,B}$ has only simple critical points, in the similar way we can define locally a holomorphic mapping $(A,B) \mapsto (a,b)$, which is the inverse of $(a,b) \mapsto (A,B)$. It follows that in fact the map $(a,b) \mapsto (A,B)$ is biholomorphic in a small neighbourhood $\mathcal{V} \subset \mathbb{C} \times \mathbb{C}$ of (a_0,b_0). But $\widetilde{f}'_{A,B}(1) = (A+2B+1)/(A+B)$, so in the parameterization (A,B) the set $\mathrm{Fix}(t)$ is a surface described by

$$B = \frac{At-A-1}{2-t}.$$

Hence, $\mathrm{Fix}(t) \cap \mathcal{V}$ is a one-dimensional complex manifold. ∎

Remark. Actually, one can prove that for every $\varrho \in \mathbb{C}$ the entire set $\mathrm{Fix}(\varrho)$ is a manifold.

Define

$$M_t = Q_t(M_N).$$

Note that $Q_0 = \mathrm{id}$, so $M_0 = M_N$. Moreover, $Q_t : \mathcal{U}' \to \mathrm{Fix}(t)$ is continuous and $f^k_{Q_t(\sigma)}|_{U_{\sigma,t}} = f^k_{\sigma,t}|_{U_{\sigma,t}}$ is hybrid equivalent to $N^k_\sigma|_{U'_\sigma}$. By the uniqueness of hybrid equivalence and the compactness of M_N, the map $Q_t|_{M_N} : M_N \to M_t$ is a homeomorphism and $Q_t(M_N) \cap Q_t(\mathcal{U}' \setminus M_N) = \emptyset$.

Lemma 4.12. *$M_t \subset \mathrm{int}_{\mathrm{Fix}(t)}\, Q_t(\mathcal{U}')$ for every $t \in [0,1)$.*

Proof. Choose a point $\sigma_0 \in \operatorname{int}_{\mathcal{U}'} M_N$ and Jordan curves γ_0, γ_1 such that $\gamma_0 \subset \operatorname{int}_{\mathcal{U}'} M_N \setminus \{\sigma_0\}$, $\gamma_1 \subset \mathcal{U}' \setminus M_N$, γ_0 and γ_1 are homotopic in $\mathcal{U}' \setminus \{\sigma_0\}$ and the index of γ_i with respect to σ_0 is equal to 1 for $i = 0, 1$. By Lemma 4.11 and the fact that $Q_t|_{M_N}$ is a homeomorphism, using topological properties of surfaces we can find a topological disc $\mathcal{W} \subset \operatorname{Fix}(t)$ containing M_t. The index of $Q_t \circ \gamma_0$ with respect to $Q_t(\sigma_0)$ is equal to 1, because Q_t is a holomorphic bijection on $\operatorname{int}_{\mathcal{U}'} M_N$ (see Theorem 3.3). Moreover, since Q_t is continuous, $Q_t|_{M_N}$ is a homeomorphism and $Q_t(M_N) \cap Q_t(\mathcal{U}' \setminus M_N) = \emptyset$, if we take γ_1 sufficiently close to M_N, then $Q_t \circ \gamma_0$ and $Q_t \circ \gamma_1$ are homotopic in $\mathcal{W} \setminus Q_t(\sigma_0)$. Hence, the index of $Q_t \circ \gamma_1$ with respect to $Q_t(\sigma_0)$ is equal to 1.

Let C_t be the component of $\mathcal{W} \setminus Q_t(\gamma_1)$ containing $Q_t(\sigma_0)$. Then C_t is a topological disc in $\mathcal{W}$ and $M_t \subset C_t$. We show that $C_t \subset Q_t(\mathcal{U}')$. Suppose the converse and take a point $(a, b) \in C_t \setminus Q_t(\mathcal{U}')$. Then the index of $Q_t \circ \gamma_0$ with respect to (a, b) is equal to 0 and the curves $Q_t \circ \gamma_0$, $Q_t \circ \gamma_1$ are homotopic in $\mathcal{W} \setminus (a, b)$, so the index of $Q_t \circ \gamma_1$ with respect to (a, b) is equal to 0. But on the other hand, both points $Q_t(\sigma_0)$ and (a, b) are in C_t, which implies that the indices of $Q_t \circ \gamma_1$ with respect to (a, b) and $Q_t(\sigma_0)$ must be equal. This is a contradiction. Therefore, $C_t \subset Q_t(\mathcal{U}')$, so $M_t \subset \operatorname{int}_{\operatorname{Fix}(t)} Q_t(\mathcal{U}')$. ∎

Using Lemmas 4.11 and 4.12 and the fact that Q_t is continuous and depends continuously on $t \in [0, 1)$ we can find topological discs $\mathcal{U}_t \subset \operatorname{Fix}(t)$, $t \in [0, 1)$, such that $M_N \subset \mathcal{U}_0 \subset \mathcal{U}'$, $M_t \subset \mathcal{U}_t \subset \operatorname{int}_{\operatorname{Fix}(t)} Q_t(\mathcal{U}')$ for $t \in (0, 1)$ and $\operatorname{cl} \mathcal{U}_t$ depends continuously on $t \in [0, 1)$ in the Hausdorff metric. Then there exists a homeomorphism from $\mathbb{D} \times [0, 1)$ onto $\bigcup_{t \in [0,1)} \mathcal{U}_t$ mapping holomorphically $\mathbb{D} \times \{t\}$ onto $\mathcal{U}_t$ for each t. Note that $\operatorname{cl} U_{\sigma,t}$ depends continuously on $\sigma \in \mathcal{U}'$, $t \in [0, 1)$ in the Hausdorff metric. These facts together with Theorem 3.3 and Proposition 3.5 give

Corollary 4.13. *Define $U_{a,b} = U_{\sigma,t}$ for $(a, b) = (a(\sigma, t), b(\sigma, t))$, $\sigma \in \mathcal{U}'$, $t \in [0, 1)$. Then $\{f^k_{a,b}|_{U_{a,b}}\}_{(a,b) \in \mathcal{U}_t}$ is a Mandelbrot-like family with the Mandelbrot-like set M_t and $\{\{f^k_{a,b}|_{U_{a,b}}\}_{(a,b) \in \mathcal{U}_t}\}_{t \in [0, t_0]}$ is a continuous path of Mandelbrot-like families for every $t_0 \in [0, 1)$.* ∎

4.3. *The construction of* $\mathbf{f}_1$. Consider a sequence $(a(\sigma_n, t_n), b(\sigma_n, t_n))_{n>0}$ for $\sigma_n \in \mathcal{U}'$ and $t_n \in [0, 1)$ such that $t_n \to 1^-$. Passing to a subsequence we can assume it converges to some $(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. It is easy to see that (a, b) is either in $\operatorname{Fix}(1)$ or in Sing. Now we show that if $\sigma_n \in M_N$, then the case $(a, b) \in \operatorname{Sing}$ is not possible. In other words, we prove

Proposition 4.14. *For every $(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$, if $(a(\sigma_n, t_n), b(\sigma_n, t_n)) \to (a, b)$ as $n \to \infty$ for $\sigma_n \in M_N$, $t_n \to 1^-$, then $(a, b) \notin \operatorname{Sing}$.*

Proof. Recall that by the definition of $U_{\sigma,t}$, if $\sigma_n \in M_N$, then for the map f_{σ_n, t_n} we have $u \in B(\xi_{\sigma_n, t_n})$ and 1 is not in the basins of $0, \infty, \xi_{\sigma_n, t_n}$.

We consider several cases. Suppose first $a = 0$, $b \neq \infty$. By (1), there exists $r > 0$ such that for sufficiently large n we have $\mathrm{cl}\, f_{\sigma_n,t_n}(\mathbb{D}_r) \subset \mathbb{D}_r$, so $\mathbb{D}_r \subset B(0)$ for f_{σ_n,t_n}. But $f_{\sigma_n,t_n}(1) = a(\sigma_n, t_n) \in \mathbb{D}_r$ for large n. This is a contradiction.

Suppose now $a = 0$, $b = \infty$. Let $b_n = b(\sigma_n, t_n)$. If $|z| < 1/|2b_n|$, then for large n we have $|(2 - b_n)z - 1| > \mathrm{const} > 0$, so by (1), $\mathrm{cl}\, f_{\sigma_n,t_n}(\mathbb{D}_{1/|2b_n|}) \subset \mathbb{D}_{1/|2b_n|}$ and $\mathbb{D}_{1/|2b_n|} \subset B(0)$ for f_{σ_n,t_n}. On the other hand, by (2), $f_{\sigma_n,t_n}(u) \in \mathbb{D}_{1/|2b_n|}$ for large n. Hence, u is in the basin of 0 for f_{σ_n,t_n}, which is impossible.

We have proved the case $a = 0$. Since $f_{1/a,1/b}$ is conjugate to $f_{a,b}$ by $z \mapsto 1/z$, this covers also the case $a = \infty$.

Suppose $a \neq 0, 1, \infty$, $b = 1$. By Lemma I.2.2, $f_{\sigma_n,t_n}(z)$ tends to az^2 uniformly in the spherical metric outside a small neighbourhood of 1. The map az^2 has a fixed point $1/a \neq 1$ of multiplier 2. Hence, the map f_{σ_n,t_n} for large n has a fixed point of multiplier close to 2. By (3), the other fixed point cannot be attracting, which is a contradiction.

Suppose now $a = b = 1$. By Lemma I.2.2, f_{σ_n,t_n} tends to the polynomial $z \mapsto z^2$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{1\}$. Let $\varphi_n^\zeta = \varphi_{\sigma_n,t_n}^\zeta$ for $\zeta \in \{0, \infty\}$. It is clear that the map f_{σ_n,t_n} on $B(\zeta)$ satisfies the conditions of Lemma 2.3 with φ_n^ζ instead of ψ_n (cf. Corollary 4.9). By this lemma, $\varphi_n^0(z) \to z$ and $\varphi_n^\infty(z) \to 1/z$ almost uniformly on $\mathbb{D}$. (Note that in this case we have $h_n(z) = h(z) = z^2$ for every n.) Let $\gamma : [0, \infty) \to \mathbb{D}$ be the curve parameterizing the segment $[1/2, 1)$ such that $\gamma(0) = 1/2$, $\gamma(1) = \sqrt{1/2}$, γ is affine on $[0, 1]$ and $(\gamma(s))^2 = \gamma(s - 1)$ for $s > 1$. Then the assumptions of the case (a) of Lemma 2.4 are fulfilled for $\gamma_n = \gamma$ and φ_n^ζ instead of ψ_n. Using this lemma, in the same way as in the proof of (23) we show that $\varphi_n^0((-1, -\sqrt{1/2}])$ and $\varphi_n^\infty((-1, -\sqrt{1/2}])$ land at the same point near -1 (we use the fact that $z \mapsto z^2$ sends -1 to 1 and $f_{\sigma_n,t_n}(z) \to z^2$ uniformly near -1). In the language of internal rays this means $l_0(1/2) = l_\infty(1/2)$ for f_{σ_n,t_n}. Hence, by the definition of $f_{\sigma,t}$, we have $l_0(1/2) = l_\infty(1/2)$ for N_{σ_n}. But this is impossible by the combinatorics of N_σ for $\sigma \in M_N$ (see e.g. Lemma 4.6 and Subsection I.4.4). In this way we have shown that the case $a = b = 1$ does not hold.

We are left with the case $a \neq 0, \infty$, $b = 0, \infty$. If $b = \infty$, then f_{σ_n,t_n} tends to $g(z) = az(2 - z)$ uniformly outside a small neighbourhood of 0. The polynomial g has a fixed point $2 - 1/a$ of multiplier $2 - 2a$. Suppose $a \neq 1/2$. Then $2 - 1/a \neq 0$, so by the uniform convergence near $2 - 1/a$, we have $\xi_{\sigma_n,t_n} \to 2 - 1/a$ and $2 - 2a = g'(2 - 1/a) = \lim_n f'_{\sigma_n,t_n}(\xi_{\sigma_n,t_n}) = 1$. This leads to a contradiction. Hence, if $b = \infty$, then $a = 1/2$. Similarly, if $b = 0$, then $a = 2$. We conclude that the only possibilities are: $(a, b) = (1/2, \infty), (2, 0)$.

Suppose $(a, b) = (1/2, \infty)$. To make use of Lemma 2.4, we change the coordinates in $\widehat{\mathbb{C}}$ by a suitable Möbius map. More precisely, let $f_n = H \circ$

$f_{\sigma_n,t_n} \circ H^{-1}$, where H exchanges 0 and ∞ and maps 1 to u (this is the map h_2 from Lemma I.3.2). Note that the f_n-critical point 1 is in the immediate basin of an attracting f_n-fixed point ξ_n of multiplier t_n and the fourth f_n-critical point u_n is in the filled-in Julia set of f_n^k as a quadratic-like map on some topological disc. Moreover, by Lemmas I.2.2 and I.3.2, f_n tends to the quadratic polynomial $g(z) = z - z^2/2$ almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{0\}$. Note that g is a quadratic polynomial with a parabolic fixed point at 0 of multiplier 1, a critical point at 1 and is conformally conjugate to $z \mapsto z^2 + 1/4$.

Let $\varphi_n^\zeta = H \circ \varphi_{\sigma_n,t_n}^\zeta$ for $\zeta \in \{\infty, \xi\}$. Note that φ_n^∞ conjugates f_n to $z \mapsto z^2$ and φ_n^ξ conjugates f_n to h_{t_n} from (26). It is easy to see that f_n on $B(\infty)$ (resp. $B(\xi_n)$) satisfies the assumptions of Lemma 2.3 with φ_n^∞ (resp. φ_n^ξ) instead of ψ_n. Using this lemma, we obtain $\varphi_n^\infty \to \varphi^\infty$ and $\varphi_n^\xi \to \varphi^0$ almost uniformly on $\mathbb{D}$ for suitable Riemann maps φ^0 onto $B(0)$ and φ^∞ onto $B(\infty)$ for the polynomial g. Moreover, $h_{t_n} \to h_1$ for h_1 from (26).

Let γ^∞ be the curve γ defined in the proof of the case $a = b = 1$. Then the conditions of the case (a) of Lemma 2.4 are satisfied for $\gamma_n = \gamma^\infty$, $h_n(z) = z^2$ and φ_n^∞ instead of ψ_n. Note that the segment $[1/2, 1)$ is h_{t_n}-invariant, contains the attracting h_{t_n}-fixed point s_{t_n} and lands at the repelling h_{t_n}-fixed point 1. It is easy to see that we can define a curve γ^{ξ_n} parameterizing $[1/2, 1)$ in such a way that the conditions of the case (b) of this lemma are satisfied for $\gamma_n = \gamma^{\xi_n}$, $h_n = h_{t_n}$ and φ_n^ξ instead of ψ_n. In the same way as in the case $a = b = 1$ we show that $\varphi_n^\infty((-1, -\sqrt{1/2}])$ and $\varphi_n^\xi((-1, -\sqrt{1/2}])$ land at the same point near 2. Recall that defining f_n we changed the coordinates exchanging $0, \infty$ and $1, u$. Hence, for the map f_{σ_n,t_n} we obtain $l_0(1/2) \in \operatorname{cl} B(\xi_{t_n})$. This is again impossible by the definition of $f_{\sigma,t}$ and the combinatorics of N_σ for $\sigma \in M_N$.

We are left with the case $(a, b) = (2, 0)$. As previously, we change the coordinates in $\widehat{\mathbb{C}}$ setting $f_n = H \circ f_{\sigma_n,t_n} \circ H^{-1}$ for the Möbius map H fixing 0 and ∞ and sending 1 to u (then $H = h_1 \circ h_2$ for h_1, h_2 from Lemma I.3.2). Again, the f_n-critical point 1 is in the immediate basin $B(\xi_n)$ of an invariant f_n-fixed point ξ_n and the fourth f_n-critical point u_n is in the suitable filled-in Julia set of a quadratic-like map. Moreover, f_n tends to g almost uniformly in the spherical metric on $\widehat{\mathbb{C}} \setminus \{0\}$ for g as in the case $(a, b) = (1/2, \infty)$.

Define φ_n^∞, φ_n^ξ, γ^∞, γ^{ξ_n} for f_n as in the case $(a, b) = (1/2, \infty)$. In the same way as in the proof of (25) we find small $\varepsilon, \varepsilon' > 0$ and $s_0 > 0$ such that for large n we have $\varphi_n^\infty(\gamma^\infty([s_0, \infty))), \varphi_n^\xi(\gamma^{\xi_n}([s_0, \infty))) \subset \mathbb{D}_\varepsilon$ and $f_n^{k-1}(\mathbb{D}_{\varepsilon'}(x)), f_n^k(\mathbb{D}_{\varepsilon'}(y)) \supset \mathbb{D}_\varepsilon$, where $x = l_\infty(1 - 2\beta)$ and $y = l_\infty(1 - 2\delta)$ for the polynomial g and the angles β, δ from the definition of U'_σ. Let U_n be the topological disc U_{σ_n,t_n} in the new coordinates. By the above facts and the definition of $U_{\sigma,t}$, we have

$$\partial f_n(U_n) \subset \mathbb{D}_{\varepsilon'}(x) \cup \mathbb{D}_{\varepsilon'}(y) \cup \varphi_n^\infty(\mathbb{D}_r) \cup \varphi_n^\xi(\mathbb{D}_r)$$

for some fixed $r < 1$. Since $x, y \neq 0$ and $0 \notin B(0) \cup B(\infty)$ for g, taking sufficiently small ε' and using the uniform convergence of φ_n^∞ and φ_n^ξ on $\mathbb{D}_r$ we get $|f_n(u_n)| > c$ for some fixed constant $c > 0$ and large n. But this is a contradiction, because $f_n \to g$, so by (2), we have $f_n(u_n) \to 0$. This ends the proof. ∎

Remark. Proposition 4.14 completes the proof of Theorem I.4.20 showing that the curve γ from this theorem does not land at a singular parameter λ.

Definition 4.15. Let M_1 be the set of all points $(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ such that (a, b) is a limit point of a sequence $(a(\sigma_n, t_n), b(\sigma_n, t_n))_{n>0}$, where $\sigma_n \in M_N$, $t_n \in [0, 1)$ and $t_n \to 1^-$.

By Proposition 4.14, M_1 is a subset of $\mathrm{Fix}(1)$.

Definition 4.16. Let $\widetilde{\mathcal{U}}_1$ be the set of all points $(a, b) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ such that $\mathrm{dist}((a, b), M_1) < \varepsilon_1$ for a fixed small $\varepsilon_1 > 0$ and (a, b) is a limit point of a sequence $(a(\sigma_n, t_n), b(\sigma_n, t_n))_{n>0}$, where $\sigma_n \in \mathcal{U}'$, $t_n \in [0, 1)$ and $t_n \to 1^-$.

By (1), it is easy to check that $\mathrm{Fix}(1)$ is described by the equation

$$(2b - 1)^2 a^2 - 2(2b^2 - 3b + 2)a + (b - 2)^2 = 0, \tag{30}$$

which for $(a, b) \notin \mathrm{Sing}$ gives locally two surfaces $a = a_1(b)$ and $a = a_2(b)$. Hence, $\mathrm{Fix}(1)$ is a one-dimensional complex manifold. Moreover, M_1 is compact by definition, so for sufficiently small ε_1 we have

$$M_1 \subset \widetilde{\mathcal{U}}_1 \subset \mathrm{Fix}(1).$$

Thus, if $(a(\sigma_n, t_n), b(\sigma_n, t_n)) \to (a, b) \in \widetilde{\mathcal{U}}_1$, then $f_{\sigma_n, t_n} \to f_{a,b}$ uniformly in the spherical metric on $\widehat{\mathbb{C}}$.

Before defining the family $\mathbf{f}_1$ we prove the following technical lemma similar to Lemma 4.8.

Lemma 4.17. *There exists $\varepsilon_0 > 0$ such that for every $\sigma_n \in \mathcal{U}'$, $t_n \to 1^-$, if $(a(\sigma_n, t_n), b(\sigma_n, t_n)) \to (a, b) \in \widetilde{\mathcal{U}}_1$ and $1 \in B(\xi_{\sigma_n, t_n})$, then*

$$|v_{\sigma_n, t_n} - 1| > \varepsilon_0,$$

where v_{σ_n, t_n} is the unique non-zero $h^\xi_{\sigma_n, t_n}$-critical point in $\mathbb{D}$.

Proof. Suppose the converse. For simplicity, set $v_n = v_{\sigma_n, t_n}$ and $G_n = G_{\sigma_n, t_n}$ for G_{σ_n, t_n} from (28). Passing to a subsequence we can assume $h^\xi_{\sigma_n, t_n} \to h$, where h is a Blaschke product with a critical point at 0 and a parabolic fixed point at 1 of multiplier 1. Since $v_n \to 1$ by assumption, we have $\deg h = 2$. This easily implies $h = h_1$ for h_1 from (26). Let

$$\gamma_n' = (\varphi^\xi_{\sigma_n, t_n})^{-1}(H_{\sigma_n, t_n}(\varphi^u_{\sigma_n}(\gamma'^u_{\sigma_n}))), \quad \gamma_n'' = (\varphi^\xi_{\sigma_n, t_n})^{-1}(H_{\sigma_n, t_n}(\varphi^u_{\sigma_n}(\gamma''^u_{\sigma_n})))$$

for the curves $\gamma'^u_{\sigma_n}$, $\gamma''^u_{\sigma_n}$ from (24) and let

$$\gamma_n = h^\xi_{\sigma_n,t_n}(\gamma''_n).$$

Then γ''_n begins at $G_n(-\widehat{r})$ and γ'_n begins at the image under G_n of the unique point of $\partial D_{t_n} \cap (0,1)$. Moreover,

$$\gamma_n = G_n([h_{t_n}(-\widehat{r}), 1) \cap D_{t_n}) \cup \gamma'_n. \tag{31}$$

In particular, γ_n begins at $G_n(h_{t_n}(-\widehat{r}))$, lands at the repelling $h^\xi_{\sigma_n,t_n}$-fixed point 1, is $h^\xi_{\sigma_n,t_n}$-invariant and contains the attracting $h^\xi_{\sigma_n,t_n}$-fixed point $G_{\sigma_n,t_n}(w_{t_n})$ of multiplier t_n. It is easy to see that we can parameterize γ_n according to the conditions of the case (b) of Lemma 2.4 with $f_n = f_{\sigma_n,t_n}$, $f = f_{a,b}$ and $\varphi^\xi_{\sigma_n,t_n}, \varphi^p_{a,b}$ instead of ψ_n, ψ.

Recall that G_n conjugates holomorphically h_{t_n} on W_{t_n} to $h^\xi_{\sigma_n,t_n}$. Let

$$D_1 = \mathbb{D}_{(1+\widehat{r})/2}((1-\widehat{r})/2), \qquad W_1 = h_1^{-1}(D_1) \cap \mathbb{D}.$$

Then $\operatorname{cl} D_{t_n}$ tends to $\operatorname{cl} D_1$ and $\operatorname{cl} W_{t_n}$ tends to $\operatorname{cl} W_1$ in the Hausdorff metric. Repeating the proof of Lemma 2.3 we show that $G_n \to G$ almost uniformly on W_1 for some holomorphic map G, in the sense that for any compact set in W_1 almost all G_n are defined on it and tend uniformly to G. Moreover, G conjugates h_1 to h_1, which easily gives $G = \mathrm{id}$. Let

$$\gamma'' = (-1, -\widehat{r}], \qquad \gamma = h_1(\gamma'') = [h_1(-\widehat{r}), 1).$$

Note that γ is backward h_1-invariant and lands at the parabolic h_1-fixed point 1. Since $G_n \to \mathrm{id}$ and any compact subset of $[h_1(-\widehat{r}), 1)$ is contained in $D_{t_n} \cap [h_{t_n}(-\widehat{r}), 1)$ for large n, it follows from (31) that $\gamma_n(s) \to \gamma(s)$ almost uniformly for a suitable parameterization of γ.

To use Lemma 2.4, we need to show

$$\operatorname{diam} \gamma_n([n,\infty)) \to 0. \tag{32}$$

To prove (32), suppose it is not true. Then in the same way as in the proof of Lemma 2.4 we can find small $\varepsilon_1, \varepsilon_2 > 0$ and $s_j, n_j \to \infty$ such that $s_j > n_j$ and

$$\varepsilon_1 < |\gamma_{n_j}(s_j) - 1| < \varepsilon_2 \quad \text{and} \quad \operatorname{dist}(\gamma_{n_j}(s_j), [0,1]) < |\gamma_{n_j}(s_j) - 1|/10$$

for large j (cf. (12)). This together with the almost uniform convergence $G_n \to \mathrm{id}$ on W_1 implies $\gamma_{n_j}(s_j) \in G_{n_j}(W_{t_{n_j}})$ and $G^{-1}_{n_j}(\gamma_{n_j}(s_j)) \in K \subset W_1$ for some compact set K independent of j. Hence,

$$G^{-1}_{n_j}(\gamma_{n_j}(s_j)) \in [h_{t_{n_j}}(-\widehat{r}), r_1]$$

for some fixed $r_1 < 1$. This is impossible, because $s_j > n_j$ and the parameterization of γ_{n_j} is such that $G^{-1}_{n_j}(\gamma_{n_j}((n_j,\infty)) \subset (w_{t_{n_j}}, 1)$ and $w_{t_{n_j}} \to 1$. Hence, (32) holds.

We have shown that γ_n, γ satisfy all assumptions of the case (b) of Lemma 2.4 with $\varphi^\xi_{\sigma_n,t_n}, \varphi^p_{a,b}$ instead of ψ_n, ψ (cf. Corollary 4.9). By this lemma, $\mathrm{cl}\,\gamma_n$ tends to $\mathrm{cl}\,\gamma = [h_1(-\widehat{r}), 1]$ in the Hausdorff metric.

Let κ_n (resp. κ) be the upper half of the circle, which is symmetric with respect to the real axis and contains $-\widehat{r}$, $h_{t_n}(-\widehat{r})$ (resp. $-\widehat{r}$, $h_1(-\widehat{r})$). Note that $\kappa_n \subset D_{t_n}$, $\kappa \subset D_1$. Define

$$\Delta_n = \gamma_n \cup G_n(\kappa_n) \cup \gamma_n'', \qquad \Delta = \gamma \cup \kappa \cup \gamma''$$

and let X_n be the component of $\mathbb{D} \setminus \Delta_n$ which does not contain 0. Note that $\gamma_n', \gamma_n'' \subset \Delta_n$ and $G_n(\partial D_{t_n} \cap \mathbb{D}^+) \subset X_n$, so by the definition of U_{σ_n,t_n},

$$X_n \supset (\varphi^\xi_{\sigma_n,t_n})^{-1}(f^k_{\sigma_n,t_n}(U_{\sigma_n,t_n} \cap B(\xi_{\sigma_n,t_n}))).$$

The definition of Δ_n implies that $(h^\xi_{\sigma_n,t_n})^j(0), (h^\xi_{\sigma_n,t_n})^j(v_n) \notin \Delta_n$ for $j = 0, \ldots, k$, so all branches of $(h^\xi_{\sigma_n,t_n})^{-k}$ are defined in a neighbourhood of Δ_n. Similarly, all branches of h_1^{-k} are defined in a neighbourhood of Δ. It is easy to see that for large n we can extend G_n by dynamics to a holomorphic map on $h_{t_n}^{-k}(D_{t_n}) \cap \mathbb{D}$, conjugating h_{t_n} to $h^\xi_{\sigma_n,t_n}$ and converging almost uniformly to id. Moreover, for $t \in [0,1]$ the map h_t^k on $\partial(h_t^{-k}(D_t) \cap \mathbb{D})$ is conjugate to $z \mapsto z^{2^k}$ on $\partial\mathbb{D}$ and the conjugation depends continuously on $t \in [0,1]$. These facts together with the definition of U_{σ_n,t_n} easily imply that there exist a branch ν_n of $(h^\xi_{\sigma_n,t_n})^{-k}$ defined in a neighbourhood of Δ_n and a branch ν of h_1^{-k} defined in a neighbourhood of Δ, such that if Y_n is the component of $\mathbb{D} \setminus \nu_n(\Delta_n)$ which does not contain 0 and Y is the component of $\mathbb{D} \setminus \nu(\Delta)$ which does not contain 0, then

$$Y_n \supset (\varphi^\xi_{\sigma_n,t_n})^{-1}(U_{\sigma_n,t_n} \cap B(\xi_{\sigma_n,t_n})) \ni v_n,$$

$1 \notin \mathrm{cl}\,Y$ and $\nu_n(G_n(\kappa_n))$ tends to $\nu(\kappa)$ in the Hausdorff metric. Since $h^\xi_{\sigma_n,t_n} \to h_1$ almost uniformly on $\mathrm{cl}\,\mathbb{D} \setminus \{1\}$ and $\mathrm{cl}\,\gamma_n$ tends to $\mathrm{cl}\,\gamma$ in the Hausdorff metric, this easily implies that ∂Y_n tends to ∂Y in the Hausdorff metric. As $1 \notin \mathrm{cl}\,Y$, this gives $\mathrm{dist}(\mathrm{cl}\,Y_n, 1) > c > 0$, so $|v_n - 1| > c$, which contradicts $v_n \to 1$. ∎

Let $(a,b) \in \widetilde{\mathcal{U}}_1$ and take $\sigma_n \in \mathcal{U}'$, $t_n \to 1^-$ such that $(a(\sigma_n,t_n), b(\sigma_n,t_n)) \to (a,b)$. By Proposition 4.14 and Lemma 2.3, $f_{a,b}$ has a parabolic fixed point $p_{a,b}$ of multiplier 1 with the unique invariant simply connected parabolic basin $B(p_{a,b})$ such that $u \in B(p_{a,b})$, $p_{\sigma_n,t_n} \to p_{a,b}$ and $(B(\xi_{\sigma_n,t_n}), u)$ tends to $(B(p_{a,b}), u)$ in the Carathéodory topology. Similarly, for $\zeta \in \{0, \infty\}$ the immediate basin $B(\zeta)$ for $f_{a,b}$ is simply connected and $(B(\zeta), \zeta)$ for the map f_{σ_n,t_n} tends to $(B(\zeta), \zeta)$ for the map $f_{a,b}$ in the Carathéodory topology.

Note that $f_{a,b}$ on $B(p_{a,b})$ is conjugate by a Riemann map to a quadratic or cubic Blaschke product $h^p_{a,b}$ with a (unique) fixed point of multiplier 1 in $\partial\mathbb{D}$. Let

$$\varphi^p_{a,b} : \mathbb{D} \to B(p_{a,b})$$

be the unique Riemann mapping such that $\varphi^p_{a,b}(0) = u$ and the $h^p_{a,b}$-fixed point of multiplier 1 is equal to 1. It is easy to check that $\varphi^p_{a,b}$ depends continuously on $(a,b) \in \widetilde{\mathcal{U}}_1$ in the uniform convergence topology.

Recall that f_{σ_n,t_n} on $B(\infty)$ is conjugate by $\varphi^\infty_{\sigma_n,t_n}$ to $h^\infty_{\sigma_n}$ for $h^\infty_{\sigma_n}$ from (17). By Corollary 4.9, it is easy to see that for $(a,b) \in \widetilde{\mathcal{U}}_1$ we can choose a Riemann map

$$\varphi^\infty_{a,b} : \mathbb{D} \to B(\infty)$$

such that $\varphi^\infty_{a,b}(0) = \infty$, $\varphi^\infty_{\sigma_n,t_n} \to \varphi^\infty_{a,b}$ almost uniformly on $\mathbb{D}$ and $\varphi^\infty_{a,b}$ depends continuously on $(a,b) \in \widetilde{\mathcal{U}}_1$.

By Lemma 4.17 we easily obtain

COROLLARY 4.18. *If $(a,b) \in \widetilde{\mathcal{U}}_1$ and $\sigma_n \in \mathcal{U}'$, $t_n \to 1^-$ are such that $(a(\sigma_n,t_n), b(\sigma_n,t_n)) \to (a,b)$, then for $\zeta \in \{\xi,\infty\}$ Lemmas* 2.3 *and* 2.4 *hold for $f_n = f_{\sigma_n,t_n}$, $f = f_{a,b}$ and $\varphi^\zeta_{\sigma_n,t_n}, \varphi^\zeta_{a,b}$ instead of ψ_n, ψ. In particular, $h^\xi_{\sigma_n,t_n} \to h^p_{a,b}$ and there exists $\sigma_{a,b} \in \mathrm{cl}\,\mathcal{U}'$ such that $h^\infty_{\sigma_n} \to h^\infty_{\sigma_{a,b}}$ and $\varphi^\infty_{a,b}$ conjugates $f_{a,b}|_{B(\infty)}$ to $h^\infty_{\sigma_{a,b}}$. Moreover, the convergence of $h^\xi_{\sigma_n,t_n}$ and $h^\infty_{\sigma_n}$ is almost uniform in the spherical metric on $\widehat{\mathbb{C}} \setminus \{z_0\}$, where $z_0 \in \partial\mathbb{D} \setminus \{1\}$.* ∎

REMARK. The parameter $\sigma_{a,b}$ is not uniquely defined, but $h^\infty_{\sigma_{a,b}}$ is unique.

Note that this corollary implies that for $(a,b) \in M_1$ we have $\deg h^p_{a,b} = \deg h_{\sigma_{a,b}} = 2$, which clearly gives $h^p_{a,b} = h_1$ for h_1 from (26) and $h^\infty_{\sigma_{a,b}}(z) = z^2$ in this case.

Now we show that

$$1 \notin f^{-(k+1)}_{a,b}(p_{a,b}) \quad \text{for every } (a,b) \in \widetilde{\mathcal{U}}_1. \tag{33}$$

By the compactness of M_1, if we take sufficiently small ε_1 in the definition of $\widetilde{\mathcal{U}}_1$, then it is sufficient to check (33) for $(a,b) \in M_1$. Suppose that (33) does not hold. Then $f^{2k+1}_{a,b}(1) = p_{a,b}$. Take σ_n, t_n as above. Note that if we take $\mathcal{U}'$ sufficiently close to M_N, then $N^{2k+1}_\sigma(1) \in N_\sigma(U'_\sigma)$ for $\sigma \in \mathcal{U}'$, so $f^{2k+1}_{\sigma_n,t_n}(1) \in f_{\sigma_n,t_n}(U_{\sigma_n,t_n})$. Moreover, since $(a,b) \in M_1$, we have $h^\xi_{\sigma_n,t_n} \to h_1$ and $h^\infty_{\sigma_n}(z) \to z^2$. In the same way as in the proof of Lemma 4.17, using Lemma 2.4 we show that for large n,

$$\partial f_{\sigma_n,t_n}(U_{\sigma_n,t_n}) \subset \mathbb{D}_\varepsilon(x) \cup \mathbb{D}_\varepsilon(y) \cup \varphi^\xi_{\sigma_n,t_n}(\mathbb{D}_r) \cup \varphi^\infty_{\sigma_n,t_n}(\mathbb{D}_r)$$

for $x, y \in f^{-k}_{a,b}(p_{a,b}) \setminus \{p_{a,b}\}$, some fixed $r < 1$ and a small $\varepsilon > 0$. This implies $|f^{2k+1}_{\sigma_n,t_n}(1) - p_{a,b}| > \mathrm{const} > 0$, which contradicts $f^{2k+1}_{a,b}(1) = p_{a,b}$. In this way we have proved (33).

Now for $(a,b) \in \widetilde{\mathcal{U}}_1$ we define curves $\Gamma^p_{a,b}, \Gamma^\infty_{a,b}$ in $\mathbb{D}$, similar to the curves $\Gamma^u_\sigma, \Gamma^\infty_\sigma$ from (24). To define the curve $\Gamma^p_{a,b}$, consider the map G_{σ_n,t_n} from (28). Recall that G_{σ_n,t_n} is a biholomorphic map from W_{t_n} into $\mathbb{D}$ conjugating h_{t_n} to $h^\xi_{\sigma_n,t_n}$. Let

$$D_1 = \mathbb{D}_{(1+\widehat{r})/2}((1-\widehat{r})/2), \qquad W_1 = h_1^{-1}(D_1) \cap \mathbb{D}.$$

Then D_1 is the horodisc tangent to $\partial\mathbb{D}$ at 1 such that $-\widehat{r} \in \partial D_1$. It is easy to check that

$$W_1 \supset (\mathrm{cl}\, D_1 \setminus \{1\}) \cup (-1,1). \tag{34}$$

Moreover, $\mathrm{cl}\, W_{t_n}$ tends to $\mathrm{cl}\, W_1$ in the Hausdorff metric. In the same way as in the proof of Lemma 2.3 we show that G_{σ_n,t_n} tends almost uniformly on W_1 to some holomorphic map $G_{a,b}$ conjugating h_1 to $h^p_{a,b}$, in the sense that for any compact set in W_1 almost all G_{σ_n,t_n} are defined on it and tend uniformly to $G_{a,b}$. Let

$$\Gamma^p_{a,b} = G_{a,b}((\partial D_1 \cap \mathbb{D}^+) \cup (-1, -\widehat{r}]).$$

By definition, $\Gamma^p_{a,b}$ is a simple arc in $\mathbb{D}$ connecting 1 to -1 (see Fig. 10).

Note that if $(a,b) \in M_1$, then $G_{a,b}$ conjugates h_1 to h_1, which easily gives $G_{a,b} = \mathrm{id}$. Hence, $\Gamma^p_{a,b} = (\partial D_1 \cap \mathbb{D}^+) \cup (-1, -\widehat{r}]$ in this case.

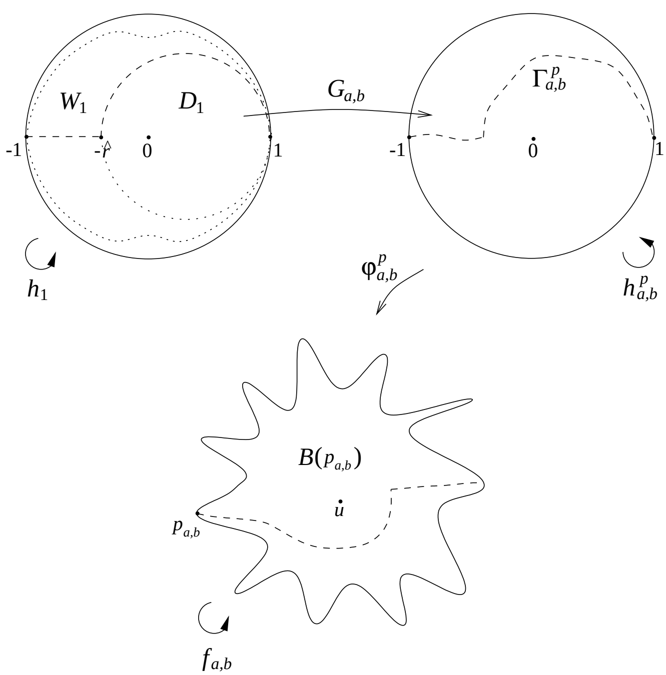

Fig. 10. The sets D_1, W_1 and the curve $\Gamma^p_{a,b}$

Recall that by Corollary 4.18, $f_{a,b}$ on $B(\infty)$ is conjugate by $\varphi^\infty_{a,b}$ to $h^\infty_{\sigma_{a,b}}$ for some $\sigma_{a,b} \in \operatorname{cl}\mathcal{U}'$. Let

$$\Gamma^\infty_{a,b} = \Gamma^\infty_{\sigma_{a,b}}$$

for $\Gamma^\infty_{\sigma_{a,b}}$ from (24). It is easy to see that $\Gamma^p_{a,b}$ and $\Gamma^\infty_{a,b}$ do not depend on the choice of the sequences $\sigma_n,\ t_n$.

Proposition 4.19. *For every $(a,b) \in \widetilde{\mathcal{U}}_1$ and $\sigma_n \in \mathcal{U}'$, $t_n \to 1^-$ such that $(a(\sigma_n,t_n), b(\sigma_n,t_n)) \to (a,b)$,*

$$\operatorname{cl}(\varphi^p_{a,b}(\Gamma^p_{a,b}) \cup \varphi^\infty_{a,b}(\Gamma^\infty_{a,b}))$$

is a Jordan curve and $\partial f^k_{\sigma_n,t_n}(U_{\sigma_n,t_n})$ tends to it in the Hausdorff metric.

The proof is split into two lemmas. The first proves that the curves $\varphi^\infty_{a,b}(\gamma''^\infty_{\sigma_{a,b}})$ and $\varphi^p_{a,b}(G_{a,b}((-1,-\widehat{r}]))$ have a common landing point and the second shows the same for $\varphi^\infty_{a,b}(\gamma'^\infty_{\sigma_{a,b}})$ and $\varphi^p_{a,b}(G_{a,b}(\partial D_1 \cap \mathbb{D}^+))$ (where $\gamma'^\infty_{\sigma_{a,b}}, \gamma''^\infty_{\sigma_{a,b}}$ are the curves from (24)).

Lemma 4.20. *For $(a,b), \sigma_n, t_n$ as in Proposition 4.19, $\varphi^\infty_{a,b}(\gamma''^\infty_{\sigma_{a,b}})$ and $\varphi^p_{a,b}(G_{a,b}((-1,-\widehat{r}]))$ land at the same point and $\operatorname{cl}(H_{\sigma_n,t_n}(\varphi^\infty_{\sigma_n}(\gamma''^\infty_{\sigma_n})) \cup H_{\sigma_n,t_n}(\varphi^u_{\sigma_n}(\gamma''^u_{\sigma_n})))$ tends to $\operatorname{cl}(\varphi^\infty_{a,b}(\gamma''^\infty_{\sigma_{a,b}}) \cup \varphi^p_{a,b}(G_{a,b}((-1,-\widehat{r}])))$ in the Hausdorff metric. Moreover, $\operatorname{cl} H_{\sigma_n,t_n}(\varphi^\infty_{\sigma_n}(\gamma'^\infty_{\sigma_n}))$ tends to $\operatorname{cl}\varphi^\infty_{a,b}(\gamma'^\infty_{\sigma_{a,b}})$ in the Hausdorff metric.*

Proof. Define γ'_n, γ''_n and γ_n as in the proof of Lemma 4.17 and let

$$\gamma'' = G_{a,b}((-1,-\widehat{r}]), \quad \gamma = h^p_{a,b}(\gamma'') = G_{a,b}([h_1(-\widehat{r}),1)).$$

In the same way as in the proof of Lemma 4.17 we show that we can parameterize γ_n, γ according to the conditions of the case (b) of Lemma 2.4 so that $\gamma_n(s) \to \gamma(s)$ almost uniformly.

To use Lemma 2.4, we need to show (32) in this case. Suppose it does not hold. In the same way as in the proof of Lemma 2.4 we find $s_n > n$ such that (after passing to a subsequence) we have $\gamma_n(s_n) \in K$ for a fixed compact set $K \subset \mathbb{D}$. Since $\mathbb{D}$ is the invariant attracting basin for the parabolic $h^p_{a,b}$-fixed point 1 of multiplier 1 and $h^\xi_{\sigma_n,t_n} \to h^p_{a,b}$ almost uniformly on $\mathbb{D}$, there exist arbitrarily large $j,m > 0$ such that

$$|(h^\xi_{\sigma_n,t_n})^j(\gamma_n(s_n)) - (h^\xi_{\sigma_n,t_n})^m(0)| < |(h^\xi_{\sigma_n,t_n})^m(0) - (h^\xi_{\sigma_n,t_n})^{m+1}(0)| \tag{35}$$

for sufficiently large n. By the definition of G_{σ_n,t_n}, for every $l \ge 0$ we have

$$(h^\xi_{\sigma_n,t_n})^l(0) = G_{\sigma_n,t_n}(h^l_{t_n}(0)).$$

Moreover, for almost all n the map G_{σ_n,t_n} is defined and has universally bounded distortion on

$$D^{(n)} = \mathbb{D}_{|h^m_{t_n}(0) - w_{t_n}|/2}(h^m_{t_n}(0)) \tag{36}$$

and $|h_{t_n}^m(0) - h_{t_n}^{m+1}(0)|/|h_{t_n}^m(0) - w_{t_n}|$ is arbitrarily small provided m is chosen sufficiently large. By this and (35), $(h_{\sigma_n,t_n}^{\xi})^j(\gamma_n(s_n)) \in G_{\sigma_n,t_n}(D^{(n)})$ for large n. But since $s_n > n$, we have $(h_{\sigma_n,t_n}^{\xi})^j(\gamma_n(s_n)) = \gamma_n(s_n')$ for some $s_n' > n$. This is a contradiction, because the definition of γ_n implies $G_{\sigma_n,t_n}(D^{(n)}) \cap \gamma_n((n,\infty)) = \emptyset$. This shows (32) in our case.

We have checked that γ_n, γ satisfy all assumptions of the case (b) of Lemma 2.4 with $h_n = h_{\sigma_n,t_n}^{\xi}$, $h = h_{a,b}^p$ and $\varphi_{\sigma_n,t_n}^{\xi}, \varphi_{a,b}^p$ instead of ψ_n, ψ. Note also that the curves $\gamma_{\sigma_n}^{\infty}, \gamma_{\sigma_{a,b}}^{\infty}$ with suitable parameterizations fulfill the conditions of the case (a) of that lemma with $h_n = h_{\sigma_n}^{\infty}$, $h = h_{\sigma_{a,b}}^{\infty}$ and $\varphi_{\sigma_n,t_n}^{\infty}, \varphi_{a,b}^{\infty}$ instead of ψ_n, ψ. By Corollary 4.18 and Lemma 2.4, both curves $\varphi_{a,b}^p(\gamma)$ and $\varphi_{a,b}^{\infty}(\gamma_{\sigma_{a,b}}^{\infty})$ land at $p_{a,b}$ and for every $\varepsilon > 0$ we can find s_0 such that

$$\varphi_{\sigma_n,t_n}^{\xi}(\gamma_n([s_0,\infty))), \varphi_{\sigma_n,t_n}^{\infty}(\gamma_{\sigma_n}^{\infty}([s_0,\infty))) \subset \mathbb{D}_{\varepsilon}(p_{a,b})$$

for large n. Note that the curves $\varphi_{\sigma_n,t_n}^{\xi}(\gamma_n)$ and $\varphi_{\sigma_n}^{\infty}(\gamma_{\sigma_n}^{\infty})$ are contained in $\partial f_{\sigma_n,t_n}(U_{\sigma_n,t_n})$ and land at p_{σ_n,t_n}.

Since $\varphi_{a,b}^p(G_{a,b}((-1,-\widehat{r}]))$ is a preimage under $f_{a,b}^{-1}$ of $\varphi_{a,b}^p(\gamma)$, it has a well-defined landing point $x \in f_{a,b}^{-1}(p_{a,b}) \setminus \{p_{a,b}\}$. Similarly, $\varphi_{a,b}^{\infty}(\gamma''^{\infty}_{\sigma_{a,b}})$ lands at a point $y \in f_{a,b}^{-1}(p_{a,b}) \setminus \{p_{a,b}\}$. By (33), $f_{a,b}$ maps biholomorphically a small neighbourhood of x and y onto a neighbourhood of $p_{a,b}$. Hence, there exists $\varepsilon' > 0$, which is arbitrarily small if ε is small enough, such that $f_{\sigma_n,t_n}(\mathbb{D}_{\varepsilon'}(x)), f_{\sigma_n,t_n}(\mathbb{D}_{\varepsilon'}(y)) \supset \mathbb{D}_{\varepsilon}(p_{a,b})$ for large n. Moreover, by the almost uniform convergence of $\varphi_{\sigma_n,t_n}^{\xi}, \varphi_{\sigma_n,t_n}^{\infty}$, the disc $\mathbb{D}_{\varepsilon'}(x)$ contains points from $H_{\sigma_n,t_n}(\varphi_{\sigma_n}^u(\gamma''^u_{\sigma_n}))$ and $\mathbb{D}_{\varepsilon'}(y)$ contains points from $\varphi_{\sigma_n,t_n}^{\infty}(\gamma''^{\infty}_{\sigma_n})$ for large n. Therefore, $H_{\sigma_n,t_n}(\varphi_{\sigma_n}^u(\gamma''^u_{\sigma_n}))$ lands at some point in $\mathbb{D}_{\varepsilon'}(x)$ and $\varphi_{\sigma_n,t_n}^{\infty}(\gamma''^{\infty}_{\sigma_n})$ lands at some point in $\mathbb{D}_{\varepsilon'}(y)$. But by the definition of $U_{\sigma,t}$, these curves are contained in $\partial f_{\sigma_n,t_n}^k(U_{\sigma_n,t_n})$ and land at the same point. This implies $x = y$, because otherwise we have a contradiction for $\varepsilon' < |x-y|/2$. In this way we have shown that the curves $\varphi_{a,b}^p(G_{a,b}((-1,-\widehat{r}]))$ and $\varphi_{a,b}^{\infty}(\gamma''^{\infty}_{\sigma_{a,b}})$ land at the same point. Moreover, by Lemma 2.4, $\operatorname{cl}\varphi_{\sigma_n,t_n}^{\xi}(\gamma_n)$ tends to $\operatorname{cl}\varphi_{a,b}^p(\gamma)$ in the Hausdorff metric, so $\operatorname{cl} H_{\sigma_n,t_n}(\varphi_{\sigma_n}^u(\gamma''^u_{\sigma_n})) = \operatorname{cl}\varphi_{\sigma_n,t_n}^{\xi}(\gamma''^u_{\sigma_n})$ tends to $\operatorname{cl}\varphi_{a,b}^p(G_{a,b}((-1,-\widehat{r}])) = \operatorname{cl}\varphi_{a,b}^p(\gamma'')$. Similarly, $\operatorname{cl}\varphi_{\sigma_n,t_n}^{\infty}(\gamma_{\sigma_n}^{\infty})$ tends to $\operatorname{cl}\varphi_{a,b}^{\infty}(\gamma_{\sigma_{a,b}}^{\infty})$, which implies that $\operatorname{cl} H_{\sigma_n,t_n}(\varphi_{\sigma_n}^{\infty}(\gamma'^{\infty}_{\sigma_n})) = \operatorname{cl}\varphi_{\sigma_n,t_n}^{\infty}(\gamma'^{\infty}_{\sigma_n})$ tends to $\operatorname{cl}\varphi_{a,b}^{\infty}(\gamma'^{\infty}_{\sigma_{a,b}}))$ and $\operatorname{cl} H_{\sigma_n,t_n}(\varphi_{\sigma_n}^{\infty}(\gamma''^{\infty}_{\sigma_n})) = \operatorname{cl}\varphi_{\sigma_n,t_n}^{\infty}(\gamma''^{\infty}_{\sigma_n})$ tends to $\operatorname{cl}\varphi_{a,b}^{\infty}(\gamma''^{\infty}_{\sigma_{a,b}})$. This ends the proof of the lemma. ∎

Note that by Lemma 2.4, $\varphi_{a,b}^{\infty}(\gamma'^{\infty}_{\sigma_{a,b}})$ lands at $p_{a,b}$. Now we show that $\varphi_{a,b}^p(G_{a,b}(\partial D_1 \cap \mathbb{D}^+))$ lands at $p_{a,b}$.

Lemma 4.21. *For $(a,b), \sigma_n, t_n$ as in Proposition 4.19, $G_{a,b}(\partial D_1 \cap \mathbb{D}^+)$ lands at 1 and $\varphi_{a,b}^p(G_{a,b}(\partial D_1 \cap \mathbb{D}^+))$ lands at $p_{a,b}$. Moreover, if $z_n \in \partial D_{t_n}$*

and $z_n \to 1$, *then* $G_{\sigma_n,t_n}(z_n) \to 1$ *and* $\varphi^{\xi}_{\sigma_n,t_n}(G_{\sigma_n,t_n}(z_n)) \to p_{a,b}$. *Furthermore*, $\operatorname{cl} H_{\sigma_n,t_n}(\varphi^u_{\sigma_n}(\Gamma^u_{\sigma_n} \setminus \gamma''^u_{\sigma_n}))$ *tends to* $\operatorname{cl} \varphi^p_{a,b}(G_{a,b}(\partial D_1 \cap \mathbb{D}^+))$ *in the Hausdorff metric.*

Proof. Let

$$J^{\pm}_{\sigma_n} = \bigcup_{j=0}^{\infty} (\nu'^u_{\sigma_n})^j((\Phi^u_{\sigma_n})^{-1}([\sqrt{\widehat{r}}\, e^{\pm i\theta_0}, \widehat{r}e^{\pm 2i\theta_0}]))$$

for the inverse branch $\nu'^u_{\sigma_n}$ of $(h^u_{\sigma_n})^{-1}$ used in the definition of the curve $\gamma'^u_{\sigma_n}$ and a small fixed $\theta_0 > 0$ (by $[x, y]$ we mean here the straight line segment between x and y). Then $J^{\pm}_{\sigma_n}$ are two backward $h^u_{\sigma_n}$-invariant curves in $\mathbb{D}$ landing at 1 and

$$\Phi^{-1}_{\sigma_n}(\{\widehat{r}e^{i\theta} : \theta \in [2\theta_0, 2\pi - 2\theta_0]\}) \cup J^+_{\sigma_n} \cup J^-_{\sigma_n} \cup \{1\}$$

is a Jordan curve. Define P_{σ_n} to be the open domain in $\mathbb{D}$ such that $0 \in P_{\sigma_n}$ and ∂P_{σ_n} is equal to this curve. It is easy to check that $h^u_{\sigma_n}(\operatorname{cl} P_{\sigma_n}) \subset \operatorname{cl} P_{\sigma_n}$. Let

$$P_n = (\varphi^{\xi}_{\sigma_n,t_n})^{-1}(H_{\sigma_n,t_n}(\varphi^u_{\sigma_n}(P_{\sigma_n}))),$$
$$J^{\pm}_n = (\varphi^{\xi}_{\sigma_n,t_n})^{-1}(H_{\sigma_n,t_n}(\varphi^u_{\sigma_n}(J^{\pm}_{\sigma_n}))).$$

Then P_n are topological discs in $\mathbb{D}$, $\partial P_n \cap \partial\mathbb{D} = \{1\}$ and $h^{\xi}_{\sigma_n,t_n}(\operatorname{cl} P_n) \subset \operatorname{cl} P_n$ (see Fig. 11).

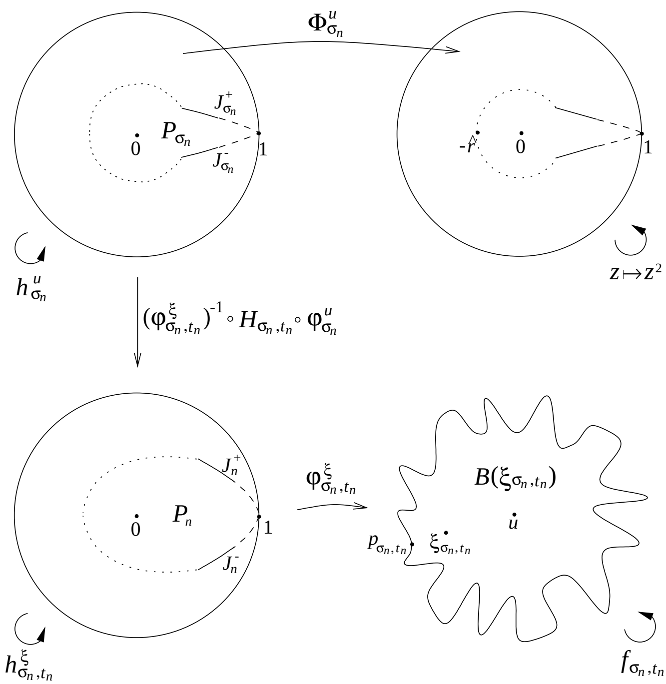

Fig. 11. The curves $J^{\pm}_n$ and the set P_n

Moreover, by the definitions of $F_{\sigma,t}$ and $G_{\sigma,t}$, there exists a sequence $y_n > 0$ such that $y_n \to y_0 > 0$ and

$$\begin{aligned} \partial P_n &= G_{\sigma_n,t_n}(\partial D_{t_n} \setminus \mathbb{D}_{y_n}(1)) \cup J_n^+ \cup J_n^- \cup \{1\}, \\ P_n &\supset G_{\sigma_n,t_n}(\partial D_{t_n} \cap \mathbb{D}_{y_n}(1)). \end{aligned} \tag{37}$$

Parameterize the curves $J_n^\pm$ according to the conditions of the case (a) of Lemma 2.4. It is easy to check that $J_n^\pm$ tend to some backward h_1-invariant curves $J^\pm$ in $\mathbb{D}$, which begin at two points of $G_{a,b}(\partial D_1 \cap \mathbb{D}_{y_0}(1))$. By Lemma 2.4, $J^\pm$ lands at 1, so

$$G_{a,b}(\partial D_1 \setminus \mathbb{D}_{y_0}(1)) \cup J^+ \cup J^- \cup \{1\}$$

is a Jordan curve. Let P be the topological disc in $\mathbb{D}$ such that $0 \in P$ and ∂P is equal to this curve. Then $\partial P \cap \partial\mathbb{D} = \{1\}$, $G_{a,b}(\partial D_1 \cap (\mathbb{D}_{y_0}(1) \setminus \{1\})) \subset P$ and $h_{a,b}^p(\operatorname{cl} P) \subset \operatorname{cl} P$. These facts easily imply that $G_{a,b}(\partial D_1 \cap \mathbb{D}^+)$ lands at 1.

Let $z_n \in \partial D_{t_n}$, $z_n \to 1$ and suppose $G_{\sigma_n,t_n}(z_n) \not\to 1$. Passing to a subsequence assume $|G_{\sigma_n,t_n}(z_n) - 1| > \varepsilon$ for a fixed $\varepsilon > 0$. By (37), we have $G_{\sigma_n,t_n}(z_n) \in P_n$ for large n. Using Lemma 2.4 for the curves $J_n^\pm$ we get $G_{\sigma_n,t_n}(z_n) \in K$ for a fixed compact set $K \in \mathbb{D}$. Hence, (35) holds for $G_{\sigma_n,t_n}(z_n)$ instead of $\gamma_n(s_n)$. Repeating the arguments from the proof of Lemma 4.20 we show that for large n,

$$(h_{\sigma_n,t_n}^\xi)^j(G_{\sigma_n,t_n}(z_n)) \in G_{\sigma_n,t_n}(D^{(n)})$$

for $D^{(n)}$ from (36), which gives $h_{t_n}^j(z_n) \in D^{(n)}$. But $h_{t_n} \to h_1$ uniformly on $\operatorname{cl}\mathbb{D}$, so $h_{t_n}^j(z_n) \to 1$. This is a contradiction, since by the definition of $D^{(n)}$, we have $\operatorname{dist}(\operatorname{cl} D^{(n)}, 1) > \text{const} > 0$. Hence, $G_{\sigma_n,t_n}(z_n) \to 1$.

By Lemma 2.4, the curves $\varphi_{a,b}^p(J^\pm)$ land at $p_{a,b}$, so $\varphi_{a,b}^p(\partial P \setminus \{1\}) \cup \{p_{a,b}\}$ is a Jordan curve. Let $S \subset \widehat{\mathbb{C}}$ be the component of the complement of this Jordan curve containing $u = \varphi_{a,b}^p(0)$. Then $S \supset \varphi_{a,b}^p(P)$ and $\partial S \subset B(p_{a,b}) \cup \{p_{a,b}\}$. Moreover, we have $\infty \notin S$ by Lemma 4.20. Now we show that the unique point from $f_{a,b}^{-1}(\infty) \setminus \{\infty\}$ is not in S. This is obvious if $\deg f_{a,b}|_{B(\infty)} = 3$. If $\deg f_{a,b}|_{B(\infty)} = 2$, then there exists a unique component $\widetilde{B}(\infty)$ of $f_{a,b}^{-1}(B(\infty)) \setminus B(\infty)$. In the same way as for Lemma 4.20 we show that some preimage under $f_{a,b}^{-(k-1)}$ of the curve $\varphi_{a,b}^p(G_{a,b}((-1, -\widehat{r}]))$ is contained in $B(p_{a,b})$ and lands at the same point as some preimage of the curve $\varphi_{a,b}^\infty(\gamma_{\sigma_{a,b}}''^\infty)$ contained in $\widetilde{B}(\infty)$. This gives $\widetilde{B}(\infty) \cap S = \emptyset$. Hence, we have $f_{a,b}^{-1}(\infty) \cap \operatorname{cl} S = \emptyset$. Moreover, $f_{a,b}(\partial S) \subset \operatorname{cl} S$ by the definition of S. Hence, by the maximum principle, $f_{a,b}(\operatorname{cl} S) \subset \operatorname{cl} S$. This implies that S does not contain points from the Julia set of $f_{a,b}$, which gives

$$S = \varphi_{a,b}^p(P) \subset B(p_{a,b}).$$

This easily implies that $\varphi_{a,b}^p(G_{a,b}(\partial D_1 \cap \mathbb{D}^+))$ lands at $p_{a,b}$.

Suppose $\varphi^{\xi}_{\sigma_n,t_n}(G_{\sigma_n,t_n}(z_n)) \not\to p_{a,b}$. Passing to a subsequence assume $\varphi^{\xi}_{\sigma_n,t_n}(G_{\sigma_n,t_n}(z_n)) \to x \neq p_{a,b}$. By (37), $\varphi^{\xi}_{\sigma_n,t_n}(G_{\sigma_n,t_n}(z_n)) \in \varphi^{\xi}_{\sigma_n,t_n}(P_n)$, so by Lemma 2.4 for the curves $J^{\pm}_n$ we get $x \in \operatorname{cl} \varphi^{p}_{a,b}(P) \setminus \{p_{a,b}\} \subset B(p_{a,b})$. This leads to a contradiction with $G_{\sigma_n,t_n}(z_n) \to 1$.

Note that $\gamma'^{u}_{\sigma_n} \subset P_{\sigma_n} \setminus (\Phi^{u}_{\sigma_n})^{-1}(\mathbb{D}_{\widehat{r}})$, so $H_{\sigma_n,t_n}(\varphi^{u}_{\sigma_n}(\gamma'^{u}_{\sigma_n})) \subset \varphi^{\xi}_{\sigma_n,t_n}(P_n \setminus G_{\sigma_n,t_n}(D_{t_n}))$ and the same arguments as previously show that

$$\text{if} \quad \widetilde{z}_n \in H_{\sigma_n,t_n}(\varphi^{u}_{\sigma_n}(\gamma'^{u}_{\sigma_n})), \quad \text{then} \quad \widetilde{z}_n \to p_{a,b}.$$

The above facts together with the almost uniform convergence imply that

$$\begin{aligned}\operatorname{cl} H_{\sigma_n,t_n}(\varphi^{u}_{\sigma_n}(\Gamma^{u}_{\sigma_n} \setminus \gamma''^{u}_{\sigma_n}))&\\ = \operatorname{cl}(\varphi^{\xi}_{\sigma_n,t_n}&(G_{\sigma_n,t_n}(\partial D_{t_n} \cap \mathbb{D}^{+})) \cup H_{\sigma_n,t_n}(\varphi^{u}_{\sigma_n}(\gamma'^{u}_{\sigma_n})))\end{aligned}$$

tends to $\operatorname{cl} \varphi^{p}_{a,b}(G_{a,b}(\partial D_1 \cap \mathbb{D}^{+}))$ in the Hausdorff metric. ■

Proof of Proposition 4.19. Lemmas 4.20 and 4.21 show that $\operatorname{cl}(\varphi^{p}_{a,b}(\Gamma^{p}_{a,b}) \cup \varphi^{\infty}_{a,b}(\Gamma^{\infty}_{a,b}))$ is a Jordan curve. Note that by almost uniform convergence, $\operatorname{cl} H_{\sigma_n,t_n}(\varphi^{\infty}_{\sigma_n}(\Gamma^{\infty}_{\sigma_n} \setminus (\gamma'^{\infty}_{\sigma_n} \cup \gamma''^{\infty}_{\sigma_n}))) = \operatorname{cl} \varphi^{\infty}_{\sigma_n,t_n}(\Gamma^{\infty}_{\sigma_n} \setminus (\gamma'^{\infty}_{\sigma_n} \cup \gamma''^{\infty}_{\sigma_n}))$ tends to $\operatorname{cl} \varphi^{\infty}_{a,b}(\Gamma^{\infty}_{a,b} \setminus (\gamma'^{\infty}_{\sigma_{a,b}} \cup \gamma''^{\infty}_{\sigma_{a,b}}))$ in the Hausdorff metric. This together with Lemmas 4.20, 4.21 and (27) implies that $\partial f^{k}_{\sigma_n,t_n}(U_{\sigma_n,t_n})$ tends to $\operatorname{cl}(\varphi^{p}_{a,b}(\Gamma^{p}_{a,b}) \cup \varphi^{\infty}_{a,b}(\Gamma^{\infty}_{a,b}))$ in the Hausdorff metric. ■

An immediate consequence of Proposition 4.19 is

Corollary 4.22. *The Jordan curve* $\operatorname{cl}(\varphi^{p}_{a,b}(\Gamma^{p}_{a,b}) \cup \varphi^{\infty}_{a,b}(\Gamma^{\infty}_{a,b}))$ *depends continuously on* $(a,b) \in \widetilde{\mathcal{U}}_1$ *in the Hausdorff metric.* ■

Now for $(a,b) \in \widetilde{\mathcal{U}}_1$ we define a topological disc $U_{a,b}$ such that $f^{k}_{a,b}$ is quadratic-like on $U_{a,b}$ and $\operatorname{cl} U_{\sigma_n,t_n}$ tends to $\operatorname{cl} U_{a,b}$ in the Hausdorff metric.

The case $(a,b) \in M_1$. Let $(a,b) \in M_1$ and take $\sigma_n \in M_N$, $t_n \to 1^{-}$ such that $(a(\sigma_n,t_n), b(\sigma_n,t_n)) \to (a,b)$. For $\zeta \in \{p_{a,b}, \infty\}$ denote by $\widetilde{B}(\zeta)$ the unique component of $f^{-1}_{a,b}(B(\zeta)) \setminus B(\zeta)$. Consider the Blaschke product h_t from (26) for $t \in [0,1]$. Note that the forward trajectory of the h_t-critical point 0 is contained in $[0,1)$, so all inverse branches of h_t^{-k} are defined on $\mathbb{D}^{+}$. Moreover, $h_t|_{\partial\mathbb{D}}$ is a degree two covering depending continuously on $t \in [0,1]$. Therefore, there exists a branch ν_t of h_t^{-k} defined on $\mathbb{D}^{+}$ such that $\nu_0(\mathbb{D}^{+}) = \{re^{2\pi i\theta} : 0 < r < 1,\ \theta \in (\beta,\delta)\}$ for β, δ from the definition of U'_{σ} and ν_t depends continuously on $t \in [0,1]$. By (29) and the definition of the sets $U_{\sigma,t}$, we have

$$\begin{aligned}&\partial U_{\sigma_n,t_n} \cap B(\xi_{\sigma_n,t_n}) = \varphi^{\xi}_{\sigma_n,t_n}(\partial \nu_{t_n}(\mathbb{D}^{+} \setminus \operatorname{cl} D_{t_n}) \cap \mathbb{D}),\\ &\partial U_{\sigma_n,t_n} \cap \widetilde{B}(\xi_{\sigma_n,t_n}) = f^{-1}_{\sigma_n,t_n}(f_{\sigma_n,t_n}(\partial U_{\sigma_n,t_n} \cap B(\xi_{\sigma_n,t_n}))) \cap \widetilde{B}(\xi_{\sigma_n,t_n}).\end{aligned}$$

Now we observe that

$$\begin{aligned}(38)\qquad &\mathrm{cl}(\varphi^p_{a,b}(\partial\nu_1(\mathbb{D}^+\setminus\mathrm{cl}\,D_1)\cap\mathbb{D})\\ &\cup(f^{-1}_{a,b}(f_{a,b}(\varphi^p_{a,b}(\partial\nu_1(\mathbb{D}^+\setminus\mathrm{cl}\,D_1)\cap\mathbb{D})))\cap\widetilde{B}(p_{a,b}))\\ &\cup\varphi^\infty_{a,b}((\varphi^\infty_{\sigma_{a,b}})^{-1}(\partial U_{\sigma_{a,b}}\cap B_{\sigma_{a,b}}(\infty)))\\ &\cup(f^{-1}_{a,b}(f_{a,b}(\varphi^\infty_{a,b}((\varphi^\infty_{\sigma_{a,b}})^{-1}(\partial U_{\sigma_{a,b}}\cap B_{\sigma_{a,b}}(\infty)))))\cap\widetilde{B}(\infty)))\end{aligned}$$

is a Jordan curve and $\partial U_{\sigma_n,t_n}$ converges to it in the Hausdorff metric. The proof is the same as the proof of Proposition 4.19 and we leave it to the reader.

DEFINITION 4.23. For $(a,b)\in M_1$ let $U_{a,b}$ be the topological disc containing 1, whose boundary is equal to the Jordan curve from (38).

By (34), we have $\mathrm{cl}\,\nu_1(\mathbb{D}^+\setminus\mathrm{cl}\,D_1)\subset\mathbb{D}^+\setminus\mathrm{cl}\,D_1$. This easily implies that $f^k_{a,b}$ is quadratic-like on $U_{a,b}$. Moreover, $\partial f^k_{a,b}(U_{a,b})$ is equal to the Jordan curve from Proposition 4.19 (see Fig. 12).

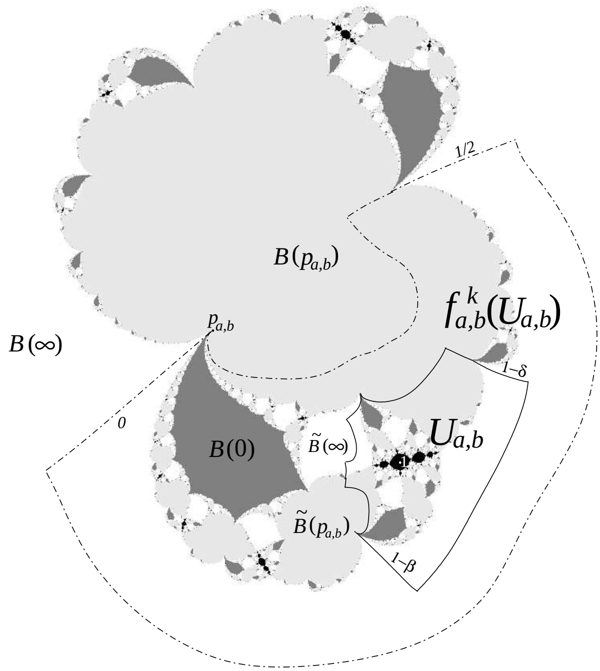

Fig. 12. The set $U_{a,b}$ for a parabolic map $f_{a,b}$

The case $(a,b)\notin M_1$. Now we extend the definition of $U_{a,b}$ for $(a,b)\in\widetilde{\mathcal{U}}_1$.

DEFINITION 4.24. For $(a,b) \in \widetilde{\mathcal{U}}_1$ let $A_{a,b}$ be the component of the complement of the Jordan curve $\mathrm{cl}(\varphi^p_{a,b}(\Gamma^p_{a,b}) \cup \varphi^\infty_{a,b}(\Gamma^\infty_{a,b}))$ from Proposition 4.19 containing $f^k_{a,b}(1)$ and let $U_{a,b}$ be the component of $f^{-k}_{a,b}(A_{a,b})$ containing 1.

By (33), if we take sufficiently small ε_1 in the definition of $\widetilde{\mathcal{U}}_1$, then for $(a,b) \in \widetilde{\mathcal{U}}_1$ the point $f^k_{a,b}(1)$ is not in $\mathrm{cl}(\varphi^p_{a,b}(\Gamma^p_{a,b}) \cup \varphi^\infty_{a,b}(\Gamma^\infty_{a,b}))$. Hence, $U_{a,b}$ is a topological disc, $f^k_{a,b}$ is quadratic-like on $U_{a,b}$ and $\mathrm{cl}\, U_{a,b}$ depends continuously on $(a,b) \in \widetilde{\mathcal{U}}_1$ by Corollary 4.22. Moreover, by Proposition 4.19, if $\sigma_n \in \mathcal{U}'$, $t_n \to 1^-$ such that $(a(\sigma_n,t_n), b(\sigma_n,t_n)) \to (a,b)$, then

$$\mathrm{cl}\, U_{\sigma_n,t_n} \text{ tends to } \mathrm{cl}\, U_{a,b} \text{ in the Hausdorff metric.} \tag{39}$$

Note that for $(a,b) \in M_1$ the definition of $U_{a,b}$ coincides with the previous one.

It is easy to check by (1) that for $\varrho \in \mathbb{C}$ we have

$$\mathrm{Fix}(\varrho) = \{(a,b) \in (\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}) \setminus \mathrm{Sing} : P(\varrho,a,b) = 0\}, \tag{40}$$

where P is a polynomial such that $P(1,a,b)$ is equal to the left-hand side of (30). Hence, every $(a_0,b_0) \in \mathrm{Fix}(1)$ has an open neighbourhood

$$\mathcal{W}_{a_0,b_0} = \mathbb{D}_{\varepsilon(a_0,b_0)}(a_0) \times \mathbb{D}_{\varepsilon(a_0,b_0)}(b_0) \subset \mathbb{C} \times \mathbb{C}$$

such that

$$\mathrm{Fix}(1) \cap \mathcal{W}_{a_0,b_0} = \{(a,b) : a = \widetilde{\omega}_1(b),\ b \in \mathbb{D}_{\varepsilon(a_0,b_0)}(b_0)\} \tag{41}$$

for some holomorphic map $\widetilde{\omega}_1$. By (40) and the implicit function theorem, for $\varrho \in \mathbb{C}$ close to 1 we have

$$\mathrm{Fix}(\varrho) \cap \mathcal{W}_{a_0,b_0} = \{(a,b) : a = \widetilde{\omega}_\varrho(b),\ b \in \mathbb{D}_{\varepsilon(a_0,b_0)}(b_0)\}$$

for some holomorphic map $\widetilde{\omega}_\varrho$ depending holomorphically on ϱ. This implies that there exists a holomorphic homeomorphism

$$\mathcal{H}_{a_0,b_0} : \mathcal{W}_{a_0,b_0} \to \mathbb{C} \times \mathbb{C}$$

such that

$$\begin{gathered} \mathcal{H}_{a_0,b_0}(\widetilde{\mathcal{U}}_1 \cap \mathcal{W}_{a_0,b_0}) \subset \{(\eta,\mu) : \eta = 1\}, \\ \mathcal{H}_{a_0,b_0}(Q_t(\mathcal{U}') \cap \mathcal{W}_{a_0,b_0}) \subset \{(\eta,\mu) : \eta = \omega_t(\mu)\}, \end{gathered} \tag{42}$$

where ω_t is a holomorphic map depending continuously on $t \le 1$ close to 1 and $\omega_1 \equiv 1$.

LEMMA 4.25. *There exists $c > 0$ such that for every $t \in (0,1)$,*

$$\begin{gathered} Q_t(\mathcal{U}') \supset \{(a,b) \in \mathrm{Fix}(t) : \mathrm{dist}((a,b), M_t) < c\}, \\ \widetilde{\mathcal{U}}_1 \supset \{(a,b) \in \mathrm{Fix}(1) : \mathrm{dist}((a,b), M_1) < c\}. \end{gathered}$$

Proof. Note that the first statement of the lemma together with Lemma 4.12 and (42) implies the second one, so it is sufficient to prove the first statement. Let

$$\mathcal{A}_n = \{\sigma \in \mathcal{U}' : N_\sigma^{kj}(1) \in U'_\sigma \text{ for } 0 \le j \le n\}.$$

It is easy to check that $\mathcal{A}_n$ is open in $\mathcal{U}'$ and

$$\partial\mathcal{A}_n \subset \{\sigma \in \mathcal{U}' : N_\sigma^{kj}(1) \in U'_\sigma \text{ for } 0 \le j \le n-1,\ N_\sigma^{kn}(1) \in \partial U'_\sigma\}. \tag{43}$$

Since $\bigcap_{n\ge 0} \mathcal{A}_n = M_N$, we can find n_0 such that if $\mathcal{A}$ is the component of $\mathcal{A}_{n_0}$ containing M_N, then $\operatorname{cl}\mathcal{A} \subset \mathcal{U}'$. It is easy to check that there exists a component γ of $\partial\mathcal{A}$ such that the index of γ with respect to a point from M_N is equal to 1. By (43) and the definition of $U_{\sigma,t}$,

$$Q_t(\gamma) \subset \{(a(\sigma,t), b(\sigma,t)) : f_{\sigma,t}^{kn_0}(1) \in \partial U_{\sigma,t}\}$$

for Q_t from Lemma 4.12. This together with (39) implies that any limit point of the sequence $Q_{t_n}(\sigma_n)$ for $\sigma_n \in \gamma$, $t_n \to 1^-$ does not belong to M_1. Hence, by the compactness of γ and the properties of Q_t, there exists $c > 0$ such that $\operatorname{dist}(Q_t(\gamma), M_t) > c$ for every $t < 1$. Note that by (42) and the topological properties of surfaces we can assume that for some $\widetilde{c} > 0$ the set $\{(a,b) \in \operatorname{Fix}(t) : \operatorname{dist}((a,b), M_t) < \widetilde{c}\}$ for every $t < 1$ is homeomorphic to an open subset of $\mathbb{C}$. Then repeating the proof of Lemma 4.12 with γ_1 replaced by γ we show that $Q_t(\mathcal{U}') \supset \{(a,b) \in \operatorname{Fix}(t) : \operatorname{dist}((a,b), M_t) < c\}$. ∎

By Lemma 4.25, if we take sufficiently small ε_1 in the definition of $\widetilde{\mathcal{U}}_1$, then $\widetilde{\mathcal{U}}_1$ is open in $\operatorname{Fix}(1)$, so we have

Corollary 4.26. *$\widetilde{\mathcal{U}}_1$ is a one-dimensional complex manifold containing M_1 and $\{f_{a,b}^k|_{U_{a,b}}\}_{(a,b)\in\widetilde{\mathcal{U}}_1}$ is an analytic family of quadratic-like maps.* ∎

Lemma 4.27. *For every $\sigma \in M_N$ the curve $\{(a(\sigma,t), b(\sigma,t))\}_{t\in[0,1)}$ lands at a point in M_1.*

Proof. Suppose that the curve has distinct limit points $(a_1,b_1), (a_2,b_2) \in M_1$. By Lemma 4.25, we can take a small ball $\mathcal{V}$ in $\mathbb{C}\times\mathbb{C}$ centred at (a_1,b_1) such that $(a_2,b_2) \notin \operatorname{cl}\mathcal{V}$ and $\operatorname{cl}\mathcal{V} \cap \operatorname{Fix}(1) \subset \widetilde{\mathcal{U}}_1$. Note that if $\mathcal{V}$ is sufficiently small, then $\{f_{a,b}^k|_{V_{a,b}}\}_{(a,b)\in\mathcal{V}}$ is an analytic family of quadratic-like maps, where $V_{a,b}$ is defined as the component of $f_{a,b}^{-k}(f_{a_1,b_1}^k(U_{a_1,b_1}))$ containing 1. It is easy to check (see [McM]) that if $(a,b) \in \mathcal{V} \cap \widetilde{\mathcal{U}}_1$, then the filled-in Julia set of $f_{a,b}^k|_{U_{a,b}}$ and $f_{a,b}^k|_{V_{a,b}}$ coincide. Similarly, by (39), if $(a,b) = (a(\sigma,t), b(\sigma,t)) \in \mathcal{V} \cap Q_t(\mathcal{U}')$, then the filled-in Julia sets of $f_{a,b}^k|_{U_{\sigma,t}}$ and $f_{a,b}^k|_{V_{a,b}}$ are the same.

By assumption, the curve $\{(a(\sigma,t), b(\sigma,t))\}_{t\in[0,1)}$ must intersect $\partial\mathcal{V}$ for infinitely many $t_n < 1$ such that $t_n \to 1^-$. Passing to a subsequence, we can assume that $(a(\sigma,t_n), b(\sigma,t_n))$ converges to some point in $\partial\mathcal{V} \cap \widetilde{\mathcal{U}}_1$. Since $\mathcal{V}$ can be arbitrarily small, this easily implies that (a_1,b_1) is a point of density of the set of limit points of the curve $\{(a(\sigma,t), b(\sigma,t))\}_{t\in[0,1)}$. By the definition of $U_{\sigma,t}$, the map $f^k_{\sigma,t}|_{U_{\sigma,t}}$ is hybrid equivalent to $z^2 + \chi(\sigma)$. By Theorem 3.3 for the family $\{f^k_{a,b}|_{V_{a,b}}\}_{(a,b)\in\mathcal{V}}$, the point (a_1, b_1) is a point of density of the set of points $(a, b) \in \mathcal{V} \cap \widetilde{\mathcal{U}}_1$ for which $f^k_{a,b}|_{V_{a,b}}$ is hybrid equivalent to $z^2 + \chi(\sigma)$. Then Theorem 3.4 implies that all maps $f^k_{a,b}|_{V_{a,b}}$ for $(a, b) \in \mathcal{V}$ are hybrid equivalent. Applying the same theorem to the family $\{f^k_{a,b}|_{U_{a,b}}\}_{(a,b)\in\widetilde{\mathcal{U}}_1}$ we conclude that all maps $f^k_{a,b}|_{U_{a,b}}$ for (a, b) in the component of $\widetilde{\mathcal{U}}_1$ containing (a_1, b_1) are hybrid equivalent. This is clearly impossible (as follows e.g. from Lemma 4.25). ∎

By Lemma 4.27, we can define a mapping

$$Q_1 : M_N \to M_1, \qquad Q_1(\sigma) = \lim_{t\to 1^-} Q_t(\sigma).$$

LEMMA 4.28. *Q_1 is a homeomorphism.*

Proof. By definition, Q_1 maps M_N onto M_1. To show that Q_1 is injective, suppose that $Q_1(\sigma_1) = Q_1(\sigma_2) = (a_1, b_1)$. Define $\mathcal{V}$ and $V_{a,b}$ as in the proof of Lemma 4.27. Then Theorem 3.3 for the family $\{f^k_{a,b}|_{V_{a,b}}\}_{(a,b)\in\mathcal{V}}$ shows $\sigma_1 = \sigma_2$. Now we prove that Q_1 is continuous. Let $\sigma_n \to \sigma$ for $\sigma_n, \sigma \in M_N$ and suppose that

$$\operatorname{dist}(Q_1(\sigma_n), Q_1(\sigma)) > c_0 > 0 \tag{44}$$

for all n. Take $0 < c_1 < c_0$. Since Q_t is continuous, for any $j > 0$ we can choose n_j such that $\operatorname{dist}(Q_{1-1/j}(\sigma_{n_j}), Q_{1-1/j}(\sigma)) < c_1$. By (44), there exists $t_j \in (1 - 1/j, 1)$ such that $\operatorname{dist}(Q_{t_j}(\sigma_{n_j}), Q_{t_j}(\sigma)) = c_1$. Passing to a subsequence, we can assume that $Q_{t_j}(\sigma_{n_j}) \to (a_1, b_1) \in M_1$ such that $\operatorname{dist}((a_1, b_1), Q_1(\sigma)) = c_1$. Arguing as in the proof of Lemma 4.27 we show that $f^k_{a_1,b_1}|_{U_{a_1,b_1}}$ is hybrid equivalent to $z^2 + \chi(\sigma)$. Since c_1 was arbitrary, it follows that $Q_1(\sigma)$ is a point of density of the set of points $(a, b) \in M_1$ for which $f^k_{a,b}|_{U_{a,b}}$ is hybrid equivalent to $z^2 + \chi(\sigma)$. As in the proof of Lemma 4.27, this leads to a contradiction.

We have shown that Q_1 is a continuous injective map from M_N onto M_1. By the compactness of M_N, Q_1 is a homeomorphism. ∎

Since Fix(1) is a one-dimensional complex manifold, using Lemma 4.28 we can find a topological disc in Fix(1) containing M_1. This implies that the map $\widetilde{\omega}_1$ from (41) can be defined globally, i.e. there exist an open neighbourhood $\mathcal{W} \subset \mathbb{C} \times \mathbb{C}$ of M_1, a topological disc $W \subset \mathbb{C}$ and a holomorphic map $\widetilde{\omega}_1$ on W such that

$$\operatorname{Fix}(1) \cap \mathcal{W} \subset \{(a, b) : a = \widetilde{\omega}_1(b),\ b \in W\}.$$

If ε_1 in the definition of $\widetilde{\mathcal{U}}_1$ is sufficiently small, then $\widetilde{\mathcal{U}}_1 \subset \mathcal{W}$ and by the implicit function theorem, there exists a biholomorphic map

$$\mathcal{H} : \mathcal{W} \to \mathbb{C} \times \mathbb{C}$$

such that

$$(45)\quad \mathcal{H}(\widetilde{\mathcal{U}}_1) \subset \{(\eta,\mu) : \eta = 1\}, \qquad \mathcal{H}(Q_t(\mathcal{U}') \cap \mathcal{W}) \subset \{(\eta,\mu) : \eta = \omega_t(\mu)\},$$

where ω_t is a holomorphic map depending continuously on $t \le 1$ close to 1 and $\omega_1 \equiv 1$. This together with Lemma 4.25 implies that the path of the topological discs $\mathcal{U}_t$, $t \in [0,1)$, from Corollary 4.13 can be extended continuously to $\mathcal{U}_t$, $t \in [0,1]$, i.e. there exist disc-equivalent Riemann surfaces $\mathcal{U}_t \subset \mathrm{Fix}(t)$, $t \in [0,1]$, such that $\mathcal{U}_0 \subset \mathcal{U}'$, $M_t \subset \mathcal{U}_t \subset Q_t(\mathcal{U}')$ for $t \in [0,1)$, $M_1 \subset \mathcal{U}_1 \subset \widetilde{\mathcal{U}}_1$ and $\mathrm{cl}\,\mathcal{U}_t$ depends continuously on $t \in [0,1]$ in the Hausdorff metric. Let

$$\mathbf{f}_t = \{f^k_{a,b}|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}_t} \quad \text{for } t \in [0,1].$$

By Corollary 4.13, (39), Corollary 4.26, Theorem 3.3 and Proposition 3.5, the families $\mathbf{f}_t$ for $t \in [0,1]$ are Mandelbrot-like with the Mandelbrot-like set M_t and $\{\mathbf{f}_t\}_{t\in[0,1]}$ is a continuous path of Mandelbrot-like families. By Proposition 3.7, the sets M_t for $t \in [0,1]$ are quasiconformally homeomorphic.

4.4. *The construction of* $\mathbf{f}^0_t, \mathbf{f}^\infty_t$. To simplify notation, in the following two subsections we will use a new holomorphic system of coordinates (η,μ) in a part of the parameter space defined by $\mathcal{H}$ from (45). We write $f_{\eta,\mu}$ for $f_{a,b}$, where $\mathcal{H}(a,b) = (\eta,\mu)$. Recall that these coordinates are defined in some open neighbourhood of $\mathcal{U}_1$ in $\mathbb{C}\times\mathbb{C}$ and in these coordinates $\mathcal{U}_1$ is contained in the plane $\{\eta = 1\}$. For simplicity, we write $\mu \in M_1$, $\mu \in \mathcal{U}_1$ instead of $(1,\mu) \in M_1$, $(1,\mu) \in \mathcal{U}_1$ respectively.

For $\mu \in \mathcal{U}_1$ denote by p_μ the parabolic $f_{1,\mu}$-fixed point of multiplier 1. For $\zeta \in \{0,\infty\}$ let $\Phi^\zeta_{\eta,\mu}$ be the Böttcher coordinates on $B(\zeta)$ for $f_{\eta,\mu}$. If $\mu \in M_1$, then $\Phi^\zeta_{1,\mu}$ is defined on $B(\zeta)$. Hence, if we fix $r_0 < 1$ close to 1 and a small $\widetilde{\varepsilon} > 0$, and take $\mathcal{U}_1$ sufficiently close to M_1, then for $\mu \in \mathcal{U}_1$ and $\eta \in \mathbb{D}_{\widetilde{\varepsilon}}(1)$ the map $(\Phi^\zeta_{\eta,\mu})^{-1}$ is defined on $\mathrm{cl}\,\mathbb{D}_{r_0}$ and $(\Phi^\zeta_{\eta,\mu})^{-1}([r_0^2, r_0])$ is contained in a small neighbourhood of p_μ. Let

$$\gamma^\zeta_{\eta,\mu}(s) = (\Phi^\zeta_{\eta,\mu})^{-1}(s) \quad \text{for } s \in [0, r_0].$$

For $\mu \in \mathcal{U}_1$ we can extend the curve $\gamma^\zeta_{1,\mu}$ taking its successive preimages by the branch of $f^{-1}_{1,\mu}$ fixing p_μ and parameterize it by $s \in [0,1)$ in such a way that $\gamma^\zeta_{1,\mu}(s^2) = f_{1,\mu}(\gamma^\zeta_{1,\mu}(s))$. Then $\gamma^\zeta_{1,\mu}$ is a simple arc in $B(\zeta)$ beginning at ζ and landing at p_μ. For $\eta \in \mathbb{D}_{\widetilde{\varepsilon}}(1)$ we can extend $\gamma^\zeta_{\eta,\mu}$ in the same way parameterizing it by $s \in [0,r]$ such that r is arbitrarily close to 1 and $\gamma^\zeta_{\eta,\mu}(r)$ is arbitrarily close to p_μ, if η is sufficiently close to 1. Since the Böttcher coordinates depend holomorphically on the mapping, the function $(\eta,\mu) \mapsto \gamma^\zeta_{\eta,\mu}(r)$ is holomorphic.

For $\mu \in \mathcal{U}_1$ consider the family $\{f_{\eta,\mu}\}_{\eta\in\mathbb{D}_{\widetilde{\varepsilon}}(1)}$. Let $\eta(\lambda) = 1 + \lambda^2$ for $\lambda \in \mathbb{D}_{\sqrt{\widetilde{\varepsilon}}}$. Since $f_{1,\mu}$ has a double fixed point p_μ, we can choose an $f_{\eta(\lambda),\mu}$-

fixed point $\xi_\mu(\lambda)$ such that $\xi_\mu(0) = p_\mu$ and $(\lambda, \mu) \mapsto \xi_\mu(\lambda)$ is holomorphic. Let

$$g_{\lambda,\mu}(z) = f_{\eta(\lambda),\mu}(z + \xi_\mu(\lambda)) - \xi_\mu(\lambda).$$

Then the family $\{g_{\lambda,\mu}\}_{\lambda\in\mathbb{D}_{\sqrt{\varepsilon}}}$ satisfies the assumptions of Theorem I.4.21. Taking $\delta_+ = \gamma^\zeta_{\eta,\mu}(r)$ in this theorem and repeating the proof of Theorem I.4.20 we show that for every $t \in (1, 1+\varepsilon]$ with some small $\varepsilon > 0$ there exists $\lambda^\zeta_\mu(t) \in \mathbb{D}_{\sqrt{\varepsilon}}$ such that the curve $\gamma^\zeta_{\eta(\lambda^\zeta_\mu(t)),\mu}$ can be extended to a simple arc in $B(\zeta)$ connecting ζ to $f_{\eta(\lambda^\zeta_\mu(t)),\mu}(u)$, parameterized by $s \in [0, 2-t]$ such that $\gamma^\zeta_{\eta(\lambda^\zeta_\mu(t)),\mu}(s^2) = f_{\eta(\lambda^\zeta_\mu(t)),\mu}(\gamma^\zeta_{\eta(\lambda^\zeta_\mu(t)),\mu}(s))$,

$$f_{\eta(\lambda^\zeta_\mu(t)),\mu}(u) = \gamma^\zeta_{\eta(\lambda^\zeta_\mu(t)),\mu}(2-t)$$

and $\lambda^\zeta_\mu(t) \to 0$ as $t \to 1^+$. Moreover, by the proof of Theorem I.4.21 (see [DH1], Proposition XI.5) and by the holomorphicity of $(\eta, \mu) \mapsto \gamma^\zeta_{\eta,\mu}(r)$, we can choose the parameter $\lambda^\zeta_\mu(t)$ so that it depends continuously on $t \in (1, 1+\varepsilon]$ and holomorphically on $\mu \in \mathcal{U}_1$.

Let $\mathcal{G}^\zeta_\mu(1) = 1$ and $\mathcal{G}^\zeta_\mu(t) = \eta(\lambda^\zeta_\mu(t))$ for $t \in (1, 1+\varepsilon]$. Then

$$\mathcal{G}^\zeta_\mu : [1, 1+\varepsilon] \to \mathbb{C}$$

is a curve such that

$$f_{\mathcal{G}^\zeta_\mu(t),\mu}(u) = \gamma^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu}(2-t)$$

for $t \in (1, 1+\varepsilon]$. In particular, $u \in B(\zeta)$ for the map $f_{\mathcal{G}^\zeta_\mu(t),\mu}$. Note that if $\Phi^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu}$ is defined in a neighbourhood of the curve $\gamma^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu}$ (e.g. if $1 \notin B(\zeta)$), then $\gamma^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu}(s) = (\Phi^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu})^{-1}(s)$ for every $s \in [0, 2-t]$, so

$$\Phi^\zeta_{\mathcal{G}^\zeta_\mu(t),\mu}(f_{\mathcal{G}^\zeta_\mu(t),\mu}(u)) = 2 - t. \tag{46}$$

For $t \in [1, 1+\varepsilon]$ let

$$\mathcal{U}^\zeta_t = \{(\mathcal{G}^\zeta_\mu(t), \mu) : \mu \in \mathcal{U}_1\}.$$

Since $\mu \mapsto \lambda^\zeta_\mu(t)$ is holomorphic, $\mu \mapsto \mathcal{G}^\zeta_\mu(t)$ is holomorphic, so $\mathcal{U}^\zeta_t$ are disc-equivalent Riemann surfaces. For $(\eta, \mu) \in \mathcal{U}^\zeta_t$ define $U_{\eta,\mu}$ to be the component of $f^{-k}_{\eta,\mu}(f^k_{1,\mu}(U_{1,\mu}))$ containing 1. If ε is sufficiently small, then $U_{\eta,\mu}$ is a topological disc, $f^k_{\eta,\mu}$ is quadratic-like on $U_{\eta,\mu}$ and

$$\mathbf{f}^\zeta_t = \{f^k_{\eta,\mu}|_{U_{\eta,\mu}}\}_{(\eta,\mu)\in\mathcal{U}^\zeta_t}$$

is an analytic family of quadratic-like maps. By Theorem 3.3 and Proposition 3.5, this is a Mandelbrot-like family and $\{\mathbf{f}^\zeta_t\}_{t\in[1,1+\varepsilon]}$ is a continuous path of Mandelbrot-like families such that $\mathbf{f}^\zeta_1 = \mathbf{f}_1$. Let M^ζ_t be the

Mandelbrot-like set in $\mathcal{U}_t^\zeta$. By Proposition 3.7, all the sets M_t^ζ are quasiconformally homeomorphic.

4.5. $B(\infty)$ *is exotic*, $B(0)$ *is not exotic.* Now we show that for $t > 1$, if $(\eta,\mu) \in \mathcal{U}_t^\infty$, then $B(\infty)$ for the map $f_{\eta,\mu}$ is not simply connected and if $(\eta,\mu) \in M_t^0$, then $B(0)$ for the map $f_{\eta,\mu}$ is simply connected.

Let $\mu \in \mathcal{U}_1$ and consider the curve

$$\operatorname{cl}(\varphi_{a,b}^p(G_{a,b}([h_1(0),1))) \cup \gamma_{1,\mu}^\infty, \tag{47}$$

where (a,b) is such that $\mathcal{H}(a,b) = (1,\mu)$ for $\mathcal{H}$ from (45) and $\gamma_{1,\mu}^\infty$ is the curve from Subsection 4.4. Then the curve (47) is contained in $B(\infty) \cup \{p_\mu\} \cup B(p_\mu)$ and connects ∞ to $f_{1,\mu}(u)$. Moreover, one of the two components of the preimage under $f_{1,\mu}^{-1}$ of this curve is equal to the Jordan curve

$$\operatorname{cl}(\varphi_{a,b}^p(G_{a,b}((-1,1))) \cup \varphi_{a,b}^\infty(\gamma_{\sigma_{a,b}}'^\infty \cup (\Phi_{\sigma_{a,b}}^\infty)^{-1}((-\widehat{r},\widehat{r})) \cup \gamma_{\sigma_{a,b}}''^\infty)) \tag{48}$$

for $\gamma_{\sigma_{a,b}}'^\infty, \gamma_{\sigma_{a,b}}''^\infty$ from (24), which is contained in $B(\infty) \cup f_{1,\mu}^{-1}(p_\mu) \cup B(p_\mu)$, contains ∞, p_μ, u and separates 0 from $f_{1,\mu}^{-1}(0) \setminus \{0\}$.

Let $(\eta,\mu) \in \mathcal{U}_t^\infty$ for $t > 1$. Recall that the curve $\gamma_{\eta,\mu}^\infty$ from Subsection 4.4 is contained in $B(\infty)$ for $f_{\eta,\mu}$ and connects ∞ to $f_{\eta,\mu}(u)$. Moreover, the proof of Theorem I.4.20 applied for the family $\{f_{\eta,\mu}\}_{\eta \in \mathbb{D}_{\widetilde{\varepsilon}}(1)}$ shows that $\gamma_{\eta,\mu}^\infty$ is arbitrarily close to the curve (47), if t is sufficiently close to 1. Taking the component of the preimage under $f_{\eta,\mu}^{-1}$ of $\gamma_{\eta,\mu}^\infty$ which is close to (48) we find a Jordan curve in $B(\infty)$ for $f_{\eta,\mu}$ separating 0 from $f_{\eta,\mu}^{-1}(0) \setminus \{0\}$. Hence, $B(\infty)$ is not simply connected for $(\eta,\mu) \in \mathcal{U}_t^\infty$.

Consider now a map $f_{1,\mu}$ for $\mu \in M_1$ and let K_μ be the filled-in Julia set of the quadratic-like map $f_{1,\mu}^k|_{U_{1,\mu}}$. By the definition of the set $U_{\sigma,t}$, it is easy to check that

$$K_\mu \ni l_\infty(\alpha), \widetilde{l}_\infty(\alpha),$$

where α is Head's angle used in the definition of U_σ' and $l_\infty(\alpha), \widetilde{l}_\infty(\alpha)$ denote the landing points of the suitable rays for the map $f_{1,\mu}$. Note that $l_\infty(\alpha)$ is a fixed repelling point for the quadratic-like map $f_{1,\mu}^k|U_{1,\mu}$. Hence, for the map $f_{\eta,\mu}$ with $(\eta,\mu) \in M_t^0$ and $t > 1$ close to 1 we have

$$K_{\eta,\mu} \ni l_\infty(\alpha), \widetilde{l}_\infty(\alpha),$$

where $K_{\eta,\mu}$ is the filled-in Julia set of $f_{\eta,\mu}^k|_{U_{\eta,\mu}}$ and $l_\infty(\alpha), \widetilde{l}_\infty(\alpha)$ are the suitable landing points for the map $f_{\eta,\mu}$. Since $K_{\eta,\mu}$ is connected, this implies that the set $B(\infty) \cup K_{\eta,\mu} \cup \widetilde{B}(\infty)$ for $f_{\eta,\mu}$ is connected, so

$$C = \operatorname{cl} \bigcup_{n=-\infty}^{\infty} f_{\eta,\mu}^n(B(\infty) \cup K_{\eta,\mu})$$

is connected. By the classification theorem, since only one $f_{\eta,\mu}$-critical point 1 is not contained in $B(0) \cup B(\infty)$, the closure of any Fatou component different from a component of the entire basin of attraction to 0 or ∞ must contain points from the forward trajectory of $1 \in K_{\eta,\mu}$. This together with the connectedness of C easily implies that the complement of the entire basin of attraction to 0 is connected. Moreover, by Theorem I.4.4, $B(0)$ is completely invariant, i.e. $B(0)$ is the entire basin of attraction to 0. We conclude that $\widehat{\mathbb{C}} \setminus B(0)$ is connected, which means that $B(0)$ is simply connected.

REMARK. Note that it is not true that $B(0)$ is simply connected for every $\mu \in \mathcal{U}_t^0$. In fact, if $1, u \in B(0)$, then there are three critical points in $B(0)$, so $B(0)$ cannot be simply connected.

Recall that an exotic basin for a cubic rational map is a non-simply connected completely invariant basin of attraction containing less than three critical points counted with multiplicity (for more information see [P1], [Ba]). Let $(\eta, \mu) \in M_t^\infty$. Then $B(\infty)$ for the map $f_{\eta,\mu}$ is not simply connected and contains two critical points counted with multiplicity. Moreover, by Theorem I.4.4, $B(\infty)$ is completely invariant. Hence, $B(\infty)$ is exotic.

Consider now $(\eta, \mu) \in M_t^0$. We have shown that $B(0)$ for the map $f_{\eta,\mu}$ is simply connected. Thus, there exists a topological disc $V \subset B(0)$ such that $0, u \in V$ and $f_{\eta,\mu}^{-1}(V)$ is a topological disc containing $\operatorname{cl} V$. If we perturb slightly $(\eta, \mu) \in \mathcal{U}_t^0$, then V has the same properties for the perturbed map $f_{\eta,\mu}$. If $1 \in B(0)$ for $f_{\eta,\mu}$, then $B(0)$ is not exotic by definition. If $1 \notin B(0)$, then $B(0) = \bigcup_{n \geq 0} f_{\eta,\mu}^{-n}(V)$, so $B(0)$ is simply connected as the union of an increasing sequence of topological discs. Hence, if we take $\mathcal{U}_1$ sufficiently close to M_1 and ε sufficiently close to 0, then $B(0)$ is not exotic for $(\eta, \mu) \in \mathcal{U}_t^0$.

4.6. *Extending the path* $\{\mathbf{f}_t^0\}$. We only sketch the proof, since it is similar to the construction of the families $\mathbf{f}_t$ for $0 < t < 1$. Return to the standard coordinates (a, b) in the parameter space and consider the family $\mathbf{f}_{1+\varepsilon}^0$. To simplify notation, parameterize the topological disc $\mathcal{U}_{1+\varepsilon}^0$ by τ in some topological disc in $\mathbb{C}$ and write f_τ instead of $f_{a,b}$ for $(a, b) \in \mathcal{U}_{1+\varepsilon}^0$.

If $\tau \in M_{1+\varepsilon}^0$, then $1 \notin B(0)$ and f_τ on $B(0)$ is conjugate to a cubic Blaschke product with a supersink. Hence, there exists a topological disc $\widetilde{V}_\tau \subset B(0)$ with smooth Jordan boundary depending continuously on $\tau \in M_{1+\varepsilon}^0$ in the Hausdorff metric such that $0, u \in \widetilde{V}_\tau$ and there exists a simply connected component V_τ of $f_\tau^{-1}(\widetilde{V}_\tau)$ such that $\operatorname{cl} \widetilde{V}_\tau \subset V_\tau$. Then f_τ is a cubic cover on the closed annulus $\operatorname{cl} V_\tau \setminus \widetilde{V}_\tau$.

Note that diminishing the sets $\mathcal{U}_t^0$, $1 < t \leq 1+\varepsilon$, we can assume that $\mathcal{U}_{1+\varepsilon}^0$ is arbitrarily close to $M_{1+\varepsilon}^0$. Hence, for $\tau \in \mathcal{U}_{1+\varepsilon}^0$ we can find a topological disc $\widetilde{V}_\tau \subset B(0)$ with smooth Jordan boundary depending continuously on $\tau \in \mathcal{U}_{1+\varepsilon}^0$ with the same properties as above.

For $\tau \in \mathcal{U}_{1+\varepsilon}^0$ let
$$\psi_\tau : \mathbb{D} \to V_\tau$$

be the unique Riemann mapping such that $\psi_\tau(0) = 0$ and $\psi'_\tau(0) > 0$. Fix $0 < r_0 < 1$. We define a C^1-smooth cubic branched cover

$$g_\tau : \mathbb{D} \to \mathbb{D}$$

depending continuously on τ such that $g_\tau(z) = z^3$ on $\mathbb{D} \setminus \mathbb{D}_{\sqrt{r_0}}$ and $g_\tau(z) = r_0\psi_\tau^{-1}(f_\tau(\psi_\tau(z/r_0)))$ on $\mathbb{D}_{r_0}$. Applying the measurable Riemann theorem for a suitable g_τ-invariant conformal structure on $\mathbb{D}$ we obtain a continuous family of quasiconformal homeomorphisms H_τ of $\mathbb{D}$, holomorphic on $\mathbb{D}_{r_0}$ such that $H_\tau(0) = 0$ and $H_\tau \circ g_\tau \circ H_\tau^{-1}$ is a cubic Blaschke product of the form

$$h_\alpha(z) = z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} \tag{49}$$

for some $\alpha = \alpha(\tau) \in \mathbb{D}$ (having no relation to Head's angle). Let v_α be the unique non-zero h_α-critical point in $\mathbb{D}$ and let Φ_α (resp. Φ_τ) be the Böttcher coordinates for h_α on $\mathbb{D}$ (resp. f_τ on $B(0)$). If we take $\mathcal{U}^0_{1+\varepsilon}$ sufficiently close to $M^0_{1+\varepsilon}$, then Φ_τ is defined in a neighbourhood of the curve $\gamma^0_{\eta,\mu}$ from Subsection 4.4 for (η, μ) corresponding to τ, so by (46) we have

$$\Phi_\tau(f_\tau(u)) = 1 - \varepsilon \quad \text{ for } \tau \in \mathcal{U}^0_{1+\varepsilon}.$$

For $\alpha \in \mathbb{D}$ set

$$\Psi(\alpha) = \Phi_\alpha(h_\alpha(v_\alpha)).$$

Since f_τ on $\widetilde{V}_\tau$ is holomorphically conjugate to $h_{\alpha(\tau)}$ by

$$G_\tau(z) = H_\tau(r_0\psi_\tau^{-1}(z)),$$

it follows from the uniqueness of the Böttcher coordinates that

$$\Phi_{\alpha(\tau)} \circ G_\tau = \Phi_\tau, \quad \Psi(\alpha(\tau)) = 1 - \varepsilon. \tag{50}$$

In the same way as in the proof of Theorem I.4.14 we show that $\alpha \mapsto \Psi(\alpha)$ is a local homeomorphism for $\alpha \in \mathbb{D} \setminus \{0\}$. Since $\tau \mapsto \alpha(\tau)$ is continuous, this together with (50) implies that in fact there exists $\alpha_0 \in \mathbb{D}$ such that

$$\alpha(\tau) = \alpha_0 \quad \text{ for every } \tau \in \mathcal{U}^0_{1+\varepsilon}.$$

It is easy to check that if we take $\mathcal{U}^0_{1+\varepsilon}$ sufficiently close to $M^0_{1+\varepsilon}$, then we can assume

$$G_\tau(\widetilde{V}_\tau) \supset \mathbb{D}_{r_1} \tag{51}$$

for some fixed $r_1 < 1$ arbitrarily close to 1.

Now define $\mathbf{f}^0_t$ for $t \in [1+\varepsilon, 1+2\varepsilon]$, setting

$$\mathcal{U}^0_t = \mathcal{U}^0_{1+\varepsilon}, \quad \mathbf{f}^0_t = \{f^k_\tau|_{U^t_\tau}\}_{\tau \in \mathcal{U}^0_t}$$

for topological discs U^t_τ such that $1 \in U^t_\tau$, f^k_τ is quadratic-like on U^t_τ, $t \mapsto \partial U^t_\tau$ is continuous in the Hausdorff metric, $U^{1+\varepsilon}_\tau = U_\tau$, $U^{t_1}_\tau \supset U^{t_2}_\tau$ for every

$t_1 < t_2$ and

$$f_\tau^k(U_\tau^{1+2\varepsilon}) \cap V_\tau = \emptyset \quad \text{ for } \tau \in \mathcal{U}^0_{1+2\varepsilon}.$$

It is clear that we can find such sets U_τ^t by taking successive preimages of the sets U_τ under $(f_\tau^k|_{U_\tau})^{-1}$. Moreover, it is obvious that $\{\mathbf{f}_t^0\}_{t\in[1+\varepsilon,1+2\varepsilon]}$ is a Mandelbrot-like path.

Fix some $r \in (|\alpha_0|, 1)$. By (51), we can assume that $v_\alpha \in \mathbb{D}_{r_1}$ for every $\alpha \in \mathbb{D}_r$, so h_α is a cubic cover on the closed annulus $\mathrm{cl}(h_\alpha^{-1}(\mathbb{D}_{r_1}) \cap \mathbb{D}) \setminus \mathbb{D}_{r_1}$. This enables us to define for $\alpha \in \mathbb{D}_r$ and $\tau \in \mathcal{U}^0_{1+2\varepsilon}$ a C^1-smooth cubic branched cover $g_{\tau,\alpha}$ of $\widehat{\mathbb{C}}$ depending continuously on τ, α such that $g_{\tau,\alpha} = f_\tau$ on $\widehat{\mathbb{C}} \setminus G_\tau^{-1}(h_\alpha^{-1}(\mathbb{D}_{r_1}) \cap \mathbb{D})$, $g_{\tau,\alpha} = G_\tau^{-1} \circ h_\alpha \circ G_\tau$ on $G_\tau^{-1}(\mathbb{D}_{r_1})$ and $g_{\tau,\alpha_0} = f_\tau$. As previously, apply the measurable Riemann theorem for suitable $g_{\tau,\alpha}$-invariant conformal structures to obtain quasiconformal homeomorphisms $H_{\tau,\alpha}$ of $\widehat{\mathbb{C}}$ such that $H_{\tau,\alpha}$ is holomorphic on $G_\tau^{-1}(\mathbb{D}_{r_1})$ and $H_{\tau,\alpha} \circ g_{\tau,\alpha} \circ H_{\tau,\alpha}^{-1} = f_{a(\tau,\alpha),b(\tau,\alpha)}$ for some continuous functions $a(\tau,\alpha), b(\tau,\alpha)$. Write $f_{\tau,\alpha}$ for $f_{a(\tau,\alpha),b(\tau,\alpha)}$ and let

$$U_{\tau,\alpha} = H_{\tau,\alpha}(U_\tau^{1+2\varepsilon}).$$

Since $f_\tau^k(U_\tau^{1+2\varepsilon}) \cap V_\tau = \emptyset$, we have $f_\tau^k|_{U_\tau^{1+2\varepsilon}} = g_{\tau,\alpha}^k|_{U_\tau^{1+2\varepsilon}}$. Hence, $U_{\tau,\alpha}$ is a topological disc containing 1 and $f_{\tau,\alpha}^k$ is quadratic-like on $U_{\tau,\alpha}$. Moreover, $f_{\tau,\alpha_0} = f_\tau$, $U_{\tau,\alpha_0} = U_\tau^{1+2\varepsilon}$ and $f_{\tau,\alpha}^k|_{U_{\tau,\alpha}}$ is hybrid equivalent to $f_\tau^k|_{U_\tau^{1+2\varepsilon}}$.

For $\tau \in \mathcal{U}^0_{1+2\varepsilon}$ and $\alpha \in \mathbb{D}_r$ let $Q(\tau,\alpha) = (a(\tau,\alpha), b(\tau,\alpha))$. Define also $Q_\alpha : \mathcal{U}^0_{1+2\varepsilon} \to \mathbb{C}\times\mathbb{C}$ and $Q_\tau : \mathbb{D}_r \to \mathbb{C}\times\mathbb{C}$ setting $Q_\alpha(\tau) = Q_\tau(\alpha) = Q(\tau,\alpha)$. Let $\Phi_{\tau,\alpha}$ be the Böttcher coordinates for $f_{\tau,\alpha}$ on $B(0)$ and let

$$\widehat{\Psi}(\tau,\alpha) = \Phi_{\tau,\alpha}(f_{\tau,\alpha}(u)).$$

It is easy to check that

$$\widehat{\Psi}(\tau,\alpha) = \Psi(\alpha).$$

Since $\alpha \mapsto \Psi(\alpha)$ is a local homeomorphism for $\alpha \neq 0$, this implies that for every $\alpha_1 \in \mathbb{D}_r \setminus \{0\}$ we have $Q_{\alpha_1}(\mathcal{U}^0_{1+2\varepsilon}) \cap Q_{\alpha_2}(\mathcal{U}^0_{1+2\varepsilon}) = \emptyset$ if α_2 is sufficiently close to α_1. Moreover, for every $\alpha \in \mathbb{D}_r$ the map Q_α is a homeomorphism on $M^0_{1+2\varepsilon}$ and $Q_\alpha(M^0_{1+2\varepsilon})$ is disjoint from $Q_\alpha(\mathcal{U}^0_{1+2\varepsilon} \setminus M^0_{1+2\varepsilon})$. Let

$$\mathcal{X} = \{(\tau,\alpha) : \tau \in \mathcal{U}^0_{1+2\varepsilon},\ \alpha \in \mathbb{D}_r \setminus \{0\}\},$$
$$\mathcal{Y} = \{(\tau,\alpha) : \tau \in M^0_{1+2\varepsilon},\ \alpha \in \mathbb{D}_r \setminus \{0\}\}.$$

We observe that $Q(\mathcal{X})$ contains an open set in $\mathbb{C}\times\mathbb{C}$ containing $Q(\mathcal{Y})$. (The proof is the same as for Lemma 4.12, where we replace the indices of curves by indices of Q restricted to suitable three-dimensional manifolds (boundaries of open subsets of $\mathbb{C}\times\mathbb{C}$) homeomorphic to the three-dimensional sphere S^3.) This implies that $\{f_{\tau,\alpha}^k|_{U_{\tau,\alpha}}\}_{(a(\tau,\alpha),b(\tau,\alpha))\in \mathrm{int}\, Q(\mathcal{X})}$ is an analytic family of quadratic-like maps. Moreover, by Theorem 3.3, for every $\tau \in M^0_{1+2\varepsilon}$ the set $Q_\tau(\mathbb{D}_r \setminus \{0\})$ is analytic. These facts imply that

$Q_\tau(\mathbb{D}_r \setminus \{0\})$ is a Riemann surface for $\tau \in M^0_{1+2\varepsilon}$. In the same way as in the proof of Theorem I.4.14 we show that the map $\widehat{\Psi}$ has no critical points in $Q_\tau(\mathbb{D}_r \setminus \{0\})$ for $\tau \in M^0_{1+2\varepsilon}$. This together with the analyticity of the sets $\{\widehat{\Psi} = \text{const}\}$ implies that the sets $Q_\alpha(\mathcal{U}^0_{1+2\varepsilon})$ for $\alpha \in \mathbb{D}_r \setminus \{0\}$ contain disc-equivalent Riemann surfaces containing $Q_\alpha(M^0_{1+2\varepsilon})$. Note that for $\alpha = 0$ the map $f_{\tau,0}$ has a double critical point at 0, so by (2), the set $Q_0(\mathcal{U}^0_{1+2\varepsilon})$ is contained in the plane $\{(a,b) : b = 1/2\}$. Repeating the proof of Lemma 4.12 we show that $Q_0(\mathcal{U}^0_{1+2\varepsilon})$ contains a topological disc containing $Q_0(M^0_{1+2\varepsilon})$. Therefore, for $\alpha \in \mathbb{D}_r$ we can find topological discs $\widetilde{\mathcal{U}}^0_\alpha \subset Q_\alpha(\mathcal{U}^0_{1+2\varepsilon})$ such that $Q_\alpha(M^0_{1+2\varepsilon}) \subset \widetilde{\mathcal{U}}^0_\alpha$, $\mathrm{cl}\,\widetilde{\mathcal{U}}^0_\alpha$ depends continuously on α in the Hausdorff metric and $\widetilde{\mathcal{U}}^0_{\alpha_0} = \mathcal{U}^0_{1+2\varepsilon}$. Then $\{f^k_{a,b}|_{U_{a,b}}\}_{(a,b)\in\widetilde{\mathcal{U}}^0_\alpha}$ is a Mandelbrot-like family for $U_{a,b} = U_{\tau,\alpha}$ with the Mandelbrot-like set $Q_\alpha(M^0_{1+2\varepsilon})$.

Now for $t \in (1+2\varepsilon, 2]$ we can define

$$\mathbf{f}^0_t = \{f^k_{a,b}|_{U_{a,b}}\}_{(a,b)\in\mathcal{U}^0_t} \quad \text{for } \mathcal{U}^0_t = \widetilde{\mathcal{U}}^0_{\alpha_0(2-t)/(1-2\varepsilon)}.$$

It is clear that the path $\{\mathbf{f}^0_t\}_{t\in[1,2]}$ is the suitable Mandelbrot-like path. This finishes the proof of Theorem 4.1.

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