

# FROM NORM DERIVATIVES TO ORTHOGONALITIES IN HILBERT $C^*$ -MODULES

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ABSTRACT. Let  $(\mathcal{X}, \langle \cdot, \cdot \rangle)$  be a Hilbert  $C^*$ -module over a  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{S}(\mathcal{A})$  be the set of states on  $\mathcal{A}$ . In this paper, we first compute the norm derivative for elements  $x$  and  $y$  of  $\mathcal{X}$  as follows

$$\rho_+(x, y) = \max \left\{ \operatorname{Re} \varphi(\langle x, y \rangle) : \varphi \in \mathcal{S}(\mathcal{A}), \varphi(\langle x, x \rangle) = \|x\|^2 \right\}.$$

We then apply it to characterize different concepts of orthogonality in  $\mathcal{X}$ . In particular, we present a simpler proof of the classical characterization of Birkhoff–James orthogonality in Hilbert  $C^*$ -modules. Moreover, some generalized Daugavet equation in the  $C^*$ -algebra  $\mathbb{B}(\mathcal{H})$  of all bounded linear operators acting on a Hilbert space  $\mathcal{H}$  is solved.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $(X, \|\cdot\|)$  be a normed space and  $X^*$  its dual topologic space. We define two mappings  $\rho_+, \rho_- : X \times X \rightarrow \mathbb{R}$  by the formulas

$$\rho_{\pm}(x, y) := \lim_{t \rightarrow 0^{\pm}} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = \|x\| \cdot \lim_{t \rightarrow 0^{\pm}} \frac{\|x + ty\| - \|x\|}{t}.$$

The convexity of the norm yields that the above definitions are meaningful. These mappings are called the *norm derivatives* and their following useful properties can be found, e.g. in [1, 12]. For every  $x$  and  $y$  in  $X$  and for every  $\alpha = |\alpha|e^{i\theta}$ ,  $\beta = |\beta|e^{i\omega}$  in  $\mathbb{C}$ , we have

- (P1)  $\rho_-(x, y) \leq \rho_+(x, y)$ ,  $|\rho_{\pm}(x, y)| \leq \|x\| \cdot \|y\|$  and  $\rho_{\pm}(x, x) = \|x\|^2$ ,
- (P2)  $\rho_{\pm}(-x, y) = \rho_{\pm}(x, -y) = -\rho_{\mp}(x, y)$ ,
- (P3)  $\rho_{\pm}(x, \alpha x + y) = \operatorname{Re} \alpha \|x\|^2 + \rho_{\pm}(x, y)$ ,
- (P4)  $\rho_{\pm}(\alpha x, \beta y) = |\alpha\beta| \rho_{\pm}(x, e^{i(\omega-\theta)}y)$ ,
- (P5)  $\rho_+(x, y) = \lim_{t \rightarrow 0^+} \rho_+(x + ty, y)$ .

If the norm on  $X$  comes from an inner product  $[\cdot, \cdot]$ , then we obtain  $\rho_+(x, y) = \rho_-(x, y) = \operatorname{Re} [x, y]$  for all  $x, y \in X$ , i.e. both  $\rho_+$  and  $\rho_-$  are perfect generalizations of inner products.

In more general cases mappings  $\rho_+$  and  $\rho_-$  are useful for applications in approximation theory and, in particular, they played a significant role in the paper [28]. For more information about the norm derivatives and their properties the reader is referred to [1, 12] (see also [8, 9, 19, 32]).

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From [12], for two elements  $x$  and  $y$  of a normed linear space  $X$ , we have

$$\rho_+(x, y) = \|x\| \max \{ \operatorname{Re} x^*(y) : x^* \in J(x) \}, \quad (1.1)$$

where  $J(x) := \{x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|\}$ . If we have additional structures on a normed linear space  $X$ , then we obtain other expressions for the norm derivative  $\rho_+$ . Therefore we survey on the well known results involving norm derivatives. So the present section has an expository character in part; however, many of the surveyed results will be essentially extended in the next section.

In the classical Banach space  $\mathcal{C}(K)$  of all continuous functions on a compact Hausdorff space  $K$ , the result for  $f, g \in \mathcal{C}(K)$  were given by Kečkić [16]:

$$\rho_+(f, g) = \|f\| \max \left\{ \operatorname{Re} \left( e^{-i \arg(f(x))} g(x) \right) : x \in M_f \right\}$$

where  $M_f := \{x \in K : |f(x)| = \|f\|\}$ .

In  $\mathbb{B}(\mathcal{H})$ , the  $C^*$ -algebra of all bounded linear operators on a Hilbert space  $(\mathcal{H}, [\cdot, \cdot])$ , and for  $T, S \in \mathbb{B}(\mathcal{H})$ , Kečkić [15] obtained the following formula:

$$\rho_+(T, S) = \inf_{\varepsilon > 0} \sup \left\{ \operatorname{Re}[Sx, Tx] : x \in \mathcal{H}_\varepsilon, \|x\| = 1 \right\},$$

where  $\mathcal{H}_\varepsilon := E_{T^*T}((\|T\| - \varepsilon)^2, \|T\|^2)$ , and  $E_{T^*T}$  stands for the spectral measure of the operator  $T^*T$ .

Norm derivatives of the space  $\mathbb{K}(\mathcal{H})$  (compact operators on  $\mathcal{H}$ ) has been studied in [14]. More precisely, for  $T, S \in \mathbb{K}(\mathcal{H})$  where  $T = U|T|$  is the polar decomposition of  $T$ , we have

$$\rho_+(T, S) = \|T\| \max \left\{ \operatorname{Re}[U^*Sx, x]; x \in \Phi, \|x\| = 1 \right\},$$

where  $\Phi$  is the characteristic subspace of  $T$  with respect to its eigenvalue  $s_1$ .

Wójcik [24], by using a different method, extended this result for compact operators between real normed spaces, i.e.  $\mathbb{K}(X, Y)$ . Moreover, similar investigations have been carried out by Wójcik [26] in  $M$ -ideals in bounded operator space  $\mathbb{B}(X, Y)$ . The main result of [26] says that if  $T, S \in \mathbb{B}(X, Y)$  and  $\operatorname{dist}(T, \mathbb{K}(X, Y)) < \|T\|$ , then

$$\rho_+(T, S) = \|T\| \max \left\{ \rho_+(Tx, Sx) : x \in \operatorname{Ext}B_X, \|Tx\| = \|T\| \right\},$$

where  $\operatorname{Ext}B_X$  denotes the set of all extremal points of the closed unit ball  $B_X$ .

Motivated by the above properties, we compute the norm derivatives in Hilbert  $C^*$ -modules. Namely, for two elements  $x$  and  $y$  of a Hilbert  $\mathcal{A}$ -module  $(\mathcal{X}, \langle \cdot, \cdot \rangle)$  we will prove that

$$\rho_+(x, y) = \max \left\{ \operatorname{Re} \varphi(\langle x, y \rangle) : \varphi \in \mathcal{S}(\mathcal{A}), \varphi(\langle x, x \rangle) = \|x\|^2 \right\},$$

where  $\mathcal{S}(\mathcal{A})$  is the set of states on  $\mathcal{A}$ . This formula enables us to characterize different concepts of Birkhoff-James orthogonality for elements of a Hilbert  $C^*$ -module. Some other related results are also discussed.

Before stating the results, we establish the notation and recall some definitions from the literature. An element  $a$  in a  $C^*$ -algebra  $\mathcal{A}$  is called *positive* (we write

$a \geq 0$ ) if  $a = b^*b$  for some  $b \in \mathcal{A}$ . A linear functional  $\varphi$  of  $\mathcal{A}$  is *positive* if  $\varphi(a) \geq 0$  for every positive element  $a \in \mathcal{A}$ . A *state* is a positive linear functional whose norm is equal to one. The symbol  $\mathcal{S}(\mathcal{A})$  denotes the set of states on  $\mathcal{A}$ .

An *inner product module* over  $\mathcal{A}$  is a (left)  $\mathcal{A}$ -module  $\mathcal{X}$  equipped with an  $\mathcal{A}$ -valued inner product  $\langle \cdot, \cdot \rangle$ , which is  $\mathbb{C}$ -linear and  $\mathcal{A}$ -linear in the first variable and has the properties  $\langle x, y \rangle^* = \langle y, x \rangle$  as well as  $\langle x, x \rangle \geq 0$  with equality if and only if  $x = 0$ . An inner product  $\mathcal{A}$ -module  $\mathcal{X}$  is called a *Hilbert  $\mathcal{A}$ -module* if it is complete with respect to the norm  $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ . For  $x \in \mathcal{X}$ , by [20, Theorem 3.3.6], there always exists a  $\varphi \in \mathcal{S}(\mathcal{A})$  such that  $\varphi(\langle x, x \rangle) = \|x\|^2$ . So, let  $\Omega_x$  denote the (nonempty) subset of the set of supporting functionals:

$$\Omega_x := \{\varphi \in \mathcal{S}(\mathcal{A}) : \varphi(\langle x, x \rangle) = \|x\|^2\} \subseteq J(x).$$

Given a positive functional  $\varphi$  on  $\mathcal{A}$ , we have the following useful version of the Cauchy–Schwarz inequality:

$$|\varphi(\langle x, y \rangle)|^2 \leq \varphi(\langle x, x \rangle)\varphi(\langle y, y \rangle) \quad (x, y \in \mathcal{X}). \quad (1.2)$$

Every  $C^*$ -algebra  $\mathcal{A}$  can be regarded as a Hilbert  $C^*$ -module over itself where the inner product is defined by  $\langle a, b \rangle := a^*b$ . By  $\mathbb{M}_n(\mathbb{C})$  we denote the  $C^*$ -algebra of all complex  $n \times n$  matrices. We shall identify  $\mathbb{B}(\mathbb{C}^n)$  and  $\mathbb{M}_n(\mathbb{C})$  in the usual way. We refer the reader to [11, 17] for more information on Hilbert  $C^*$ -modules.

A concept of orthogonality in a Hilbert  $\mathcal{A}$ -module  $\mathcal{X}$  can be defined with respect to the  $\mathcal{A}$ -valued inner product in a natural way, that is, two elements  $x$  and  $y$  of  $\mathcal{X}$  are orthogonal, in short  $x \perp y$ , if  $\langle x, y \rangle = 0$ . There are many different ways how one can extend this notion, see [3] and the references therein. One of them is the Birkhoff–James orthogonality: we say that  $x$  and  $y$  are Birkhoff–James orthogonal, and we write  $x \perp_B y$ , if  $\|x\| \leq \|x + \lambda y\|$  for all  $\lambda \in \mathbb{C}$ . A well known characterization of the Birkhoff–James orthogonality is due to Arambašić and Rajić (see [2]):

$$x \perp_B y \quad \Leftrightarrow \quad (\exists \varphi \in \Omega_x : \varphi(\langle x, y \rangle) = 0). \quad (1.3)$$

It has been proved also by Bhattacharyya and Grover (cf. [5]). The statement of this nice characterization is so simple, and its existing proofs so extremely long, that one is easily seduced into an effort to find a simpler, shorter proof. The present work is the result of our attempt. It is worth mentioning that those four mathematicians applied a faithful representation  $\pi: \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$  (see [11, Theorem 2.6.1]) and linking algebra of  $\mathcal{X}$  (see [17]). But we do not use this strong tool. We apply norm derivatives. In this paper we would like to present a simpler proof of this nice result and we demonstrate the power of the norm derivatives. We hope that it sheds new light on this intricate geometric structure of Hilbert  $C^*$ -modules and will provide the great applications of the mappings  $\rho_{\pm}$  in the future.

## 2. MAIN RESULTS: NORM DERIVATIVES IN HILBERT $C^*$ -MODULES

In this section, we first compute the norm derivatives in Hilbert  $C^*$ -modules. Then, as an application of our results, we get an explicit formula for the norm derivatives  $\rho_{\pm}$  of certain elements in Hilbert  $C^*$ -modules. Moreover, we apply

our results to give some solutions of the generalized Daugavet equation in the operator space  $\mathbb{B}(\mathcal{H})$ .

We start our work with the following lemma.

**Lemma 2.1.** [22] *Suppose that  $X$  is a real normed space. Let  $D \subseteq X$  be a dense and star-shaped subset (i.e.  $\alpha D \subseteq D$  for all  $\alpha > 0$ ). Let  $M$  be a closed affine hyperplane (i.e.  $\text{codim} M = 1$ ) such that  $0 \notin M$ . Then  $\text{cl}(M \cap D) = M$ .*

It is worth mentioning that Lemma 2.1 played a significant role in the papers [22] and [23]. Its proof can be found in [22]. Here, this lemma will be a helpful tool, again.

We say that  $X$  is *smooth at point  $x_o$*  if there is a unique functional  $f \in J(x_o)$ . Now, we consider a set  $\mathcal{N}_{sm}(X) := \{x \in X : X \text{ is smooth at } x\} \cup \{0\}$ . In particular, we have

$$x_o \in \mathcal{N}_{sm}(X) \quad \Leftrightarrow \quad \rho_+(x_o, \cdot) = \rho_-(x_o, \cdot) = \|x\| \text{Re } f(\cdot), \quad f \in J(x_o). \quad (2.1)$$

It is known that if  $\dim X < \infty$ , then  $\mathcal{N}_{sm}(X)$  is a dense, star-shaped subset of  $X$  – cf. [22] or [1]. Thus we can rewrite Lemma 2.1 as

**Lemma 2.2.** *Suppose that  $Z$  is a two-dimensional real normed space. Let  $\{x, y\} \subseteq Z$  be a linearly independent subset. If we consider a line  $M$  spanned by the vectors  $x, y$  (i.e.  $M := \{x + ty \in Z : t \in \mathbb{R}\}$ ), then  $\text{cl}(M \cap \mathcal{N}_{sm}(Z)) = M$ .*

We are now in a position to prove the main result of this paper.

**Theorem 2.3.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . Then*

$$\rho_+(x, y) = \max \left\{ \text{Re } \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \right\}.$$

*Proof.* We may and shall assume that  $x \neq 0$  otherwise the statement trivially holds. Let  $\varphi \in \Omega_x$ , that is,  $\varphi \in \mathcal{S}(\mathcal{A})$  and  $\varphi(\langle x, x \rangle) = \|x\|^2$ . It is easy to check that  $\frac{1}{\|x\|} \varphi(\langle x, \cdot \rangle) \in J(x)$ . Thus the property (1.1) yields  $\|x\| \text{Re } \frac{1}{\|x\|} \varphi(\langle x, y \rangle) \leq \rho'_+(x, y)$ . Hence  $\text{Re } \varphi(\langle x, y \rangle) \leq \rho_+(x, y)$ . Passing to the supremum over  $\varphi \in \Omega_x$  we get

$$\sup \left\{ \text{Re } \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \right\} \leq \rho_+(x, y).$$

To complete the proof, we must find  $\varphi \in \Omega_x$  such that  $\text{Re } \varphi(\langle x, y \rangle) = \rho_+(x, y)$ . Then we will be able to write "max" instead of "sup". We consider two cases. First assume that  $x$  and  $y$  are linearly dependent, i.e.  $y = \alpha x$  with some number  $\alpha$ . Fix  $\varphi \in \Omega_x$ . Then

$$\begin{aligned} \rho_+(x, y) &= \rho_+(x, \alpha x) \stackrel{(P3)}{=} \text{Re}(\alpha \|x\|^2) = \text{Re}(\alpha \varphi(\langle x, x \rangle)) \\ &= \text{Re } \varphi(\langle x, \alpha x \rangle) = \text{Re } \varphi(\langle x, y \rangle). \end{aligned}$$

So, the first case is complete. Now, suppose that  $\{x, y\}$  is a linearly independent subset. Let us define a real subspace  $Z \subseteq \mathcal{X}$  by the formula

$$Z := \{\alpha x + \beta y \in \mathcal{X} : \alpha, \beta \in \mathbb{R}\}.$$

Put  $M := \{x + ty \in Z : t \in \mathbb{R}\}$ . It follows from Lemma 2.2 that  $\text{cl}(M \cap \mathcal{N}_{sm}(Z)) = M$ . This equality yields, in particular, that there exists a sequence  $\{t_n : n = 1, 2, \dots\}$  in  $(0, +\infty)$  such that  $t_n \rightarrow 0^+$  and  $x + t_n y \in \mathcal{N}_{sm}(Z)$ .

Let us define for the moment a closed subalgebra  $\mathcal{A}_o \subseteq \mathcal{A}$  spanned by the vectors  $\langle x, x \rangle$ ,  $\langle y, y \rangle$ ,  $\langle x, y \rangle$  and  $\langle y, x \rangle$ . In particular, we have

$$\langle x + t_n y, x + t_n y \rangle, \langle x + t_n y, y \rangle \in \mathcal{A}_o.$$

Moreover, it is easily seen that  $\mathcal{A}_o$  is separable. Also, there exist states  $\varphi_1, \varphi_2, \varphi_3, \dots \in \mathcal{S}(\mathcal{A}_o)$  such that

$$\varphi_n(\langle x + t_n y, x + t_n y \rangle) = \|x + t_n y\|^2. \quad (2.2)$$

Now our attention must focus on the real two-dimensional space  $Z$ . Fix  $n \in \mathbb{N}$ . It is not difficult to see that  $\frac{1}{\|x + t_n y\|} \text{Re } \varphi_n(\langle x + t_n y, \cdot \rangle)|_Z : Z \rightarrow \mathbb{R}$  is a  $\mathbb{R}$ -linear functional and

$$\frac{1}{\|x + t_n y\|} \text{Re } \varphi_n(\langle x + t_n y, \cdot \rangle)|_Z \in J(x + t_n y)|_Z. \quad (2.3)$$

The space  $Z$  is smooth at  $x + t_n y$ . Thus, from (2.1) and (2.3) it is known that

$$\rho_+(x + t_n y, y) = \text{Re } \varphi_n(\langle x + t_n y, y \rangle). \quad (2.4)$$

By Alaoglu's Theorem, we know that the closed unit ball  $B_{\mathcal{A}_o^*}$  is weakly\* compact. It is easy to check that subset  $\mathcal{S}(\mathcal{A}_o) \subseteq \mathcal{A}_o^*$  is a norm-closed convex subset of the weakly\* compact ball of  $B_{\mathcal{A}_o^*}$ . Therefore  $\mathcal{S}(\mathcal{A}_o)$  is weakly\* compact. Since  $\mathcal{A}_o$  is separable,  $\mathcal{S}(\mathcal{A}_o)$  is weakly\* sequentially compact. Thus, there are an element  $\varphi_o \in B_{\mathcal{A}_o^*}$  and a subsequence  $(\varphi_{n_k})_{k=1}^\infty \subseteq B_{\mathcal{A}_o^*}$  such that  $\varphi_{n_k} \xrightarrow{w^*} \varphi_o$ . Since  $\langle x + t_{n_k} y, y \rangle \xrightarrow{\|\cdot\|} \langle x, y \rangle$ , we conclude that

$$\text{Re } \varphi_{n_k}(\langle x + t_{n_k} y, y \rangle) \longrightarrow \text{Re } \varphi_o(\langle x, y \rangle). \quad (2.5)$$

Similarly, since  $\langle x + t_{n_k} y, x + t_{n_k} y \rangle \xrightarrow{\|\cdot\|} \langle x, x \rangle$ , it follows that

$$\varphi_{n_k}(\langle x + t_{n_k} y, x + t_{n_k} y \rangle) \longrightarrow \varphi_o(\langle x, x \rangle). \quad (2.6)$$

Combining the three conditions (P5)-(2.4)-(2.5) we deduce that

$$\text{Re } \varphi_o(\langle x, y \rangle) = \rho_+(x, y).$$

Furthermore, combining (2.2) with (2.6) yields  $\varphi_o(\langle x, x \rangle) = \|x\|^2$ .

Now we back to the  $C^*$ -algebra  $\mathcal{A}$  and we go to the dual normed space  $\mathcal{A}^*$ . Namely, since  $\varphi_o : \mathcal{A}_o \rightarrow \mathbb{C}$  is a state on  $\mathcal{A}_o$ , it follows that there exists a state  $\varphi$  on  $\mathcal{A}$  such that  $\varphi|_{\mathcal{A}_o} = \varphi_o$  (see e.g. [10, p.259]). From this we get  $\text{Re } \varphi(\langle x, y \rangle) = \rho_+(x, y)$  and  $\varphi(\langle x, x \rangle) = \|x\|^2$ . The proof is completed.  $\square$

Now we are able to calculate a formula for the norm derivative  $\rho_-$  in Hilbert  $C^*$ -modules.

**Theorem 2.4.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . Then*

$$\rho_-(x, y) = \min \left\{ \text{Re } \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \right\}.$$

*Proof.* Applying (P2) and Theorem 2.3 we get

$$\begin{aligned}\rho_-(x, y) &= -\rho_+(x, -y) = -\max \left\{ \operatorname{Re} \varphi(\langle x, -y \rangle) : \varphi \in \Omega_x \right\} \\ &= \min \left\{ \operatorname{Re} \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \right\},\end{aligned}$$

and we are done.  $\square$

Recall that every  $C^*$ -algebra  $\mathcal{A}$  can be regarded as a Hilbert  $C^*$ -module over itself with the inner product  $\langle a, b \rangle := a^*b$ . Thus, as a consequence of Theorem 2.3 (and Theorem 2.4) we have the following result.

**Corollary 2.5.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra, and  $a, b \in \mathcal{A}$ . Then*

$$\begin{aligned}\rho_+(a, b) &= \max \left\{ \operatorname{Re} \varphi(a^*b) : \varphi \in \Omega_a \right\}, \\ \rho_-(a, b) &= \min \left\{ \operatorname{Re} \varphi(a^*b) : \varphi \in \Omega_a \right\}.\end{aligned}$$

Let  $\mathbb{T}(\mathcal{H})$  be trace-class operators on a Hilbert space  $(\mathcal{H}, [\cdot, \cdot])$ . It is well known that (see e.g. [20, Theorem 4.2.1]) every state  $\varphi$  of  $\mathbb{K}(\mathcal{H})$  is of the form  $a \rightarrow \operatorname{tr}(pa)$  for some positive trace one operator  $p \in \mathbb{T}(\mathcal{H})$ . Therefore, as an immediate consequence of Theorem 2.3 and Theorem 2.4, we have the following result.

**Proposition 2.6.** *Let  $\mathcal{X}$  be a Hilbert  $\mathbb{K}(\mathcal{H})$ -module, and  $x, y \in \mathcal{X}$ . Then*

$$\begin{aligned}\rho_+(x, y) &= \max \left\{ \operatorname{Re} \operatorname{tr}(p\langle x, y \rangle) : p \in \mathbb{P}(\mathcal{H}), \operatorname{tr}(p\langle x, x \rangle) = \|x\|^2 \right\}, \\ \rho_-(x, y) &= \min \left\{ \operatorname{Re} \operatorname{tr}(p\langle x, y \rangle) : p \in \mathbb{P}(\mathcal{H}), \operatorname{tr}(p\langle x, x \rangle) = \|x\|^2 \right\},\end{aligned}$$

where  $\mathbb{P}(\mathcal{H}) = \{p \in \mathbb{T}(\mathcal{H}) : p \text{ is positive trace one}\}$ .

We continue this section by applying our results to get an explicit formula for the norm derivatives  $\rho_\pm$  of certain elements in Hilbert  $C^*$ -modules.

**Theorem 2.7.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x \in \mathcal{X}$ . Then*

$$\rho_+(x, x\langle x, x \rangle) = \|x\|^4 = \rho_-(x, x\langle x, x \rangle). \quad (2.7)$$

*Proof.* Fix  $\varphi \in \mathcal{S}(\mathcal{A})$  such that  $\varphi(\langle x, x \rangle) = \|x\|^2$ . We may suppose that  $\mathcal{A}$  is a Banach algebra with identity (by going to extensions  $\tilde{\mathcal{A}}$  and  $\tilde{\varphi} = \varphi|_{\tilde{\mathcal{A}}}$  if necessary; see [10, pp.194, 259]). So, let  $e$  be the identity of  $\mathcal{A}$ . Then

$$\|x\|^4 = |\varphi(\langle x, x \rangle e)|^2 \stackrel{(1.2)}{\leq} \varphi(\langle x, x \rangle^2) \varphi(e^2) \leq \varphi(\langle x, x \rangle^2) \leq \|\langle x, x \rangle^2\| \leq \|x\|^4.$$

This implies

$$\varphi(\langle x, x \rangle^2) = \|x\|^4. \quad (2.8)$$

Therefore we obtain the following equalities

$$\begin{aligned}\operatorname{Re} \varphi(\langle x, \|x\|^2 x - x\langle x, x \rangle \rangle) &= \|x\|^2 \operatorname{Re} \varphi(\langle x, x \rangle) - \operatorname{Re} \varphi(\langle x, x\langle x, x \rangle \rangle) \\ &= \|x\|^2 \operatorname{Re} \|x\|^2 - \operatorname{Re} \varphi(\langle x, x \rangle \langle x, x \rangle) \\ &\stackrel{(2.8)}{=} \|x\|^2 \|x\|^2 - \|x\|^4 = 0.\end{aligned}$$

Now we apply Theorem 2.3 (resp. Theorem 2.4). Namely, since  $\varphi$  was arbitrarily chosen from  $\Omega_x$ , passing to the maximum (resp. minimum) over  $\varphi \in \Omega_x$ , we get

$$\rho_+(x, \|x\|^2x - x\langle x, x \rangle) = 0 \quad \text{and} \quad \rho_-(x, \|x\|^2x - x\langle x, x \rangle) = 0.$$

So, from (P3) and (P2) we obtain, respectively,  $\|x\|^2 \cdot \|x\|^2 - \rho_-(x, \langle x, x \rangle) = 0$  and  $\|x\|^2 \cdot \|x\|^2 - \rho_+(x, x\langle x, x \rangle) = 0$ , and we may consider (2.7) as shown.  $\square$

An element  $x$  in a normed space  $(X, \|\cdot\|)$  is called *norm-parallel* to another element  $y \in X$  (see [30] and the references therein), denoted by  $x \parallel y$ , if  $\|x + \xi y\| = \|x\| + \|\xi y\|$  for some complex unit  $\xi$ . In the framework of inner product spaces, the norm-parallel relation is exactly the usual vectorial parallel relation, that is,  $x \parallel y$  if and only if  $x$  and  $y$  are linearly dependent. In the setting of normed linear spaces, two linearly dependent vectors are norm-parallel, but the converse is false in general. Many characterizations of the norm-parallelism for operators spaces  $\mathbb{B}(X, Y)$  and elements of an arbitrary Hilbert  $C^*$ -module were given in [7], [25], [30] and [31].

**Theorem 2.8.** *Suppose that  $\mathcal{X}$  is a Hilbert  $\mathcal{A}$ -module, and let  $x \in \mathcal{X}$ ,  $\alpha, \beta \in (0, +\infty)$ . Then*

$$\|\alpha x + \beta x\langle x, x \rangle\| = \alpha\|x\| + \beta\|x\|^3. \quad (2.9)$$

*In particular,  $x$  is norm-parallel to  $x\langle x, x \rangle$ .*

*Proof.* It is clear that we may assume  $x \neq 0$ . Then we have

$$\begin{aligned} \alpha\|x\|^2 + \beta\|x\|^4 &\stackrel{(2.7)}{=} \alpha\|x\|^2 + \beta\rho_+(x, x\langle x, x \rangle) \stackrel{(P3, P4)}{=} \rho_+(x, \alpha x + \beta x\langle x, x \rangle) \\ &\stackrel{(P1)}{\leq} \|x\| \cdot \|\alpha x + \beta x\langle x, x \rangle\| \leq \|x\| \cdot (\alpha\|x\| + \beta\|x\langle x, x \rangle\|) \\ &\leq \|x\| \cdot (\alpha\|x\| + \beta\|x\| \cdot \|\langle x, x \rangle\|) = \alpha\|x\|^2 + \beta\|x\|^4. \end{aligned}$$

Thus the string of inequalities becomes a string of equalities and we obtain  $\alpha\|x\|^2 + \beta\|x\|^4 = \|x\| \cdot \|\alpha x + \beta x\langle x, x \rangle\|$ . Dividing by  $\|x\|$ , we have (2.9).  $\square$

In the context of bounded linear operators on normed spaces, the well-known *Daugavet equation*  $\|I + T\| = 1 + \|T\|$  is a particular case of parallelism. We refer to the book [21] and more recent the paper [27] for motivations, history, various aspects and problems connected with the Daugavet equation. It is worth mentioning that a generalized Daugavet equation  $\|T + S\| = \|T\| + \|S\|$  is one useful property in solving a variety of problems in approximation theory (cf. [27]).

A consequence of Theorem 2.8 is established in the next result.

**Corollary 2.9.** *Let  $T \in \mathbb{B}(\mathcal{H})$ . Then*

$$\|T + TT^*T\| = \|T\| + \|T\|^3.$$

*Moreover, if  $\text{dist}(T, \mathbb{K}(\mathcal{H})) < \|T\|$  or  $\text{dist}(TT^*T, \mathbb{K}(\mathcal{H})) < \|TT^*T\|$ , then there exists a unit vector  $x_o \in \mathcal{H}$  such that  $\frac{T}{\|T\|}x_o = \frac{TT^*T}{\|TT^*T\|}x_o$ ,  $\|Tx_o\| = \|T\|$  and  $\|TT^*Tx_o\| = \|TT^*T\|$ .*



*Proof.* It follows from (2.9) that  $\|T + TT^*T\| = \|T\| + \|T\|^3$ . In addition, it is not difficult to check that  $\|TT^*T\| = \|T\|^3$ . Therefore both  $T$  and  $TT^*T$  satisfy the generalized Daugavet equation

$$\|T + TT^*T\| = \|T\| + \|TT^*T\|. \quad (2.10)$$

Now, suppose that  $\text{dist}(T, \mathbb{K}(\mathcal{H})) < \|T\|$  or  $\text{dist}(TT^*T, \mathbb{K}(\mathcal{H})) < \|TT^*T\|$ . Recall that  $\mathbb{K}(\mathcal{H})$  is an  $M$ -ideal in  $\mathbb{B}(\mathcal{H})$  (see [25]). Moreover,  $\mathcal{H}$  is strictly convex. Thus these information are legitimates to apply [27, Theorem 4.4]. So, it follows from [27, Theorem 4.4] and (2.10) that there exists a unit vector  $x_o \in \mathcal{H}$  such that  $\frac{T}{\|T\|}x_o = \frac{TT^*T}{\|TT^*T\|}x_o$ ,  $\|Tx_o\| = \|T\|$  and  $\|TT^*Tx_o\| = \|TT^*T\|$ .  $\square$

### 3. AN APPLICATION: ORTHOGONALITIES IN HILBERT $C^*$ -MODULES

In a normed linear space  $(X, \|\cdot\|)$ , for two vectors  $x, y \in X$ , one can consider the *Birkhoff–James orthogonality* (see [6, 13]) defined by

$$x \perp_B y \quad :\Leftrightarrow \quad \forall \lambda \in \mathbb{C} \quad \|x\| \leq \|x + \lambda y\|.$$

We will consider also the  *$r$ -Birkhoff–James orthogonality*, defined by

$$x \perp_B^r y \quad :\Leftrightarrow \quad \forall \alpha \in \mathbb{R} \quad \|x\| \leq \|x + \alpha y\|.$$

Notice that  $\perp_B \subseteq \perp_B^r$ , but the converse is false in general. For example, let us take  $X = \mathbb{C}^2$  and let  $x = (1, 0)$ ,  $y = (i, 0)$ . Then for all  $\alpha \in \mathbb{R}$  we have

$$\|x + \alpha y\| = \|(1 + \alpha i, 0)\| = \sqrt{1 + \alpha^2} \geq 1 = \|x\|.$$

Hence  $x \perp_B^r y$ . But  $x \not\perp_B y$  since for  $\lambda := i$  we have  $\|x + \lambda y\| = 0 < 1 = \|x\|$ .

We will use the following characterizations of both orthogonality relations. Namely, for arbitrary  $x, y \in X$ , we have (see [1, 14]):

$$x \perp_B y \quad \Leftrightarrow \quad \inf_{0 \leq \theta < 2\pi} \rho_+(x, e^{i\theta}y) \geq 0 \quad (3.1)$$

and

$$x \perp_B^r y \quad \Leftrightarrow \quad \rho_-(x, y) \leq 0 \leq \rho_+(x, y). \quad (3.2)$$

When  $X = \mathbb{M}_n(\mathbb{C})$  and  $T, S \in X$ , a very tractable condition of the Birkhoff–James orthogonality was found by Bhatia and Šemrl in [4]. They showed that  $T \perp_B S$  if and only if there exists a unit vector  $x \in \mathbb{C}^n$  such that

$$\|Tx\| = \|T\| \quad \text{and} \quad [Tx, Sx] = 0. \quad (3.3)$$

Later Bhattacharyya and Grover [5] showed that  $T \perp_B^r S$  if and only if there exists a unit vector  $x \in \mathbb{C}^n$  such that

$$\|Tx\| = \|T\| \quad \text{and} \quad \text{Re}[Tx, Sx] = 0. \quad (3.4)$$

To summarize, the papers [4], [5] and conditions (3.3), (3.4) motivate the next theorem. In other words, we now obtain a characterization of real version of the Birkhoff–James orthogonality in Hilbert  $C^*$ -modules in terms of states of the underlying  $C^*$ -algebra.

**Theorem 3.1.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . The following statements are equivalent:*



- (i)  $x \perp_B^r y$ ,
- (ii) there exists  $\varphi \in \Omega_x$  such that  $\operatorname{Re} \varphi(\langle x, y \rangle) = 0$ .

*Proof.* We assume that  $x \neq 0$  (otherwise the result is trivial). First, suppose that  $x \perp_B^r y$ . By (3.2) we have  $\rho_-(x, y) \leq 0 \leq \rho_+(x, y)$ . So, by Theorem 2.3, there exist  $\varphi_1, \varphi_2 \in \Omega_x$  such that

$$\operatorname{Re} \varphi_1(\langle x, y \rangle) \leq 0 \leq \operatorname{Re} \varphi_2(\langle x, y \rangle).$$

It follows from the above inequality that for some  $\lambda_o \in [0, 1]$  we have

$$\lambda_o \operatorname{Re} \varphi_1(\langle x, y \rangle) + (1 - \lambda_o) \operatorname{Re} \varphi_2(\langle x, y \rangle) = 0. \quad (3.5)$$

Since  $S(\mathcal{A})$  is convex,  $\lambda_o \varphi_1 + (1 - \lambda_o) \varphi_2 \in S(\mathcal{A})$ . Put  $\varphi := \lambda_o \varphi_1 + (1 - \lambda_o) \varphi_2$ . We get then  $\varphi \in S(\mathcal{A})$  and  $\varphi(\langle x, x \rangle) = \|x\|^2$ . Hence  $\varphi \in \Omega_x$ . Also, by (3.5),  $\operatorname{Re} \varphi(\langle x, y \rangle) = 0$ .

Now we prove the implication (ii)  $\Rightarrow$  (i). Assume that there exists  $\varphi \in \Omega_x$  such that  $\operatorname{Re} \varphi(\langle x, y \rangle) = 0$ . Fix arbitrarily  $\alpha \in \mathbb{R}$ . We have

$$\begin{aligned} \|x\|^2 &= \varphi(\langle x, x \rangle) = \operatorname{Re} \varphi(\langle x, x \rangle) + \alpha \operatorname{Re} \varphi(\langle x, y \rangle) \\ &= \operatorname{Re} \varphi(\langle x, x + \alpha y \rangle) \leq |\varphi(\langle x, x + \alpha y \rangle)| \\ &\leq \|\langle x, x + \alpha y \rangle\| \leq \|x\| \cdot \|x + \alpha y\|. \end{aligned}$$

Thus  $\|x\|^2 \leq \|x\| \cdot \|x + \alpha y\|$  and so  $\|x\| \leq \|x + \alpha y\|$ . Hence  $x \perp_B^r y$ .  $\square$

Finally, we are able to give a simple proof of (1.3) using the map  $\rho_+$ , i.e. Theorem 2.3. Let us recall again: unlike [2, 5], we did not apply a faithful representation  $\pi: \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$  and linking algebra of  $\mathcal{X}$ . Therefore the proofs of Theorem 2.3 and Theorem 3.2 are simpler and shorter than the proofs in [2, 5]. So, we now show that there is another (and easier) way to get the celebrated result (1.3).

**Theorem 3.2.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . The following statements are mutually equivalent:*

- (i)  $x \perp_B y$ ,
- (ii) there exists  $\varphi \in \Omega_x$  such that  $\varphi(\langle x, y \rangle) = 0$ ,
- (iii)  $\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2 m(y)$  for all  $\lambda \in \mathbb{C}$ ,

where  $m(y) := \inf \{ \varphi(\langle y, y \rangle) : \varphi \in \mathcal{S}(\mathcal{A}) \}$ .

*Proof.* (i)  $\Rightarrow$  (ii) Let  $x \perp_B y$ . Then, by Theorem 2.3 and (3.1), we get

$$\inf_{0 \leq \theta < 2\pi} \max \left\{ \operatorname{Re} e^{i\theta} \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \right\} \geq 0. \quad (3.6)$$

It is easy to see that the set  $E := \{ \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \}$  is convex and hence its closure is a closed convex set. Therefore, by (3.6), the set  $E$  has such a position in the complex plane that it must contain at least one value with positive real part, under all rotations around the origin. Thus  $E$  must contain zero, and so there is a  $\varphi \in \Omega_x$  such that  $\varphi(\langle x, y \rangle) = 0$ .

(ii) $\Rightarrow$ (iii) Suppose (ii) holds. Then, for every  $\lambda \in \mathbb{C}$ , we have

$$\begin{aligned} \|x + \lambda y\|^2 &\geq \varphi(\langle x + \lambda y, x + \lambda y \rangle) \\ &= \varphi(\langle x, x \rangle) + 2\operatorname{Re}(\lambda\varphi(\langle x, y \rangle)) + |\lambda|^2\varphi(\langle y, y \rangle) \\ &= \|x\|^2 + |\lambda|^2\varphi(\langle y, y \rangle). \end{aligned}$$

Therefore

$$\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2\varphi(\langle y, y \rangle),$$

which yields  $\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2m(y)$  for all  $\lambda \in \mathbb{C}$ .

(iii) $\Rightarrow$ (i) The implication is trivial.  $\square$

As a natural generalization of the notion of Birkhoff–James orthogonality, the concept of strong Birkhoff–James orthogonality, which involves modular structure of a Hilbert  $C^*$ -module was introduced in [3]. When  $x$  and  $y$  are elements of a Hilbert  $\mathcal{A}$ -module  $\mathcal{X}$ , we consider the *strong Birkhoff–James orthogonality*:

$$x \perp_B^s y \quad :\Leftrightarrow \quad \forall_{a \in \mathcal{A}} \|x\| \leq \|x + ya\|.$$

One can easily observe that  $x \perp y \implies x \perp_B^s y \implies x \perp_B^r y$ , while the converses do not hold in general (see [3]).

In the next result we establish characterizations of the strong Birkhoff–James orthogonality for elements of a Hilbert  $C^*$ -module based on norm derivatives. We will apply our new tools - Theorems 2.3, 2.4 and 3.1.

**Theorem 3.3.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . The following statements are mutually equivalent:*

- (i)  $x \perp_B^s y$ ,
- (ii)  $\rho_-(x, ya) \leq 0$  for all  $a \in \mathcal{A}$ ,
- (iii)  $\rho_+(x, ya) \geq 0$  for all  $a \in \mathcal{A}$ .

*Proof.* (i) $\Rightarrow$ (ii) Let  $x \perp_B^s y$ . Then  $x \perp_B^r y \langle y, x \rangle$ . So, by Theorem 3.1, there exists  $\varphi_o \in \Omega_x$  such that  $\operatorname{Re}\varphi_o(\langle x, y \langle y, x \rangle \rangle) = 0$ . Hence  $\operatorname{Re}\varphi_o(\langle x, y \rangle \langle y, x \rangle) = 0$ . Thus  $\varphi_o(\langle x, y \rangle \langle y, x \rangle) = 0$ , since  $\varphi_o(\langle x, y \rangle \langle y, x \rangle) \in \mathbb{R}$ . Therefore,

$$|\operatorname{Re}\varphi_o(\langle x, ya \rangle)|^2 \leq |\varphi_o(\langle x, y \rangle a)|^2 \stackrel{(1.2)}{\leq} \varphi_o(\langle x, y \rangle \langle y, x \rangle) \varphi_o(a^*a) = 0$$

for each  $a \in \mathcal{A}$ . This implies  $\operatorname{Re}\varphi_o(\langle x, ya \rangle) = 0$ . Hence

$$\min \left\{ \operatorname{Re}\varphi(\langle x, ya \rangle) : \varphi \in \Omega_x \right\} \leq 0$$

and by Theorem 2.4 it follows that  $\rho_-(x, ya) \leq 0$  for all  $a \in \mathcal{A}$ .

(ii) $\Rightarrow$ (iii) By the condition (ii),  $\rho_-(x, y(-a)) \leq 0$  for all  $a \in \mathcal{A}$ . Hence, by (P2),  $\rho_+(x, ya) = -\rho_-(x, y(-a)) \geq 0$  for all  $a \in \mathcal{A}$ .

(iii) $\Rightarrow$ (i) Suppose (iii) holds. We may assume that  $x \neq 0$  otherwise (i) trivially holds. So, fix  $a \in \mathcal{A}$ . By Theorem 2.3, there exists a state  $\varphi_a$  in  $\Omega_x$  such that

$\operatorname{Re} \varphi_a(\langle x, ya \rangle) \geq 0$ . Then we have

$$\begin{aligned} 0 &\leq \operatorname{Re} \varphi_a(\langle x, ya \rangle) = \operatorname{Re} \varphi_a(\langle x, x + ya \rangle) - \operatorname{Re} \varphi_a(\langle x, x \rangle) \\ &\leq |\varphi_a(\langle x, x + ya \rangle)| - \|x\|^2 \leq \|\langle x, x + ya \rangle\| - \|x\|^2 \\ &\leq \|x\| \cdot \|x + ya\| - \|x\|^2 = \|x\| \cdot (\|x + ya\| - \|x\|). \end{aligned}$$

This implies that  $\|x\| \leq \|x + ya\|$ . Hence  $x \perp_B^s y$ .  $\square$

We recall that (see [12, 18]) two elements  $x$  and  $y$  of a normed linear space  $X$  are  $\rho$ -orthogonal if  $\rho(x, y) := \frac{\rho_+(x, y) + \rho_-(x, y)}{2} = 0$ , and in this case we write  $x \perp_\rho y$ . It is worth mentioning that the notion of  $\rho$ -orthogonality may be a strong tool. Indeed, the open problem posed in [1] was solved in the paper [29] and the concept of  $\rho$ -orthogonality played a significant role. For facts about the  $\rho$ -orthogonality in normed linear spaces, we refer the reader to [8, 9, 19, 32].

If  $x$  and  $y$  are elements of a Hilbert  $\mathcal{A}$ -module  $\mathcal{X}$ , then we have

$$x \perp y \implies x \perp_\rho y \implies x \perp_B^r y. \quad (3.7)$$

Indeed, if  $x \perp y$ , then  $\langle x, y \rangle = 0$ . Thus, for every  $\varphi \in \Omega_x$ , we have  $\operatorname{Re} \varphi(\langle x, y \rangle) = 0$  and so by Theorem 2.3 and Theorem 2.4 we obtain  $\rho_-(x, y) = \rho_+(x, y) = 0$ . Hence  $\rho(x, y) = 0$ , and thus  $x \perp_\rho y$ .

Further, if  $x \perp_\rho y$ , then  $\rho_-(x, y) + \rho_+(x, y) = 0$  and therefore  $\rho_-(x, y) \leq 0 \leq \rho_+(x, y)$  by (P1). It follows (3.2) from that  $x \perp_B^r y$ .

As an immediate consequence of Theorem 2.3 and Theorem 2.4 we obtain a characterization of the  $\rho$ -orthogonality in Hilbert  $C^*$ -modules as follows.

**Theorem 3.4.** *Let  $\mathcal{X}$  be a Hilbert  $\mathcal{A}$ -module, and  $x, y \in \mathcal{X}$ . The following statements are equivalent:*

- (i)  $x \perp_\rho y$ ,
- (ii)  $\max \{ \operatorname{Re} \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \} = \max \{ -\operatorname{Re} \varphi(\langle x, y \rangle) : \varphi \in \Omega_x \}$ .

*Remark 3.5.* Notice that the converses in (3.7) do not hold in general. For example, consider  $\mathbb{M}_2(\mathbb{C})$  as a Hilbert  $\mathbb{M}_2(\mathbb{C})$ -module and let  $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

and  $R = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ . Then simple computations show that

$$\rho_+(T, S) = -\rho_-(T, S) = -\rho_-(T, R) = 1 \quad \text{and} \quad \rho_+(T, R) = 0.$$

Hence  $T \perp_\rho S$ . But  $T \not\perp S$ , since  $\langle T, S \rangle = S \neq 0$ . Also,  $T \not\perp_\rho R$  but

$$\|T + \alpha R\| = \left\| \begin{bmatrix} 1 - \alpha & 0 \\ 0 & 1 \end{bmatrix} \right\| = \max\{|1 - \alpha|, 1\} \geq 1 = \|T\|$$

for all  $\alpha \in \mathbb{R}$ . Therefore  $T \perp_B^r R$ .

To end the work we show that the orthogonalities  $\perp_\rho$  and  $\perp_B^s$  are incomparable.

Indeed, since for  $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  we have  $\|T + SC\| = 0 < 1 = \|T\|$ , we get

$T \not\perp_B^s S$ . Furthermore, for every  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$  we have  $\|T + RA\| = \left\| \begin{bmatrix} 1-a & -b \\ 0 & 1 \end{bmatrix} \right\| \geq 1 = \|T\|$ , whence  $T \perp_B^s R$ . Therefore  $\perp_\rho \not\subseteq \perp_B^s$  and  $\perp_\rho \not\supseteq \perp_B^s$ .

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