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FROM ONE REEB ORBIT TO TWO

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Abstract

We show that every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits. We also show that if there are only finitely many embedded Reeb orbits, then their symplectic actions are not all integer multiples of a single real number; and if there are exactly two embedded Reeb orbits, then the product of their symplectic actions is less than or equal to the contact volume of the manifold. The proofs use a relation between the contact volume and the asymptotics of the amount of symplectic action needed to represent certain classes in embedded contact homology, recently proved by the authors and V. Ramos.

1. Statement of results

Let Y be a closed oriented three-manifold. Recall that a *contact form* on Y is a 1-form λ on Y such that $\lambda \wedge d\lambda > 0$. A contact form λ determines the *contact structure* $\xi := \operatorname{Ker}(\lambda)$, and the *Reeb vector field* R characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. A *Reeb orbit* is a closed orbit of the vector field R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$, modulo reparametrization. The Reeb orbit γ is *nondegenerate* if the linearized Reeb flow along γ does not have 1 as an eigenvalue, and the contact form λ is called nondegenerate if all Reeb orbits are nondegenerate.

The three-dimensional Weinstein conjecture, first proved in full generality by Taubes [**19**], asserts that any contact form on a closed three-manifold has at least one Reeb orbit. It is interesting to try to improve the lower bound on the number of Reeb orbits. In fact, it seems that the only known examples of contact forms on closed three-manifolds with only finitely many embedded Reeb orbits are certain contact forms on S^3 and lens spaces with exactly two embedded Reeb orbits, for example the standard contact form on an irrational ellipsoid [**11**, Ex. 1.8] and quotients thereof by cyclic group actions. It is shown in [**14**, Thm. 1.3] that any nondegenerate contact form on a closed three-manifold Y has

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at least two embedded Reeb orbits; and if Y is not S^3 or a lens space, then every nondegenerate contact form has at least three embedded Reeb orbits. The main theorem of the present paper asserts that one can prove the existence of at least two embedded Reeb orbits without the nondegeneracy assumption:

Theorem 1.1. *Every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits.*

For example, Theorem 1.1 has the following implication for Hamiltonian dynamics. Recall that if Y is a hypersurface in a symplectic manifold (X, ω), then the *characteristic foliation* on Y is the rank one foliation $L_Y := \operatorname{Ker}(\omega|_{TY})$, and a *closed characteristic* in Y is an embedded loop in Y tangent to L_Y. If Y is a regular level set of a smooth function $H : X \to \mathbb{R}$, then closed characteristics on Y are the same as unparametrized embedded closed orbits of the Hamiltonian vector field X_H on Y. Now consider $X = \mathbb{R}^4$ with the standard symplectic form $\omega = \sum_{i=1}^2 dx_i\, dy_i$. If Y is a compact hypersurface in $\mathbb{R}^4$ which is star-shaped, meaning that it is tranverse to the radial vector field, then

$$\lambda = \frac{1}{2}\sum_{i=1}^{2}(x_i dy_i - y_i dx_i)$$

restricts to a contact form on Y, and the unparametrized embedded Reeb orbits are the same as the closed characteristics. (The contact forms that arise this way correspond to the contact forms on S^3 for the tight contact structure.) Thus Theorem 1.1 applied to S^3 implies the following:

Corollary 1.2. *Every smooth compact star-shaped hypersurface in $\mathbb{R}^4$ has at least two closed characteristics.*

Previously, Hofer-Wysocki-Zehnder showed in [**7**, Thm. 1.1] that every strictly convex hypersurface in $\mathbb{R}^4$ has either two or infinitely many closed characteristics, and in [**8**, Cor. 1.10] that every nondegenerate contact form on S^3 giving the tight contact structure has either two or infinitely many embedded Reeb orbits, provided that all stable and unstable manifolds of the hyperbolic periodic orbits intersect transversally. In higher dimensions, Wang [**22**] has shown that there are at least $\left\lfloor \frac{n+1}{2} \right\rfloor + 1$ closed characteristics on every compact strictly convex hypersurface Σ in $\mathbb{R}^{2n}$. It has long been conjectured that there are at least n closed characteristics on every compact convex hypersurface in $\mathbb{R}^{2n}$, cf. [**4**, Conj. 1]. After the first version of this paper appeared, alternate proofs of Corollary 1.2 were given in [**6**, **17**].

Similarly, applying Theorem 1.1 to the unit cotangent bundle of S^2 recovers the result of Bangert-Long [**1**] that every (not necessarily reversible) Finsler metric on S^2 has at least two closed geodesics.

The method used to prove Theorem 1.1 yields a slightly more general result. To state it, define the *symplectic action* of a Reeb orbit γ by

$$\mathcal{A}(\gamma) := \int_\gamma \lambda.$$

We then have:

Theorem 1.3. *Let (Y, λ) be a closed contact three-manifold having only finitely many embedded Reeb orbits $\gamma_1, \ldots, \gamma_m$. Then their symplectic actions $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_m)$ are not all integer multiples of a single real number.*

Remark 1.4. If λ has infinitely many embedded Reeb orbits, then their symplectic actions can all be integer multiples of a single real number, for example in a prequantization space, or in an ellipsoid $(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} = 1) \subset \mathbb{C}^2$ with a_1/a_2 rational. Theorem 1.3 (and its proof) does extend to contact forms with infinitely many embedded Reeb orbits if they are isolated in the free loop space.

To state one more result, if λ is a contact form on a closed oriented three-manifold Y, define the *volume* of (Y, λ) by

$$\operatorname{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda. \tag{1.1}$$

One can ask whether there exists a Reeb orbit with an upper bound on its symplectic action in terms of the volume of (Y, λ), for example with symplectic action less than or equal to the square root of the volume. One might also expect that in most cases there are at least three embedded Reeb orbits. The following theorem asserts that at least one of these two statements always holds:

Theorem 1.5. *Let (Y, λ) be a closed contact three-manifold. Then either:*

- *λ has at least three embedded Reeb orbits, or*
- *λ has exactly two embedded Reeb orbits, and their symplectic actions T, T' satisfy $TT' \leq \operatorname{vol}(Y, \lambda)$.*

2. Embedded contact homology and volume

To prepare for the proofs of Theorems 1.1, 1.3, and 1.5, we need to recall some notions from embedded contact homology (ECH). For more about ECH, see [**11**] and the references therein.

2.1. Definition of embedded contact homology. If λ is a nondegenerate contact form on a closed three-manifold Y, then for each $\Gamma \in H_1(Y)$ the *embedded contact homology* with $\mathbb{Z}/2$ coefficients, which we denote by $ECH_*(Y, \lambda, \Gamma)$, is defined. (ECH can actually be defined over $\mathbb{Z}$, see [**13**], but $\mathbb{Z}/2$ coefficients are sufficient for the applications in this paper). This is the homology of a chain complex $ECC(Y, \lambda, \Gamma, J)$ generated by finite sets $\alpha = \{(\alpha_i, m_i)\}$ such that the α_i are distinct embedded Reeb orbits, $m_i = 1$ when α_i is hyperbolic, and

$$\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y).$$

Here a Reeb orbit γ is called *hyperbolic* if the linearized Reeb flow around γ has real eigenvalues. The J that enters into the chain complex is an $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times Y$ that sends the two-plane field $\xi = \operatorname{Ker}(\lambda)$ to itself, rotating it positively with respect to $d\lambda$, and satisfies $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate on $\mathbb{R} \times Y$. The chain complex differential ∂ counts certain mostly embedded J-holomorphic curves in $\mathbb{R} \times Y$. Specifically, if α and β are two chain complex generators, then the differential coefficient $\langle \partial\alpha, \beta\rangle \in \mathbb{Z}/2$ is a count of J-holomorphic curves in $\mathbb{R} \times Y$, modulo translation of the $\mathbb{R}$ coordinate, that are asymptotic as currents to $\mathbb{R} \times \alpha$ as $s \to \infty$ and to $\mathbb{R} \times \beta$ as $s \to -\infty$. The curves are required to have *ECH index* 1. The ECH index is a certain function of the relative homology class of the curve, explained e.g. in [**9**]; we do not need to recall the definition here. If J is generic, then ∂ is well-defined and $\partial^2 = 0$, as shown in [**12**, **13**].

The ECH index induces a relative $\mathbb{Z}/d$-grading on $ECH_*(Y, \lambda, \Gamma)$, where d denotes the divisibility of $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ in $H^2(Y; \mathbb{Z})$ mod torsion, see [**9**, §2.8]. Here $\mathrm{PD}(\Gamma)$ denotes the Poincare dual of Γ.

2.2. The isomorphism with Seiberg-Witten Floer cohomology. Although a priori the homology of the chain complex $ECC(Y, \lambda, \Gamma, J)$ might depend on J, in fact it does not. This follows from a theorem of Taubes [**20**] asserting that when Y is connected, there is a canonical isomorphism between embedded contact homology and a version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [**16**]. The precise statement is that there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$-modules

$$ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + PD(\Gamma)), \tag{2.1}$$

where $\mathfrak{s}_\xi$ is the spin-c structure determined by the oriented two-plane field ξ, see e.g. [**16**, Lem. 28.1.1]. (The isomorphism (2.1) holds using $\mathbb{Z}$ or $\mathbb{Z}/2$ coefficients.) In particular, there is a well-defined relatively graded $\mathbb{Z}/2$-module $ECH(Y, \xi, \Gamma)$. By summing over all $\Gamma \in H_1(Y)$, one also obtains a well-defined $\mathbb{Z}/2$-module $ECH(Y, \xi)$.

2.3. Filtered ECH. If $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the ECH chain complex, define the *symplectic action* of α by

$$\mathcal{A}(\alpha) := \sum_i m_i \mathcal{A}(\alpha_i) = \sum_i m_i \int_{\alpha_i} \lambda.$$

It follows from the conditions on J that the ECH differential decreases the symplectic action. Hence, for any real number L, one can define the *filtered ECH*, denoted by $ECH^L(Y,\lambda,\Gamma)$, to be the homology of the subcomplex of ECC spanned by generators with action strictly less than L.

It is shown in [**15**, Thm. 1.3] that $ECH^L(Y,\lambda,\Gamma)$ does not depend on the choice of generic J required to define the chain complex differential. On the other hand, $ECH^L(Y,\lambda,\Gamma)$, for fixed Y, Γ and L, does depend on the contact form λ and not just on the contact structure ξ.

As with the usual ECH, one can take the direct sum of the filtered ECH for all $\Gamma \in H_1(Y)$ to obtain a $\mathbb{Z}/2$ module $ECH^L(Y,\lambda)$.

2.4. The U map. If Y is connected, there is a degree -2 map

$$U : ECH(Y,\lambda,\Gamma) \to ECH(Y,\lambda,\Gamma). \tag{2.2}$$

It is induced by a chain map U_z which is defined similarly to the differential ∂, but instead of counting ECH index 1 curves modulo translation, it counts J-holomorphic curves of ECH index 2 passing through $(0,z) \in \mathbb{R}\times Y$, where z is a base point in Y which is not contained in any Reeb orbit. The connectedness of Y implies that the induced map (2.2) does not depend on z. (When Y is disconnected there is one U map for each component.) For details see [**14**, §2.5] or [**11**, §3.8].

There is an analogous U map on Seiberg-Witten Floer cohomology, and it is shown in [**21**, Thm. 1.1] that this agrees with the U map on ECH under the isomorphism (2.1).

2.5. Minimum symplectic action needed to represent a class. Let $0 \neq \sigma \in ECH(Y,\xi)$. We now recall from [**10**] the definition of a real number $c_\sigma(Y,\lambda)$, which roughly speaking is the minimum symplectic action needed to represent the class σ.

If λ is nondegenerate, then $c_\sigma(Y,\lambda)$ is the infimum over L such that σ is in the image of the inclusion-induced map $ECH^L(Y,\lambda) \to ECH(Y,\xi)$. Note that for any J as needed to define the chain complex $ECC(Y,\lambda,J)$, there exists a cycle θ in the chain complex representing the class σ, such that every chain complex generator α that appears in θ satisfies $\mathcal{A}(\alpha) \le c_\sigma(Y,\lambda)$, and $c_\sigma(Y,\lambda)$ is the smallest number with this property. We call a cycle θ as above an *action-minimizing representative* of σ.

If λ is degenerate, one defines

$$c_\sigma(Y,\lambda) = \lim_{n\to\infty} c_\sigma(Y,f_n\lambda), \tag{2.3}$$

where $f_n : Y \to \mathbb{R}$ are positive smooth functions such that the contact form $f_n\lambda$ is nondegenerate and $\lim_{n\to\infty} f_n = 1$ in the C^0 topology.

The numbers $c_\sigma(Y,\lambda)$ then satisfy the following axioms:

(Monotonicity) If $f : Y \to \mathbb{R}$ is a smooth function with $f > 1$, then $c_\sigma(Y,\lambda) \le c_\sigma(Y,f\lambda)$.

(Scaling) If $\kappa > 0$ is a constant then $c_\sigma(Y,\kappa\lambda) = \kappa c_\sigma(Y,\lambda)$.

(Continuity) If $f_n : Y \to \mathbb{R}$ are positive smooth functions with $\lim_{n\to\infty} f_n = 1$ in the C^0 topology, then $\lim_{n\to\infty} c_\sigma(Y, f_n\lambda) = c_\sigma(Y,\lambda)$.

To see that (2.3) is well-defined and to prove the above axioms, one can first show that the Monotonicity and Scaling axioms hold for nondegenerate contact forms, see [**10**, §4]. It then follows from this that the definition (2.3) does not depend on the sequence $\{f_n\}$, and that the Monotonicity, Scaling, and Continuity axioms hold without any nondegeneracy assumption.

2.6. Asymptotics and volume. In [**3**], the following result was established relating the asymptotics of the numbers $c_\sigma(Y,\lambda)$ to the contact volume (1.1). If $\Gamma \in H_1(Y)$ is such that $c_1(\xi) + 2PD(\Gamma) \in H^2(Y;\mathbb{Z})$ is torsion, then we know from §2.1 that $ECC(Y,\xi,\Gamma)$ has a relative $\mathbb{Z}$-grading. Choose any normalization of this to an absolute $\mathbb{Z}$-grading, and denote the grading of a generator x by $I(x) \in \mathbb{Z}$. We then have:

Theorem 2.1. [**3**, Thm. 1.3] *Let (Y,λ) be a closed connected contact three-manifold, let $\Gamma \in H_1(Y)$, suppose that $c_1(\xi)+2PD(\Gamma) \in H^2(Y,\mathbb{Z})$ is torsion, and choose an absolute $\mathbb{Z}$-grading I on $ECH(Y,\xi,\Gamma)$. Let $\{\sigma_k\}_{k=1,2,\ldots}$ be a sequence of nonzero homogeneous elements of $ECH(Y,\xi,\Gamma)$ satisfying $\lim_{k\to\infty} I(\sigma_k) = \infty$. Then*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} = \mathrm{vol}(Y,\lambda). \tag{2.4}$$

To prove Theorems 1.1 and 1.3, we just need the following weaker result:

Corollary 2.2. *Let (Y,λ) be a closed connected contact three-manifold. Then there exist nonzero classes $\{\sigma_k\}_{k\geq 1}$ in $ECH(Y,\xi)$ such that*

$$U\sigma_{k+1} = \sigma_k \tag{2.5}$$

for all $k \geq 1$, and

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)}{k} = 0. \tag{2.6}$$

Proof. We can always find a class $\Gamma \in H_1(Y)$ such that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y;\mathbb{Z})$ is torsion. It follows from the isomorphism (2.1) of $ECH(Y,\xi,\Gamma)$ with Seiberg-Witten Floer cohomology, together with known properties of the latter [**16**, Lem. 33.3.9, Cor. 35.1.4], that there exists a sequence $\{\sigma_k\}_{k\geq 1}$ of nonzero homogeneous elements of $ECH(Y,\xi,\Gamma)$ satisfying (2.5). Since the U map has degree -2, we have $I(\sigma_{k+1}) = I(\sigma_k) + 2$. Hence, Theorem 2.1 applies to give (2.4), which then implies (2.6). q.e.d.

Remark 2.3. The analysis in [**3**] is not required for Corollary 1.2, because it was already shown in [**10**] that Theorem 2.1 holds for any contact form on S^3 giving the tight contact structure. In particular, it follows from [**10**, Rmk. 3.3, Prop. 4.5] that Theorem 2.1 holds for the boundary of an ellipsoid in $\mathbb{R}^4$, and it then follows from [**10**, Prop. 8.6(b)] that Theorem 2.1 holds for any other contact form giving the same contact structure.

3. The key lemma

The key to the proofs of Theorems 1.1, 1.3, and 1.5 is the following:

Lemma 3.1. *Let Y be a closed connected three-manifold and let λ be a (possibly degenerate) contact form on Y with kernel ξ. Assume that λ has only finitely many embedded Reeb orbits $\gamma_1, \ldots, \gamma_m$. Then:*

(a) If $0 \neq \sigma \in ECH(Y,\xi)$, then $c_\sigma(Y,\lambda)$ is a nonnegative integer linear combination of $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_m)$.

(b) If $\sigma \in ECH(Y,\xi)$ and $U\sigma \neq 0$, then $c_{U\sigma}(Y,\lambda) < c_\sigma(Y,\lambda)$.

Proof. Fix a nonzero class $\sigma \in ECH(Y,\xi)$ and write $L = c_\sigma(Y,\lambda)$. Choose open tubular neighborhoods N_i of the Reeb orbits γ_i whose closures are disjoint, and let $N = \bigcup_{i=1}^m N_i$. Fix a point $z \in Y \setminus \overline{N}$ for use in defining the U map. By shrinking the tubular neighborhoods N_i if necessary, we may assume that:

(i) If γ is a Reeb trajectory intersecting both z and $\overline{N}$ then $\int_\gamma \lambda \geq L + 3$.

Next, choose a sequence of smooth functions $\{f_n : Y \to \mathbb{R}^{>0}\}$ such that:

(ii) $f_n|_{Y\setminus N} \equiv 1$,

(iii) The contact form $f_n\lambda$ is nondegenerate,

(iv) $\lim_{n\to\infty} f_n = 1$ in the C^1 topology, and

(v) Every Reeb orbit of $f_n\lambda$ with symplectic action less than $L+1$ is contained in some N_i, and has symplectic action within $1/n$ of an integer multiple of $\mathcal{A}(\gamma_i)$.

(The reason we can obtain condition (v) is that otherwise there would be a sequence f_n such that each $f_n\lambda$ has a Reeb orbit of action less than $L+1$ not contained in N, or a Reeb orbit in N_i of action $< L+1$ whose action is not within ε of an integer multiple of $\mathcal{A}(\gamma_i)$ for some n-independent $\varepsilon > 0$. Then a subsequence of these Reeb orbits would converge to a Reeb orbit of λ which could not be a multiple of one of the Reeb orbits γ_i.)

It follows from conditions (iii), (iv) and (v) and the Continuity axiom for c_σ that if n is sufficiently large, then $c_\sigma(Y, f_n\lambda) < L+1$, and $c_\sigma(Y, f_n\lambda)$ is within distance M/n of an integer linear combination of $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_m)$, where $M = 2L/\min(\mathcal{A}(\gamma_i))$.

To prove (b), continue to fix the above data, and assume that $U\sigma \neq 0$. For each n, choose a generic almost complex structure J_n on $\mathbb{R}\times Y$ as needed to define the filtered ECH chain complex $ECC^{L+1}(Y, f_n\lambda, J_n)$

and the chain map U_z on it. Specifically, we need J_n to satisfy the genericity conditions listed in the first paragraph of [**13**, §10], for J_n-holomorphic curves counted by ∂ or U_z whose positive ends have total action less than $L+1$. These conditions on J_n can all be achieved by perturbing J_n near the Reeb orbits of action less than $L+1$. So by condition (v) above, we can arrange that the almost complex structures J_n agree with a fixed almost complex structure J_0 on $\mathbb{R}\times(Y\setminus N)$.

We know from the proof of (a) that if n is sufficiently large then $c_\sigma(Y, f_n\lambda) < L+1$, so we can choose an action-minimizing representative θ_n of σ in $ECC^{L+1}(Y, f_n\lambda)$.

Claim. There exists $\delta > 0$ and a positive integer n_0 such that if $n \geq n_0$ and C_n is a J_n-holomorphic curve counted by $U_z\theta_n$, then $\int_{C_n} d(f_n\lambda) \geq \delta$.

The Claim implies (b), because it implies that if $n \geq n_0$ then $c_{U\sigma}(Y, f_n\lambda) \leq c_\sigma(Y, f_n\lambda) - \delta$, and so by the Continuity axiom $c_{U\sigma}(Y,\lambda) \leq c_\sigma(Y,\lambda) - \delta$.

Proof of Claim: Recall that the conditions on J_n imply that if C_n is any J_n-holomorphic curve, then $d(f_n\lambda)$ is pointwise nonnegative on C_n, with equality only where the tangent space to C_n is the span of the $\mathbb{R}$ direction and the Reeb direction (or where C_n is singular, although below C_n will be a curve counted by $U_z\theta_n$ and these do not have singularities). In particular, $\int_{C_n} d(f_n\lambda) \geq 0$. Consequently, if the Claim is false, then we can find an increasing sequence $\{n_i\}_{i\geq 1}$ of positive integers, and for each i a J_{n_i}-holomorphic curve C_{n_i} counted by $U_z\theta_{n_i}$, such that $\lim_{i\to\infty}\int_{C_{n_i}} d(f_{n_i}\lambda) = 0$.

We now use the following proposition, which is a special case of a result of Taubes [**18**, Prop. 3.3]:

Proposition 3.2. *Let (X,ω) be a compact symplectic 4-manifold with boundary with a compatible almost complex structure J. Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of compact J-holomorphic curves in X with boundary contained in ∂X, and suppose that there exists $E>0$ such that $\int_{C_i}\omega < E$ for all i. Then one can pass to a subsequence such that:*

(Convergence as currents) The curves $\{C_i\}$ converge weakly as currents to a compact J-holomorphic curve C_0 with boundary in ∂X such that $\int_{C_0}\omega \leq E$, and

(Pointwise convergence)

$$\lim_{i\to\infty}\left(\sup_{x\in C_i}\operatorname{dist}(x, C_0) + \sup_{x\in C_0}\operatorname{dist}(x, C_i)\right) = 0.$$

We apply the above proposition to the intersections of the holomorphic curves C_{n_i} with $X = [-1,1]\times(Y\setminus N)$, with the symplectic form $\omega = d(e^s\lambda)$. To see why we have the necessary upper bound on ω to apply the proposition, given i, choose $s_+ \in [1,2]$ and $s_- \in [-2,-1]$ such that C_{n_i} is tranverse to $\{s_\pm\}\cap Y$. Then since $d(e^s f_{n_i}\lambda)$ and $d(f_{n_i}\lambda)$ are pointwise nonnegative on C_{n_i}, we have an upper bound

$$\begin{aligned}\int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))}\omega &\le \int_{C_{n_i}\cap([s_-,s_+]\times Y)} d(e^s f_{n_i}\lambda)\\ &= e^{s_+}\int_{C_{n_i}\cap(\{s_+\}\times Y)} f_{n_i}\lambda - e^{s_-}\int_{C_{n_i}\cap(\{s_-\}\times Y)} f_{n_i}\lambda\\ &< e^2(L+1).\end{aligned}$$

So we can pass to a subsequence such that $C_{n_i}\cap([-1,1]\times(Y\setminus N))$ converges in the sense of Proposition 3.2 to a (possibly multiply covered) J_0-holomorphic curve C_0 in $[-1,1]\times(Y\setminus N)$. By the "pointwise convergence" condition, the curve C_0 contains the point $(0,z)$, since each C_{n_i} does.

Since C_0 is J_0-holomorphic, it follows that $d\lambda$ is pointwise nonnegative on C_0, with equality only where C_0 is singular or the tangent space of C_0 is the span of the $\mathbb{R}$ direction and the Reeb direction. In particular,

$$\int_{C_0} d\lambda \ge 0. \tag{3.1}$$

In fact, the inequality (3.1) must be strict. Otherwise C_0, regarded as a current, is invariant under translation of the $[-1,1]$ coordinate on $[-1,1]\times(Y\setminus N)$. It follows that $C_0\cap(\{0\}\times(Y\setminus N))$ is tangent to the Reeb vector field for λ. In particular, $C_0\cap(\{0\}\times(Y\setminus N))$, regarded as a subset of Y, contains a Reeb trajectory for λ passing through z with endpoints on $\partial\overline{N}$. So by (i) above,

$$\int_{C_0\cap(\{0\}\times(Y\setminus N))}\lambda \ge L+3.$$

By the convergence of currents above, it follows that

$$\int_{C_{n_i}\cap(\{s\}\times(Y\setminus N))} f_{n_i}\lambda \ge L+2 \tag{3.2}$$

whenever i is sufficiently large and $s\in[-1,1]$ is such that C_{n_i} is transverse to $\{s\}\times Y$. When this transversality holds, we orient $C_{n_i}\cap(\{s\}\times Y)$, regarded as a submanifold, by the "$\mathbb{R}$-direction first" convention. The conditions on J_{n_i} imply that $f_{n_i}\lambda$ is pointwise nonnegative on this oriented one-manifold, so it follows from (3.2) that

$$\int_{C_{n_i}\cap(\{s\}\times Y)} f_{n_i}\lambda \ge L+2. \tag{3.3}$$

But this is impossible, because the left hand side of (3.3) must be less than or equal to the maximum symplectic action of a generator in θ_{n_i}, which is less than $L+1$. This contradiction proves that the inequality (3.1) is strict.

Given this, let $\delta = \frac{1}{2}\int_{C_0} d\lambda > 0$. It follows from the convergence of currents that if i is sufficiently large then

$$\begin{aligned}\int_{C_{n_i}} d(f_{n_i}\lambda) &\geq \int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))} d(f_{n_i}\lambda)\\ &= \int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))} d\lambda\\ &\geq \int_{C_0} d\lambda \;-\delta\\ &= \delta.\end{aligned}$$

This contradicts our assumption that $\lim_{i\to\infty}\int_{C_{n_i}} d(f_{n_i}\lambda) = 0$ and thus completes the proof of the Claim, and with it Lemma 3.1. q.e.d.

Remark 3.3. In the above argument we cannot quote the SFT compactness theorem from [**2**], because that result assumes both a genus bound (which one does not have in ECH) as well as nondegeneracy of the contact form. This is why we use Taubes's approach via currents. Although this is only applicable in four dimensions, if one has a genus bound then one can cite [**5**] for similar arguments in higher dimensions.

4. Proofs of theorems

Proof of Theorem 1.1. This follows from Theorem 1.3. q.e.d.

Proof of Theorem 1.3. Suppose that λ has only finitely many embedded Reeb orbits and suppose that their symplectic actions are all integer multiples of a single real number $T > 0$. Let $\{\sigma_k\}_{k\geq 1}$ be any sequence satisfying (2.5). Then by Lemma 3.1, we have $c_{\sigma_k}(Y,\lambda) = n_k T$ where $\{n_k\}_{k\geq 1}$ is a strictly increasing sequence of nonnegative integers. It follows that

$$\liminf_{k\to\infty}\frac{c_{\sigma_k}(Y,\lambda)}{k} \geq T, \tag{4.1}$$

so that (2.6) cannot hold. This contradicts Corollary 2.2. q.e.d.

Proof of Theorem 1.5. Suppose there are fewer than three embedded Reeb orbits. We know from Theorem 1.1 that Y is connected and there are exactly two embedded Reeb orbits; denote their symplectic actions by T and T'.

Let $\{\sigma_k\}_{k\geq 1}$ be a sequence of homogeneous classes satisfying (2.5). By Lemma 3.1, we have $c_{\sigma_k}(Y,\lambda) = n_k T + n_k' T'$ where n_k and n_k' are nonnegative integers such that $n_{k+1}T + n_{k+1}'T' > n_k T + n_k' T'$. It follows from this that

$$\lim_{k\to\infty}\frac{c_{\sigma_k}(Y,\lambda)^2}{k} \geq 2TT'. \tag{4.2}$$

To see this, note that if we fix k and write $L = c_{\sigma_k}(Y,\lambda) = n_kT + n_{k'}T'$, then k is less than or equal to the number of pairs of nonnegative integers (x,y) with $xT + yT' \le L$, which is the number of lattice points in the triangle enclosed by the line $Tx + T'y = L$ and the x and y axes, which is $L^2/(2TT') + O(L)$, compare [**10**, §3.3]. On the other hand, since the U map has degree -2, we have

$$\lim_{k\to\infty} \frac{I(\sigma_k)}{k} = 2. \tag{4.3}$$

Putting (4.2) and (4.3) into (2.4) gives $\mathrm{vol}(Y,\lambda) \ge TT'$. q.e.d.

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