From proofs to focused proofs: a modular proof of Focalization in Linear Logic

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Abstract. Probably the most significant result concerning cut-free sequent calculus proofs in linear logic is the completeness of focused proofs. This completeness theorem has a number of proof theoretic applications — e.g. in game semantics, Ludics, and proof search — and more computer science applications — e.g. logic programming, call-by-name/value evaluation. Andreoli proved this theorem for first-order linear logic 15 years ago. In the present paper, we give a new proof of the completeness of focused proofs in terms of proof transformation. The proof of this theorem is simple and modular: it is first proved for MALL and then is extended to full linear logic. Given its modular structure, we show how the proof can be extended to larger systems, such as logics with induction. Our analysis of focused proofs will employ a proof transformation method that leads us to study how focusing and cut elimination interact. A key component of our proof is the construction of a *focalization graph* which provides an abstraction over how focusing can be organized within a given cut-free proof. Using this graph abstraction allows us to provide a detailed study of atomic *bias assignment* in a way more refined that is given in Andreoli's original proof. Permitting more flexible assignment of bias will allow this completeness theorem to help establish the completeness of a number of other automated deduction procedures. Focalization graphs can be used to justify the introduction of an inference rule for *multifocus* derivation: a rule that should help us better understand the relations between sequentiality and concurrency in linear logic.

1 Introduction

Linear Logic was introduced 20 years ago by Girard and since then it has led to many developments in proof theory, computational logic, and programming language theory. Much proof theoretic analyses and applications of linear logic have concentrated on the nature and dynamics of cut-elimination via the geometry of interactions, game semantics, interactions, etc. Less has been studied about the structure of cut-free proofs themselves: the main result in that area is probably the completeness of focused proofs due to Andreoli [3, 4]. This completeness theorem has a number of applications in computer science: for example, focused proofs have been used to design and formalize logic programming languages [2, 20], to formalize proof systems that allow for both forward-chaining and backward-chaining [15, 19], and should be behind the dualities between call-by-name and call-by-value evaluation in the λ-calculus [6]. The structure of focused proofs is also a key ingredient in the development of Polarized Logic [17, 18] and Ludics [13].

Andreoli's result, however, is wrapped up in one theorem about one logic. This seems an unfortunate situation for a number of reasons.

- Various extensions to linear logic are known (based on higher-order quantification [11], induction and co-induction [5], different kinds of exponentials [7, 12, 16], etc.) and it is likely that one will want to know if focusing can be proved for them.
- When examining the issues behind the assignment of polarity to literals (a necessary *annotation* step needed to define focused proofs), it is clear that there is a lot of flexibility allowed in providing such annotations, certainly more than what is technically allowed in Andreoli's proof system.
- Other logics exhibit focusing behaviors. In particular, there are focused proof systems for classical logic, namely the LKQ/LKT [8] and LK_p^η [9], and for intuitionistic logic, namely, the LJT [14], LJQ calculus [14, 10], and LJF [19].
- In [4], focusing is not seen as a process. There appears to be advantages to consider the process of transforming proofs into focused proofs: mixing this process with the process of doing cut-elimination should also be rather interesting.

These reasons suggest that the notions surrounding the "completeness of focused proofs" is both more general and more flexible than what is captured in the original theorem and its proof. Thus, we take on the task in this paper of attempting to develop an approach to proving focusing results by getting after the essential conditions for "focalization" to hold and by analyzing those conditions more broadly. By analogy, once the importance of cut-elimination was appreciated, Gentzen single cut-elimination theorem was analyzed in ways to uncover the essentially features that now allow researchers to prove cut-elimination for a number of logics.

This paper is organized as follows. In the next Section, we state some basic definitions and results for linear logic, including the original focused proof system (Figure 3). In Section 3, we present the key elements of our methodology, in particular, the *focalization graph* and a flexible *bias assignment* scheme, on the multiplicative and additive subset of linear logic (MALL). Section 4 considers how this methodology can account for additional structure within linear logic, including the exponentials and quantifiers. In Section 5, we briefly consider adding to the sequent calculus proofs the *multifocus* inference rule. Finally, we conclude in Section 6.

2 Linear Logic Preliminaries

The formulas of LL are made from literals which are atoms $(a, b, \dots)$ or negations of atoms $(a^\perp, b^\perp, \dots)$ and multiplicative $(\otimes, \mathbin{\rotatebox[origin=c]{180}{\&}}, \mathbf{1}, \perp)$, additive $(\oplus, \&, \mathbf{0}, \top)$ and exponential $(!, ?)$ connectives as well as (first-order) quantifiers $(\exists, \forall)$, following the grammar:

$$\begin{aligned} F ::= {} & a \mid F \otimes F \mid F \oplus F \mid \mathbf{1} \mid \mathbf{0} \mid \exists x.F \mid\ !F \\ & a^\perp \mid F \mathbin{\rotatebox[origin=c]{180}{\&}} F \mid F \,\&\, F \mid \perp \mid \top \mid \forall x.F \mid\ ?F \end{aligned}$$

For notational convenience we will write $A^\perp$ for the negation normal form of A (that is, where negations have only atomic scope) and we will work with one-sided sequents. We give in Figure 1 the inference rules for Linear Logic. The initial rule can be restricted to

$$\frac{}{\vdash A, A^{\perp}}\ \mathit{initial} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ \mathit{cut}$$

$$\frac{}{\vdash \mathbf{1}}\ \mathbf{1} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \quad \frac{\vdash \Gamma, A_1}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus_1 \quad \frac{\vdash \Gamma, A_2}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus_2 \qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x.A}\ \exists$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \with B}\ \with \quad \frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, A[c/x]}{\vdash \Gamma, \forall x.A}\ \forall$$

$$\frac{\vdash ?\Gamma, B}{\vdash ?\Gamma, !B}\ ! \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?B}\ ?w \quad \frac{\vdash \Gamma, ?B, ?B}{\vdash \Gamma, ?B}\ ?c \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, ?B}\ ?d \qquad \text{provided } c \text{ is new}$$

Fig. 1. Inference rules for LL

literals without a loss of completeness. We shall assume this restriction to atomic initial rules in the following. In Figure 2 we give an example of a sequent proof.

The logical connectives of linear logic can be divided into two sets: the *asynchronous* connectives ($\top, \bot, \with, \parr, ?, \forall$) and the *synchronous* connectives ($\mathbf{1}, \mathbf{0}, \otimes, \oplus, !, \exists$) (they are de Morgan duals of the asynchronous connectives). Reading the rules bottom-up, the rules for the asynchronous connectives are invertible (their application is independent from the context) whereas the synchronous have rules for which application depends on the surrounding context. Formulas built with a topmost asynchronous connective are also called negative, the one built with synchronous connective are positive.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash q, q^{\perp}}\ \mathit{ini} \quad \dfrac{}{\vdash r, r^{\perp}}\ \mathit{ini}}{\vdash q \otimes r, q^{\perp}, r^{\perp}}\ \otimes}{\vdash q \otimes r, q^{\perp} \parr r^{\perp}}\ \parr \quad \dfrac{}{\vdash s, s^{\perp}}\ \mathit{ini}}{\vdash q \otimes r, s \otimes (q^{\perp} \parr r^{\perp}), s^{\perp}}\ \otimes}{\vdash p \oplus (q \otimes r), s \otimes (q^{\perp} \parr r^{\perp}), s^{\perp}}\ \oplus \quad \dfrac{}{\vdash \mathbf{1}}\ \mathbf{1}}{\vdash p \oplus (q \otimes r), s \otimes (q^{\perp} \parr r^{\perp}), s^{\perp} \otimes \mathbf{1}}\ \otimes$$

Fig. 2. Example of a LL proof

The search for a focused proof can utilize this division of inference rules. If we read inference rules from conclusion to premiss, we can apply invertible rules in any order (no the need for backtracking) and when only synchronous rules are available we can focus on a certain formula and its positive subformulas. Such a chain of synchronous rules, usually called a *focused phase*, terminates when it reaches an asynchronous formula. Proof search can then alternate between applications of asynchronous rules and chains of synchronous rules.

A second aspect of focused proofs is that the synchronous/asynchronous classification of non-atomic formulas must be extended to atomic formulas. The arbitrary assignment of positive (synchronous) and negative (asynchronous) *bias* to atomic formulas must be made before the notion of focused proof is complete. How this bias is assigned does not affect the existence of a focused proof but does impact the size and shape of the resulting focused proofs. We shall sometimes think of such an assignment of bias to atomic formulas as an *annotation* of the atoms in the formula.

The focusing proof system for linear logic, presented in Figure 3, contains two kinds of sequents. In the sequent $\Psi : \Delta \Uparrow L$, the "zones" Ψ and Δ are multisets and L is

$$
\frac{\Psi : \Delta \Uparrow L}{\Psi : \Delta \Uparrow \bot, L}\ \bot \qquad
\frac{\Psi : \Delta \Uparrow F, G, L}{\Psi : \Delta \Uparrow F \parr G, L}\ \parr \qquad
\frac{\Psi, F : \Delta \Uparrow L}{\Psi : \Delta \Uparrow ?F, L}\ ?
$$

$$
\frac{}{\Psi : \Delta \Uparrow \top, L}\ \top \qquad
\frac{\Psi : \Delta \Uparrow F, L \quad \Psi : \Delta \Uparrow G, L}{\Psi : \Delta \Uparrow F \mathbin{\&} G, L}\ \& \qquad
\frac{\Psi : \Delta \Uparrow B[y/x], L}{\Psi : \Delta \Uparrow \forall x.B, L}\ \forall
$$

$$
\frac{}{\Psi : \cdot \Downarrow \mathbf{1}}\ \mathbf{1} \qquad
\frac{\Psi : \Delta_1 \Downarrow F \quad \Psi : \Delta_2 \Downarrow G}{\Psi : \Delta_1, \Delta_2 \Downarrow F \otimes G}\ \otimes \qquad
\frac{\Psi : \cdot \Uparrow F}{\Psi : \cdot \Downarrow\ !F}\ !
$$

$$
\frac{\Psi : \Delta \Downarrow F_1}{\Psi : \Delta \Downarrow F_1 \oplus F_2}\ \oplus_1 \qquad
\frac{\Psi : \Delta \Downarrow F_2}{\Psi : \Delta \Downarrow F_1 \oplus F_2}\ \oplus_2 \qquad
\frac{\Psi : \Delta \Downarrow B[t/x]}{\Psi : \Delta \Downarrow \exists x.B}\ \exists
$$

$$
\frac{\Psi : \Delta, F \Uparrow L}{\Psi : \Delta \Uparrow F, L}\ R\Uparrow \qquad
\frac{}{\Psi : K^{\bot} \Downarrow K}\ I_1 \qquad
\frac{\Psi : \Delta \Downarrow F}{\Psi : \Delta, F \Uparrow \cdot}\ D_1
$$

$$
\frac{\Psi : \Delta \Uparrow F}{\Psi : \Delta \Downarrow F}\ R\Downarrow \qquad
\frac{}{\Psi, K^{\bot} : \cdot \Downarrow K}\ I_2 \qquad
\frac{\Psi, F : \Delta \Downarrow F}{\Psi, F : \Delta \Uparrow \cdot}\ D_2
$$

Fig. 3. The Σ_3 focused proof system of [4] for linear logic. *The provisos on the rules are the following: In $\forall$-rule variable y is not free in the conclusion. In $R \Uparrow F$ is not asynchronous while in $R \Downarrow F$ is either asynchronous or a negative literal. In I_1 and I_2, K is a positive literal. In D_1 and D_2, F is not a negative literal.*

a list. This sequent encodes the usual one-sided sequent $\vdash\ ?\,\Psi, \Delta, L$ (here, we assume the natural coercion of lists into multisets). This sequent will also satisfy the invariant that requires Δ to contain only literals and synchronous formulas. In the sequent $\Psi : \Delta \Downarrow F$, the zone Ψ is a multiset of formulas and Δ is a multiset of literals and synchronous formulas, and F is a single formula.

The main result about focused proofs is that they are complete for all of linear logic. The following theorem was proved in [4].

Theorem 1. *Given Ψ a set of formulas, Γ a multiset of non-asynchronous formulas and Δ an arbitrary list of formulas, $\vdash ?\Psi, \Gamma, \Delta$ is provable in LL if and only if the sequent $\Psi : \Gamma \Uparrow \Delta$ is provable in Σ_3 proof system (given in figure 3).*

3 Focalization in MALL

In this part we will prove Focalization for MALL only in order to deal with a smaller system when introducing our proof technique. We will later extend the result to full LL. In doing so, we are driven by the will for simplicity but also by the particular interest for focalization in MALL for it is the system on which are built the basic objects of Ludics [13], the designs. It is actually the initial motivation of our work: finding a simpler and shorter proof of Focalization for MALL for Ludics purpose.

But still, our main concern is simplicity and that is why we first consider cut-free MALL proofs and we intend to demonstrate that Focalization is actually a fairly simple result, although the size of Σ_3 often makes it difficult to grasp.

3.1 Permutation of rules in LL

The sequential structure of sequent calculus proofs records the precise ordering of the application of inference rules, even when that ordering is not particularly important or when other orders result in similar proofs. Such sequentialization is responsible for not only an explosion in the space of proofs but also for the possibility of providing a precise analysis of the relationship between proof rules. In other words, what makes it difficult to determine if two sequent proofs are essentially the same or different is what provides us with powerful analysis tools for developing an approach to "causation" *a la* focalization. Systems like proof nets which get rid of the first difficulty have trouble when it comes to checking whether a proof structure is a proof net or whether a link in a proof net depends on another link.

Definition 1 (Permutation of inference rules). *We define two notions of permutability: (i)* α/β**-permutability:** *there is an* α/β*-permutability if, given a sequent* $\mathcal{S}$ *containing two formulas A and B, then for any proof* Π *of* $\mathcal{S}$ *starting with the* α *rule (on formula A) right before the* β *rule (on formula B) is applied, there exists a proof* Π' *of* $\mathcal{S}$ *where the two rules have been exchanged: the* β *rule comes first, immediately followed by the* α *rule (there is of course a degenerated case for rules with no premiss, like* $\top$*). (ii)* $\alpha|\beta$**-permutability:** *we speak of* $\alpha|\beta$*-permutability when there is both* α/β*-permutability and* β/α*-permutability.*

Given two sets of inference rules $\mathcal{N}$ *and* $\mathcal{P}$*, we say that, with respect to these two sets,* $\mathcal{P}$ *has* **weak permutability** *if given two rules* α,β *of* $\mathcal{P}$ *we have* $\alpha|\beta$*-permutability. We say that* $\mathcal{N}$ *has* **full permutability** *when it has weak permutability and when in addition for any pair of rules* $(\alpha,\beta) \in \mathcal{P} \times \mathcal{N}$*, we have* α/β*-permutability.*

Proposition 1 (Permutabilities of linear logic inference rules). *Let* $\mathcal{N}$ *be the set of inference rules attached to the MALL asynchronous connectives and* $\mathcal{P}$ *be the set of inference rules attached to the MALL synchronous connectives.* $\mathcal{N}$ *has full permutability while* $\mathcal{P}$ *has weak permutability.*

The proof is trivial either by introducing cuts and then reducing them or by doing small steps permutations. Notice that the synchronous connectives do not have full permutability: sequent $\vdash a^\perp \parr b^\perp, a \otimes b$ has no cut-free proof that begins with a $\otimes$-rule.

3.2 Focalization Graph

The introduction of the *Focalization Graph* structure brings us to the heart of our result. The acyclicity of the graph will be crucial in establishing focalization.

Definition 2. *A MALL sequent containing at least a negative non-literal formula is* **negative***. It is* **positive** *when it contains no negative non-literal formula and at least one positive non-literal formula. Otherwise it is* **atomic***.*

Definition 3 (Positive Trunks). *Given a MALL proof* Π *of a positive sequent* $\mathcal{S}$ *we define the* **Positive Trunk** Π^+ *as the maximal prefix of the tree* Π *containing only positive rules, that is the tree starting at the root of* Π *and whose leaves are the bottom* *sequents of the first non-positive rules encountered on every branch of the tree, if such a rule exists. The* **Border** *of a Positive Trunk is the set of its leaves. The border contains only negative or atomic sequents. The* **Active Formulas** *of a Positive Trunk Π^+ are the formulas which are principal formulas of a rule occurring in Π^+. They are the formulas of the base that are decomposed into subformulas within the considered Trunk.*

Remark 1. When addressing the case of the exponentials, we will see that we can add a condition to shorten a branch in the positive trunk, this condition can also be regarded as expressing the fact that the rule for ! is bipolarized, being both positive and negative.

We now define a relation on occurrences of formulas involved in Π: $F \prec G$ iff G is a subformula (or sub-occurrence) of F in the precise sense that occurrence G is obtained from the decomposition of F along a branch of Π.

$$\dfrac{\dfrac{\dfrac{\boxed{\vdash q\otimes r, q^\perp \parr r^\perp} \quad \boxed{\vdash s, s^\perp}}{\vdash q\otimes r, s\otimes(q^\perp \parr r^\perp), s^\perp}\otimes}{\vdash p\oplus(q\otimes r), s\otimes(q^\perp\parr r^\perp), s^\perp}\oplus \qquad \dfrac{}{\vdash \mathbf{1}}\mathbf{1}}{\vdash p\oplus(q\otimes r), s\otimes(q^\perp\parr r^\perp), s^\perp\otimes\mathbf{1}}\otimes$$

Fig. 4. Positive Trunk associated to figure 2

Definition 4 ($\prec$-relation). *The* **suboccurrence relation** *(written $\prec$) on occurrences of formulas appearing in Π is the reflexive and transitive closure of the binary relation $\prec^1$ defined by $F \prec^1 G$ if there exists in Π a rule α with conclusion sequent $\mathcal{S}$ and premisses $(\mathcal{S}_i)_{i\in I}$ such that F is the principal formula of $\mathcal{S}$ and G is a subformula of F produced by the rule α in some of the $\mathcal{S}_i$.*

If $F \prec G$ we will say that G is a **$\prec$-subformula** *of F or a* **descendent** *of F.*

The following lemma will help us proving our main result:

Lemma 1. *Let Π^+ be a Positive Trunk with root $\mathcal{S}$ and border $\mathcal{B}$. For any $\mathcal{S}' \in \mathcal{B}$ the relation $\prec$ defines a one-to-one function from $\mathcal{S}'$ to $\mathcal{S}$.*

Proof. We actually prove a stronger result: the result holds for any sequent appearing in the trunk, not only for sequents in $\mathcal{B}$.

The result is proved by induction on the height of the considered sequent in Π^+:
• The base case is trivial since the considered sequent is $\mathcal{S}$ itself (recall $\prec$ is reflexive).
• Suppose the result is true for a given height $n \leq h(\Pi^+)$ and suppose $n+1 \leq h(\Pi^+)$. Let $\mathcal{S}_{n+1}$ be a sequent of height $n+1$ and let α be the rule of which $\mathcal{S}_{n+1}$ is a premiss and call $\mathcal{S}_n$ its conclusion. By induction hypothesis $\mathcal{S}_n$ satisfies the condition. We can define a one-to-one function ι_n from $\mathcal{S}_{n+1}$ (as set of occurences of formulas) to $\mathcal{S}_n$ as follows: let G be a formula of $\mathcal{S}_{n+1}$. if $F \prec^1 G$ for some $F \in \mathcal{S}_n$ then fix F to be the image of G by ι_n. If no such formula exists, then an occurrence of G is also present in $\mathcal{S}_n$ then associate the two occurrences of G. The function built in this way is one-to-one thanks to the fact that every MALL positive rule produce at most (and actually exactly) one subformula of the principal formula in every premiss of the rule. Composing the function we just defined with the one-to-one function provided by the induction hypothesis we see that $\mathcal{S}_{n+1}$ satisfies the condition.
By induction we get the result we expected. □

Lemma 2. *A formula which is not active in the Positive Trunk appears in exactly one sequent of the border.*

An active formula F to which no branching rule is applied in Π^+ (speak of a non-branching formula wrt. Π^+) is such that there exists exactly one formula G in one of the sequents of the border which is $\prec$-related with F: $F \prec G$.

To a Positive Trunk we associate a graph as follows:

Definition 5 (Focalization graph). *Given a Positive Trunk Π+ we define the* **Focalization Graph** *$\mathcal{G}$ to be the graph whose vertices are the active formulas of the Trunk and such that there is an edge from the F to G iff there is a sequent $\mathcal{S}'$ in the border containing a negative $\prec$-subformula F' of F and a positive $\prec$-subformula G' of G.*

Example 1. The Focalization graph associated with our example proof is:

$$s^\perp \otimes \mathbf{1} \qquad s \otimes (q^\perp \parr r^\perp) \longrightarrow p \oplus (q \otimes r)$$

This graph is acyclic. In the follwing we will show that it is true in general and this will be crucial for focalization.

Lemma 3. *If $\mathcal{S}'$ and $\mathcal{S}''$ are sequents occurring in different branches of Π^+, then there is at most one formula in the root of Π^+ which has $\prec$-subformulas in both $\mathcal{S}'$ and $\mathcal{S}''$.*

Proof. If this was not the case, let $\mathcal{S}' \wedge \mathcal{S}''$ be their highest predecessor in the tree. This sequent would necessarily have at least two formulas that would be $\prec$-subformulas of the same formula in the root which is impossible thanks to lemma 1. □

Proposition 2. *The Focalization Graphs are acyclic.*

Proof. We prove the result by *reductio ad absurdum*.
Let $\mathcal{S}$ be a positive sequent with a proof Π. Let Π^+ be the corresponding positive trunk and $\mathcal{G}$ the associated Focalization Graph. Suppose that $\mathcal{G}$ has a cycle and consider such a cycle of minimal length ($F_1 \to F_2 \to \cdots \to F_n \to F_1$) in $\mathcal{G}$ and let us consider $\mathcal{S}_1, \ldots, \mathcal{S}_n$ sequents of the border justifying the arrows of the cycle.

Thanks to lemma 3 these sequents are actually uniquely defined. With the same idea we can immediately notice that the cycle is necessarily of length $n \geq 2$ since two $\prec$-subformulas of the same formula can never be in the same sequent in the border of the positive trunk, thanks to lemma 1.

Let $\mathcal{S}_0$ be $\bigwedge_{i=1}^n \mathcal{S}_i$ be the highest sequent in Π such that all the $\mathcal{S}_i$ are leaves of the tree rooted in $\mathcal{S}_0$. We will obtain the contradiction by studying $\mathcal{S}_0$ and we will reason by case on the rule applied to this sequent $\mathcal{S}_0$:

• the rule cannot be a $\mathbf{1}$ rule since this rule produces no premiss and thus we would have an empty cycle which is non-sens. Any rule with no premiss would lead to the same contradiction.

• If the rule is one of the $\oplus$-rules, then the premiss $\mathcal{S}'_0$ of the rule would also satisfy the condition required for $\mathcal{S}_0$ (all the $\mathcal{S}_i$ would be part of the proof tree rooted in $\mathcal{S}'_0$) contradicting the maximality of $\mathcal{S}_0$. If the rule is any other non-branching rule, maximality of $\mathcal{S}_0$ would also be contradicted.

• Thus the rule shall be branching: it shall be a $\otimes$-rule. Write $\mathcal{S}_L$ and $\mathcal{S}_R$ for the left and right premisses of $\mathcal{S}_0$. Let $G = G_L \otimes G_R$ be the principal formula in $\mathcal{S}_0$ and let F be the active formula of the Trunk such that $F \prec G$. There are two possibilities:
(i) either $F \in \{F_1, \ldots, F_n\}$ and F is the only formula of the cycle having at the same time $\prec$-subformulas in the left premiss and in the right premiss,
(ii) or $F \notin \{F_1, \ldots, F_n\}$ and no formula of the cycle has $\prec$-subformulas in both premisses.

Let thus I_L (resp. I_R) be the sets of indices of the active formulas of the root $\mathcal{S}$ having ($\prec$-related) subfomulas only in the left (resp. right) premiss. Clearly neither I_L nor I_R is empty since it would contradict the maximality of $\mathcal{S}_0$. Indeed if $I_L = \emptyset$, then $\mathcal{S}_R$ satisfies the condition of being dominated by all the $\mathcal{S}_i$, $1 \leq i \leq n$ and $\mathcal{S}_0$ is not maximal anymore. By definition of the two sets of indices we have of course $I_L \cap I_R = \emptyset$ and the only formula of the cycle possibly not in $I_L \cup I_R$ is F if we are in the case (i): all other formulas in the cycle have their index either in I_L or in I_R.

As a consequence there must be an arrow in the cycle (and thus in the graph) from a formula in I_L to a formula in I_R (or the opposite). Let $i \in I_L$ and $j \in I_R$ be such indexes (say for instance $F_i \to F_j$ in $\mathcal{G}$) and let $\mathcal{S}'$ be the sequent of the border responsible for this edge. $\mathcal{S}'$ contains F'_i and F'_j and by definition of the sets I_L and I_R, $\mathcal{S}'$ cannot be in the tree rooted in $\mathcal{S}_0$ which is in contradiction with the way we constructed $\mathcal{S}_0$.

Then there cannot be any cycle in the graph. □

3.3 Pre-Focalization process

What the previous result actually tells us is that the Focalization Graph has a source, a formula that is not pointed to by any other formula in the graph, that is a formula such that whenever a sequent of the border contains one of its $\prec$-subformulas F, the subformula is not positive or the sequent is positive. To put things in other terms, there is a positive active formula in the root sequent whose positive layer of connective is completely decomposed during the Positive Trunk, independentely of any focusing discipline. This can be regarded as a kind of implicit focusing result. In some sense that tells us there is a formula which is already implicitely focused in the positive trunk.

Thanks to full permutability of the negatives, weak permutability of the positives and the acyclicity of the focalization graphs we know that, given a MALL proof Π of a sequent $\mathcal{S}$, we can transform it to another proof satisfying the following conditions:

Pre-Focalization Process:

1. **Asynchronous phase:** thanks to full permutability of negatives, if $\mathcal{S}$ is negative then we can permute down all the negative rules so that Π is transformed to a proof Π' where the bottom part of the proof tree is made only of negative rules up to the point where the branches of the tree reach positive or atomic sequents;
2. **Synchronous phase:** if $\mathcal{S}$ is positive, the associated Focalization Graph allows us to select a source of the graph, let us say P, as a focus and thanks to weak permutability, we can have the positive rules on $\prec$-subformulas of P permuted down so that Π is transformed into a tree Π' for which the maximal prefix containing only

rules applied to P and its positive $\prec$-subformulas decomposes P up to its negative or literal subformulas. We are thus left with negative or atomic sequents, or positive sequents where the subformulas of P are literals.
3. **if $\mathcal{S}$ is atomic**, we can only apply an initial rule and thus close the tree.

This process is clearly terminating thanks to easy arguments on the complexity/size of the considered sequents in terms of number of polarity layers, for instance.

3.4 Dealing with Bias Assignments.

The method described in the previous section shows a proof transformation technique that results almost in focused proofs but not exactly. Indeed we will now see that Andreoli's system forces more cosntraints on the proofs in that the use of the initial rule is more constrained. We shall now generalize our technique to capture exactly Andreoli's focussing disciplin as well as a more general focusing disciplin with a different management of the atoms. The freedom we get on Bias Assignment can be crucial for several applications in proof search.

In Σ_3, the initial rule has two versions, I_1 and I_2 (see figure 3). The initial rule can be applied only during a focusing phase on *positive literals*. In particular, the sequent $\vdash a^\perp \oplus \mathbf{0}, a \oplus \mathbf{0}$ would have only one focused proof whereas the technique of the Focalization Graph presented previously would have led to two different focused proofs. Andreoli system adds more constraints to the proof search while remaining complete. We now introduce Bias Assignments in order to treat this.

Definition 6. *Given a provable sequent $\mathcal{S}$, we call $\mathcal{P}_\mathcal{S}$ (for available positions for $\mathcal{S}$) the set containing all the branches of all possible proof trees for $\mathcal{S}$. We write $\mathcal{O}_\mathcal{S}$ for the set of occurrences of literal occurring in $\mathcal{S}$.*

Definition 7 (Bias assignment $\mathcal{B}_\mathcal{S}$). *A bias assignment for a provable sequent $\mathcal{S}$, written $\mathcal{B}_\mathcal{S}$, is a partial function from $\mathcal{P}_\mathcal{S} \times \mathcal{O}_\mathcal{S}$ to $\{-;+\}$*

Example 2. We give here some examples of typical bias assignments:
• The bias assignment which is defined ***nowhere*** corresponds to the previous situation.
• ***Andreoli's bias assignment***. $\mathcal{B}^{\Sigma_3}$ is the function defined as: for any atom a, $\mathcal{B}^{\Sigma_3}(\_, a) = +$ and $\mathcal{B}^{\Sigma_3}(\_, a^\perp) = -$. More generally the bias assignments may not be sensitive to their first component and give the same polarity to different occurrences of the same litteral. In that case, we speak of an ***atom-based*** bias assignment.
• We can consider bias assignments which are *sensitive* to the position in the tree where the considered literal is. For such assignments b, $b(p, a)$ may be different from $b(q, a)$. In this case we speak of an ***occurrence-based*** bias assignment. We can consider coherence conditions on the assignments. For instance, moving upwards on a branch, we may want to ensure that the polarity won't change once it is set: if p and q are two branches, p being an extension of q and if $b(q, a) \searrow$ then $b(p, a) \searrow$ and $b(p, a) = b(q, a)$. But on the other hand we may also want to consider totally arbitrary assignments.

Definition 8 ($\mathcal{B}$-Focalization Graphs). *Given a positive sequent $\mathcal{S}$, a proof Π of $\mathcal{S}$ and a bias assignment $\mathcal{B}$ for $\mathcal{S}$, we define the $\mathcal{B}$-Focalization Graph $\mathcal{G}^\mathcal{B}_\mathcal{S}$ as in the previous* *subsection but considering as negative formulas the literals which are assigned polarity $-$ in a sequent $\mathcal{S}'$ of the border and as positive formulas the literals which are assigned polarity $+$. The literals for which $\mathcal{B}$ is not defined in $\mathcal{S}'$ are treated as before: they do not contribute to the graph.*

The bias assignment results in more arcs in the Focalization Graph. For instance, with $\mathcal{B}^{\Sigma_3}$ our example of figure 2 has the following focalization graph:

$$s^{\perp} \otimes \mathbf{1} \longrightarrow s \otimes (q^{\perp} \parr r^{\perp}) \longrightarrow p \oplus (q \otimes r)$$

This might also produce cycles. The following proposition ensures it does not:

Proposition 3. *Given a positive sequent $\mathcal{S}$ and a proof Π of $\mathcal{S}$, whatever bias assignment $\mathcal{B}$ we choose, the $\mathcal{B}$-Focalization Graph $\mathcal{G}^{\mathcal{B}}_{\mathcal{S}}$ is acyclic.*

It is essentially sufficient to notice that adding these arcs will have no effect on the arguments we used previously since they were only concerned with the splitting structure of the branching rules. We can now state our main results concerning Focalization:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash q, q^{\perp}}\,ini \quad \dfrac{}{\vdash r, r^{\perp}}\,ini}{\vdash q \otimes r, q^{\perp}, r^{\perp}}\,\otimes}{\vdash p \oplus (q \otimes r), q^{\perp}, r^{\perp}}\,\oplus}{\vdash p \oplus (q \otimes r), q^{\perp} \parr r^{\perp}}\,\parr \quad \dfrac{}{\vdash s, s^{\perp}}\,ini}{\vdash p \oplus (q \otimes r), s \otimes (q^{\perp} \parr r^{\perp}), s^{\perp}}\,\otimes \quad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash p \oplus (q \otimes r), s \otimes (q^{\perp} \parr r^{\perp}), s^{\perp} \otimes \mathbf{1}}\,\otimes$$

Fig. 5. focalized proof of figure 2

Theorem 2 ($\mathcal{B}$-Focalization for MALL). *Let $\mathcal{S}$ be a MALL sequent. To any proof Π of $\mathcal{S}$ and bias assignment $\mathcal{B}$, we can associate a new proof satisfying the following constraints depending on the sequent $\mathcal{S}$:*
(i) if it is a negative sequent starts by decomposing negative formulas;
(ii) when a positive sequent is encountered, a positive formula is chosen as a focus and is hereditarily decomposed until its negative or literal subformulas are found. if the subformula is negative we use the previous item, if the formula is a litteral, the behaviour depends on the bias which is assigned to the literal.

Theorem 3 (Andreoli's Focalization for MALL). *If we consider the bias assignment $\mathcal{B}^{\Sigma_3}$, the focalization process produces proofs which are focused in Andreoli's Σ_3 sense.*

4 Focalization for full LL and larger extensions

Our analysis was first restricted to the case of cut-free propositional MALL, mainly for simplicity purposes. We now extend the result to richer fragments of Linear Logic and present how to treat the cut, the exponentials and the quantifiers.

4.1 Quantifiers

The proof in the previous section can be directly adapted to the quantifiers: they are connectives with non-branching rules and with the appropriate permutabilities (full-permutabilities for the $\forall$ which is negative and weak-permutability for the $\exists$ which is positive). The first-order case is thus treated trivially. The higher-order case requires some additional care for Bias Assignments in order to verify that bias assignments are still meaningful in this case but our abstract definition of Bias Assignments allows us to define the needed constraints on bias assignments. The details are beyond the scope of this paper.

4.2 MALL with cut

Dealing with the cut-rule in an analysis of focusing is not critical when one is driven by completeness purposes only. But since we want to study a dynamic process of focalization, adressing the cut becomes important and even crucial. For instance, we may be interested in studying how the cut-reduction and the focalization process interact.

Our solution is inspired by what Andreoli does [4] but is slightly simplified. The basic idea is to notice that the cut-rule is very similar to a $\otimes$ rule: replacing a cut rule on A in a proof Π of $\vdash \Gamma$ results in an object which is *almost*[1] a proof of $\vdash \Gamma, A \otimes A^{\perp}$.

In fact we do not even need to use the proof itself. We will simply use this analogy in order to find how to adapt the Focalization graph to proofs with cuts. Our analogy simply suggests to treat the cut rule as a positive and, as a consequence, positive trunks may contain cut rules and the Focalization Graph will have new vertices of the form $Cut(A)$. The relation $\prec$ is extended in a straightforward way ($A \prec Cut(A)$ and $A^{\perp} \prec Cut(A)$) and the edges are created with the same conditions as we did in the previous section.

As before, we can prove that the Focalization graph is acyclic and then:

Theorem 4. *The Focalization Graph method produces focused proofs from MALL proofs with cuts.*

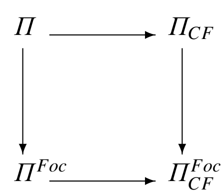

Fig. 6.

We think that the difference between our approach and Andreoli's starts really to make sense at this point: we always stayed in the same proof system, LL, and we worked by proof transformation. In our mind Focalization is really a process for transforming proofs. The interaction between this process and other transformation processes, like cut-reduction for instance shall now be studied.

Pushing this discussion further would be beyond the scope of this paper, but we would like to give an idea of the kind of question we can now try to adress: Given a proof Π in MALL with cuts, two processes are available: focalization and cut-reduction. Do the two processes commutes? Are we in the situation described by figure 6 where vertical arrows correspond to Focalization process while horizontal arrows correpond to the cut-reduction?

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[1] It is only almost a proof since the $\&$-rule, the ! rule and the $\forall$-rule may cause trouble. Andreoli fixes this by considering the formula $?A \otimes A^{\perp}$ instead of $A \otimes A^{\perp}$ which is fine for $\&$ and ! but inefficient for the $\forall$ quantifier...

In our setting, we will get a proof of $\vdash \Gamma, A \otimes A^{\perp}$: we are only interested in the cut rules which are performed within the positive trunk. We can easily check that if Π^{+} is a positive trunk for $\vdash \Gamma$ containing a cut rule on A then replacing the cut rule with a tensor rule on $A \otimes A^{\perp}$ leads straightforwardly to a positive trunk Π'^{+} on $\vdash \Gamma, A \otimes A^{\perp}$.

4.3 Exponentials

As it comes to exponentials we cannot carry our construction as straightforwardly as we did for the cut since it is not possible to attribute a polarity to the exponentials in a simple way: they do not have the right permutation rules in order to have full or weak permutability. In order to extend our result we have to adapt the sequent calculus in a way which is pretty similar to what is done by Andreoli with his dyadic sequents [4]. For this change not to seem too *ad hoc* we quickly justify this by considering the !-rule (see figure 1). This rule has a peculiar shape because, contrarily to other inference rules, it depends on the toplevel structure of every formula in the context: one formula has to be banged while all the other shall be question-marked. This indicates a special level of knowledge about the sequent structure which is not the usual one we use in sequent calculus. This is reflected in the way the !-rule is implemented in linear logic programming systems or by the boxing construction in Proof Nets.

We actually see two kinds of operations performed with the !-rule: (i) classifying F as a question-marked formula on the one hand and removing the ! on $!G$ when $!G$ is the only non-question-marked formula in the sequent. This can be reflected by the paradoxical example following: considering $\vdash\ ?\Gamma, F, !G$, can you apply the !-rule to this sequent? There could be two answers: “it depends on F” or “no, at least not yet”. Both answers carry the same idea that ! can be applied only if F is $?F'$ but they are different from the operational point of view: the second answer suggests that there is some more work to do in order to apply the !-rule: F should first be recognized as $?F'$. This remark suggests to introduce a separate context that will store those formulas that have been recognized as having a “?”: $\vdash \Gamma \mid \Delta$. The two operations discussed earlier and dereliction now become the following rules:

$$\frac{\vdash \Gamma, A \mid \Delta}{\vdash \Gamma \mid\ ?A, \Delta}\ ? \qquad \frac{\vdash \Gamma \mid A}{\vdash \Gamma \mid\ !A}\ ! \qquad \frac{\vdash \Gamma, A \mid A, \Delta}{\vdash \Gamma, A \mid \Delta}\ der$$

We then have to adapt all the usual MALL rules in the obvious way.

?-rule will be considered as negative whereas ! and dereliction will be considered as positive. We can now extend the positive trunks to LL proofs with exponentials:

Definition 9 (Exponential Positive Trunk). *Given a positive sequent $\mathcal{S}$ and a proof Π of $\mathcal{S}$, an exponential positive trunk (or positive trunk for short) for a positive sequent is a maximal subtree of Π containing only positive rules and such that* !*-rules produce leaves of the tree (the branches are cut as soon as a* !*-rule is applied).*

The reader may be surprised by the fact that the branches of the positive trunk are cut as soon as a ! rule is encoutered. This is reminiscent of the bipolar character of the exponentials: the ? is decomposed into two rules (one negative, the other positive) and for its dual connective, the !, the rule is positive but the focusing phase is stopped.

In order to build the Focalization graph, we first notice that each *der*-rule in the positive trunk produces an occurrence of a formula, say A, that might be chosen as a focus. We have to distinguish such occurrences and to do so we will index them as (A, i). The index i will refer to the place in the tree where the dereliction rule has been applied. Notice also the $\prec$-relation is straightforwardly extended to exponential sequents.

Definition 10. *Let $\vdash \Gamma \mid \Delta$ be a positive sequent*[2], *Π be a proof of the sequent and Π^+ be the associated (Exponential) Positive Trunk. The* **Exponential Focalization Graph** *extends the definition of standard Focalization graphs as follows:*
(i) *The vertices of the graph are the active formulas of Δ and the active occurrence of formulas in Γ, ie. of the form (A, i).*
(ii) *The arcs are given by the sequents of the border in the same way as usually (including the bias assignment if any)*[3].

The following allows us to extend our Focalization result to the exponential setting:

Proposition 4. *The Exponential Focalization Graphs are acyclic.*

4.4 Further extensions.

The proof we presented is modular in the sense that it relies on a series of simple results which can be adapted to richer settings. It is what is done in [5] in order to extend Focalization to an extension to LL with Fixpoints. We shall consider in future works other extensions. In particular, non-commutative logics and light logics should be good candidates to test the methodology of this paper.

5 Multi-Focalization

The question of Multi-Focalization naturally arises from the structure of Focalization Graphs. Indeed, the only two ingredients needed in our proof are (i) appropriate permutability properties (full and weak permutabilities) and (ii) the acyclicity of the Focalization Graph $\mathcal{G}$ which ensures us of the existence of a source which can be taken as a focus in the proof we are building.

In this last section we consider briefly this question of multi-focalization although most details on an analysis on Multi-Focalization are beyond the scope of this paper and will be postponed to future work. We only intend to introduce this notion and outline what could be the first step to a general theory of multifocalization.

We know that $\mathcal{G}$ has a source, but nothing forbids $\mathcal{G}$ to have multiple sources. In such a case, we would have several formulas (say $F_1, \dots, F_k$) for which the topmost positive layer of connectives is totally decomposed within the positive trunk. Weak permutability allows to conclude that the proof Π can be transformed to a proof where the bottom part of the tree is made only of positive rules on the F_i's and their subformulas up to a point where all the F_i's are turned to negative formulas (or literals).

This is enough to consider a notion of multifocalization and this leads us to associated sequent rules that we are currently investigating with Kaustuv Chaudhuri and which can be presented in a Σ_3 inspired sequent presentation as

$$\frac{\Psi, F_1, \dots F_k : \Delta \Downarrow F_1^{i_1}, \dots F_k^{i_k}, F'_1, \dots F'_l}{\Psi, F_1, \dots F_k : \Delta, F'_1, \dots F'_l \Uparrow \cdot}\ \mathit{MultiFoc}$$

[2] Straightforward extension of the one for MALL sequents.
[3] We do not need to take care of the premisses of !-rules since these sequents contain exactly *one* subformula of an active formula of the root: A is the only formula in the linear part of this sequent of the border.

with the proviso that during a multifocusing section, only positive rules can be applied: the negative rules that could be present would be frozen until all the positive formulas under focus have been decomposed.

There is much to do in order to understand precisely this notion of Multi-Focalization but we can already draw some comments:

- completeness is not an issue for multi-focalization since it extends focalization;
- more interesting would be to understand how to obtain proofs which have been multifocalized as much as possible. In particular, is there such an interesting notion of maximality in the world of multi-focused proofs?
- clearly, multi-focused proofs have a taste of concurrency: having F and G as foci actually means that we are focussing on the two formulas at the same time, even though we keep the sequent syntax. It would thus be pretty interesting to compare this with works on concurrent or asynchronous games [1];
- This notion of Multifocalization might have interesting consequences for proof search allowing, for instance, to detect failures of the proof search earlier.

6 Conclusion and Future Works

We have presented a new proof of the completeness of focused proofs for linear logic. We first focused on MALL fragment in which rather elementary considerations of the permutability of inference rules allowed us to define a focalization graph. The fact that such a graph is acyclic allows us to build sequent calculus proofs. There are many possibilities for building such proof: a flexible bias assignment mechanism allows edges to be added to the focalization graph, which, in turn, constrains the space of sequent calculus proofs that can be produced. The techniques developed for MALL can be lifted directly to providing focusing results much stronger logics, in particular, full first-order and higher-order linear logic and linear logic with fixed points. Given the centrality of the focalization graph and since such graphs may have more than one source, we have also considered adding to a focused proof system the multifocusing inference rule that can capture such multiplicity of foci.

The structure of Focalization Graph we introduced in this paper and the consideration of Focalization as a process for transforming proofs suggest we study several developments for future works:

- The interaction between Focalization process and cut-reduction shall be made clear;
- We would like to extend our results to richer logics such as non-commutative logics or light logics as a test for our methodology;
- We would be interested in adapting focalization result directly to logics such as LJ;
- The study of Multi-Focalization is a direction that seems to be fruitful and to relate focalization with interesting topics of concurrent view of proofs;
- In a more applied setting, we should pursue the classification of Bias Assignments since it seems to be meaningful for applications in proof search and other settings;

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