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FROM SCALAR TO VECTOR OPTIMIZATION

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Abstract. Initially, second-order necessary optimality conditions and sufficient optimality conditions in terms of Hadamard type derivatives for the unconstrained scalar optimization problem $\varphi(x) \to \min$, $x \in \mathbb{R}^m$, are given. These conditions work with arbitrary functions $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}}$, but they show inconsistency with the classical derivatives. This is a base to pose the question whether the formulated optimality conditions remain true when the "inconsistent" Hadamard derivatives are replaced with the "consistent" Dini derivatives. It is shown that the answer is affirmative if φ is of class $\mathcal{C}^{1,1}$ (i.e., differentiable with locally Lipschitz derivative).

Further, considering $\mathcal{C}^{1,1}$ functions, the discussion is raised to unconstrained vector optimization problems. Using the so called "oriented distance" from a point to a set, we generalize to an arbitrary ordering cone some second-order necessary conditions and sufficient conditions given by Liu, Neittaanmäki, Křížek for a polyhedral cone. Furthermore, we show that the conditions obtained are sufficient not only for efficiency but also for strict efficiency.

Keywords: scalar and vector optimization, $\mathcal{C}^{1,1}$ functions, Hadamard and Dini derivatives, second-order optimality conditions, Lagrange multipliers.

MSC 2000: 90C29, 90C30, 49J52

1. Introduction

In this paper we use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ for the extended real line. Let m be a positive integer and $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ a given function. Recall that the domain of φ is the set $\operatorname{dom}\varphi = \{x \in \mathbb{R}^m : \varphi(x) \neq \pm\infty\}$. The problem to find the local minima of φ (in general a nonsmooth function) is written down by

$$\varphi(x) \to \min, \quad x \in \mathbb{R}^m.$$

Optimality conditions in nonsmooth optimization are based on various definitions of directional derivatives. In the sequel, we give definitions of the first and second-order Hadamard and Dini derivatives. In terms of Hadamard derivatives Ginchev [10] gives second-order necessary optimality conditions and sufficient optimality conditions. We choose them as a starting point for our discussion because of the amazing property that they possess, namely they could be applied for quite arbitrary functions $\varphi\colon\ \mathbb{R}^m \to \overline{\mathbb{R}}$, while in contrast the known conditions in nonsmooth optimization usually assume in advance certain regularity of the optimized function φ, the usual prerequisite being that φ is a locally Lipschitz function with some additional properties.

The following "complementary principle" in nonsmooth optimization is welcome: *if the optimized function is a smooth one, then the optimality conditions should reduce to known classical optimality conditions.* Recall that the notion of "complementary principle" in physics is used in the sense that the laws of classical physics are obtained as limits from those of relativistic physics. Obviously, similar demand in optimization is not imperative and the inconsistency is not a ground to reject a meaningful theory. Nevertheless, consistency complies with our natural expectations.

Having in mind the above remark, in Sections 2, 3 and 4 we show that the defined second-order Hadamard derivatives do not coincide with the classical ones in the case of $\mathcal{C}^2$ function, while there is coincidence for the Dini derivatives. Taking into account that the classical second-order conditions involve rather the Dini derivatives, in the spirit of the "complementary principle", we pose the natural question whether the formulated second-order conditions remain true if the inconsistent Hadamard derivatives are replaced by the consistent Dini ones. In Example 1 we show that, in general, the answer is negative. This observation leads to another problem, namely to find a class $\mathcal{F}$ of functions for which the optimality conditions in Dini derivatives are true. We show that the class of $\mathcal{C}^{1,1}$ functions solves affirmatively this problem, while the same is not true for the class $\mathcal{C}^{0,1}$ (here $\mathcal{C}^{k,1}$ denotes the class of functions which are k times Fréchet differentiable with locally Lipschitz kth derivative). Considering $\mathcal{C}^{1,1}$ functions we move towards vector optimization problems. In Section 5, we give scalar characterizations of efficiency in terms of the "oriented distance function" from a point to a set. Section 6 generalizes (in the case of unconstrained vector optimization problems) to an arbitrary convex, closed and pointed ordering cone, the second-order optimality conditions obtained by Liu, Neittaanmäki, Křížek [22] for polyhedral cones. There, we state also sufficient conditions for strict efficiency [5]. Finally, Section 7 is devoted to some comparison with the results obtained by Guerraggio, Luc [15], and Bolintenéanu, El Maghri [6].

2. Directional derivatives and second-order conditions

Denote the unit sphere and the open unit ball in $\mathbb{R}^m$ by $S = \{x \in \mathbb{R}^m : \|x\| = 1\}$ and $B = \{x \in \mathbb{R}^m : \|x\| < 1\}$, respectively. Given $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}}$, $x^0 \in \operatorname{dom}\varphi$ and $u \in S$ (actually the same definitions hold for $u \in \mathbb{R}^m \setminus \{0\}$) we define the first and second-order lower directional Hadamard derivatives (for brevity we say just Hadamard derivatives) as follows. The first-order Hadamard derivative $\varphi'_H(x^0, u)$ takes values in $\overline{\mathbb{R}}$ and is defined by

$$\varphi'_H(x^0, u) = \liminf_{(t,v)\to(0+,u)} \frac{1}{t}(\varphi(x^0 + tv) - \varphi(x^0)).$$

Note that the difference on the right-hand side is well defined, since due to $x^0 \in \operatorname{dom}\varphi$ only $\varphi(x^0 + tv)$ could possibly take infinite values. The second-order Hadamard derivative $\varphi''_H(x^0, u)$ is defined only if the first-order derivative $\varphi'_H(x^0, u)$ takes a finite value. Then we put

$$\varphi''_H(x^0, u) = \liminf_{(t,v)\to(0+,u)} \frac{2}{t^2}(\varphi(x^0 + tv) - \varphi(x^0) - t\,\varphi'_H(x^0, u)).$$

The expression in the parentheses has sense, since only $\varphi(x^0 + tv)$ possibly takes an infinite value.

The first and second-order lower Dini directional derivatives (we call them just Dini derivatives) are defined similarly for $u \in S$ (and for arbitrary $u \in \mathbb{R}^m$) with the only difference that there is no variation in the directions. For the first-order Dini derivative we put

$$\varphi'_D(x^0, u) = \liminf_{t\to 0+} \frac{1}{t}(\varphi(x^0 + tu) - \varphi(x^0)).$$

The second-order Dini derivative $\varphi''_D(x^0, u)$ is defined only if the first-order derivative $\varphi'_D(x_0, u)$ takes a finite value; then

$$\varphi''_D(x^0, u) = \liminf_{t\to 0+} \frac{2}{t^2}(\varphi(x^0 + tu) - \varphi(x^0) - t\,\varphi'_D(x^0, u)).$$

The first-order derivatives $\varphi'_H(x^0, u)$ and $\varphi'_D(x^0, u)$ are considered, for instance, in Demyanov and Rubinov [9], who proposed the names of Hadamard and Dini derivatives. We use these names for the second-order derivatives given above, because the same type of convergence as in [9] is used. The definitions of the second-order derivatives look natural in the framework of what comes out if one solves the classical Taylor expansion formula of the second order for a twice differentiable function with respect to the second-order derivative. Hence, the second-order Hadamard and Dini derivatives are Peano-type derivatives [11], [27].

However, let us mention that another second-order directional derivative of Hadamard type can be obtained if in the above definition of the second-order Hadamard derivative the term $-t\varphi'_H(x^0, u)$ is replaced by $-t\varphi'_H(x^0, v)$. Although, because of the type of convergence used, such a derivative could also pretend to the name second-order Hadamard derivative, here it is beyond our interest, since with this new understanding of the second-order Hadamard derivative, Theorem 1 cited below fails to be true. Let us also remark that in Theorem 1 we assume that $u \in S$. When the Hadamard derivatives are restricted to $u \in S$, we may assume that by the lim inf in their definition, the convergence $(t, v) \to (0+, u)$ is such that v remains on S. Obviously, such a restriction does not change the values of $\varphi'_H(x^0, u)$ and $\varphi''_H(x^0, u)$.

Recall that $x^0 \in \mathbb{R}^m$ is said to be a local minimizer (we prefer to say simply minimizer) of φ, if for some neighbourhood U of x^0 $\varphi(x) \geqslant \varphi(x^0)$ holds for all $x \in U$. A minimizer is strong if this inequality is strong for $x \in U \setminus \{x^0\}$. It is said to be an isolated minimizer of order k (k is a positive integer) if there exists a constant $A > 0$ such that $\varphi(x) \geqslant \varphi(x^0) + A\|x - x^0\|^k$, $x \in U$. Obviously, each isolated minimizer is a strong minimizer. The concept of an isolated minimizer has been popularized by Auslender [4].

The next theorem states second-order necessary conditions and sufficient conditions in terms of Hadamard derivatives and is a particular case of a result proved in Ginchev [10].

Theorem 1. *Let $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ be an arbitrary function.*

(Necessary Conditions) *Let $x^0 \in \operatorname{dom}\varphi$ be a minimizer of φ. Then for each $u \in S$ the following two conditions hold:*

$$(\mathrm{N}'_H) \qquad \varphi'_H(x^0, u) \geqslant 0,$$

$$(\mathrm{N}''_H) \qquad \textit{if } \varphi'_H(x^0, u) = 0 \quad \textit{then} \quad \varphi''_H(x^0, u) \geqslant 0.$$

(Sufficient Conditions) *Let $x^0 \in \operatorname{dom}\varphi$. If for each $u \in S$ one of the following two conditions hold:*

$$(\mathrm{S}'_H) \qquad \varphi'_H(x^0, u) > 0,$$

$$(\mathrm{S}''_H) \qquad \varphi'_H(x^0, u) = 0 \quad \textit{and} \quad \varphi''_H(x^0, u) > 0,$$

then x^0 is a strong minimizer of φ. Moreover, these conditions are necessary and sufficient for x^0 to be an isolated minimizer of the second order.

3. Consistency with the classical derivatives

The astonishing property of Theorem 1 is that it works with quite arbitrary functions $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}}$. Further, the sufficiency characterizes the isolated minimizers of the second order of such functions, that is, the sufficient conditions are both necessary and sufficient for a point to be an isolated minimizer. The next example illustrates an application of this theorem.

E x a m p l e 1. Consider

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}, \quad \varphi(x_1, x_2) = \max(-2x_1^2 + x_2, x_1^2 - x_2). \tag{1}$$

Then the point $x^0 = (0, 0)$ is not a minimizer, which is seen by observing that the necessary conditions from Theorem 1 are not satisfied. In fact, for nonzero $u = (u_1, u_2)$ we have

$$\varphi'_H(x^0, u) = \varphi'_D(x^0, u) = \begin{cases} u_2, & u_2 > 0, \\ 0, & u_2 = 0, \\ -u_2, & u_2 < 0, \end{cases}$$

$$\varphi''_H(x^0, u) = -4u_1^2 \quad \text{and} \quad \varphi''_D(x^0, u) = 2u_1^2 \quad \text{for } u_2 = 0.$$

Therefore, $\varphi'_H(x^0, u) \geqslant 0$ for all $u \in S$, that is, the first-order necessary condition N'_H is satisfied. However, for $u = (u_1, u_2) = (\pm 1, 0)$ we have $\varphi'_H(x^0, u) = 0$ and $\varphi''_H = -4u_1^2 < 0$, hence the second-order necessary condition N''_H is not satisfied.

In Example 1 we calculated the Dini derivatives for the purpose of comparison. It falls into eyes immediately that the second-order derivatives, in general, are different: $\varphi''_H(x^0, u) \neq \varphi''_D(x^0, u)$. The next simple example shows that such a difference occurs even for $\mathcal{C}^2$ functions. The genesis of this difference is in the definition of the Hadamard derivatives, where in the lim inf the convergence $(t, v) \to (0+, u)$ means an independent convergence $t \to 0+$ and $v \to u$.

E x a m p l e 2. For the function

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}, \quad \varphi(x_1, x_2) = x_1,$$

and nonzero $u = (u_1, u_2)$ we have $\varphi'_H(x, u) = \varphi'_D(x, u) = u_1$, whereas $\varphi''_H(x, u) = -\infty$ differs from $\varphi''_D(x, u) = 0$.

Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be twice differentiable at x. In this case the gradient of φ at x is denoted by $\varphi'(x)$ and the Hessian by $\varphi''(x)$. Then the classical first and second-order directional derivatives of φ are

$$\varphi'(x,u) = \lim_{t\to 0+} \frac{1}{t}(\varphi(x+tu) - \varphi(x)) = \varphi'(x)u$$

and

$$\varphi''(x,u) = \lim_{t\to 0+} \frac{2}{t^2}(\varphi(x+tu) - \varphi(x) - t\,\varphi'(x,u)) = \varphi''(x)(u,u).$$

The classical optimality conditions can be obtained from Theorem 1 by replacing the Hadamard derivatives by the classical directional derivatives. Actually, simple reasoning shows that in this case the first-order conditions are replaced by the assumption for stationarity $\varphi'(x^0,u) = 0$ for all $u \in S$ or equivalently by $\varphi'(x^0) = 0$.

It is easily seen that for a twice differentiable at x^0 functions $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ the classical first and second-order directional derivatives coincide with the Dini derivatives of the first and second order, respectively, i.e. $\varphi'(x)u = \varphi'_D(x,u)$ and $\varphi''(x)(u,u) = \varphi''_D(x,u)$. Therefore, the following problem arises naturally as an attempt to generalize the classical optimality conditions in a consistency preserving way.

P r o b l e m 1. Determine a class $\mathcal{F}$ of functions $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ such that Theorem 1 with Hadamard derivatives replaced by the respective Dini derivatives holds true for all functions $\varphi \in \mathcal{F}$.

The classical second-order conditions show that the class of twice differentiable functions solves this problem. We show in Section 4 that the class of $\mathcal{C}^{1,1}$ functions also solves this problem, while this is not true for the class of $\mathcal{C}^{0,1}$ functions.

Actually, Problem 1 concerns only the sufficient conditions as one sees from the following remark.

R e m a r k 1. The necessary conditions for Dini derivatives remain true for an arbitrary class $\mathcal{F}$, which follows from the following reasoning. If x^0 is a minimizer for $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and $u \in S$ then $t^0 = 0$ is a minimizer for the function $\varphi_u\colon \mathbb{R} \to \mathbb{R}$, $\varphi_u(t) = \varphi(x^0 + tu)$. We can write the necessary conditions from Theorem 1 for the function φ_u. The Hadamard derivatives for φ_u in direction 1 however coincide with the respective Dini derivatives for φ in direction u, whence we see that the necessary conditions for Dini derivatives are satisfied.

4. Optimization of $\mathcal{C}^{1,1}$ functions

The function in Example 1 satisfies the "sufficient conditions" in terms of Dini instead of Hadamard derivatives as is seen from the Dini derivatives calculated there. Hence, the sufficiency with Dini instead of Hadamard derivatives does not hold for arbitrary functions φ, which is why we need to restrict the class $\mathcal{F}$ of the functions considered in order to get a solution of Problem 1.

Recall that in practical optimization the classes $\mathcal{C}^{0,1}$ and $\mathcal{C}^{1,1}$ play an important role. A function $\varphi\colon\ \mathbb{R}^m \to \mathbb{R}$ is said to be of class $\mathcal{C}^{0,1}$ on $\mathbb{R}^m$ if it is locally Lipschitz on $\mathbb{R}^m$. It is said to be of class $\mathcal{C}^{1,1}$ on $\mathbb{R}^m$ if it is differentiable at each point $x \in \mathbb{R}^m$ and its gradient $\varphi'(x)$ is locally Lipschitz on $\mathbb{R}^m$. Similarly one defines functions of class $\mathcal{C}^{0,1}$ or $\mathcal{C}^{1,1}$ having as their domain an open set $X \subset \mathbb{R}^m$. The case of an open proper subset $X \subset \mathbb{R}^m$ does not introduce new elements in our discussion, which is why we confine to $X = \mathbb{R}^m$.

Let us underline that for a $\mathcal{C}^{1,1}$ function φ in the definition of $\varphi''_D(x^0,u)$ in Section 2 the term $\varphi'_D(x^0,u)$ is replaced by $\varphi'(x^0)u$. In the sequel we discuss whether we get a solution of Problem 1 by taking for $\mathcal{F}$ one of the classes $\mathcal{C}^{0,1}$ and $\mathcal{C}^{1,1}$. The function in Example 1 is of class $\mathcal{C}^{0,1}$ being the maximum of $\mathcal{C}^2$ functions. Therefore the class $\mathcal{C}^{0,1}$ does not solve the posed problem. We show, however, in Theorem 2 that the class $\mathcal{C}^{1,1}$ is a solution of this problem. For the proof we need the following lemma.

Lemma 1. *Let $\varphi\colon\ \mathbb{R}^m \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$ function. Let φ' be Lipschitz with a constant L in $x^0 + r\,\mathrm{cl}\,B$, where $x^0 \in \mathbb{R}^m$ and $r > 0$. Then, for $u, v \in \mathbb{R}^m$ and $0 < t < \min(r/\|u\|, r/\|v\|)$ we have*

$$(2)\quad \Big|\frac{2}{t^2}(\varphi(x^0+tv)-\varphi(x^0)-t\varphi'(x^0)v) - \frac{2}{t^2}(\varphi(x^0+tu)-\varphi(x^0)-t\varphi'(x^0)u)\Big| \leqslant L(\|u\|+\|v\|)\|v-u\|$$

and, consequently,

$$(3)\quad |\varphi''_D(x^0,v)-\varphi''_D(x^0,u)| \leqslant L(\|u\|+\|v\|)\|v-u\|.$$

For $v = 0$ we get

$$\Big|\frac{2}{t^2}(\varphi(x^0+tu)-\varphi(x^0)-t\varphi'(x^0)u)\Big| \leqslant L\|u\|^2$$

and

$$|\varphi''_D(x^0,u)| \leqslant L\|u\|^2.$$

In particular, if $\varphi'(x^0) = 0$ *then inequality* (2) *implies*

$$\left|\frac{2}{t^2}(\varphi(x^0+tv) - \varphi(x^0+tu))\right| \leqslant L(\|u\| + \|v\|)\|v-u\|. \tag{4}$$

P r o o f. For $t \in (0, r)$ we have

$$\begin{aligned}
&\frac{2}{t^2}(\varphi(x^0+tv) - \varphi(x^0) - t\varphi'(x^0)v) \\
&\qquad = \frac{2}{t^2}\big((\varphi(x^0+tv) - \varphi(x^0+tu)) - t\varphi'(x^0)(v-u)\big) \\
&\qquad\quad + \frac{2}{t^2}(\varphi(x^0+tu) - \varphi(x^0) - t\varphi'(x^0)u) \\
&\qquad = \frac{2}{t}(v-u)\int_0^1 (\varphi'(x^0+(1-s)tu+stv) - \varphi'(x^0))\,\mathrm{d}s \\
&\qquad\quad + \frac{2}{t^2}(\varphi(x^0+tu) - \varphi(x^0) - t\varphi'(x^0)u) \\
&\qquad \leqslant 2L\|v-u\|\int_0^1 \|(1-s)u+sv\|\,\mathrm{d}s \\
&\qquad\quad + \frac{2}{t^2}(\varphi(x^0+tu) - \varphi(x^0) - t\varphi'(x^0)u) \\
&\qquad \leqslant 2L\|v-u\|\int_0^1 ((1-s)\|u\|+s\|v\|)\,\mathrm{d}s \\
&\qquad\quad + \frac{2}{t^2}(\varphi(x^0+tu) - \varphi(x^0) - t\varphi'(x^0)u) \\
&\qquad = L(\|u\|+\|v\|)\|v-u\| + \frac{2}{t^2}(\varphi(x^0+tu) - \varphi(x^0) - t\varphi'(x^0)u).
\end{aligned}$$

Interchanging u and v, we get inequality (2) and as a particular case also (4). Inequality (3) is obtained from (2) after passing to the limit. □

Theorem 2. *Let* $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ *be a* $\mathcal{C}^{1,1}$ *function.*

(N e c e s s a r y C o n d i t i o n s) *Let* x^0 *be a minimizer of* φ. *Then* $\varphi'(x^0) = 0$ *and for each* $u \in S$ *we have* $\varphi''_D(x^0, u) \geqslant 0$.

(S u f f i c i e n t C o n d i t i o n s) *Let* $x^0 \in \mathbb{R}^m$ *be a stationary point, that is* $\varphi'(x^0) = 0$. *If for each* $u \in S$ *we have* $\varphi''_D(x^0, u) > 0$ *then* x^0 *is an isolated minimizer of the second order for* φ. *Conversely, each isolated minimizer of the second order satisfies these sufficient conditions.*

P r o o f. Necessity is satisfied according to Remark 1. Since φ is differentiable, we have only to observe that $\varphi'(x^0, -u) = \varphi'(x^0)(-u) = -\varphi'(x^0)(u) = -\varphi'(x^0, u)$, whence if both $\varphi'(x^0, -u) \geqslant 0$ and $\varphi'(x^0, u) \geqslant 0$ hold, we get $\varphi'(x^0, u) = \varphi'(x^0)u = 0$. If the equality $\varphi'(x^0)u = 0$ holds for all $u \in S$, then $\varphi'(x^0) = 0$.

Now we prove sufficiency. Let φ' be Lipschitz with a constant L in the ball $x^0 + r\,\mathrm{cl}\,B$, $r > 0$. Let $u \in S$ and $0 < 3\varepsilon(u) < \varphi''_D(x^0, u)$. Choose $0 < \delta(u) < r$ such that for $0 < t < \delta(u)$ we have

$$\frac{2}{t^2}(\varphi(x^0 + tu) - \varphi(x^0) - t\varphi'(x^0)u) > \varphi''_D(x^0, u) - \varepsilon(u).$$

Put also $U(u) = u + (\varepsilon(u)/2L)B$ and let $v \in U(u) \cap S$. Then applying Lemma 1, we get

$$\begin{aligned} \frac{2}{t^2}(\varphi(x^0 + tv) - \varphi(x^0)) &= \frac{2}{t^2}(\varphi(x^0 + tu) - \varphi(x^0)) + \frac{2}{t^2}(\varphi(x^0 + tv) - \varphi(x^0 + tu)) \\ &\geqslant \frac{2}{t^2}(\varphi(x^0 + tu) - \varphi(x^0) - t\varphi'(x^0)u) - 2L\|v - u\| \\ &\geqslant \varphi''_D(x^0, u) - 2\varepsilon(u) > \varepsilon(u) > 0. \end{aligned}$$

Therefore,

$$\varphi(x^0 + tv) \geqslant \varphi(x^0) + \frac{1}{2}(\varphi''_D(x^0, u) - 2\varepsilon(u))t^2.$$

The compactness of S yields that $S \subset U(u^1) \cup \ldots \cup U(u^n)$ for some $u^1, \ldots, u^n \in S$. Put $\delta = \min \delta(u^i)$, $A = \min \frac{1}{2}(\varphi''_D(x^0, u^i) - 2\varepsilon(u^i))$. The above chain of inequalities implies that $\varphi(x) \geqslant \varphi(x^0) + A\|x - x^0\|^2$ holds for $\|x - x^0\| < \delta$. Therefore, x^0 is an isolated minimizer of the second order.

Conversely, if x^0 is an isolated minimizer of the second order for a $\mathcal{C}^{1,1}$ function φ, then from the necessary conditions $\varphi'(x^0) = 0$. Further, for some $A > 0$, $t > 0$ sufficiently small and $u \in S$, we have $\varphi(x^0 + tu) \geqslant \varphi(x^0) + At^2$, whence

$$\frac{2}{t^2}(\varphi(x^0 + tu) - \varphi(x^0) - t\varphi'(x^0)u) \geqslant 2A$$

and consequently, $\varphi''_D(x^0, u) \geqslant 2A > 0$. □

Remark 2. If $\varphi''_D(x^0, u)$ is defined as the set of all cluster points of $(2/t^2)(\varphi(x^0 + tu) - \varphi(x^0) - t\varphi'(x^0)u)$ as $t \to 0+$, then Lemma 1 and Theorem 2 can be restated in terms of this new definition, since the same proof can be applied to this new interpretation. This new point of view fits better to the vector case studied in Section 6.

If $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is a $\mathcal{C}^{1,1}$ function, then φ' is locally Lipschitz and according to the Rademacher theorem the Hessian φ'' exists almost everywhere. Then, the second-order subdifferential of φ at x^0 is defined by

$$\partial^2\varphi(x^0) = \mathrm{cl\,conv}\{\lim \varphi''(x^i)\colon\ x^i \to x^0,\ \varphi''(x^i)\ \text{ exists}\}.$$

The $\mathcal{C}^{1,1}$ functions in optimization and the second-order optimality conditions have been introduced in Hiriart-Urruty, Strodiot, Hien Nguen [16]. Thereafter an intensive study of various aspects of $\mathcal{C}^{1,1}$ functions has been undertaken (see for instance Klatte, Tammer [19], Liu [20], Yang, Jeyakumar [31], Yang [29], [30], Liu, Křížek [21], Liu, Neittaanmäki, Křížek [22]). The Taylor expansion formula and necessary conditions for $\mathcal{C}^{k,1}$ functions, i.e., functions having the kth order locally Lipschitz derivative, have been generalized in Luc [24]. The following result is proved in Guerraggio, Luc [15].

Theorem 3. *Let $\varphi\colon\ \mathbb{R}^m \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$ function.*

(Necessary Conditions) *Let x^0 be a minimizer of φ. Then $\varphi'(x^0) = 0$ and for each $u \in S$ there exists $\zeta \in \partial^2\varphi(x^0)$ such that $\zeta(u,u) \geqslant 0$.*

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^m$. If $\varphi'(x^0) = 0$ and if $\zeta(u,u) > 0$ for all $u \in S$ and $\zeta \in \partial^2\varphi(x^0)$, then x^0 is a minimizer of φ.*

For functions of class $\mathcal{C}^2$ Theorem 3 obviously coincides with the classical second-order conditions. However, already for twice differentiable but not $\mathcal{C}^2$ functions the hypothesis of the sufficient conditions of Theorem 3 fails to be true, which is seen in the next example.

Example 3. The function $\varphi\colon\ \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x^4 \sin \dfrac{1}{x} + ax^2, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is twice differentiable but is not a $\mathcal{C}^2$ function. Its first and second derivatives are given by

$$\varphi'(x) = \begin{cases} 4x^3 \sin \dfrac{1}{x} - x^2 \cos \dfrac{1}{x} + 2ax, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$\varphi''(x) = \begin{cases} 12x^2 \sin \dfrac{1}{x} - 6x \cos \dfrac{1}{x} - \sin \dfrac{1}{x} + 2a, & x \neq 0, \\ 2a, & x = 0. \end{cases}$$

For $a > 0$ the point $x^0 = 0$ is an isolated minimizer of the second order. The sufficient conditions at x^0 of Theorem 2 are satisfied. At the same time $\partial^2\varphi(0) = [-1+2a,\, 1+2a]$ and, therefore, for $0 < a < 1/2$ the hypotheses of the sufficient conditions of Theorem 3 are not satisfied, although x^0 is an isolated minimizer of second order.

5. Scalar characterization of vector optimality concepts

Our purpose from now on is to generalize the result of Theorem 2 from scalar to vector optimization. In this section we introduce optimality concepts for vector optimization problems and give some scalar characterizations.

We consider a vector function $f\colon\ \mathbb{R}^m \to \mathbb{R}^n$. We denote by $\|\cdot\|$, S, B and $\langle\cdot,\cdot\rangle$, respectively the norm, the unit sphere, the open unit ball and the scalar product, both in the domain and the image space, since from the context it will be clear which of the two spaces is considered.

Further, C is a given pointed closed convex cone in $\mathbb{R}^n$. We deal with the minimization problem

$$f(x) \to \min, \quad x \in \mathbb{R}^m. \tag{5}$$

There are different concepts of solutions of this problem. A point x^0 is said to be a weakly efficient (efficient) point, if there is a neighbourhood U of x^0 such that if $x \in U$ then $f(x) - f(x^0) \notin -\operatorname{int} C$ (respectively, $f(x) - f(x^0) \notin -(C \setminus \{0\})$). A point x^0 is said to be properly efficient if there exists a pointed closed convex cone $\tilde{C}$ such that $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$ and x^0 is a weakly efficient point with respect to $\tilde{C}$. In this paper the weakly efficient, the efficient and the properly efficient points will be called w-minimizers, e-minimizers and p-minimizers, respectively.

Each p-minimizer is an e-minimizer, which follows from the implication $f(x) - f(x^0) \notin -\operatorname{int} \tilde{C} \Rightarrow f(x) - f(x^0) \notin -(C \setminus \{0\})$, a consequence of $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$. Each e-minimizer is a w-minimizer, which follows from the implication $f(x) - f(x^0) \notin -(C \setminus \{0\}) \Rightarrow f(x) - f(x^0) \notin -\operatorname{int} C$, a consequence of $\operatorname{int} C \subset C \setminus \{0\}$.

Let us point out that we do not assume in advance that $\operatorname{int} C \neq \emptyset$. If $\operatorname{int} C = \emptyset$, then according to our definition each point x^0 is a w-minimizer. In the case $\operatorname{int} C = \emptyset$ we can define x^0 to be a relative weakly efficient point, and call it an rw-minimizer, if $f(x) - f(x^0) \notin -\operatorname{ri} C$. Here $\operatorname{ri} C$ stands for the relative interior of C. However, in the sequel we will not use rw-minimizers.

For a cone $K \subset \mathbb{R}^n$ its positive polar cone K' is defined by $K' = \{\xi \in \mathbb{R}^n :\ \langle \xi, y\rangle \geqslant 0 \ \text{for all}\ y \in K\}$. The cone K' is closed and convex. It is well known that $K'' := (K')' = \operatorname{cl} \operatorname{co} K$, see e.g. Rockafellar [28, Chapter III, § 15]. In particular, for the closed convex cone C we have $C' = \{\xi \in \mathbb{R}^n :\ \langle \xi, y\rangle \geqslant 0 \ \text{for all}\ y \in C\}$ and $C = C'' = \{y \in \mathbb{R}^n :\ \langle \xi, y\rangle \geqslant 0 \ \text{for all}\ \xi \in C'\}$.

The relation of the vector optimization problem (5) to some scalar optimization problem can be obtained in terms of the positive polar cone of C.

Proposition 1. *A point $x^0 \in \mathbb{R}^m$ is a w-minimizer of $f\colon\ \mathbb{R}^m \to \mathbb{R}^n$ with respect to a pointed closed convex cone C if and only if x^0 is a minimizer of the scalar* *function*

$$(6)\qquad \varphi(x) = \max\{\langle \xi, f(x) - f(x^0)\rangle \colon \ \xi \in C', \|\xi\| = 1\}.$$

P r o o f. If $\operatorname{int} C = \emptyset$ then each point $x^0 \in \mathbb{R}^m$ is a w-minimizer. At the same time, since C' contains at least one pair of opposite unit vectors $\hat{\xi}$, $-\hat{\xi}$, for each $x \in \mathbb{R}^m$ we have

$$\varphi(x) \geqslant \max(\langle \hat{\xi}, f(x) - f(x^0)\rangle, -\langle \hat{\xi}, f(x) - f(x^0)\rangle) = |\langle \hat{\xi}, f(x) - f(x^0)\rangle| \geqslant 0 = \varphi(x^0),$$

i.e., φ has a minimum at x^0.

Assume now that $\operatorname{int} C \neq \emptyset$ and let x^0 be a w-minimizer. Let U be the neighbourhood from the definition of the w-minimizer and fix $x \in U$. Then $f(x) - f(x^0) \notin -\operatorname{int} C \neq \emptyset$. By the well known Separation Theorem there exists $\xi_x \in \mathbb{R}^n$, $\|\xi_x\| = 1$, such that $\langle \xi_x, f(x) - f(x^0)\rangle \geqslant 0$ and $\langle \xi_x, -y\rangle = -\langle \xi_x, y\rangle \leqslant 0$ for all $y \in C$. The latter inequality shows that $\xi_x \in C'$ and the former shows that $\varphi(x) \geqslant \langle \xi_x, f(x) - f(x^0)\rangle \geqslant 0 = \varphi(x^0)$. Thus $\varphi(x) \geqslant \varphi(x^0)$, $x \in U$, and therefore, x^0 is a minimizer of φ.

Let now x^0 be a minimizer of φ. Choose a neighbourhood U of x^0 such that $\varphi(x) \geqslant \varphi(x^0)$, $x \in U$, and fix $x \in U$. Then there exists $\xi_x \in C'$, $\|\xi_x\| = 1$, such that $\varphi(x) = \langle \xi_x, f(x) - f(x^0)\rangle \geqslant \varphi(x^0) = 0$ (here we use the compactness of the set $\{\xi \in C' \colon \ \|\xi\| = 1\}$. From $\xi_x \in C'$ it follows that $\langle \xi_x, -y\rangle < 0$, $y \in \operatorname{int} C$. Therefore, $f(x) - f(x^0) \notin -\operatorname{int} C$. Consequently, x^0 is w-minimizer. □

We call x^0 a strong e-minimizer if there is a neighbourhood U of x^0 such that $f(x) - f(x^0) \notin -C$ for $x \in U \setminus \{x^0\}$. Obviously, each strong e-minimizer is an e-minimizer. The following characterization of the strong e-minimizers holds (the proof is omitted, since it nearly repeats the reasoning from the proof of Proposition 1).

Proposition 2. *A point $x^0 \in \mathbb{R}^m$ is a strong e-minimizer of $f\colon \mathbb{R}^m \to \mathbb{R}^n$ with respect to a pointed closed convex cone C if and only if x^0 is a strong minimizer of the scalar function* (6).

The p-minimizers admit the following characterization.

Proposition 3. *A point $x^0 \in \mathbb{R}^m$ is a p-minimizer of $f\colon \mathbb{R}^m \to \mathbb{R}^n$ with respect to a pointed closed convex cone C if and only if there exists a nontrivial closed convex cone $\tilde{C}'$ such that $\tilde{C}' \setminus \{0\} \subset \operatorname{int} C'$ and x^0 is a minimizer of the scalar function*

$$(7)\qquad \tilde{\varphi}(x) = \max\{\langle \xi, f(x) - f(x^0)\rangle \colon \ \xi \in \tilde{C}', \ \|\xi\| = 1\}.$$

P r o o f. Let us mention that in this case we put the positive polar of $\tilde{C}'$ for the cone $\tilde{C}$ required in the definition of a p-minimizer, i.e., $\tilde{C} := (\tilde{C}')'$. Then, since $\tilde{C}'$ is a closed convex cone, it follows that $\tilde{C}'$ is the positive polar cone of $\tilde{C}$. Indeed, we have $\tilde{C}' = ((\tilde{C}')')' = (\tilde{C})'$. This observation justifies the consistency of the notation. The inclusion $\tilde{C}' \setminus \{0\} \subset \operatorname{int} C'$ is equivalent to $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$ and according to Proposition 1 the point x^0 is a w-minimizer of f with respect to $\tilde{C}$ if and only if x^0 is a minimizer of the function (7). □

Proposition 1 claims that the statement x^0 is a *w-minimizer of* (5) is equivalent to the statement x^0 *is a minimizer of the scalar function* (6). Applying some first or second-order sufficient optimality conditions to check the latter, we usually get more, namely that x^0 *is an isolated minimizer of the first and second order of* (6), respectively. It is natural now to introduce the following concept of optimality for the vector problem (5):

Definition 1. We say that x^0 is an isolated minimizer of order k for a vector function f if it is an isolated minimizer of order k for the scalar function (6).

To interpret geometrically the property that x^0 is a minimizer for f of certain type we introduce the so called oriented distance. Given a set $A \subset Y := \mathbb{R}^n$, then the distance from $y \in \mathbb{R}^n$ to A is given by $d(y, A) = \inf\{\|a - y\| \colon a \in A\}$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, Y \setminus A)$. Saying *oriented distance* one may think of a generalization of the well-known oriented distance from a point to an oriented plane with a given normal (here we rather prefer to relate this oriented distance to the half-space with a given outer normal). The function D is introduced in Hiriart-Urruty [17], [18] and is used later in Ciligot-Travain [7], Amahroq, Taa [3], Miglierina [25], Miglierina, Molho [26]. Zaffaroni [32] gives different notions of efficiency and uses the function D for their scalarization and comparison. Ginchev, Hoffmann [14] use the oriented distance to study approximation of set-valued functions by single-valued ones and in the case of a convex set A show the representation $D(y, A) = \sup\limits_{\|\xi\|=1} \big(\inf\limits_{a \in A} \langle \xi, a\rangle - \langle \xi, y\rangle\big)$. In particular, from this representation, provided C is a convex cone and taking into account

$$\inf_{a \in C} \langle \xi, a\rangle = \begin{cases} 0, & \xi \in C', \\ -\infty, & \xi \notin C', \end{cases}$$

we get easily

$$D(y, C) = \sup_{\|\xi\|=1,\, \xi \in C'} (-\langle \xi, y\rangle), \quad D(y, -C) = \sup_{\|\xi\|=1,\, \xi \in C'} (\langle \xi, y\rangle).$$

In particular, putting $y = f(x) - f(x^0)$, we obtain a representation of the function (6) in terms of the oriented distance: $\varphi(x) = D(f(x) - f(x^0), -C)$. Now Propositions 1–3 can be easily reformulated in terms of the oriented distance and in the new formulation they are geometrically more evident. With the assumptions made there, we get the conclusions

$$\begin{aligned} x^0 \text{ is a } w\text{-minimizer} &\Leftrightarrow D(f(x)-f(x^0),-C) \geqslant 0 \quad \text{for } x \in U,\\ x^0 \text{ is a strict } e\text{-minimizer} &\Leftrightarrow D(f(x)-f(x^0),-C) > 0 \quad \text{for } x \in U \setminus \{x^0\},\\ x^0 \text{ is a } p\text{-minimizer} &\Leftrightarrow D(f(x)-f(x^0),-\tilde{C}) \geqslant 0 \quad \text{for } x \in U. \end{aligned}$$

The definition of the isolated minimizers gives

$$x^0 \text{ is an isolated minimizer of order } k \Leftrightarrow D(f(x)-f(x^0),-C) \geqslant A\|x-x^0\|^k \quad \text{for } x \in U.$$

Now we see that the isolated minimizers (of a positive order) are strong e-minimizers. The next proposition gives a relation between the p-minimizers and the isolated minimizers of the first order (the proof can be found in [13]).

Proposition 4. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz at x^0. If x^0 is an isolated minimizer of the first order, then x^0 is a p-minimizer (with respect to the same pointed closed convex cone C).*

Let us note that the proposition fails to be true if we replace the property of x^0 being an isolated minimizer of the first order by x^0 being an isolated minimizer of the second order. In fact, the property of x^0 being an isolated minimizer of the second order can be considered to be a generalization of the property of x^0 being a p-minimizer. Namely, isolated minimizers of the second order are related to strictly efficient points.

Definition 2 (see [5]). A point x^0 is said to be locally strictly efficient provided there exists a neighborhood U of x^0 such that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$(f(x)-f(x^0)) \cap (\delta B - C) \subseteq \varepsilon B, \quad \forall x \in U.$$

We will refer to (locally) strictly efficient points as to s-minimizers of f. It is known [33] that each p-minimizer is also an s-minimizer and each s-minimizer is an e-minimizer. Hence, strictly efficient points form an intermediate class between the efficient and properly efficient points. The following proposition can be found in [8].

Proposition 5. *Let f be a continuous function. If x^0 is an isolated minimizer of the second order for f, then x^0 is an s-minimizer.*

Let C be a proper closed convex cone with $\operatorname{int} C \neq \emptyset$. Then its positive polar C' is a pointed closed convex cone. Recall that a set Ξ is a base for C' if Ξ is convex with $0 \notin \Xi$ and $C' = \operatorname{cone} \Xi := \{y\colon\ y = \lambda\xi,\ \lambda \geqslant 0,\ \xi \in \Xi\}$. The fact that C' is pointed closed convex and the finite dimensional setting imply that C' possesses a compact base Ξ and

$$0 < \alpha = \min\{\|\xi\|\colon\ \xi \in \Xi\} \leqslant \max\{\|\xi\|\colon\ \xi \in \Xi\} = \beta < +\infty. \tag{8}$$

Further, let us assume that Ξ_0 is a compact set such that $\Xi = \operatorname{conv} \Xi_0$. With help of Ξ_0 we define the function

$$\varphi_0(x) = \max\{\langle \xi, f(x) - f(x^0)\rangle\colon\ \xi \in \Xi_0\}. \tag{9}$$

Proposition 6. *Propositions* 1–5 *and Definition* 1 *remain true provided the function* (6) (*or equivalently the oriented distance $D(f(x) - f(x^0), -C)$*) *is replaced by the function* (9).

P r o o f. From (8) we get in a routine way the inequalities

$$\alpha\varphi(x) \leqslant \varphi_0(x) \leqslant \beta\varphi(x) \quad \text{if } \varphi(x) \geqslant 0$$

and

$$\beta\varphi(x) \leqslant \varphi_0(x) \leqslant \alpha\varphi(x) \quad \text{if } \varphi(x) < 0$$

whence we see that in Propositions 1–5 and Definition 1 the same properties are possessed both by φ and φ_0 (e.g. in Proposition 1 the point x^0 is a w-minimizer iff x^0 is a minimizer of φ, which to due the shown inequalities is equivalent to x^0 being a minimizer of φ_0). □

Corollary 1. *In the important case $C = \mathbb{R}^n_+$ the function* (6) *can be replaced by the maximum of the coordinates*

$$\varphi_0(x) = \max_{1 \leqslant i \leqslant n} (f_i(x) - f_i(x^0)). \tag{10}$$

P r o o f. Clearly, $C' = \mathbb{R}^n_+$ has a base $\Xi = \operatorname{conv} \Xi_0$, where $\Xi_0 = \{e^1, \ldots, e^n\}$ are the unit vectors on the coordinate axes. With this set we get immediately that the function (9) is in fact (10). □

More generally, the cone C is said to be polyhedral if $C' = \operatorname{cone} \Xi_0$ with a finite set of nonzero vectors $\Xi_0 = \{\xi_1, \dots, \xi_k\}$. In this case, similarly to Corollary 1, the function (9) is the maximum of a finite number of functions

$$\varphi_0(x) = \max_{1\leqslant i\leqslant k} \langle \xi_i, f_i(x) - f_i(x^0)\rangle.$$

The reduction of the vector optimization problem to a scalar one allows to make conclusions from the scalar case, which is demonstrated by the next example.

E x a m p l e 4. Consider the optimization problem (5) for

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (-2x_1^2 + x_2, \, x_1^2 - x_2) \tag{11}$$

with respect to $C = \mathbb{R}^2_+$. Then the scalar function (10) for $x^0 = (0, 0)$ reduces to the function φ from Example 1. Therefore, by virtue of Theorem 1 it can be established in terms of Hadamard derivatives that x^0 is not a minimizer (the second-order necessary conditions are not satisfied). Similar second-order "necessary conditions" in Dini derivatives are satisfied, but they do not imply that x^0 is a minimizer.

This example is a source of some speculations. The function f in (11) possesses a continuous second-order Fréchet derivative, but for $x^0 = (0, 0)$ the respective scalar function φ in (1) is only $\mathcal{C}^{0,1}$ and consequently, it does not allow application of "more smooth" optimality conditions, say the ones like those of Theorem 2. This observation shows that even a smooth vector optimization problem exhibits a nonsmooth nature, that is, it suffers the nonsmooth effect of the corresponding scalar representation. Further, in order to take advantage of the differentiability of the vector function, it is better to formulate optimality conditions directly to the vector problem instead of to the scalar representation established in Proposition 1. The next section is devoted to this task.

6. The vector problem

This section generalizes the second-order conditions from Section 4 from scalar to vector optimization. The scalar experience suggests us to deal with Dini derivatives, because of the inconsistency of the Hadamard ones. Although the literature on the second-order theory in vector optimization is rather limited, lately there has been growing interest in the subject, see e.g. Aghezzaf [1], Bolintenéanu, El Maghri [6], Guerraggio, Luc [15], Liu, Neittaanmäki, Křížek [22]. Our main result is Theorem 5 which, for an unconstrained vector optimization problem, turns out to be an extension of Theorems 3.1 and 3.3 in [22]. We will devote a subsequent paper to the case of a constrained problem.

6.1 Optimality conditions in primal form

Saying that $y \in \mathbb{R}^n$, we accept that this point is $y = (y_1, \dots, y_n)$. Similarly, a point $x \in \mathbb{R}^m$ is $x = (x_1, \dots, x_m)$, a point $\xi \in \mathbb{R}^n$ is $\xi = (\xi_1, \dots, \xi_n)$, and a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is $f = (f_1, \dots, f_n)$.

We say that a vector function f is $\mathcal{C}^{1,1}$ if all of its components are $\mathcal{C}^{1,1}$. Equivalently, f is $\mathcal{C}^{1,1}$ if it is Fréchet differentiable with locally Lipschitz derivative f'.

Wishing, like in Theorem 2, to exploit Dini derivatives, we need first to define them for a vector function $f\colon \mathbb{R}^m \to \mathbb{R}^n$. We confine in fact to a $\mathcal{C}^{1,1}$ function f. The first Dini derivative as in the scalar case is the usual directional derivative

$$f'_D(x, u) = \lim_{t \to 0+} \frac{1}{t}(f(x + tu) - f(x)) = f'(x^0)u.$$

The second derivative $f''_D(x^0, u)$ was introduced in [22] and is defined as the set of the cluster points of $(2/t^2)(f(x^0 + tu) - f(x^0) - tf'(x^0)u)$ when $t \to 0+$, or in other words as the Kuratowski upper limit set

$$f''_D(x^0, u) = \operatorname*{Limsup}_{t \to 0+} \frac{2}{t^2}(f(x^0 + tu) - f(x^0) - tf'(x^0)u).$$

This definition is convenient for the vector case, but differs from the definition of the second-order Dini derivative in the scalar case, which was commented in Remark 2.

The next theorem gives second-order necessary conditions for the vector optimization problem (5).

Theorem 4. *Assume that $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^{1,1}$ function minimized with respect to a pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$.*

(N e c e s s a r y C o n d i t i o n s) *Let x^0 be a w-minimizer of f. Then for each $u \in S$ the following two conditions are satisfied:*

$$(\mathrm{N}'_p) \qquad f'(x^0)u \notin -\operatorname{int} C,$$

$$(\mathrm{N}''_p) \qquad \text{if } f'(x^0)u \in -(C \setminus \operatorname{int} C) \text{ then } \operatorname{conv}\{y, \operatorname{im} f'(x^0)\} \cap (-\operatorname{int} C) = \emptyset \quad \text{for all } y \in f''_D(x^0, u).$$

The notions, such as directional derivatives, stated straight in terms of the image space $\mathbb{R}^n$, are called primal. The subscript p in N'_p and N''_p refers to conditions (necessary of the first and second order, respectively) stated in primal concepts. We confine ourselves here only to necessary conditions. In the next subsection we formulate necessary conditions and sufficient conditions in dual concepts, such as the elements of the positive polar cone. We call here dual the concepts stated in terms of the dual space to the image space $\mathbb{R}^n$.

P r o o f. We prove first N'_p. Since x^0 is a w-minimizer, we have $t^{-1}(f(x^0 + tu) - f(x^0)) \notin -\operatorname{int} C$ for $0 < t$ sufficiently small, whence passing to a limit we get $f'(x^0)u \notin -\operatorname{int} C$.

Now we prove N''_p. Assume on the contrary that for some $u \in S$ we have $f'(x^0)u \in -(C \setminus \operatorname{int} C)$ and there exists $y(u) \in f''_D(x^0, u)$ such that

$$\operatorname{conv}\{y(u), \operatorname{im} f'(x^0)\} \cap (-\operatorname{int} C) \neq \emptyset. \tag{12}$$

According to the definition of $f''_D(x^0, u)$, there exists a sequence $t_k \to 0+$ such that $\lim_k y^k(u) = y(u)$, where for $v \in \mathbb{R}^m$ we put

$$y^k(v) = \frac{2}{t_k^2}(f(x^0 + t_k v) - f(x^0) - t_k f'(x^0)v).$$

Condition (12) shows that there exist $\overline{w} \in \mathbb{R}^m$ and $\bar{\lambda} \in (0, 1)$ such that

$$(1 - \bar{\lambda})y(u) + \bar{\lambda} f'(x^0)\overline{w} \in -\operatorname{int} C.$$

Let $v^k \to u$. For k “large enough” we have, as in Lemma 2 below, that

$$\|y^k(v^k) - y^k(u)\| \leqslant L(\|u\| + \|v^k\|)\|v^k - u\|,$$

and hence,

$$\begin{aligned} \|y^k(v^k) - y(u)\| &\leqslant \|y^k(v^k) - y^k(u)\| + \|y^k(u) - y(u)\| \\ &\leqslant L(\|u\| + \|v^k\|)\|v^k - u\| + \|y^k(u) - y(u)\| \to 0 \end{aligned}$$

as $k \to +\infty$ (here L denotes a Lipschitz constant for f').

For $k = 1, 2, \ldots$, let v^k be such that $\overline{w} = 2(1 - \bar{\lambda})(t_k\bar{\lambda})^{-1}(v^k - u)$, i.e. $v^k = u + \frac{1}{2}\bar{\lambda}(1 - \bar{\lambda})^{-1} t_k \overline{w}$ and hence, $v^k \to u$. For every k we have

$$\begin{aligned} &f(x^0 + t_k v^k) - f(x^0) \\ &\quad = t_k f'(x^0)u + t_k f'(x^0)(v^k - u) + \frac{1}{2}t_k^2 y(u) + o(t_k^2) \\ &\quad = t_k f'(x^0)u + \frac{1}{2(1 - \bar{\lambda})}t_k^2\Big((1 - \bar{\lambda})y(u) + \frac{2(1 - \bar{\lambda})}{t_k} f'(x^0)(v^k - u)\Big) + o(t_k^2) \\ &\quad = t_k f'(x^0)u + \frac{1}{2(1 - \bar{\lambda})}t_k^2\Big((1 - \bar{\lambda})y(u) + \bar{\lambda} f'(x^0)\Big(\frac{2(1 - \bar{\lambda})}{t_k\bar{\lambda}}(v^k - u)\Big)\Big) + o(t_k^2) \\ &\quad = t_k f'(x^0)u + \frac{1}{2(1 - \bar{\lambda})}t_k^2((1 - \bar{\lambda})y(u) + \bar{\lambda} f'(x^0)\overline{w}) + o(t_k^2) \in -C - \operatorname{int} C + o(t_k^2). \end{aligned}$$

For k large enough, the last set in the previous chain of relations is contained in $-\operatorname{int} C$ and this contradicts the fact that x^0 is a w-minimizer. $\square$

Now we apply Theorem 4 to Example 4. We minimize the $\mathcal{C}^2$ vector function $f(x) = (-2x_1^2 + x_2, x_1^2 - x_2)$ with respect to the cone $\mathbb{R}^2_+$. Simple calculations give $f'(x)u = (-4x_1u_1 + u_2, 2x_1u_1 - u_2)$, $f''_D(x,u) = f''(x)(u,u) = (-4u_1^2, 2u_1^2)$. If $x_1 \neq 0$ then $\operatorname{im} f'(x) = \mathbb{R}^2$ and condition N'_p is not satisfied. Therefore, only the points $x^0 = (0, x_2^0)$, $x_2^0 \in \mathbb{R}$, could possibly be w-minimizers. On each such point $\operatorname{im} f'(x^0) = \{x\colon\ x_1 + x_2 = 0\}$ and condition N'_p is satisfied. Further, $f'(x^0) = (u_2, -u_2) \in -(C \setminus \operatorname{int} C)$ if and only if $u_2 = 0$. Then $u = (u_1, 0) \in S$ implies $u_1 = \pm 1$. In each of these cases $f''(x^0)(u,u) = (-4, 2)$. Now $(3, -3) \in \operatorname{im} f'(x^0)$ (instead of $(3,-3)$ we can use any point from $\operatorname{im} f'(x^0) \setminus \{0\}$) and $\frac{1}{2}(-4,2) + \frac{1}{2}(3,-3) = (-\frac{1}{2}, -\frac{1}{2}) \in -\operatorname{int} C$. Therefore, condition N''_p is not satisfied. Consequently, on the basis of Theorem 4 we conclude that the function in Example 4 does not possess w-minimizers.

6.2. Optimality conditions in dual form: Lagrange multipliers

In this subsection we establish necessary optimality conditions and sufficient optimality conditions in dual terms, that is in terms of vectors from the dual space, which as usually are called Lagrange multipliers. Further, the subscript d in, say, N'_d, stands for *dual*. The next theorem is our main result.

Theorem 5. *Assume that $f\colon\ \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^{1,1}$ function minimized with respect to a pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$ and let $\Delta(x) = \{\xi \in \mathbb{R}^n :\ \xi f'(x) = 0,\ \|\xi\| = 1\}$.*

(Necessary Conditions) *Let x^0 be a w-minimizer of f. Then for each $u \in S$ the following two conditions are satisfied:*

(N'_d) $$\Delta(x^0) \cap C' \neq \emptyset,$$

(N''_d) *if* $f'(x^0)u \in -(C \setminus \operatorname{int} C)$ *then* $\displaystyle\min_{y \in f''_D(x^0,u)} \max\{\langle \xi, y\rangle\colon\ \xi \in C' \cap \Delta(x^0)\} \geqslant 0.$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^m$ and let condition N'_d hold. Suppose further that for each $u \in S$ one of the following two conditions is satisfied:*

(S'_d) $$f'(x^0)u \notin -C,$$

(S''_d) $f'(x^0)u \in -(C \setminus \operatorname{int} C)$ *and* $\displaystyle\min_{y \in f''_D(x^0,u)} \max\{\langle \xi, y\rangle\colon\ \xi \in C' \cap \Delta(x^0)\} > 0.$

Then x^0 is an isolated minimizer of the second order for f.

These conditions are not only sufficient, but also necessary for x^0 to be an isolated minimizer of the second order for f.

Proof of the Necessary Conditions. Since N'_p holds, the linear subspace of $\mathbb{R}^n$, $\operatorname{im} f'(x^0)$, does not intersects $-\operatorname{int} C$.

According to the Separation Theorem, there exists a nonzero vector $(\xi, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle \xi, f'(x^0)u\rangle \geqslant \alpha$ for $u \in \mathbb{R}^m$ and $\langle \xi, y\rangle \leqslant \alpha$ for $y \in -C$. Since both $\operatorname{im} f'(x^0)$ and $-C$ are cones, we get easily $\alpha = 0$ and hence, $\xi \neq 0$. Now the second inequality gives $\xi \in C'$. Further, $\langle \xi, f'(x^0)\rangle = 0$, since

$$0 \leqslant \langle \xi, f'(x^0)(-u)\rangle = -\langle \xi, f'(x^0)u\rangle \leqslant 0 \quad \text{for all } u \in \mathbb{R}^m,$$

which holds only if $\langle \xi, f'(x^0)u\rangle = 0$. Since we can assume $\|\xi\| = 1$, we see that $\xi \in \Delta(x^0) \cap C'$, i.e. $\Delta(x^0) \cap C' \neq \emptyset$.

Denote by $\Delta_+(x^0)$ the set of all $\xi \in \mathbb{R}^n$, $\|\xi\| = 1$, such that $\langle \xi, f'(x^0)u\rangle \geqslant 0$ for all $u \in \mathbb{R}^m$ and $\langle \xi, y\rangle \leqslant 0$ for all $y \in -C$. We have shown that $\Delta_+(x^0) = \Delta(x^0) \cap C'$ and if x^0 is a w-minimizer, then $\Delta_+(x^0) \neq \emptyset$. The notation $\Delta_+(x^0)$ instead of $\Delta(x^0) \cap C'$ points out the underlying separation property.

We prove now the necessity of Condition N''_d. Let $u \in S$ be such that $f'(x^0)u \in -(C \backslash \operatorname{int} C)$ and $\bar{y} \in f''_D(x^0, u)$. We must show that $\max\{\langle \xi, \bar{y}\rangle \colon \xi \in \Delta(x^0) \cap C'\} \geqslant 0$. According to Theorem 4, the set $A = \operatorname{conv}\{\bar{y}, \operatorname{im} f'(x^0)\}$ is separated from $-C$. Therefore, like in the proof of N'_d, there exists a vector $\xi_{\bar{y}} \in \mathbb{R}^n$, $\|\xi_{\bar{y}}\| = 1$, such that

$$\langle \xi_{\bar{y}}, y\rangle \geqslant 0 \quad \text{for } y \in \operatorname{im} f'(x^0), \quad \langle \xi_{\bar{y}}, \bar{y}\rangle \geqslant 0, \tag{13}$$

$$\langle \xi_{\bar{y}}, y\rangle \leqslant 0 \quad \text{for } y \in -C. \tag{14}$$

The first inequality in (13) and inequality (14) show that $\xi_{\bar{y}} \in \Delta(x^0) \cap C'$, whence

$$\max\{\langle \xi, \bar{y}\rangle \colon \xi \in \Delta(x^0) \cap C'\} \geqslant \langle \xi_{\bar{y}}, \bar{y}\rangle \geqslant 0.$$

□

Further, we set $\Gamma = \{\xi \in C' \colon \|\xi\| = 1\}$, which is a compact set as the intersection of the closed cone C' with the unit sphere. Therefore, the maximum $D(y, -C) = \max\{\langle \xi, y\rangle \colon \xi \in \Gamma\}$, $y \in \mathbb{R}^n$, is attained. In Section 5 we called $D(y, -C)$ the oriented distance from y to $-C$. In particular, $D(f'(x^0)u, -C) = \max\{\langle \xi, f'(x^0)u\rangle \colon \xi \in \Gamma\}$ is the oriented distance from $f'(x^0)u$ to $-C$, which appears in conditions N'_d and S'_d. Condition N'_d can be written as $D(f'(x^0)u, -C) \geqslant 0$, which is equivalent to $f'(x^0)u \notin -\operatorname{int} C$. We can write the condition $f'(x^0)u \in -(C \backslash \operatorname{int} C)$ appearing in N''_d and S''_d also into the dual form $D(f'(x^0)u, -C) = 0$ or $\max\{\langle \xi, f'(x^0)u\rangle \colon \xi \in \Gamma\} = 0$.

To prove the sufficient conditions in Theorem 5 we need some lemmas. The first generalizes Lemma 1 to vector functions.

Lemma 2. *Let* $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *be a* $\mathcal{C}^{1,1}$ *function. Let* f' *be Lipschitz with a constant* L *in* $x^0 + r\,\mathrm{cl}\,B$, *where* $x^0 \in \mathbb{R}^m$ *and* $r > 0$. *Then for* $u, v \in \mathbb{R}^m$ *and* $0 < t < \min(r/\|u\|, r/\|v\|)$ *we have*

$$\Big\|\frac{2}{t^2}(f(x^0+tv)-f(x^0)-tf'(x^0)v)-\frac{2}{t^2}(f(x^0+tu)-f(x^0)-tf'(x^0)u)\Big\| \\ \leqslant L(\|u\|+\|v\|)\|v-u\|.$$

In particular, for $v = 0$ *we get*

$$\Big\|\frac{2}{t^2}(f(x^0+tu)-f(x^0)-tf'(x^0)u)\Big\| \leqslant L\|u\|^2.$$

We skip the proof, since with obvious changes of notation it repeats the proof of Lemma 1. Let us mention that the function defined by $\varphi(x) = \langle \xi, f(x)\rangle$ satisfies the conditions of Lemma 1, whence it satisfies the estimates obtained there. We use this function in the proof of sufficiency. The next lemma gives some of the properties of the Dini derivative.

Lemma 3. *Let* $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *be as in Lemma* 2. *Then* $\sup\limits_{y\in f''_D(x^0,u)} \|y\| \leqslant L\|u\|^2$ *and hence, for all* $u \in \mathbb{R}^m$, *the set* $f''_D(x^0,u)$ *is compact. For each* $y_u \in f''_D(x^0,u)$, $u \in \mathbb{R}^m$, *there exists a point* $y_v \in f''_D(x^0,v)$, $v \in \mathbb{R}^m$, *such that*

$$\|y_u - y_v\| \leqslant L(\|u\|+\|v\|)\|v-u\|.$$

Consequently, the set-valued function $f''_D(x^0,\cdot)$ *is locally Lipschitz (and hence continuous) with respect to the Hausdorff distance in* $\mathbb{R}^n$.

P r o o f. The inequality $\sup\limits_{y\in f''_D(x^0,u)} \|y\| \leqslant L\|u\|^2$ follows from the estimate in Lemma 2. The closedness of $f''_D(x^0,u)$ is a direct consequence of its definition, whence $f''_D(x^0,u)$ is compact. The remaining assertions also follow straightforward from Lemma 2. □

P r o o f of the sufficient conditions. We prove that if x^0 is not an isolated minimizer of the second order for f, then there exists $u^0 \in S$ for which neither of the conditions S'_d and S''_d is satisfied.

Choose a monotone decreasing sequence $\varepsilon_k \to 0+$. Since x^0 is not an isolated minimizer of the second order, there exist sequences $t_k \to 0+$ and $u^k \in S$ such that

$$D(f(x^0+t_ku^k)-f(x^0),-C) = \max_{\xi\in\Gamma}\langle \xi, f(x^0+t_ku^k)-f(x^0)\rangle < \varepsilon_k t_k^2. \tag{15}$$

Passing to a subsequence, we may assume $u^k \to u^0$.

We prove that S'_d is not satisfied at u^0. Let $\varepsilon > 0$. We claim that there exists k_0 such that for all $\xi \in \Gamma$ and all $k > k_0$ the following inequalities hold:

$$\left\langle \xi, \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) \right\rangle < \frac{1}{3}\varepsilon, \tag{16}$$

$$\left\langle \xi, f'(x^0)u^k - \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) \right\rangle < \frac{1}{3}\varepsilon, \tag{17}$$

$$\langle \xi, f'(x^0)(u^0 - u^k) \rangle < \frac{1}{3}\varepsilon. \tag{18}$$

Inequality (16) follows from (15). Inequality (17) follows from the Fréchet differentiability of f:

$$\begin{aligned} &\left\langle \xi, f'(x^0)u^k - \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) \right\rangle \\ &\qquad \leqslant \left\| \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) - f'(x^0)u^k \right\| < \frac{1}{3}\varepsilon, \end{aligned}$$

which is true for all sufficiently small t_k. Inequality (18) follows from

$$\langle \xi, f'(x^0)(u^0 - u^k) \rangle \leqslant \|f'(x^0)\| \, \|u^0 - u^k\| < \frac{1}{3}\varepsilon,$$

which is true for $\|u^k - u^0\|$ "small enough". Now we see that for arbitrary $\xi \in \Gamma$ and $k > k_0$ we have

$$\begin{aligned} \langle \xi, f'(x^0)u^0 \rangle &= \left\langle \xi, \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) \right\rangle \\ &\quad + \left\langle \xi, f'(x^0)u^k - \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) \right\rangle + \langle \xi, f'(x^0)(u^0 - u^k) \rangle \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$

whence $D(f'(x^0)u^0, -C) = \max_{\xi \in \Gamma} \langle \xi, f'(x^0)u^0 \rangle < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see that $D(f'(x^0)u^0, -C) \leqslant 0$. The geometrical meaning of the proved inequality is $f'(x^0)u^0 \in -C$. Thus, condition S'_d is not satisfied.

Now we prove that S''_d is not satisfied at u^0. We assume that $f'(x^0)u^0 \in -(C \backslash \operatorname{int} C)$ (otherwise the first assertion in condition S''_d would not be satisfied).

Recall that the sequences $\{t_k\}$ and $\{u^k\}$ are such that $t_k \to 0+$, $u^k \to u^0$, $u^k \in S$, and inequality (15) holds. We have

$$\lim_k \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) = f'(x^0)u^0,$$

which follows easily from the Fréchet differentiability of f and the following chain of inequalities, true for arbitrary $\varepsilon > 0$ and sufficiently large k:

$$\begin{aligned}
&\left\| \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) - f'(x^0)u^0 \right\| \\
&\qquad \leqslant \left\| \frac{1}{t_k}(f(x^0 + t_k u^k) - f(x^0)) - f'(x^0)u^k \right\| + \|f'(x^0)\| \, \|u^k - u^0\| \\
&\qquad \leqslant \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

Let $0 < t_k < r$, where $r > 0$ is such that f' is Lipschitz with a constant L in $x^0 + r \operatorname{cl} B$. Passing to a subsequence, we may assume that

$$y^{k,0} = \frac{2}{t_k^2}(f(x^0 + t_k u^0) - f(x^0) - t_k f'(x^0)u^0) \to y^0.$$

Obviously, $y^0 \in f''_D(x^0, u^0)$ according to the definition of the second-order Dini derivative. Put

$$y^k = \frac{2}{t_k^2}(f(x^0 + t_k u^k) - f(x^0) - t_k f'(x^0)u^k).$$

Lemma 2 implies $\|y^k - y^{k,0}\| \leqslant L(\|u^0\| + \|u^k\|)\|u^k - u^0\|$, whence $y^k \to y^0$. Let $\bar{\xi} \in \Delta(x^0) \cap C'$. We have

$$\begin{aligned}
\langle \bar{\xi}, y^k \rangle &= \frac{2}{t_k^2}\langle \bar{\xi}, f(x^0 + t_k u^k) - f(x^0) - t_k f'(x^0)u^k \rangle = \frac{2}{t_k^2}\langle \bar{\xi}, f(x^0 + t_k u^k) - f(x^0) \rangle \\
&\leqslant \frac{2}{t_k^2} \max_{\xi \in \Gamma} \langle \xi, f(x^0 + t_k u^k) - f(x^0) \rangle = \frac{2}{t_k^2} D(f(x^0 + t_k u^k) - f(x^0), -C) \\
&< \frac{2}{t_k^2} \varepsilon_k t_k^2 = 2\varepsilon_k.
\end{aligned}$$

Passing to the limit, we get $\langle \bar{\xi}, y^0 \rangle \leqslant 0$. Since $y^0 \in f''_D(x^0, u^0)$ and $\bar{\xi} \in \Delta(x^0) \cap C'$ is arbitrary, we get

$$\min_{y \in f''_D(x^0, u^0)} \max\{\langle \xi, y \rangle : \ \xi \in C'' \Delta(x^0)\} \leqslant 0.$$

Therefore, condition S''_d is not satisfied at u^0.

The reversal of the sufficient conditions. Let x^0 be an isolated minimizer of the second order, which means that there exist $r > 0$ and $A > 0$ such that

$$D(f(x^0 + tu) - f(x^0), -C) > At^2 \quad \text{for all } 0 < t < r \text{ and } u \in S.$$

Since x^0 is also a w-minimizer, one and only one of conditions N'_p and the first part of condition S''_d can hold for a given vector $u \in \mathbb{R}^n$. Suppose that N'_p and consequently S'_d does not hold. Hence $f'(x^0)u \in -(C \setminus \operatorname{int} C)$. Let $t_k \to 0+$ be a sequence such that

$$\frac{2}{t_k^2}\big(f(x^0 + t_k u) - f(x^0) - t_k f'(x^0)u\big) \to y^0 \in f''_D(x^0, u)$$

and let $w \in \mathbb{R}^m$ be chosen arbitrarily. We have

$$D\big(f(x^0 + t_k u + t_k^2 w) - f(x^0), -C\big) \geqslant A\|t_k u + t_k^2 w\|^2 \tag{19}$$

and therefore

$$\begin{aligned} &D(f(x^0 + t_k u + t_k^2 w) - f(x^0) - t_k f'(x^0)u, -C) \\ &\geqslant D(f(x^0 + t_k u + t_k^2 w) - f(x^0), -C) \geqslant A\|t_k u + t_k^2 w\|^2. \end{aligned}$$

Further we get

$$\begin{aligned} &\frac{2}{t_k^2}(f(x^0 + t_k u + t_k^2 w) - f(x^0) - t_k f'(x^0)u) \\ &= \frac{2}{t_k^2}(f(x^0 + t_k(u + t_k w)) - f(x^0) - t_k f'(x^0)(u + t_k w) + t_k^2 f'(x^0)w). \end{aligned}$$

Since $f \in C^{1,1}$, on the base of Lemma 2 we see that

$$\frac{2}{t_k^2}(f(x^0 + t_k(u + t_k w)) - f(x^0) - t_k f'(x^0)(u + t_k w)) \to y^0$$

and passing to the limit we obtain

$$\frac{2}{t_k^2}(f(x^0 + t_k u + t_k^2 w) - f(x^0) - t_k f'(x^0)u) \to y^0 + f'(x^0)w.$$

From (19) we get $D(y^0 + f'(x^0)w, -C) \geqslant A\|u\|^2 > 0$, and since w is arbitrary,

$$\inf_{w \in \mathbb{R}^m} D(y^0) + f'(x^0)w, -C) = D(y^0 + \operatorname{im} f'(x^0), -C) \geqslant 2A\|u\|^2.$$

This implies $0 \notin \operatorname{cl}(y^0 + \operatorname{im} f'(x^0) + C)$. Since $-C$ and $y^0 + \operatorname{im} f'(x^0)$ are convex sets, according to Theorem 11.4 in [28] the last inclusion amounts to say that these two sets are strongly separated, i.e. there exists a vector $\xi \in \mathbb{R}^n$ such that

$$\inf\{\langle \xi, y\rangle \colon\ y \in y^0 + \operatorname{im} f'(x^0)\} > \sup\{\langle \xi, y\rangle \colon\ y \in -C\}.$$

Let $\beta = \sup\{\langle \xi, y\rangle \colon y \in -C\}$. Since $-C$ is a cone, we have $\langle \xi, \lambda y\rangle \leqslant \beta$, for all $y \in -C$, for all $\lambda > 0$. This implies $\beta \geqslant 0$ and $\langle \xi, y\rangle \leqslant 0$ for every $y \in -C$. Hence $\beta = 0$ and $\xi \in C'$. Further, from $\inf\{\langle \xi, y\rangle \colon y \in y^0 + \operatorname{im} f'(x^0)\} > 0$, we get easily $\xi f'(x^0) = 0$. Indeed, otherwise we would find a vector $w \in \mathbb{R}^m$ such that $\langle \xi, f'(x^0)w\rangle < 0$ and hence we would have

$$\langle \xi, y^0 + \lambda f'(x^0)w\rangle > 0, \quad \forall \lambda \geqslant 0,$$

but this is impossible, since the left-hand side tends to $-\infty$ as $\lambda \to +\infty$. This completes the proof. □

Theorems 3.1 and 3.3 in Liu, Neittaanmäki, Křížek [22] are of the same type as Theorem 5. The latter is however more general and has several advantages. Theorem 5 contrary to [22] concerns arbitrary and not only polyhedral cones C. In Theorem 5 the conclusion in the sufficient conditions part is that x^0 is an *isolated minimizer of the second order* while the conclusion in [22] is only that the reference point is an e-minimizer. The property of being an isolated minimizer is more essential for the solution of the vector optimization problem than the property of being an e-minimizer (efficient point). The isolated minimizers are s-minimizers and for such points stability (well-posedness) takes place, that is a small (with respect to the sup norm) perturbation of the objective function results in a small move of the minimizer (compare with Auslender [4]). Finally, in Theorem 5 we give a reversal of the sufficient conditions, showing that they are also necessary for the reference point to be an isolated minimizer of the second order, while in [22] a reversal is absent.

Corollary 2. *In the case $n = 1$ Theorem 5 obviously transforms into Theorem 2.*

In the previous section we treated Example 4 with help of Theorem 4. Now we demonstrate the solution of the same problem with help of Theorem 5. The function $f(x) = (-2x_1^2 + x_2, x_1^2 - x_2)$ is optimized with respect to $C = \mathbb{R}^2_+$. If

$$\langle \xi, f'(x^0)u\rangle = \xi_1(-4x_1u_1 + u_2) + \xi_2(2x_1u_1 - u_2) = 2x_1(-2\xi_1 + \xi_2)u_1 + (\xi_1 - \xi_2)u_2 = 0$$

for arbitrary u_1, u_2, then $\xi_1 = \xi_2$ and $x_1 = 0$. The latter shows that $x^0 = (0, x_2^0)$ are the only points for which condition N'_d could be satisfied, and then $\xi_1 = \xi_2 = 1/\sqrt{2}$. Then $f'(x^0)u = (u_2, -u_2) \in -C$ iff $u_2 = 0$, whence $u = (u_1, 0) = (\pm 1, 0)$. Now $f''_D(x^0, u) = f''(x^0)(u, u) = (-4u_1^2, 2u_1^2) = (-4, 2)$ and

$$\min_{y \in f''_D(x^0, u^0)} \max\{\langle \xi, y\rangle \colon \xi \in C' \cap \Delta(x^0)\} = -4\xi_1 + 2\xi_2 = \frac{-4+2}{\sqrt{2}} = -\sqrt{2} < 0.$$

Therefore, for $x^0 = (0, x_2^0)$ and $u = (\pm 1, 0)$ we have $f'(x^0)u \in -(C \setminus \operatorname{int} C)$ but condition N''_d is not satisfied. On this basis we conclude that f does not possess w-minimizers.

E x a m p l e 5. The function

$$f\colon \mathbb{R} \to \mathbb{R}^2, \quad f(x) = \begin{cases} (-2x^2, x^2), & x \geqslant 0, \\ (x^2, -2x^2), & x < 0, \end{cases}$$

is $\mathcal{C}^{1,1}$. If f is optimized with respect to the cone $C = \mathbb{R}^2_+$, then $x^0 = 0$ is an isolated minimizer of the second order, which can be verified on the basis of the sufficient conditions of Theorem 5.

Indeed, f is $\mathcal{C}^{1,1}$, which follows from

$$f'(x) = \begin{cases} (-4x, 2x), & x > 0, \\ (0,0), & x = 0, \\ (2x, -4x), & x < 0. \end{cases}$$

At $x^0 = 0$ condition N'_d is satisfied, since for all $\xi \in \mathbb{R}^2$ we have $\langle \xi, f'(x^0)u \rangle = 0$, $u \in \mathbb{R}$, whence

$$\Delta(x^0) \cap C' = \{\xi \in C'\colon \|\xi\| = 1\} = \{\xi \in \mathbb{R}^2\colon \xi_1 \geqslant 0,\ \xi_2 \geqslant 0,\ \xi_1^2 + \xi_2^2 = 1\}.$$

Fix $u \in S = \{-1,\, 1\}$; then $f'(x^0)u = (0,0) \in -(C \setminus \operatorname{int} C)$. The second-order Dini derivative is

$$f''_D(x^0, u) = \begin{cases} (-4u^2, u^2), & u > 0, \\ (2u^2, -4u^2), & u < 0. \end{cases}$$

For $u = 1$ we get

$$\min_{y \in f''_D(x^0,u)} \max\{\langle \xi, y \rangle\colon\ \xi \in C' \cap \Delta(x^0)\}$$
$$= \max\{-4\xi_1 + 2\xi_2\colon\ \xi_1 \geqslant 0,\ \xi_2 \geqslant 0,\ \xi_1^2 + \xi_2^2 = 1\} = 2 > 0,$$

which verifies condition S''_d. Similarly, S''_d is satisfied also for $u = -1$.

Obviously, Theorem 5 remains true if $\Delta(x)$ is replaced by $\{\xi \in \mathbb{R}^n\colon\ \xi f'(x) = 0,\ \xi \in \Xi\}$, where Ξ is a compact base of C'. The particular case of $C = \mathbb{R}^n_+$ and $\Xi = \operatorname{conv}\{\xi^1, \dots \xi^n\}$, where $\xi^i_j = 1$ for $i = j$ and $\xi^i_j = 0$ for $i \neq j$, gives a proof of the following Corollary 3 and answers affirmatively the conjecture formulated in [12].

Corollary 3. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a $\mathcal{C}^{1,1}$ function minimized with respect to the cone $C = \mathbb{R}^n_+$.*

(Necessary Conditions) *Let x^0 be a w-minimizer of f. Then:*

a) *The set $D \subset \mathbb{R}^n$ consisting of all ξ such that $\xi \in \mathbb{R}^n_+$, $\sum\limits_{j=1}^{n} \xi_j = 1$ and $\xi f'(x^0) = 0$, is nonempty.*

b) *For each $u \in S$ such that $f'(x^0)u \in -(\mathbb{R}^n_+ \setminus \operatorname{int} \mathbb{R}^n_+)$ we have*

$$\inf_{y \in f''_D(x^0,u)} \sup_{\xi \in D} \langle \xi, y \rangle \geqslant 0.$$

(Sufficient Conditions) *Assume that for $x^0 \in \mathbb{R}^m$ the Necessary Condition* a) *holds. Suppose further that for each $u \in S$ one of the following two conditions is satisfied:*

c) $\max\limits_{1 \leqslant i \leqslant n} (f'(x^0)u)_i > 0$ (*here the subscript i stands for the ith coordinate*),

d) $\max\limits_{1 \leqslant i \leqslant n} (f'(x^0)u)_i = 0$ *and* $\inf\limits_{y \in f''_D(x^0,u)} \sup\limits_{\xi \in D} \langle \xi, y \rangle > 0$.

Then x^0 is an isolated minimizer of the second order for f.

These conditions are not only sufficient, but also necessary for x^0 to be an isolated minimizer of the second order for f.

As in the scalar case, if a vector function f is $\mathcal{C}^2$, then it is also $\mathcal{C}^{1,1}$. However, if f is only twice differentiable at x^0, it need not be $\mathcal{C}^{1,1}$. For a scalar function, it was shown that the second-order optimality conditions of Theorem 2 hold also under the hypotheses of twice differentiability at x^0. Also in the vector case, when f is twice differentiable at x^0, one can prove conditions analogous to those of Theorem 5, observing that $f''_D(x,u) = f''(x)(u,u)$ where $f''(x)$ is the Hessian of f.

7. Comparison results

The next Theorem 6 is from Guerraggio, Luc [15] (see [15, Theorems 5.1 and 5.2]). It generalizes Theorem 3 to the vector case and gives second-order optimality conditions for $\mathcal{C}^{1,1}$ vector functions in terms of the Clarke second-order subdifferential, defined as follows. Since f' is Lipschitz, according to Rademacher's Theorem, the Hessian f'' exists almost everywhere. Then the second-order subdifferential of f at x^0 is defined by

$$\partial^2 f(x^0) = \operatorname{cl}\operatorname{conv}\{\lim f''(x^i)\colon\ x^i \to x^0,\ \ f''(x^i) \text{ exists}\}.$$

Theorem 6. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a $\mathcal{C}^{1,1}$ function minimized with respect to the closed pointed convex cone C with $\operatorname{int} C \neq \emptyset$.*

(Necessary Conditions) *Assume that x^0 is a w-minimizer of f. Then the following conditions hold for each $u \in S$:*

a) $\xi f'(x^0) = 0$ *for some* $\xi \in C' \setminus \{0\}$,

b) *if* $f'(x^0)(u) \in -(C \setminus \operatorname{int} C)$ *then* $\partial^2 f(x^0)(u,u) \cap (-\operatorname{int} C)^c \neq \emptyset$.

(Sufficient Conditions) *Assume that for each $u \in S$ either of the following two conditions is satisfied:*

c) $\xi f'(x^0) = 0$ *for some* $\xi \in \operatorname{int} C'$,

d) *if* $u \in \ker f'(x^0)$ *then* $\partial^2 f(x^0)(u,u) \subset \operatorname{int} C$.

Then x^0 is an e-minimizer for f.

In order to compare the necessary conditions of Theorem 6 and Theorem 5 we observe that Theorem 6 does not work with Example 4. We check that the necessary conditions of Theorem 6 are satisfied at $x^0 = (0,0)$ and therefore, on this basis the suspicion that x^0 is a w-minimizer cannot be rejected (in the previous section we have shown that this is not the case when dealing with Theorem 5). Indeed, for the function $f(x_1,x_2) = (-2x_1^2 + x_2,\, x_1^2 - x_2)$ we have

$$f'(x) = \begin{bmatrix} -4x_1 & 1 \\ 2x_1 & -1 \end{bmatrix}, \quad f_1''(x) = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}, \quad f_2''(x) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $x^0 = (0,0)$ we have

$$\xi f'(x^0)u = \langle \xi, f'(x^0)u \rangle = (\xi_1 - \xi_2)u_2 \equiv 0 \Leftrightarrow \xi_1 - \xi_2 = 0$$

and condition a) holds for, say, $\xi = (1,1)$. We have $f''(x^0)u = (u_2, -u_2) \in -(C \setminus \operatorname{int} C)$ only if $u_2 = 0$. For $u = (u_1,u_2)$ with $u_2 = 0$ we have $\partial^2 f(x^0)(u,u) = (-4u_1^2, 2u_1^2) \notin -\operatorname{int} C$.

In order to compare the sufficient conditions of Theorem 6 and Theorem 5 we observe that Theorem 6 does not work with Example 5. We check that the sufficient conditions of Theorem 6 are not satisfied at $x^0 = 0$ and therefore, on this basis it does not follow that x^0 is an e-minimizer (in the previous section we have shown that Theorem 5 implies that x^0 is an isolated minimizer of the second order, hence an e-minimizer). We have $\xi f'(x^0) = 0$ for all $\xi \in \mathbb{R}^2$, hence, condition c) is satisfied. The second-order subdifferential at x^0 is the segment $\partial^2 f(x^0) = [(-4,2),(2,4)]$. Although $u \in \ker f'(x^0)$ for all $u \in \mathbb{R} \setminus \{0\}$, it is not true that $\partial^2 f(x^0)(u,u) = [(-4u^2, 2u^2),(2u^2,4u^2)] \subset \operatorname{int} C$ (even more, $\partial^2 f(x^0)(u,u)$ does not intersect $C = \mathbb{R}^2_+$).

The foundations of the Lagrange multipliers technique and a unified approach to programming, calculus of variations and optimal control are presented in Alexeev, Tikhomirov, Fomin [2]. Bolintenéanu, El Maghri [6] generalized some second-order conditions from [2] to vector optimization. Their results concern constrained problems in Banach spaces. For the sake of comparison in Theorem 7 we restrict these results (see Bolintenéanu, El Maghri [6, Theorems 3.1 and 4.2]) to unconstrained problems in finite dimensional spaces. We list first some assumptions and notation.

The optimization problem (5) minimized with respect to $C = \mathbb{R}^n_+$ is considered with a twice Fréchet differentiable vector function $f\colon \mathbb{R}^m \to \mathbb{R}^n$. The Lagrangian of this problem is given by

$$L\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, \quad L(x,\xi) = \langle \xi, f(x) \rangle.$$

It is assumed that there is no point x^0 such that x^0 is a minimizer for all f_j, $j = 1, \dots, n$ (we call it assumption "H"). This assumption has some technical consequences, namely if x^0 is a w-minimizer, then

$$K_w(x^0) \neq \emptyset \quad \text{and} \quad 1 \leqslant |J| < n \quad \text{for all } J \in K_w(x^0).$$

The respective definitions are the following:

$J_w(x^0)$ is the set of all $J \subset \{1, 2, \dots, n\}$ such that in any neighbourhood U of x^0 there exists a point $x^J \in U$ such that $f_J(x^J) - f_J(x^0) \in -\operatorname{int} \mathbb{R}^{|J|}_+$. Here $f_J = (f_j)_{j \in J}$ is the restriction of f to those indices which belong to J and $|J|$ denotes the cardinality of J. Further,

$$K_w(x^0) = \arg \min_{J \in J_w(x^0)} |J|.$$

In Theorem 7 below, condition b) involves the constraint qualification $(CQ)^2_w$. We say that x^0 verifies $(CQ)^2_w$ with respect to $\xi^0 \in \Omega_w(J,k)$ if the operator $(f'_{K_+}(x^0), -1_{K_+}) \in \mathcal{L}(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^{|K_+|})$ is surjective. Here $K_+ = \{j \in J \cup \{k\}\colon\ \xi^0_j > 0\}$, $1_{K_+} = (1, 1, \dots, 1) \in \mathbb{R}^{|K_+|}$ and $\mathcal{L}(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^{|K_+|})$ is the space of the linear operators from $\mathbb{R}^m \times \mathbb{R}$ into $\mathbb{R}^{|K_+|}$.

Theorem 7. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be of class $\mathcal{C}^2$, minimized with respect to $C = \mathbb{R}^n_+$, and let assumption* "H" *hold.*

(Necessary Conditions) *Assume that x^0 is a w-minimizer and choose arbitrarily $J \subset K_w(x^0)$ and $k \in \{1, \dots, n\}$. Then there exist Lagrange multipliers*

$$\xi^0 \in \Omega_w(J,k) := \Big\{ \xi \in C'\colon \sum_{j \in J \cup \{k\}} \xi_j = 1 \quad \textit{and} \quad \xi_j = 0 \ \textit{ for } j \notin J \cup \{k\} \Big\},$$

such that

a) $L'_x(x^0, \xi^0) = 0$.

b) *Assume* x^0 *verifies* $(CQ)^2_w$ *with respect to* $\xi^0 \in \Omega_w(J,k)$ *and condition* a) *is fulfilled. Then*

$$L''_{xx}(x^0,\xi^0)(u,u) \geqslant 0 \quad \text{for all } u \in \ker f'_{J\cup\{k\}}(x^0).$$

(Sufficient Conditions) *Let* $x^0 \in \mathbb{R}^m$. *Suppose that there exists* $\xi^0 \in C' \setminus \{0\}$ *such that*

c) $L'_x(x^0,\xi^0) = 0$,

d) $L''_{xx}(x^0,\xi^0)(u,u) \geqslant 0$ *for all* $u \neq 0$.

Then x^0 *is a* w*-minimizer.*

We apply Theorem 7 to Example 4. In that case $f(x) = (-2x_1^2 + x_2, x_1^2 - x_2)$. The Lagrange function is

$$L(x,\xi) = (-2\xi_1 + \xi_2)x_1^2 + (\xi_1 - \xi_2)x_2.$$

The Jacobian $L'_x(x,\xi)$ is given by

$$\frac{\partial}{\partial x_1} L(\hat{x},\hat{\xi}) = (-4\xi_1 + 2\xi_2)x_1, \quad \frac{\partial}{\partial x_2} L(\hat{x},\hat{\xi}) = \xi_1 - \xi_2.$$

Therefore, the pairs (x^0,ξ^0) satisfying condition a) are given by $x_1^0 = 0$, $\xi_1^0 = \xi_2^0 = 1/2$.

For the Hessian $L''_{xx}(x^0,\xi^0)$ we have

$$L''_{xx}(x^0,\xi^0) = \begin{pmatrix} -4\xi_1 + 2\xi_2 & 0 \\ 0 & 0 \end{pmatrix}$$

and $L''_{xx}(x,\xi)(u,u) = (-4\xi_1 + 2\xi_2)u_1^2$. Further, $f'(x)u = (-4x_1u_1 + u_2, 2x_1u_1 - u_2)$. Therefore, for the distinguished pairs (x^0,ξ^0) and for $u \neq 0$ we have $f'(x^0)u = (u_2, -u_2) = 0$ iff $u_2 = 0$, $u_1 \neq 0$. In this case, however, $L''_{xx}(x^0,\xi^0)(u,u) = -2\xi_1^0 u_1^2 < 0$. Therefore, condition b) is not satisfied and consequently x^0 is not a w-minimizer.

We have shown that in principle Theorem 7 can reject the suspicion that the function in Example 4 has w-minimizers. From the practical point of view the check of some of the conditions may cause difficulties (in this example we have omitted the details). Provided that assumption "H" is satisfied, one has to check separately that this is the case. At last, generalizing results from scalar to vector optimization, one would rather try to avoid constraint qualifications on the objective function, since they are usually absent in the scalar case.

Obviously, if a minimizer x^0 can be recognized on the basis of the sufficient conditions of Theorem 7, then x^0 is necessarily a solution of the linearly scalarized problem

$$\varphi(x) = \langle \xi^0, f(x) \rangle \to \min, \quad \xi^0 \in C' \setminus \{0\}. \tag{20}$$

Each solution of the linearly scalarized problem is a w-minimizer. The converse is true for C-convex functions (see Luc [23]), but in general these concepts are different. Theorem 5 in contrast to Theorem 7 allows on the one hand to treat $\mathcal{C}^{1,1}$ problems which are more general than $\mathcal{C}^2$ ones, on the other hand, as is seen from Example 5, it recognizes minimizers which are no solutions of any linearly scalarized problem (20).

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