From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds

Constantin Călin and Mircea Crasmareanu

July 23, 2009
Dedicated to the memory of Neculai Papaghiuc 1947-2008

Abstract

The Eisenhart problem of finding parallel tensors is solved for the symmetric case in the regular f-Kenmotsu framework. In this way, the Olszack-Rosca example of Einstein manifolds provided by f-Kenmotsu manifolds via locally symmetric Ricci tensors is recovered as well as a case of Killing vector fields. Some other classes of Einstein-Kenmotsu manifolds are presented. Our result is interpreted in terms of Ricci solitons and special quadratic first integrals.

2000 Math. Subject Classification: 53C40; 53C55; 53C12; 53C42.
Key words: f-Kenmotsu manifold; parallel second order covariant tensor field; irreducible metric; Einstein space; Ricci soliton.

Introduction

In 1923, Eisenhart [9] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [18] proved that a second order parallel symmetric non-degenerated tensor α in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the the problem of finding linear connections making parallel a given tensor field; a problem which was considered by Wong in [35]. Also, the former question implies topological restrictions namely if the (pseudo) Riemannian manifold M admits a parallel symmetric $(0, 2)$ tensor field then M is locally the direct product of a number of (pseudo) Riemannian manifolds, [36] (cited by [37]). Another situation where the parallelism of α is involved appears in the theory of totally geodesic maps, namely, as is point out in [22, p. 114], $\nabla\alpha = 0$ is equivalent with the fact that $1 : (M, g) \to (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy work locally, Ramesh Sharma gives in [26] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this *Eisenhart problem* in contact geometry [27]-[29], for example for K-contact manifolds in [28]. Since then, several other studies appeared in various contact manifolds: nearly Sasakian [33], (para) P-Sasakian [32], [6] and [19], α-Sasakian [5]. Another framework was that of quasi-constant curvature in [13]. Also, contact metrics with nonvanishing ξ-sectional curvature are studied in [10].

Returning to contact geometry, an important class of manifolds are introduced by Kenmotsu in [15] and generalized by Olszack and Rosca in [21]. Recently, there is an increasing flow of papers in this direction e.g. that of our professor N. Papaghiuc [23]-[24] to whom we dedicate this short note. Motivated by this fact, we studied the case of f-Kenmotsu manifolds satisfying a special condition called by us *regular* and show that a symmetric parallel tensor field of second order must be a constant multiple of the Riemannian metric. There are three remarks regarding our result:
i) it is in agreement with what happens in all previously recalled contact geometries for the symmetric case,
ii) it is obtained in the same manner as in Sharma's paper [26],
iii) yields a class of Einstein manifolds already indicated by Olszack and Rosca but with a more complicated proof.
Let us point out also that the anti-symmetric case appears without proof in [20].

Our main result is connected with the recent theory of Ricci solitons, a subject included in the Hamilton-Perelman approach (and proof) of Poincaré Conjecture. Ricci solitons in contact geometry were first studied by Ramesh Sharma in [11] and [30]; also the preprint [34] is available in arxiv. In these papers the K-contact and (k, μ)-contact (including Sasakian) cases are treated; thus our treatment for the Kenmotsu variant of almost contact geometry seems to be new.

Our work is structured as follows. The first section is a very brief review of Kenmotsu geometry and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in a f-Kenmotsu manifold and several situations yielding Einstein manifolds are derived. Also, the relationship with the Ricci solitons is pointed out. The last section offers a dynamical picture of the subject via Killing vector fields and quadratic first integrals of a special type.

1 f-Kenmotsu manifolds. Ricci solitons

Let M be a real $2n+1$-dimensional differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g):

$$\begin{array}{ll} (a) \quad \varphi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \eta \circ \varphi = 0, \\ (d) \quad \varphi(\xi) = 0, \quad (e) \quad \eta(X) = g(X, \xi), \\ (f) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{array} \tag{1.1}$$

for any vector fields $X, Y \in \mathcal{X}(M)$ where I is the identity of the tangent bundle TM, φ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. Throughout the paper all objects are differentiable of class C^∞.

We say that $(M, \varphi, \xi, \eta, g)$ is an *f-Kenmotsu manifold* if the Levi-Civita connection of g satisfy [20]:

$$(\nabla_X \varphi)(Y) = f(g(\varphi X, Y)\xi - \varphi(X)\eta(Y)), \tag{1.2}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$ holds. A $f =$ *constant* $\equiv \beta > 0$ is called *β-Kenmotsu manifold* with the particular case $f \equiv 1$-Kenmotsu manifold which is a usual *Kenmotsu manifold* [15].

In a general f-Kenmotsu manifold we have, [21]:

$$\nabla_X \xi = f(X - \eta(X)\xi), \tag{1.3}$$

and the curvature tensor field:

$$R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X) + Y(f)\varphi^2 X - X(f)\varphi^2 Y \tag{1.4}$$

while the Ricci curvature and Ricci tensor are, [16]:

$$S(\xi, \xi) = -2n(f^2 + \xi(f)) \tag{1.5}$$

$$Q(\xi) = -2nf^2\xi - \xi(f)\xi - (2n-1)gradf. \tag{1.6}$$

In the last part of this section we recall the notion of Ricci solitons according to [30, p. 139]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that:

$$\mathcal{L}_V g + 2S + 2\lambda g = 0. \tag{1.7}$$

The Ricci soliton is said to be *shrinking, steady* or *expanding* according as λ is negative, zero or positive.

2 Parallel symmetric second order tensors and Ricci solitons in f-Kenmotsu manifolds

Fix α a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ i.e. $\nabla\alpha = 0$. Applying the Ricci identity $\nabla^2\alpha(X, Y; Z, W) - \nabla^2(X, Y; W, Z) = 0$ we obtain the relation (1.1) of [26, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0, \tag{2.1}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (1.4) results in:

$$f^2[\eta(X)\alpha(Y,\xi) - \eta(Y)\alpha(X,\xi)] + Y(f)\alpha(\varphi^2X,\xi) - X(f)\alpha(\varphi^2Y,\xi) = 0, \tag{2.2}$$

by the symmetry of α. With $X = \xi$ we derive:

$$[f^2 + \xi(f)][\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi)] = 0$$

and supposing $f^2 + \xi(f) \neq 0$ it results:

$$\alpha(Y,\xi) = \eta(Y)\alpha(\xi,\xi). \tag{2.3}$$

Let us call a *regular f-Kenmotsu manifold* a f-Kenmotsu manifold with $f^2 + \xi(f) \neq 0$ and remark that β-Kenmotsu manifolds are regular.

Differentiating the last equation covariantly with respect to X we have:

$$\alpha(\nabla_X Y,\xi) + f[\alpha(X,Y) - \eta(X)\eta(Y)\alpha(\xi,\xi)] = X(\eta(Y))\alpha(\xi,\xi), \tag{2.4}$$

which means via (2.3) with $Y \to \nabla_X Y$:

$$f[\alpha(X,Y) - \eta(X)\eta(Y)\alpha(\xi,\xi)] = [X(g(Y,\xi)) - g(\nabla_X Y,\xi)]\alpha(\xi,\xi) =$$

$$= g(Y,\nabla_X\xi)\alpha(\xi,\xi) = f[g(X,Y) - \eta(X)\eta(Y)]\alpha(\xi,\xi). \tag{2.5}$$

From the positiveness of f we deduce that:

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y) \tag{2.6}$$

which together with the standard fact that the parallelism of α implies the $\alpha(\xi,\xi)$ is a constant, via (2.3) yields:

Theorem *A symmetric parallel second order covariant tensor in a regular f-Kenmotsu manifold is a constant multiple of the metric tensor. In other words, a regular f-Kenmotsu metric is irreducible which means that* *the tangent bundle does not admits a decomposition $TM = E_1 \oplus E_2$ parallel with respect of the Levi-Civita connection of g.*

Corollary 1 *A locally Ricci symmetric ($\nabla S \equiv 0$) regular f-Kenmotsu manifold is an Einstein manifold.*

Remarks 1) The particular case of dimension three and β-Kenmotsu of our theorem appears in Theorem 3.1 from [7, p. 2689]. The above corollary has been proved by Olszack and Rosca in another way.
2) In [2] it is shown the equivalence of the following statements for an Kenmotsu manifold:
i) is Einstein,
ii) is locally Ricci symmetric,
iii) is Ricci semi-symmetric i.e. $R \cdot S = 0$ where:

$$(R(X,Y) \cdot S)(X_1, X_2) = -S(R(X,Y)X_1, X_2) - S(X_1, R(X,Y)X_2).$$

The same implication iii) $\rightarrow$ i) for Kenmotsu manifolds is Theorem 1 from [14, p. 438]. But we have the implication iii) $\rightarrow$ i) in the more general framework of regular f-Kenmotsu manifols since $R \cdot S = 0$ means exactly (2.1) with α replaced by S. Every semisymmetric manifold, i. e. $R \cdot R = 0$, is Ricci-semisymmetric but the converse statement is not true. In conclusion:

Proposition 1 *A Ricci-semisymmetric, particularly semisymmetric, regular f-Kenmotsu manifold is Einstein.*

Another class of spaces related to the Ricci tensor was introduced in [31]; namely a Riemannian manifold is a *special weakly Ricci symmetric space* if there exists a 1-form ρ such that:

$$(\nabla_X S)(Y,Z) = 2\rho(X)S(Y,Z) + \rho(Y)S(Z,X) + \rho(Z)S(X,Y). \qquad (2.7)$$

The same condition was sometimes called *generalized pseudo-Ricci symmetric manifold* ([12]) or simply *pseudo-Ricci symmetric manifold* ([4]). By taking $X = Y = Z = \xi$ yields:

$$\xi(S(\xi,\xi)) = 4\rho(\xi)S(\xi,\xi) \qquad (2.8)$$

and then for a β-Kenmotsu manifold we get $\rho(\xi) = 0$. Returning to (2.7) with $Y = Z = \xi$ will result in $\rho(X) = 0$ for every vector field X and thus lead to a generalization of Theorem 3.3. in [1, p. 96]:

Proposition 2 *A β-Kenmotsu manifold which is special weakly Ricci symmetric is an Einstein space.*

We close this section with applications of our Theorem to Ricci solitons:

Corollary 2 *Suppose that on a regular f-Kenmotsu manifold the* $(0,2)$*-type field $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yield a Ricci soliton. In particular, if the given regular f-Kenmotsu manifold is Ricci-semisymmetric or semisymmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.*

Naturally, two situations appear regarding the vector field V: $V \in span\xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives:

$$\mathcal{L}_\xi g(X, Y) = 2f(g(X, Y) - \eta(X)\eta(Y)) = 2fg(\varphi X, \varphi Y). \tag{2.9}$$

A general expression of S is known by us only for the the 3-dimensional case and η-Einstein Kenmotsu manifolds. Let us treat these situations in the following manner:

I) [8, p. 251]:

$$S(X, Y) = \left(\frac{r}{2} + \xi(f) + f^2\right) g(X, Y) -$$

$$-\left(\frac{r}{2} + \xi(f) + 3f^2\right) \eta(X)\eta(Y) - Y(f)\eta(X) - X(f)\eta(Y) \tag{2.10}$$

where r is the scalar curvature. Then, for a 3-dimensional f-Kenmotsu manifold we obtain:

$$\alpha := (\mathcal{L}_\xi g + 2S)(X, Y) = (r + 2\xi(f) + 2f + 2f^2)g(X, Y) -$$

$$-(r + 2\xi(f) + 2f + 6f^2)\eta(X)\eta(Y) - 2Y(f)\eta(X) - 2X(f)\eta(Y) \tag{2.11}$$

while, for β-Kenmotsu:

$$\alpha(X, Y) = (r + 2\beta + 2\beta^2)g(\varphi X, \varphi Y) - 4\beta^2\eta(X)\eta(Y), \tag{2.12}$$

$$(\nabla_Z \alpha)(X, Y) = Z(r)g(\varphi X, \varphi Y) -$$

$$-\beta(r + 2\beta + 6\beta^2)[\eta(X)g(\varphi Y, \varphi Z) + \eta(Y)g(\varphi X, \varphi Z)]. \tag{2.13}$$

Substituting $Z = \xi, X = Y \in (span\xi)^\perp$, and respectively $X = Y = Z \in (span\xi)^\perp$ in (2.13), we derive that r is a constant, provided α is parallel. Thus, we can state the following:

Proposition 3 *A 3-dimensional β-Kenmotsu Ricci soliton (g, ξ, λ) is expanding and with constant scalar curvature.*

Proof $\lambda = -\frac{1}{2}\alpha(\xi, \xi) = 2\beta^2$. $\Box$

At this point we remark that the Ricci solitons of almost contact geometry studied in [30] and [34] in relationship with the Sasakian case are shrinking and this observation is in accordance with the diagram of Chinea from [3] that Sasakian and Kenmotsu are opposite sides of the trans-Sasakian moon. Also, the expanding character may be considered as a manifestation of the fact that a β-Kenmotsu manifold can not be compact.

II) Recall that the metric g is called *η-Einstein* if there exists two real functions a, b such that the Ricci tensor of g is:

$$S = ag + b\eta \otimes \eta.$$

For an η-Einstein Kenmotsu manifold we have, [14, p. 441]:

$$S(X,Y) = \left(\frac{r}{2n} + 1\right) g(X,Y) - \left(\frac{r}{2n} + 2n + 1\right) \eta(X)\eta(Y) \tag{2.14}$$

and then:

$$\alpha(X,Y) = \left(\frac{r}{n} + 4\right) g(X,Y) - \left(\frac{r}{n} + 4 + 4n\right) \eta(X)\eta(Y) \tag{2.15}$$

$$(\nabla_Z \alpha)(X,Y) = \frac{1}{n} Z(r) g(\varphi X, \varphi Y) -$$

$$- \left(\frac{r}{n} + 4n + 4\right) [\eta(Y) g(\varphi X, \varphi Z) + \eta(X) g(\varphi Y, \varphi Z)]. \tag{2.16}$$

Proposition 4 *An η-Einstein Kenmotsu Ricci soliton (g, ξ, λ) is expanding and with constant scalar curvature, thus Einstein.*

Proof $\lambda = -\frac{1}{2}\alpha(\xi, \xi) = 2n$. The same computation as in Proposition 3 implies constant scalar curvature. $\Box$

3 The dynamical point of view

We begin this section with a straightforward consequence of the main Theorem, which also appears in the Olzack-Rosca paper, and is related to the last part of Corollary 2:

Corollary 3 *An affine Killing vector field in a β-Kenmotsu manifold is Killing. As consequence, that scalar provided by the Ricci soliton (g, V) of a Ricci-semisymmetric β-Kenmotsu manifold is $\lambda = -S(V, V)$.*

Proof (inspired by [10, p. 504]) Fix $X \in \mathcal{X}(M)$ an affine Killing vector field: $\nabla \mathcal{L}_X g = 0$. From Theorem it follows that X is *conformal Killing* i.e. $\mathcal{L}_X g = cg$; more precisely X is *homothetic* since c is a constant. Lie differentiating the identity (1.5) along X and using $\mathcal{L}_X S = 0$ (since X is homothetic) and equation (1.6) we obtain $g(\mathcal{L}_X \xi, \xi) = 0$. Hence $c = (\mathcal{L}_X g)(\xi, \xi) = -2g(\mathcal{L}_X \xi, \xi) = 0$. Thus X is Killing. □

Let us present another dynamical picture of our results. Let (M, ∇) be a m-dimensional manifold endowed with a symmetric linear connection. A *quadratic first integral* (QFI on short) for the geodesics of ∇ is defined by $\mathcal{F} = a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$ with a symmetric 2-tensor field $a = (a_{ij})$ satisfying the *Killing-type equations*:

$$a_{ij:k} + a_{jk:i} + a_{ki:j} = 0, \tag{3.1}$$

where, as usual, the double dot means the covariant derivative with respect to ∇.

The QFI defined by a is called *special* (SQFI) if $a_{ij:k} = 0$ and the maximum number of linearly independent SQFI a pair (M, ∇) can admit is $\frac{m(m+1)}{2}$; a flat space will admit this number. In [17, p. 117] it is shown that a non-flat Riemannian manifold may admit as many as $M_S(m) = 1 + \frac{(m-2)(m-1)}{2}$ linearly independent SQFI. Therefore, for an almost contact manifold ($m = 2n+1$) the maximum number of SQFI is $M_S(2n+1) = 1 + n(2n-1) > 1$.

Our main result implies that for a regular f-Kenmotsu manifold the number of SQFI is exactly 1 and the only SQFI is the *kinetic energy* $\mathcal{F} = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$. So:

Proposition 5 *There exist almost contact manifolds which does not admit* $M_S(2n+1)$ *SQFI.*

It remains as an open problem to find examples of almost contact metrics with exactly $M_S(2n+1)$ SQFI.

Acknowledgement Special thanks are offered to Gheorghe Pitis for some useful remarks as well as sending us his book [25], a source of several references. Also, we are very indebted to Marian-Ioan Munteanu, Rosihan M. Ali Dato and the referees who pointed out major improvements.

References

[1] A. Nesip; G. Ali; Ö. Erdal, *On special weakly Ricci-symmetric Kenmotsu manifolds*, Sarajevo J. Math., 3(15)(2007), no. 1, 93-97. MR2327508 (2008c:53084)

[2] T. Q. Binh; L. Tamássy; U. C. De; M. Tarafdar, *Some remarks on almost Kenmotsu manifolds*, Math. Pannon., 13(2002), no. 1, 31-39. MR1888318 (2002k:53156)

[3] D. Chinea, *On horizontally conformal* (ϕ, ϕ')*-holomorphic submersions*, Houston J. Math., 34(2008), no. 3, 721-737. MR2448378

[4] M. C. Chaki, *On pseudo Ricci symmetric manifolds*, Bulgar. J. Phys., 15(1988), no. 6, 526-531. MR1028590 (90k:53071)

[5] L. S. Das, *Second order parallel tensors on* α*-Sasakian manifold*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 23(2007), no. 1, 65-69 (electronic). MR2322902

[6] U. C. De, *Second order parallel tensors on* P*-Sasakian manifolds*, Publ. Math. Debrecen, 49(1996), no. 1-2, 33-37. MR1416302 (97e:53089)

[7] U. C. De; A. K. Mondal, *On 3-dimensional normal almot contact metric manifolds satisfying certain curvature conditions*, Commun. Korean Math. Soc., 24(2009), no. 2, 265-275.

[8] U. C. De; M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J., 43(2003), no. 2, 247-255. MR1982228 (2004d:53049)

[9] L. P. Eisenhart, *Symmetric tensors of the second order whose first covariant derivatives are zero*, Trans. Amer. Math. Soc. 25 (1923), no. 2, 297-306. MR1501245

[10] A. Ghosh; R. Sharma, *Some results on contact metric manifolds*, Ann. Global Anal. Geom., 15(1997), no. 6, 497-507. MR1608675 (99d:53029)

[11] A. Ghosh; R. Sharma; J. T. Cho, *Contact metric manifolds with* η*-parallel torsion tensor*, Ann. Global Anal. Geom., 34(2008), no. 3, 287-299. MR2434858

[12] S. K. Jana; A. A. Shaikh, *On quasi-conformally flat weakly Ricci symmetric manifolds*, Acta Math. Hungar., 115(2007), no. 3, 197214. MR2317216 (2008a:53042)

[13] X.-q. Jia, *Second order parallel tensors on quasi-constant curvature manifolds*, Chinese Quart. J. Math., 17(2002), no. 2, 101-105. MR1951350 (2003h:53046)

[14] J.-B. Jun; U. C. De; Goutam Pathak, *On Kenmotsu manifolds*, J. Korean Math. Soc. 42 (2005), no. 3, 435-445. MR2134708 (2006d:53044)

[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J., 24(1972), 93-103. MR0319102 (47 #7648)

[16] J.-S. Kim; R. Prasad; M. M. Tripathi, *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc. 39 (2002), no. 6, 953-961. MR1932790 (2003f:53066)

[17] J. Levine; G. H. Katzin, *On the number of special quadratic first integrals in affinely connected and Riemannian spaces*, Tensor (N.S.), 19(1968), 113-118. MR0224022 (36 #7069)

[18] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish*, Ann. of Math., (2) 27(1925), no. 2, 91-98. MR1502714

[19] Z. Li, *Second order parallel tensors on P-Sasakian manifolds with a coefficient k*, Soochow J. Math., 23(1997), no. 1, 97-102. MR1436425 (97k:53044)

[20] V. Mangione, *Harmonic maps and stability on f-Kenmotsu manifolds*, Internat. J. Math. Math. Sci., 2008, Art. ID 798317, 7 pp. MR2377360 (2008m:53161)

[21] Z. Olszak; R. Rosca, *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen, 39(1991), no. 3-4, 315-323. MR1154263 (93c:53021)

[22] C. Oniciuc, *Nonlinear connections on tangent bundle and harmonicity*, Ital. J. Pure Appl. Math., 6(1999), 109-122 (2000). MR1758536 (2001e:53026)

[23] N. Papaghiuc, *Semi-invariant submanifolds in a Kenmotsu manifold*, Rend. Mat., (7) 3(1983), no. 4, 607-622. MR0759118 (85i:53024)

[24] N. Papaghiuc, *On the geometry of leaves on a semi-invariant $\xi^{\perp}$-submanifold in a Kenmotsu manifold*, An. Stiint. Univ. "Al. I. Cuza" Iaşi, 38(1992), no. 1, 111-119. MR1282989 (95b:53071)

[25] Gh. Pitis, *Geometry of Kenmotsu manifolds*, Publishing House of Transilvania University of Braşov, Braşov, 2007. MR2353263 (2008i:53117)

[26] R. Sharma, *Second order parallel tensor in real and complex space forms*, Internat. J. Math. Math. Sci., 12(1989), no. 4, 787-790. MR1024982 (91f:53035)

[27] R. Sharma, *Second order parallel tensors on contact manifolds. I*, Algebras Groups Geom., 7(1990), no. 2, 145-152. MR1109567 (92b:53041)

[28] R. Sharma, *Second order parallel tensors on contact manifolds. II*, C. R. Math. Rep. Acad. Sci. Canada, 13(1991), no. 6, 259-264. MR1145119 (93b:53026)

[29] R. Sharma, *On the curvature of contact metric manifolds*, J. Geom., 53(1995), no. 1-2, 179-190. MR1337435 (96d:53031)

[30] R. Sharma, *Certain results on K-contact and (k, μ)-contact manifolds*, J. Geom., 89(2008), 138-147. MR2457028

[31] H. Singh; Q. Khan, *On special weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen, 58(2001), no. 3, 523-536. MR1831059 (2002f:53076)

[32] D. Tarafdar; U. C. De, *Second order parallel tensors on P-Sasakian manifolds*, Northeast. Math. J., 11(1995), no. 3, 260-262. MR1387530 (97c:53075)

[33] M. Tarafdar; A. Mayra, *On nearly Sasakian manifold*, An. Stiint. Univ. "Al. I. Cuza" Iaşi, 45(1999), no. 2, 291-294. MR1811732 (2001k:53086)

[34] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, arXiv:0801.4222.

[35] Y.-c. Wong, *Existence of linear connections with respect to which given tensor fields are parallel or recurrent*, Nagoya Math. J., 24(1964), 67-108. MR0174015 (30 #4222)

[36] H. Wu, *Holonomy groups of indefinite metrics*, Pacific J. Math., 20(1967), 351-392. MR0212740 (35 #3606)

[37] G. Zhao, *Symmetric covariant tensor fields of order 2 on pseudo-Riemannian manifolds*, Viena Preprint ESI 479 (1997). Available at http://www.esi.ac.at/preprints/esi479.ps

Department of Mathematics,
Technical University ”Gh.Asachi”
Iaşi, 700049
Romania
e-mail: c0nstc@yahoo.com

Faculty of Mathematics
University ”Al. I.Cuza”
Iaşi, 700506
Romania
e-mail: mcrasm@uaic.ro
http://www.math.uaic.ro/∼mcrasm