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SURVEY PAPER

FROM THE HYPER-BESSEL OPERATORS OF DIMOVSKI TO THE GENERALIZED FRACTIONAL CALCULUS

Virginia Kiryakova

Dedicated to my teacher Professor Ivan Dimovski
on the occasion of his 80th anniversary

Abstract

In 1966 Ivan Dimovski introduced and started detailed studies on the Bessel type differential operators B of arbitrary (integer) order $m \geq 1$. He also suggested a variant of the Obrechkoff integral transform (arising in a paper of 1958 by another Bulgarian mathematician Nikola Obrechkoff) as a Laplace-type transform basis of a corresponding operational calculus for B and for its linear right inverse integral operator L. Later, the developments on these linear singular differential operators appearing in many problems of mathematical physics, have been continued by the author of this survey who called them hyper-Bessel differential operators, in relation to the notion of hyper-Bessel functions of Delerue (1953), shown to form a fundamental system of solutions of the IVPs for $By(t) = \lambda y(t)$. We have been able to extend Dimovski's results on the hyper-Bessel operators and on the Obrechkoff transform due to the happy hint to attract the tools of the special functions as Meijer's G-function and Fox's H-function to handle successfully these matters. These author's studies have lead to the introduction and development of a theory of generalized fractional calculus (GFC) in her monograph (1994) and subsequent papers, and to various applications of this GFC in other topics of analysis, differential equations, special functions and integral transforms.

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Here we try briefly to expose the ideas leading to this GFC, its basic facts and some of the mentioned applications.

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1. Introduction to fractional calculus

1.1. Definitions of classical FC

The classical fractional calculus can be thought as an upgrade of the Calculus, originated as early as in 1695 (the famous l'Hospital's letter to Leibnitz). Since then, many known mathematicians and applied scientists contributed to the development of this "strange" calculus, but the first book and the first conference dedicated specially to that topic appeared 279 years (1974−1695 = 279) after the mentioned correspondence. And this year we are marking 40 years (2014 − 1974 = 40) of these two events, when there are published more than 100 monographs and topical selections on the area of FC and its applications. The detailed history, theory and its various applications, by the years of 1987-1993 was presented in the "FC Encyclopedia" [42], and currently - in several recent surveys as [47], and the posters at http://www.math.bas.bg/∼fcaa.

The classical FC is based on several (equivalent or alternative) definitions for the operators of integration and differentiation of arbitrary (including real fractional or complex) order, as continuations of the classical integration and differentiation operators and their integer order powers ($n \in \mathbb{N}$), namely - the n-fold integration

$$R^n f(t) = \int_0^t dt_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-2}} d\tau_{n-1} \int_0^{\tau_{n-1}} f(\tau_n) d\tau_n = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1.1}$$

and n-th order derivatives $D^n f(t) = f^{(n)}(t)$. The so-called *Riemann-Liouville (R-L) integration of arbitrary order* $\delta > 0$ is defined by analogy with the above expression by means of replacing $(n-1)!$ by $\Gamma(\delta)$:

$$R^\delta f(t) = D^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau) d\tau = t^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(t\sigma) d\sigma. \tag{1.2}$$

This definition concerns integrations of (real part) positive orders and could not be used directly for a differentiation ($\Re\delta < 0$). However, a little trick is helpful for a suitable interpretation. For noninteger $\delta > 0$ we set $n := [\delta] + 1$ (the smallest integer greater than δ), then we can define properly the *R-L fractional derivative* by means of the differ-integral expression

$$D^\delta f(t) := D^n\, D^{\delta-n} f(t) = \left(\frac{d}{dt}\right)^n R^{n-\delta} f(t)$$
$$= \left(\frac{d}{dt}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f(\tau) d\tau \right\}, \qquad (1.3)$$

since $n-\delta>0$. In suitable functional spaces,

$$D^\delta R^\delta f(t) = f(t), \quad \text{i.e. the inversion formula holds: } \left\{R^\delta\right\}^{-1} = D^\delta .$$

An interesting fact, to compare this with the classical calculus, follows from the formula

$$D^\delta \{t^\alpha\} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\delta)}\, t^{\alpha-\delta}, \quad \delta>0,\ \alpha>-1,$$

whence, for $\alpha=0$ we obtain:

$$D^\delta\{c\} = c\, \frac{t^{-\delta}}{\Gamma(1-\delta)}\, ,$$

i.e. a R-L fractional derivative of a constant is zero *only* for positive integer values $\delta = n = 1,2,3,\ldots$, and not for arbitrary $\delta \notin \mathbb{N}_+$. To improve this situation, and also - for more important reasons - to be able to consider problems where the initial values are given by integer order derivatives instead of fractional order integrals or derivatives, an alternative definition is often used, the so-called *Caputo derivative* of the form

$$^*D^\delta f(t) := D^{\delta-n}\, D^n\, f(t) = R^{n-\delta} f^{(n)}(t)$$
$$= \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f^{(n)}(\tau) d\tau \right\}. \qquad (1.4)$$

The essence of the mathematical problem for defining integrals and derivatives of fractional order laying on the base of FC consists in the following: for each function $f(t)$ of sufficiently large class and for each number δ (rational, irrational, complex), to set up a correspondence to a function $g(t) = D^\delta f(t)$ satisfying the conditions (*axioms of FC*):

- If $f(t)$ is an analytic function of t, the derivative $D^\delta f(t)$ is an analytic function of t and δ.
- The operation D^δ gives the same result as the usual differentiation of order n, when $\delta = n$ is a positive integer, and the same effect as the n-fold integration, if $\delta = -n$ is a negative integer (i.e. $D^{-n} = R^n$). Moreover, $D^\delta f(t)$ should vanish at the initial point $t=0$ (or $t=c$) together with its first $(n-1)$ derivatives.
- The operator of order $\delta = 0$ should be the identity operator.
- The fractional operators are linear:

$$D^\delta\{af(t)+bg(t)\} = aD^\delta f(t) + bD^\delta g(t).$$

- For fractional integrations of arbitrary orders $\alpha > 0, \beta > 0$ ($\Re\alpha > 0, \Re\beta > 0$) the additive index law (semigroup property) holds:
  $D^{-\alpha}D^{-\beta}f(t) = D^{-(\alpha+\beta)}f(t)$, i.e. $R^{\alpha}R^{\beta} = R^{\beta}R^{\alpha} = R^{\alpha+\beta}$,
  if the denotation $R^{\delta}f(t) := D^{-\delta}f(t)$, $\Re\delta > 0$ is used in the case of derivative of negative (or with negative real part) order.

It is easily seen that the above mentioned fractional integrals and derivatives satisfy all the above conditions, and in particular, coincide with the repeated (n-fold) integration in the Dirichlet formula in (1.1) and with $(d/dt)^n$.

Along with the R-L fractional integration operator (1.2), several modifications and generalizations are widely used in FC. The most useful of them seem to be the *Erdélyi-Kober (E-K) integration operator* (see e.g. Sneddon [44]) whose general form

$$I_{\beta}^{\gamma,\delta}f(t) = t^{-\beta(\gamma+\delta)}\int_0^t \frac{(t^{\beta}-\tau^{\beta})^{\delta-1}}{\Gamma(\delta)}\,\tau^{\beta\gamma}f(\tau)\,d(\tau^{\beta})$$
$$= \int_0^1 \frac{(1-\sigma)^{\delta-1}\sigma^{\gamma}}{\Gamma(\delta)}\,f(t\sigma^{\frac{1}{\beta}})\,d\sigma, \quad \gamma \in \mathbb{R}, \text{ with arbitrary } \beta > 0, \tag{1.5}$$

is *used essentially in our works on the generalized FC*. Initially, the E-K operator was introduced with $\beta = 2$.

1.2. Attempts for generalized fractional calculi

Several authors, like Love [32], Saxena [43], Kalla and Saxena [18], Saigo [39, 40], McBride [35], also Tricomi, Sprinkhuizen-Kuiper, Koornwinder, etc., have studied and used different modifications (mainly in 60's-70's) of the so-called *hypergeometric operators of fractional integration*

$$\mathcal{H}f(t) = \frac{\mu t^{-\gamma-1}}{\Gamma(1-\delta)}\int_0^t {}_2F_1\left(\delta, \beta+m; \eta;\, a(\frac{\tau}{t})^{\mu}\right)\tau^{\gamma}f(\tau)d\tau, \tag{1.6}$$

involving the Gauss hypergeometric function.

An example of *fractional integration operators involving other special functions*, is given by the operators of Lowndes [33]:

$$I_{\lambda}(\eta,\nu+1)f(t) = \frac{2^{\nu+1}}{\lambda^{\nu}}\,t^{-(\nu+\eta+1)}\int_0^t \tau^{2\eta+1}(t^2-\tau^2)^{\frac{\nu}{2}}J_{\nu}(\lambda\sqrt{t^2-\tau^2})f(\tau)d\tau,$$

related to the second order Bessel type differential operator

$$B_{\eta} = t^{-2\eta-1}(d/dt)t^{2\eta+1}(d/dt).$$

One of the most general fractional integration operators of R-L type (1.2) can be obtained when the *kernel-function is an arbitrary Meijer G-function*, as in Kalla [15], also in Parashar, Rooney, etc.:

$$\mathcal{I}_G f(t) = t^{-\gamma-1}\int_0^t G_{p,q}^{m,n}\left[a(\frac{\tau}{t})^r \,\middle|\, \begin{matrix}(a_j)_1^p\\(b_k)_1^q\end{matrix}\right]\tau^{\gamma}f(\tau)d\tau, \tag{1.7}$$

or its further generalization, the *Fox H-function*, as in Kalla [14], also in Srivastava and Buschman [45], and others:

$$\mathcal{I}_H f(t) = t^{-\gamma-1} \int_0^t H^{m,n}_{p,q}\left[a(\frac{\tau}{t})^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] \tau^\gamma f(\tau) d\tau. \qquad (1.8)$$

In his papers [15, 16] of years 1970-1979, Kalla suggested that all the above operators of R-L type can be considered as "*generalized operators of fractional integration*" of the general form:

$$\mathcal{I} f(t) = t^{-\gamma-1} \int\limits_0^t \Phi(\frac{\tau}{t})\, \tau^\gamma f(\tau) d\tau = \int\limits_0^1 \Phi(\sigma)\, \sigma^\gamma f(z\sigma) d\sigma, \qquad (1.9)$$

where the kernel $\Phi(\sigma)$ is an arbitrary continuous function so that the above integral makes sense in sufficiently large functional spaces. Kalla established some general properties of (1.9), analogous to those of the classical fractional integrals, and studied some special cases. By suitable choices of the kernel-function Φ, the operators (1.9) can be shown to include all other known fractional integrals as particular cases.

However, taking an arbitrary G- or H-function in the kernel of (1.9) does not allow to develop a theory of a generalized fractional calculus and to think about any possible applications. Thus, the very particular, or the very general choice of the kernel special function, prevented the other authors to develop further their operators' theory beyond *publishing some few papers on them, containing formal manipulations.*

Let us provide shortly the definitions of the two mentioned generalized hypergeometric functions. More details on them can be found in the books [12], [46], [38] and other newer ones. By a *Fox's H-function* we mean the generalized hypergeometric function defined by a contour integral

$$\begin{aligned} H^{m,n}_{p,q}(\sigma) &= H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right] \\ &= H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)\, \sigma^s ds, \end{aligned} \qquad (1.10)$$

where the integrand has the form

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - B_k s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + B_k s) \prod\limits_{j=n+1}^{p} \Gamma(a_j - A_j s)},$$

and $\mathcal{L}$ is a suitable contour in $\mathbb{C}$; the orders $(m, n; p, q)$ are nonnegative integers such that $0 \le m \le q,\ 0 \le n \le q$; the parameters $A_j, j = 1, \dots, p$ and $B_k, k = 1, \dots, q$ are *positive* and $a_j, j = 1, \dots, p,\ b_k, k = 1, \dots, q$ are arbitrary complex numbers such that

$$A_j(b_k + l) \ne B_k(a_j - l' - 1); \quad l,\ l' = 0, 1, 2, \dots\ ; \quad j = 1, \dots, p,\ k = 1, \dots, q.$$

In particular, when all $A_j = B_k = 1$, we obtain the so-called *Meijer's G-function*:

$$H^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} (a_j,1)_1^p \\ (b_k,1)_1^q \end{array} \right.\right] = G^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right.\right],$$

that is,

$$G^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m}\Gamma(b_k-s)\ \prod\limits_{j=1}^{n}\Gamma(1-a_j+s)}{\prod\limits_{k=m+1}^{q}\Gamma(1-b_k+s)\ \prod\limits_{j=n+1}^{p}\Gamma(a_j-s)}\,\sigma^s ds. \tag{1.11}$$

Note that the known special functions all can be presented in terms of the G- and H-functions.

It was a lucky hint for the author (myself) to make a very proper choice of a peculiar Meijer's G-function or Fox's H-function, namely of the form $\Phi(\sigma) = G^{m,n}_{n,m+n}[\sigma]$, $\Phi(\sigma) = H^{m,n}_{n,m+n}[\sigma]$ ($n = 0$ in case of R-L type only), so to introduce and consider operators of form (1.9) for which a full theory (*Generalized Fractional Calculus*, GFC) could be developed and various applications to different areas of analysis, differential equations, problems of mathematical physics, etc. have been demonstrated. The wide applicability of the GFC theory is hidden in the fact that *our basic operators of generalized fractional integration* (first introduced in [20], [21], [22]):

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) := \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{array}{c} (\gamma_k+\delta_k)_1^m \\ (\gamma_k)_1^m \end{array} \right.\right] f(z\sigma)d\sigma = \prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k} f(t), \tag{1.12}$$

happen to be compositions of finite number of commutable E-K operators (1.5), while their operational rules and their theory could be easier derived by using the special functions theory (G- and H-functions) in the representation of the form (1.9).

More details on the historical aspects of introduction of the GFC, its concepts and examples can be seen in our previous paper [26].

It is a pleasure to express my personal gratitude to *Prof. Dimovski who was scientific advisor of both my M.Sc. thesis* [19] *of 1975 and Ph.D. thesis* [20] *of 1986*, and a colleague for 40 years. Thus, *all my research had been influenced by the starting point to deal with the hyper-Bessel operators and our further collaboration*. He introduced me to the field of integral transforms, proposing me as a task of the M.Sc. thesis, to continue his studies on the Obrechkoff integral transform. The relationship of its kernel function to Meijer's G-function generated my further interest to special functions. And the hyper-Bessel operators and their fractional powers, expressed in terms of integral operators involving G-functions gave rise to the theory of the generalized operators of integration and differentiation of fractional (multi-)order, the GFC, developed in my monograph [23].

2. Dimovski's hyper-Bessel operators and the Obrechkoff integral transform

In 1958 Nikola Obrechkoff proposed a far reaching generalization of the Laplace and Meijer transforms that later could be related to the hyper-Bessel differential operators. Since his paper [36] appeared originally in Bulgarian (and only recently also as English translation), it became known to the specialists abroad after a long delay, due to the works of Dimovski and our joint works, and their citations in international journals. Meanwhile, many special cases of the Obrechkoff transform were rediscovered by different authors. Such integral transforms were proposed, for example, by Ditkin and Prudnikov (1963), Botashev (1965), Krätzel (1965-67), Betancor (1989), Mendez (1988), etc. (see details in [23, Ch.3], [10], and other our works).

The original Obrechkoff integral transform from the paper [36] had the form

$$F(s) = \int\limits_0^\infty \Phi(ts)\, f(t) dt \tag{2.1}$$

with a kernel $\Phi(s)$ given by the integral representation

$$\Phi(s) = \int\limits_0^\infty \cdots \int\limits_0^\infty u_1^{\beta_1} \dots u_p^{\beta_p} \ \exp\left(-u_1 - \cdots - u_p - \frac{s}{u_1 \dots u_p} \right) du_1 \dots du_p. \tag{2.2}$$

Obrechkoff established that the above kernel-function satisfies a kind of (what we call it now) hyper-Bessel differential equation of order $p+1$:

$$t^{\beta_p+1} \frac{d}{dt} t^{-\beta_p+\beta_{p-1}+1} \frac{d}{dt} \cdots t^{-\beta_2+\beta_1+1} \frac{d}{dt} t^{-\beta_1} \frac{d}{dt} \Phi(t) = (-1)^{p-1} \Phi(t),$$

and studied its asymptotic properties. Of this, it is close to realize that the transform (2.1) can be used as a transform basis for an operational calculus for the Bessel type differential operator

$$B = \frac{d}{dt} t^{\beta_1} \frac{d}{dt} t^{\beta_2-\beta_1-1} \cdots \frac{d}{dt} t^{\beta_p-\beta_{p-1}-1} \frac{d}{dt} t^{-\beta_p-1}, \tag{2.3}$$

as a kind of "adjoint" to the previously written differential operator.

After Obrechkoff's death, since 1966, Dimovski [3], [4] developed a Mikusinski-type approach to an operational calculus for a variant of (2.3), namely for the most *general differential operator of Bessel type* of arbitrary (integer) order $m > 1$ (called later as *hyper-Bessel operator*, see [23]):

$$B = t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} \cdots t^{\alpha_{m-1}} \frac{d}{dt} t^{\alpha_m} \ , \quad 0 < t < \infty \ , \tag{2.4}$$

with real indices $\alpha_k, k = 1, ..., m$ and $\beta = m - (\alpha_0 + \alpha_1 + \cdots + \alpha_m) > 0$.

Further, in the period 1968-1974, he duplicated his "algebraic" approach, by using a slight modification of Obrechkoff's transform as a *transform basis of the operational calculus for the Bessel type differential operator* (2.4), see e.g. [5]. Dimovski was the first to use the notion "Obrechkoff transform" for (2.1), accepted later in his and mine papers and now used also by many other authors. Along with developing the relation of (2.1) to the hyper-Bessel operators (2.4) and some of its basic operational properties and inversion formulas (summaries of these can be found in [23, Ch.3], [10]), we have found also a close relation of these operators and of the Obrechkoff transform to the Meijer G-functions (e.g. [8], [9], [20]) and to the generalized fractional calculus ([20], [21], etc.) It seems that it remained unknown for Obrechkoff himself that functions rather close to his kernel (2.2) in (2.1) had been introduced by Erdélyi [11], in terms of the generalized hypergeometric functions. In my case, I myself discovered that this kernel-function is a case of Meijer's G-function, by considering the form of its Mellin transformation and reading the book of Bateman-Erdélyi [12, Vol.1]. It was the crucial moment for the new developments in this field.

Practically, Dimovski proposed and considered a *modification* of the Obrechkoff transform (2.1), suitable for building an operational calculus for the Bessel-type differential operators in their general form (2.4).

Definition **2.1**. Let $m > 1$ be an integer. By means of $\{\alpha_k, k = 0, 1, \dots, m\}$, we define the set of parameters $\{\gamma_k, k = 1, \dots, m;\ \beta > 0\}$:
$\beta = m - (\alpha_0 + \alpha_1 + \alpha_m); \quad \gamma_k = \dfrac{1}{\beta}\left(\alpha_k + \dots + \alpha_m - m + k\right),\ k = 1, \dots, m.$
Since

$$\begin{aligned} B &= t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\dots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m} \\ &= t^{-\beta-\beta\gamma_1+1}\frac{d}{dt}t^{\beta\gamma_1-\beta\gamma_2+1}\frac{d}{dt}\dots t^{\beta\gamma_{m-1}-\beta\gamma_m+1}\frac{d}{dt}t^{\beta\gamma_m} \\ &= t^{-\beta}(t^{-\beta\gamma_1+1}\frac{d}{dt}t^{\beta\gamma_1})(t^{-\beta\gamma_2+1}\frac{d}{dt}t^{\beta\gamma_2})\dots(t^{-\beta\gamma_m+1}\frac{d}{dt}t^{\beta\gamma_m}) \\ &= t^{-\beta}(t\frac{d}{dt}+\beta\gamma_1)\dots(t\frac{d}{dt}+\beta\gamma_m), \end{aligned}$$

the hyper-Bessel differential operators (2.4) can be alternatively defined by a representation

$$B = t^{-\beta}\prod_{k=1}^{m}\left(t\frac{d}{dt}+\beta\gamma_k\right) = t^{-\beta}Q_m(t\frac{d}{dt}), \quad 0 < t < \infty, \tag{2.5}$$

which is symmetric with respect to the zeros $\mu_k = -\beta\gamma_k, k = 1, \dots, m$ of the m-th degree polynomial $Q_m(\mu)$. Thus, for convenience, one can suppose that the γ_k's are arranged in a nondecreasing order, say as:

$$\gamma_1 \le \gamma_2 \le \dots \le \gamma_m.$$

To get a full impression on the nature of the hyper-Bessel differential operators, let us mention also that usually they appear in the problems of mathematical physics in the more conventional form

$$B = t^{-\beta}\left[t^m \frac{d^m}{dt^m} + a_1 t^{m-1}\frac{d^{m-1}}{dt^{m-1}} + \cdots + a_m\right],$$

which is equivalent to representations (2.4),(2.5), if we take

$$a_{m-k} = \sum_{j=0}^{m}\left[\frac{(-1)^j}{j!(k-j)!}\prod_{i=1}^{m}(\beta\gamma_i + k - j)\right], \quad k = 0, 1, \ldots, m-1.$$

Thus, each Bessel-type differential operator of order $m > 1$ can be defined uniquely by means of one of the $(m+1)$-parameters' sets: $(\alpha_0, \alpha_1, ..., \alpha_m)$, $(\beta;\ \gamma_1, ..., \gamma_m)$, $(\beta;\ a_1, ..., a_m)$, resp. in the above 3 forms.

Definition **2.2**. Let $\beta > 0,\ \gamma_1 \le \gamma_2 \le \cdots \le \gamma_m$ be real numbers and

$$\begin{aligned} K(s) \ &= \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{m-1} u_k^{\gamma_k - \gamma_m - 1}\right] \\ &\times\ \exp\left(-u_1 - \cdots - u_{m-1} - \frac{s}{u_1 \ldots u_{m-1}}\right)\ du_1 \ldots du_{m-1}. \end{aligned} \tag{2.6}$$

The following modification of the original Obrechkoff transform (2.1):

$$\mathcal{O}\{s\} = \mathcal{O}\{f(t); s\} = \beta \int_0^\infty K\left[(ts)^\beta\right]\, t^{\beta(\gamma_m+1)-1}\, f(t)dt \tag{2.7}$$

is further called, for simplicity, by the same name *"Obrechkoff transform"*.

However, from the point of view of the special functions, I have considered and worked with the Obrechkoff transform in its new alternative form as a G-transform, that is with a G-function as a kernel (see [19], [20], [23], etc).

Definition **2.3**. The G-transformation

$$\mathcal{O}\{f(t); s\} = \beta s^{-\beta(\gamma_m+1)+1} \int_0^\infty G_{0,m}^{m,0}\left[(ts)^\beta \,\middle|\, \begin{matrix} - \\ (\gamma_k - \frac{1}{\beta} + 1)_1^m \end{matrix}\right] f(t)dt \tag{2.8}$$

is said to be an *Obrechkoff integral transform*, corresponding to hyper-Bessel operators (2.4), (2.5).

This new definition allows us to simplify considerably most of the calculations and proofs of the properties of the Obrechkoff transform, by using the known theory of the G-functions, see e.g. [13], as a differential law, real and complex inversion formulas, convolution and Abel-type theorems, images of many sample functions, etc.

Note that the Obrechkoff transform includes, among the mentioned special cases studied in 70s-80s, also the classical Laplace and Meijer transforms as very special cases with $m = 1, m = 2$, serving for operational calculi for the 1st order operator d/dt and the 2nd order Bessel differential operator.

The notion *hyper-Bessel integral operator* L is used for the linear right inverse operator of B, defined by means of the IVP

$$By(t) = f(t), \quad \lim_{t\to+0} B_i\, y(t) = 0,\ i = 1,\dots,m,$$

where

$$B_i = t^{\alpha_i}\frac{d}{dt}t^{\alpha_{i+1}}\dots\frac{d}{dt}t^{\alpha_m} = t^{\beta\gamma_i}\prod_{j=i+1}^{m}\left(t\frac{d}{dt}+\beta\gamma_j\right)$$

denote the *Bessel-type initial conditions*. This integral operator has the following explicit form used in the works of Dimovski:

$$y(t) = Lf(t) = \frac{t^\beta}{\beta^m}\int_0^1\cdots\int_0^1\left[\prod_{k=1}^{m}t_k^{\gamma_k}\right]\ f\left[t(t_1\dots t_m)^{1/\beta}\right]dt_1\dots dt_m. \quad (2.9)$$

It is considered usually in the space C_α of power-weighted continuous functions of the form

$$C_\alpha^{(k)} := \left\{f(t) = t^p\widetilde{f}(t),\ p>\alpha,\ \widetilde{f}\in C^{(k)}[0,\infty)\right\},\quad C_\alpha^{(0)} := C_\alpha, \quad (2.10)$$

and $L : C_\alpha \mapsto C_{\alpha+\beta}\subset C_\alpha$.

Again using the Meijer G-function, in our works we have considered and started to use it in the more concise form

$$Lf(t) = \frac{t^\beta}{\beta^m}\int_0^1 G_{m,m}^{m,0}\left[\sigma\,\middle|\,\begin{matrix}(\gamma_k+1)_1^m\\(\gamma_k)_1^m\end{matrix}\right] f(t\sigma^{1/\beta})d\sigma. \quad (2.11)$$

In the paper [4] of 1968, Dimovski considered also *fractional powers of the integral operator* L, namely the operators $L^\lambda,\ \lambda>0$. To express them, he used his *notion of convolution* (basic one in the monograph [7]) defined for the hyper-Bessel integral operators as follows ([3]):

$$f * g(t) := T_\nu\left(f\circ g\right)(t), \quad \nu > \max_{1\le k\le m}\gamma_k,$$

where

$$T_\nu f(t) = \frac{t^{\nu\beta}}{\prod_{k=1}^m\Gamma(\nu-\gamma_k)} \quad (2.12)$$
$$\times\int_0^1\dots\int_0^1 f[t(t_1\dots t_m)^{1/\beta}]\prod_{k=1}^m\left[t^{2\gamma_k}(1-t_k)^{\nu-\gamma_k-1}\right]dt_1\dots dt_m,$$

and

$$f \circ g(t) = t^{\beta} \int_0^1 \dots \int_0^1 f[t(t_1 \dots t_m)^{1/\beta}]\, g[t((1-t_1)\dots(1-t_m))^{1/\beta}]$$
$$\times \prod_{k=1}^{m} [t_k(1-t_k)]^{\gamma_k}\, dt_1 \dots dt_m.$$

Namely, Dimovski represented the fractional powers of L as

$$L^{\lambda} f = \{l_{\lambda}\} * f, \quad \text{with} \quad l_{\lambda} = \left\{ \frac{t^{\beta(\lambda-\delta-1)}}{\prod_{k=1}^{m} \Gamma(\lambda-\delta+\gamma_k)} \right\}, \tag{2.13}$$

where $\delta \geq \max_k \gamma_k$. And he proved that under this definition, the semi-group property of FC is satisfied: $L^{\lambda} L^{\mu} = L^{\lambda+\mu}$, $\lambda > 0, \mu > 0$, $L^n = L \cdot L \cdots L$.

However, from our point of view based on the G-functions, we were able to find a representation similar in its form to (2.11),

$$L^{\lambda} f(t) = \frac{t^{\beta}}{\beta^m} \int_0^1 G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\lambda)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] f(t\sigma^{1/\beta}) d\sigma. \tag{2.14}$$

This representation of the fractional powers of the hyper-Bessel operators is first published in our paper joint with Dimovski [9] and coincides with the results proposed by McBride [35] found in completely different way.

Then my step was to think about what was to be if I replace the parameters in the upper row of the above kernel G-function

$$(\gamma_1+\lambda, \gamma_2+\lambda, \dots, \gamma_m+\lambda) \quad \text{by} \quad (\gamma_1+\delta_1, \gamma_2+\delta_2, \dots, \gamma_m+\delta_m)$$

with arbitrary and different $\delta_1 > 0, \delta_2 > 0, \dots, \delta_m > 0$? *This led me to the idea of the operators of fractional integration of multi-order (vector order)* $(\delta_1, \delta_2, \dots, \delta_m)$, whose theory (GFC) was developed in [20] and [23].

3. Generalized fractional calculus

3.1. Basic definitions of the operators of GFC, [23]

Definition **3.1**. Let $m \geq 1$ be integer, $\beta > 0$, $\gamma_1, \dots, \gamma_m$ and $\delta_1 \geq 0$, $\dots, \delta_m \geq 0$ be arbitrary real numbers. By a *generalized (multiple, m-tuple) Erdélyi-Kober (E-K) operator of integration of multi-order* $\delta = (\delta_1, \dots, \delta_m)$ we mean an integral operator of the form

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] f(t\sigma^{\frac{1}{\beta}})\, d\sigma, \quad \text{if} \quad \sum_{k=1}^{m} \delta_k > 0, \tag{3.1}$$

and $I_{\beta,m}^{(\gamma_k),(0,0,\dots,0)} f(t) = f(t)$. Then, each operator of the form

$$\mathcal{I} f(t) = t^{\beta\delta_0}\, I_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) \quad \text{with arbitrary} \quad \delta_0 \geq 0,$$

is said to be a *generalized (m-tuple) operator of fractional integration of Riemann-Liouville type*, or briefly: a *generalized (R.-L.) fractional integral.*

Generalizing further the operators of fractional, calculus, in Kiryakova [22], [17], [23, Ch.5] we introduced also operators involving classes of Fox's H-functions instead of the G-functions in (3.1). They are called in the same way, namely *generalized (multiple) E.-K. operators (fractional integrals)*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = \begin{cases} \int\limits_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{array}{c} (\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right.\right] f(t\sigma) d\sigma, & \text{if } \sum\limits_{k=1}^m \delta_k > 0, \\ f(z), \quad \text{if } \delta_1 = \delta_2 = \cdots = \delta_m = 0. \end{cases} \tag{3.2}$$

Thus, along with the *multi-order of integration* $(\delta_1, ..., \delta_m)$ and the *multi-weight* $(\gamma_1, ..., \gamma_m)$, we introduced also a *multi-parameter* $(\beta_1 > 0, ..., \beta_m > 0)$ with different β_k's, instead of the same $\beta > 0$ in the case with a G-function. Note that the operator (3.2) involving a H-function reduces to its simpler form (3.1),

$$\text{for} \quad \beta_1 = \beta_2 = ... = \beta_m = \beta > 0: \quad I_{(\beta,\beta,...,\beta),m}^{(\gamma_k),(\delta_k)} = I_{\beta,m}^{(\gamma_k),(\delta_k)}.$$

DEFINITION **3.2**. With the same parameters as in previous definition, and taking the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer,} \\ [\delta_k]+1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \qquad k = 1, \dots, m, \tag{3.3}$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} t \frac{d}{dt} + \gamma_r + j\right)\right]. \tag{3.4}$$

Then, we define the *multiple (m-tuple) Erdélyi-Kober fractional derivative* of multi-order $\delta = (\delta_1 \geq 0, \dots, \delta_m \geq 0)$ and of *R-L type* by means of the differ-integral operator:

$$\begin{aligned} D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) &= D_\eta\, I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \\ &= D_\eta \int\limits_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{array}{c} (\gamma_k+\eta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right.\right] f(t\sigma)\, d\sigma. \end{aligned} \tag{3.5}$$

In the case of equal β_k's, we obtain simpler representations involving the Meijer G-function, corresponding to generalized fractional integral (3.1):

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = D_\eta \, I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \tag{3.6}$$
$$= \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta} t\frac{d}{dt} + \gamma_r + j\right)\right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t).$$

More generally, all differ-integral operators of the form

$$\mathcal{D}f(t) = D_{\beta,m}^{(\gamma_k),(\delta_k)} t^{-\beta\delta_0} f(t) = t^{-\beta\delta_0} D_{\beta,m}^{(\gamma_k-\delta_0),(\delta_k)} f(t) \quad \text{with} \quad \delta_0 \geq 0,$$

are called *generalized (multiple, multi-order) fractional derivatives.*

Let us note that recently, in our joint paper with Luchko [30], we have introduced also the *Caputo type generalized fractional derivative*, as the integro-differential operator

$$^*D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) := I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} D_\eta f(t) \tag{3.7}$$
$$= \int_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\gamma_k+\eta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right.\right] \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} t\frac{d}{dt} + \gamma_r + j\right) f(t\sigma)\right] d\sigma,$$

where the parameters are the same as in the previous definitions and the order of the auxiliary differential operator D_η is interchanged with the multiple E-K fractional integration.

3.2. Basic operational rules in GFC

Let us mention that the main *functional spaces* discussed in our works on GFC are the same weighted spaces of continuous functions C_α as for the hyper-Bessel operators, (2.10), also - these of Lebesgue integrable or analytic functions with power weights, $L_{\alpha,p}(0,\infty)$ and resp. $H_\alpha(\Omega)$, Ω being a starlike domain in $\mathbb{C}$ containing the zero point. We suppose, in principle the following parameters' conditions as satisfied:

$$\gamma_k \geq -\frac{\alpha}{\beta_k} - 1, \; \delta_k \geq 0, \quad k = 1, ..., m. \tag{3.8}$$

Let us state first the *basic result* for the generalized fractional integrals (3.2) *suggesting their alternative name "multiple (m-tuple)" E-K fractional integrals.*

PROPOSITION **3.1**. (Composition/Decomposition theorem) *Under the conditions* (3.8), *the classical E-K fractional integrals* (1.5)*:* $I_{\beta_k}^{\gamma_k,\delta_k}, k = 1, \ldots, m,$ *commute in the spaces* C_α, $L_{\alpha,p}$, H_α, *and their product*

$$I_{\beta_m}^{\gamma_m,\delta_m}\left\{I_{\beta_{m-1}}^{\gamma_{m-1},\delta_{m-1}} \ldots \left(I_{\beta_1}^{\gamma_1,\delta_1} f(t)\right)\right\} f(t) = \left[\prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}\right] f(t)$$

$$= \int_0^1 \dots (m) \int_0^1 \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f\left(t \sigma_1^{\frac{1}{\beta_1}} \dots \sigma_m^{\frac{1}{\beta_m}} \right) d\sigma_1 \dots d\sigma_m \quad (3.9)$$

can be represented as an m-tuple E-K operator (3.2)*, i.e. by means of a single integral involving the H-function:*

$$\left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k, \delta_k} \right] f(t) = I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t)$$

$$= \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] f(t\sigma) d\sigma, \ f \in C_\alpha \ (\textit{resp. } L_{\alpha,p}, H_\alpha).$$

Conversely, under the same conditions, each multiple E-K operator of form (3.2) *can be represented as a product* (3.9).

Here we mention only briefly the basic operational rules for the operators of GFC, from Kiryakova [23], that have been proven by using the single integral representation (3.2) and the tools of G- and H-function, much easier than to exploit complicated repeated integrations (and differentiations).

Lemma **3.1**. *The multiple E-K fractional integral* (3.2) *preserves the power functions in* C_α*, with* $\alpha \geq \max_k [-\beta(\gamma_k + 1)]$ (*this means* (3.8) *holds*), *up to a constant multiplier:*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{t^p\} = c_p t^p, \ p > \alpha, \quad \text{where} \quad c_p = \prod_{k=1}^m \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)},$$

and it is an invertible mapping $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} : C_\alpha \mapsto C_\alpha^{(\eta_1 + \dots + \eta_m)} \subset C_\alpha$.

Analogously, under the same conditions, (3.2) maps the class $H_\alpha(\Omega)$ into itself, preserving the power functions (up to constant multipliers like above) and the image of a power series has the same radius of convergence.

It is also shown that (3.2) has a Mellin type convolutional representation, based on its Mellin image. Another expected result is the following.

Lemma **3.2**. *Under conditions* (3.8) *the generalized fractional integral* $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t)$ *exists almost everywhere on* $(0,\infty)$ *and it is a bounded linear operator from the Banach space* $L_{\alpha,p}$ *into itself. More exactly,*

$$\left\| I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\|_{\alpha,p} \leq h_{\alpha,p} \, \|f\|_{\alpha,p}, \quad \textit{i.e.} \quad \left\| I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\| \leq h_{\alpha,p}$$

with $h_{\alpha,p} = \prod_{k=1}^m \Gamma(\gamma_k - \frac{\alpha}{p\beta_k} + 1) \ / \ \Gamma(\gamma_k + \delta_k - \frac{\alpha}{p\beta_k} + 1) < \infty.$

We list below the following operational rules confirming that the operators of our GFC satisfy the axioms of FC.

PROPOSITION **3.2**. *Suppose conditions* (3.8) *hold. Then, in* C_α, $L_{\alpha,p}$, H_α, *the following basic operational rules hold:*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left\{\lambda f(ct)+\eta g(ct)\right\} = \lambda\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f\right\}(ct)+\eta\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} g\right\}(ct)$$

(*bilinearity of* (3.2));

$$I^{(\gamma_1,\dots,\gamma_s,\gamma_{s+1},\dots,\gamma_m),(0,\dots,0,\delta_{s+1},\dots,\delta_m)}_{(\beta_1,\dots,\beta_m),m} f(t) = I^{(\gamma_{s+1},\dots,\gamma_m)(\delta_{s+1},\dots,\delta_m)}_{(\beta_{s+1},\dots,\beta_m),m-s} f(t)$$

(*if* $\delta_1=\delta_2=\cdots=\delta_s=0$, *then the multiplicity reduces to* $(m-s)$);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} z^\lambda f(t) = z^\lambda I^{(\gamma_k+\frac{\lambda}{\beta_k}),(\delta_k)}_{(\beta_k),m} f(t), \quad \lambda\in\mathbb{R}$$

(*generalized commutability with power functions*);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} f(t) = I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t)$$

(*commutability of operators of form* (3.2));

$$\text{the left-hand side of above} = I^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n} f(t)$$

(*compositions of* m*-tuple and* n*-tuple integrals* (3.2) *give* $(m+n)$*-tuple integrals of same form*);

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m} f(t), \text{ if } \delta_k>0,\ \sigma_k>0,\ k=1,\dots,m$$

(*law of indices, product rule or semigroup property*);

$$\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\right\}^{-1} f(t) = I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m} f(t)$$

(*formal inversion formula*).

The above inversion formula follows from the index law for $\sigma_k=-\delta_k<0$, $k=1,\dots,m$ and the definition for zero multi-order of integration, since:

$$I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = I^{(\gamma_k),(0,\dots,0)}_{\beta_k,m} f(t) = f(t).$$

But symbols (3.2) have not yet been defined for negative multi-orders of integration $-\delta_k<0$, $k=1,\dots,m$. The problem is to propose an appropriate meaning for them and hence to avoid the appearance of divergent integrals. The situation is the same as in the classical case when the R-L and E-K operators of fractional order $\delta>0$ can be inverted by appealing to an additional differentiation of suitable integer order $\eta=[\delta]+1$. In the case of GFC, it was resolved by means of the auxiliary differential operator D_η as in (3.4), used in the definitions of the generalized fractional derivatives of R-L or Caputo type, (3.5), (3.6), (3.7).

3.3. Examples of operators of GFC

Example 1. In the case $m = 1$, the "multiple" E-K operators and the generalized fractional integrals and derivatives reduce to the *classical ("single") E-K operators* (1.5), resp. E-K derivatives, and for $\gamma = 0$ - to the R-L integral, R-L and Caputo derivatives.

Example 2. For $m = 2$ the operators of the GFC reduce to the *hypergeometric operators* (1.6), since $G^{2,0}_{2,2}$ can be expressed via the Gauss hypergeometric function function, see [23, p.18].

Example 3. For $m = 3$ the kernel-function $G^{3,0}_{3,3}$ with special parameters gives the so-called *Horn's (Appell's) F_3-function*, see [23, p.21]. Operators with such kernel have been considered by Marichev [34], Saigo et al. [41], and appear as interesting cases of the generalized fractional integrals and derivatives:

$$\mathcal{F}f(t) = \int_0^t \frac{(t-\tau)^{c-1}}{\Gamma(c)} F_3(a, a', b, b', 1-\frac{t}{\tau}, 1-\frac{\tau}{t})\, f(\tau) d\tau$$
$$= t^c\, I_{1,3}^{(a,b,c-a'-b'),(b,c-a'-b,a')}\, f(t).$$

Example 4. Let $m > 1$ be arbitrary integer, but all $\delta_k = 1$, $\beta_k = \beta > 0, k = 1, ..., m$. Then, the operators of form (3.1)-(3.2) and (3.5)-(3.6):

$$Lf(t) = ct^{\beta} I_{\beta,m}^{(\gamma_k),(1,...,1)}\, f(t), \quad Bf(t) = (1/c) D_{\beta,m}^{(\gamma_k),(1,...,1)} t^{-\beta}\, f(t) \qquad (3.10)$$

are the *hyper-Bessel integral operators*, resp. *hyper-Bessel differential operators*! That is, the hyper-Bessel operators of order m giving us the hint for the appropriate definitions of the operators of the GFC, appear as their special cases when the integration and differentaition is of integer "multi-order" $(1, ..., 1)$, see [23, Ch.3]. Let us mention that in such case the R-L and Caputo type generalized "fractional" derivatives both coincide with the hyper-Bessel differential operators B.

Example 5. A more general case than Example **4** gives *fractional indices analogues of the hyper-Bessel operators.* Let $\mu_1, ..., \mu_m$ be arbitrary real and $\rho_1 > 0, ..., \rho_m > 0$. With these parameters, for a power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{convergent in} \quad \{|t| < R\} \subset \mathbb{C}, \qquad (3.11)$$

we consider the following *Gelfond-Leontiev (G-L) operator of generalized integration*

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(t) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k+1)/\rho_1)...\Gamma(\mu_m + (k+1)/\rho_1)}\, t^{k+1}, \quad (3.12)$$

and resp. the *G-L generalized differentiation*,

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(t) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k-1)/\rho_1)...\Gamma(\mu_m + (k-1)/\rho_1)} t^{k-1}. \tag{3.13}$$

It happens that their analytical continuations in starlike complex domains Ω are generalized fractional integrals and derivatives of the form

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(t) = t\, I_{(\rho_k),m}^{(\mu_k-1),(1/\rho_k)} f(t), \tag{3.14}$$

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(t) = t^{-1}\, D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)} f(t) - \left[\prod_{k=1}^{m} \frac{\Gamma(\mu_k)}{\Gamma(\mu_k - 1/\rho_k)} \right] \frac{f(0)}{t}. \tag{3.15}$$

Evidently, for parameters taken as $\mu_k = \gamma_k + 1$, $\forall \rho_k = 1, k = 1, ..., m$ these G-L operators coincide with the hyper-Bessel operators L and resp. B (for functions with $a_0 = 0$) with parameter $\beta = 1$.

Example 6. Many linear integration and differentiation operators used in geometric functions theory, *in studies on classes of univalent functions*, are GFC operators, see for example [23, Ch.5].

A more extensive list of numerous other particular cases of the GFC operators, including *transmutation operators*, can be found in [23] and other our papers. Let us note, for example, that the operator T_ν (2.12) in Dimovski's convolution for the hyper-Bessel operator L, the generalizations of the Poisson and Sonine transformations related to same operator, are also generalized fractional integrals.

4. Applications of GFC back to special functions and integral transforms

In this part we survey in very short way some applications to the theory of special functions and integral transforms that appear together with development of the GFC, and therefore might be seen as aside effect from the introduction of the hyper-Bessel operators dy Dimovski in 60s. Namely, based on the GFC with the G- and H-functions, we have been able to introduce and study some new classes of special functions and of Laplace type integral transforms, as: the multi-index Mittag-Leffler functions and the fractional Obrechkoff transform. Additionally, we had the idea to represent most of the special functions as GFC operators of 3 basic elementary functions and accordingly to this, to classify them in 3 main classes.

4.1. Multi-index Mittag-Leffler functions

Let us go back to the notion of the generalized Gelfond-Leontiev integration and differentiation operators introduced in [13] in 1951, mentioned in the above Example **5**. In the general case, to the series (3.11) the G-L operators put in correspondence the series

$$\widetilde{\mathcal{I}}f(t)=\sum_{k=0}^{\infty}a_k\,\frac{\varphi_{k+1}}{\varphi_k}\,t^{k+1},\quad \widetilde{\mathcal{D}}f(t)=\sum_{k=1}^{\infty}a_k\,\frac{\varphi_{k-1}}{\varphi_k}\,t^{k-1}, \tag{4.1}$$

having the same radius of convergence R, where the multipliers are formed by means of the coefficients φ_k of an entire function $\varphi(\lambda)=\sum\limits_{k=0}^{\infty}\varphi_k\,\lambda^k,\ |t|<\infty$, of order $\rho>0$ and type $\sigma\neq 0$ such that $\lim\limits_{k\to\infty}k^{1/\rho}\sqrt[k]{|\varphi_k|}=(\sigma e\rho)^{1/\rho}$. The entire function $\varphi(\lambda)$ is said to be a *generating function of the G-L operators*. To make the matter more popular, have in mind that for $\varphi(\lambda)=\exp(\lambda)$ one gets as $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{D}}$ the conventional integration and differentiation operators. In earlier papers joint with Dimovski, we have considered G-L operators generated by the Mittag-Leffler (M-L) function and they appeared as examples of the E-K operators, details can be seen in [23, Ch.2]. The form of operators (3.12) and (3.13) suggested us to consider the entire function

$$\varphi(\lambda)=E_{(\frac{1}{\rho_k}),(\mu_k)}(\lambda)=\sum_{k=0}^{\infty}\varphi_k\lambda^k=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\mu_1+k/\rho_1)\ldots\Gamma(\mu_m+k/\rho_m)}\,. \tag{4.2}$$

We called it "*multi-index Mittag-Leffler (M-L) function*" and studied its properties in a series of papers, for example [25], [28], [29], etc.

The same function, called "*vector-indexed M-L function*" happened to be introduced also in some works by Luchko et al., for example in the book Yakubovich-Luchko [48], the paper [31], etc. as a tool to represent explicitly the solutions of some fractional order differential equations. It is important to mention also that these authors have also studied the hyper-Bessel operators of Dimovski, his convolutions and applied them to develop operational methods for fractional order operators (see Luchko [31]), further results on generalized Obrechkoff transform, etc.

Let us note that the multi-index M-L functions can be seen as fractional indices analogues of the so-called *hyper-Bessel functions of Delerue* (on their own side being multi-index analogues of the classical Bessel function) that are eigenfunctions for the hyper-Bessel differential operators. Precisely, m-terms set of them with permutated indices form a fundamental system of solutions of the hyper-Bessel differential equation $By(t)=\lambda y(t)$, for details seen [23, Ch.3] and some information given also in the paper by Paneva-Konovska [37] in the same FCAA issue (see there (1.1)-(1.2) and the remark after (1.4)). To compare, let us mention the fact that the "new" *multi-index M-L functions provide solutions to more general differential equations of multi-order* $(1/\rho_1,...,1/\rho_m)$ instead of the integer multi-order $(1, ..., 1)$ of B. Namely, it is proven that $y(t) = E_{(\frac{1}{\rho_k}),(\mu_k)}(\lambda t)$ solves the equation $\mathcal{D}_{(\mu_k),(\rho_k)} y(t) = \lambda y(t)$, with the Caputo-type generalized fractional derivative in the sense of the G-L differentiation (3.13). See for example, our paper [1].

4.2. Fractional indices analogue of the Obrechkoff transform

The introduction of the G-L operators (3.12)-(3.13) generated by the multi-index M-L function (4.2) motivated us to look for a Laplace-type integral transformation that would play for these generalized fractional differentiations and integrations the same role as the Laplace transform for the conventional differentiation and integration, as the Meijer transform for the Bessel differential operator ($m = 2$) and as the Obrechkoff transform for the hyper-Bessel operators (arbitrary $m > 1$). Thus, we introduced the following *fractional indices analogue of the Obrechkoff transform,* called initially in our papers as *multiple Borel-Dzrbashjan transform*:

$$\mathcal{B}(s) = \mathcal{B}_{(\rho_k),(\mu_k)} \left\{ f(t); s \right\} = \frac{1}{s} \int\limits_0^\infty H^{m,0}_{0,m} \left[st \left| \begin{array}{c} - \\ (\mu_k, 1/\rho_k) \end{array} \right. \right] \frac{f(t)}{t} dt. \quad (4.3)$$

We skip here the details given in the papers [25], [2], and mention only that (4.3) is closely related to the multi-index M-L function:

$$\mathcal{B}_{(\rho_k),(\mu_k)} \left\{ E_{(1/\rho_k),(\mu_k)}(t); s \right\} = \frac{1}{s-1},$$

and to the G-L operators generated by it. For example, the "integration" and "differentiation" laws of the corresponding operational calculus hold:

$$\mathcal{B}_{(\rho_i),(\mu_i)} \left\{ L_{(\rho_i),(\mu_i)} f(z); s \right\} = \frac{1}{s} \; \mathcal{B}_{(\rho_i),(\mu_i)} \left\{ f(z); s \right\}$$

and

$$\mathcal{B}_{(\rho_i),(\mu_i)} \left\{ D_{(\rho_i),(\mu_i)} f(z); s \right\} = s \; \mathcal{B}_{(\rho_i),(\mu_i)} \left\{ f(z); s \right\} \; - \; f(0) \; [\prod_{i=1}^m \Gamma(\mu_i)],$$

together with the found convolution, complex and real inversion formulas for this H-transform. *Evidently, for* $\forall \rho_k = 1, k = 1, ..., m$ *it reduces to the Obrechkoff transform as G-transform* (2.8), and so, to its special cases.

4.3. Transmutation operators

As already mentioned, the operator T_ν (2.12) in Dimovski's convolution for the hyper-Bessel operator L can be written as a generalized fractional integral with a G-function in the kernel instead of the repeated integrations. So is for the generalizations of the Poisson and Sonine transformations proposed by Dimovski in relation to the hyper-Bessel operators that we have called later as *Poisson-Sonine-Dimovski transformations.* As transmutation operators, they can be, and were, effectively used to reduce the unknown solutions of problems for hyper-Bessel operators (as convolutions, solutions of equations, etc) to the known ones for the simpler operators of same order m as d/dt^m and the m-fold integration. See details in our works as [23, Ch.3, Ch.4], [27], etc. Especially, the Poisson-Dimovski transformation and its generalization - as operators of GFC, have been applied also to find new integral representations for the hyper-Bessel functions and for the multi-index M-L functions, appearing as generalizations of the classical Poisson integral representing the Bessel function by means of the cosine function. On this matter, along with the stuff in [23], see some recent papers as [28], [29].

The operators of GFC, used as transmutation operators have been used also to construct new integral transforms of Laplace type with G- and H-functions in the kernels, when applying them on the Laplace transform. As a matter of fact, such one is the Obrechkoff transform itself, as generated from the Laplace transform by means of the Sonine-Dimovski transform. For more, see [23, Ch.3, Ch.5], [10], [2], etc.

Finally, we like separately to emphasize another important application of the GFC operators as transmutation operators. To be short, we present the basic proposition from the papers [24] and [29], whose origins are contained in [23, Ch.4].

PROPOSITION **4.1.** *All the generalized hypergeometric functions* ${}_pF_q(t)$, *that is all the classical special functions, can be considered as generalized (q-multiple) fractional integrals* (3.1), *or/and their respective generalized fractional derivatives* (3.6), *of one of the following 3 basic elementary functions, depending on whether* $p < q$, $p = q$ *or* $p = q + 1$*:*

$$\cos_{q-p+1}(t) \quad (\textit{if } p < q)\ ,\ t^{\alpha} \exp t \quad (\textit{if } p = q)\ ,\ t^{\alpha} (1-t)^{\beta} \quad (\textit{if } p = q+1). \tag{4.4}$$

Further, for the so-called "*special functions of FC*" including the Wright generalized hypergeometric functions ${}_p\Psi_q$ as Fox H-functions, analogous assertion is proved in [29], where GFC operators with H-function are used.

This treatment allows *to think about the special functions as generalized fractional integrals and derivatives of 3 simplest basic functions*, and to consider them as belonging to one of the *3 basic classes*: generalized hypergeometric functions (g.h.f.) of cosine-Bessel type (when $p < q$, as are the cos, $\cos_m$, Bessel function, hyper-Bessel functions ${}_0F_m$), g.h.f. of exponential-confluent type (when $p=q$, as are the exp, ${}_1F_1$, ${}_pF_p$) and g.h.f. of Gauss type (when $p = q+1$, as are the beta-distribution, the Gauss ${}_2F_1$-function, ${}_{p+1}F_p$).

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev Str., Block 8
Sofia 1113 – BULGARIA
e-mail: virginia@diogenes.bg

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