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Front fluctuations for the stochastic Cahn–Hilliard equation

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Abstract. We consider the Cahn–Hilliard equation in one space dimension, perturbed by the derivative of a space and time white noise of intensity $\varepsilon^{1/2}$, and we investigate the effect of the noise, as $\varepsilon \to 0$, on the solutions when the initial condition is a front that separates the two stable phases. We prove that, given $\gamma < \frac{2}{3}$, with probability going to one as $\varepsilon \to 0$, the solution remains close to a front for times of the order of $\varepsilon^{-\gamma}$, and we study the fluctuations of the front in this time scaling. They are given by a one dimensional continuous process, self similar of order $\frac{1}{4}$ and non-Markovian, related to a fractional Brownian motion and for which a couple of representations are given.

1 Introduction

The kinematics of phase segregation for binary alloys can be described by the Cahn–Hilliard equation [Cahn and Hilliard (1958, 1959); Cahn (1961)],

$$\partial_t u = -\Delta\left(\frac{1}{2}\Delta u - V'(u)\right), \tag{1.1}$$

where Δ is the Laplacian and $V : \mathbb{R} \to \mathbb{R}$ is a symmetric double well potential which, for the sake of concreteness, is chosen as

$$V(u) = \frac{1}{4}(u^2 - 1)^2. \tag{1.2}$$

The scalar field $u = u(x,t)$ is an order parameter and represents the relative concentrations of the two species. The space homogeneous stationary solutions $u = \pm 1$ are to be interpreted as the pure phases of the alloy. In contrast with the Allen–Cahn flow [Allen and Cahn (1979)], the evolution governed by (1.1) locally conserves the mass $\int dx\, u(x,t)$. Indeed, (1.1) can be viewed as the H^{-1} gradient flow of the van der Waals free energy functional,

$$\mathcal{F}(u) = \int dx \left[\frac{1}{4}|\nabla u|^2 + V(u)\right]. \tag{1.3}$$

In particular, the critical points of (1.3) with mass conservation constraint are stationary solutions to (1.1). Moreover, when (1.1) is considered in a bounded domain, its solutions converge, apart from exceptional initial conditions, to a minimizer of (1.3) with the mass fixed by the initial datum and the yet unspecified

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boundary condition. By introducing an appropriate scaling parameter, the main issues from a heuristic point of view are the following ones. First, an early, relatively fast, stage of the dynamics (referred to as spinodal decomposition), in which the flow (1.1) approaches a critical point of (1.3) by realizing a local separation of the pure phases ± 1. A later, slow, stage of the evolution toward the minimizer, taking place in a small neighborhood of the unstable manifold of the critical point.

Let us focus on the one dimensional case. For a bounded domain, a detailed analysis of the slow evolution of patterns of the rescaled Cahn–Hilliard equation is given by Alikakos et al. (1991) and Bates and Xun (1994, 1995). More precisely, in a neighborhood of a stationary solution to (1.1) having a given number of transition layers, the exponentially slow speed of the layer motion is determined. In addition, the existence of an unstable invariant manifold attracting solutions exponentially fast in the scaling parameter is established.

Analogous issues can be posed when the equation (1.1) is considered on the whole line without scaling parameter. By interpreting (1.3) as an action functional, it is easy to show that (1.1) admits "droplet-shaped" stationary solutions, that is, profiles of the form $u_{\text{droplet}}(x) = g(x - x_0)$ with $x_0 \in \mathbb{R}$ and g symmetric, monotone for $x > 0$, and exponentially approaching its asymptotic value. The function g can be determined up to an arbitrary parameter which plays the role of the mass constraint. The linear stability of such stationary solutions is determined by the spectrum of the linearization of (1.1) around u_{droplet}. By translation invariance, zero is an eigenvalue, which is responsible for the exponentially small motion in finite but large domains [Alikakos et al. (1991); Bates and Xun (1994, 1995)]. Those results suggest that the remainder of the spectrum is bounded away from zero, but we are not aware if this has been proven.

Simpler stationary solutions are the "kink-shaped" profiles $\overline{m}_{x_0}$, which describe a transition between the pure phases ± 1 at $\pm\infty$ with "center" $x_0 \in \mathbb{R}$. By translation invariance $\overline{m}_{x_0}(x) = \overline{m}(x - x_0)$ with $\overline{m} = \overline{m}_0$. For the specific choice (1.2) of the potential, $\overline{m}(x) = \tanh(x)$. Its linear stability has been analyzed by Bricmont et al. (1999), Howard (2007), and Carlen et al. (2001). Again zero is a simple eigenvalue but, in contrast both to the droplet and to the kink for the Allen–Cahn dynamics, it is an accumulation point of the spectrum. Hence, the manifold

$$\mathcal{M} = \{\overline{m}_{x_0}, x_0 \in \mathbb{R}\} \tag{1.4}$$

is not exponentially attracting for the Cahn–Hilliard flow. Sharp estimates on the actual rate of convergence are proven in the works of Bricmont et al. (1999) and Howard (2007): roughly speaking, an initial datum close to $\mathcal{M}$ relaxes to a front in $\mathcal{M}$ with a diffusive behavior. We remark that this is due to the fact that the domain is unbounded.

Both from a conceptual and a modelling point of view, the addition of a small random forcing term to (1.1) appears natural. Clearly, the random force should preserve the local mass and, under suitable assumptions, can be taken to be Gaussian.

We thus consider the stochastic partial differential equation,

$$\partial_t u = -\Delta\left(\frac{1}{2}\Delta u - V'(u)\right) + \sqrt{\varepsilon}\nabla(a_\varepsilon \dot{W}), \tag{1.5}$$

where $\dot{W}$ is a space–time white noise, a_ε is a convenient space cut-off, and $\varepsilon > 0$. In the framework of dynamical critical phenomena this is, with the choice (1.2), a model for the evolution with conserved order parameter [Model B in the review by Hohenberg and Halperin (1977)]. In spite of the short-scale singularity of the forcing term and of the unbounded domain, in the one dimensional case this equation has a meaningful mild formulation for suitable a_ε (see the next section). We refer to the works of Antonopoulou and Karali (2011); Cardon-Weber (2001); Da Prato and Debussche (1996) for existence results on stochastic perturbations of the Cahn–Hilliard equation in bounded domains. The effect of the noise on the motion of the transition layers analyzed in the aforementioned works by Alikakos et al. (1991); Bates and Xun (1994, 1995), is studied by Antonopoulou et al. (2012). It is there shown that the random fluctuations dominate the exponentially slow deterministic dynamics and an effective system of stochastic ordinary differential equations for the motion of the layers is derived.

The purpose of the present paper is to analyze the fluctuations of the kink profile $\overline{m}$ due to the random noise in (1.5), in the limit $\varepsilon \to 0$. Let us first review the corresponding results for the stochastic non-conservative Allen–Cahn equation $\partial_t u = \frac{1}{2}\Delta u - V'(u) + \sqrt{\varepsilon}\dot{W}$. If the initial datum is $\overline{m}$, in the works by Brassesco et al. (1995, 1998); Funaki (1995), it is shown that the solution at times $\varepsilon^{-1}t$ stays close to $\overline{m}_{\zeta_\varepsilon(t)}$ for some random process $\zeta_\varepsilon(t)$ which converges to a Brownian motion as $\varepsilon \to 0$. To explain heuristically this result, let us regard the random forcing term as a source of independent small kicks, which we decompose along the directions parallel and orthogonal to $\mathcal{M}$. The orthogonal component is exponentially damped by the deterministic drift, while the parallel component, associated to the zero eigenvalue of the linearization around $\overline{m}_\zeta$, is not contrasted and, by independence, sums up to a Brownian motion.

For the Cahn–Hilliard dynamics, this picture has to be completely modified, taking into account the following two related effects. The local mass conservation, which implies that fluctuations of the interface center can occur only in infinite volume, as the extra mass has "to come from infinity," and the absence of a spectral gap, which implies that perturbations in the direction orthogonal to $\mathcal{M}$ cannot be neglected. More precisely, the projection of the noise in the direction parallel to $\mathcal{M}$ vanishes, but its perpendicular component is not exponentially damped and gives rise, with a suitable delay, to the front fluctuations. On heuristic grounds, we expect that the increments of the resulting process are not independent but negatively correlated. Indeed, after a fluctuation in a given direction, the extra mass is reabsorbed, causing a successive fluctuation in the opposite direction. This mechanism is slower than that for the Allen–Cahn dynamics, and therefore a finite displacement of the interface should occur at times of the order ε^{-2}. On the other hand, it is not at all clear whether the kink-like shape of the solution survives until so long times. If this is the case, as the limiting process is anyway the sum of approximately Gaussian increments, we expect it to be a self-similar of order $\frac{1}{4}$ non-Markovian Gaussian process.

In this paper we show that, given $\gamma < \frac{2}{3}$, with probability going to 1 as $\varepsilon \to 0$, the solution to the equation (1.5) with initial condition $\overline{m}$ stays close to $\overline{m}_{Z_\varepsilon(t)}$ up to times of order $\varepsilon^{-\gamma}$, and we describe the statistics of the kink fluctuations in this regime. More precisely, we prove that

$$u\big(\cdot, \varepsilon^{-\gamma} t\big) \approx \overline{m}_{Z_\varepsilon(t)}, \qquad Z_\varepsilon(t) \approx \varepsilon^{1/2-\gamma/4} Z(t),$$

where the process Z is the odd part of a two sided fractional Brownian motion with self-similarity parameter $\frac{1}{4}$, which is indeed a non-Markovian process, with negatively correlated increments. As previously remarked, the above result suggests that the order of the time needed to obtain a finite displacement of the front should be ε^{-2}, but the analysis required to reach this scaling is not clear.

Actually, the heuristic picture discussed before has been substantiated for the stochastic phase field equation by Bertini et al. (2002). In that case, due to the weak coupling between the order parameter and the phase field, there is a sharp separation between the instantaneous noise kicks and their delayed contribution to the front propagation; the analysis can thus be carried out up to the longer time scale.

We notice that, in contrast to the kink fluctuations, the picture describing the fluctuations of the droplet for (1.5) on the whole line should be analogous to that of the kink for the Allen–Cahn equation. In particular, the droplet fluctuations should become of order one at times $\varepsilon^{-1} t$ and converge to a Brownian motion, due to the fact that the mass conservation is not present here, and the droplet can move freely.

2 Notation and results

We consider, for each $\varepsilon > 0$, the process $u(x,t)$, $x \in \mathbb{R}$ and $t > 0$, solution to the initial value problem for the Cahn–Hilliard equation with a conservative stochastic perturbation,

$$\begin{cases} \partial_t u = -\partial_x^2\Big(\dfrac{1}{2}\partial_x^2 u - V'(u)\Big) + \sqrt{\varepsilon}\,\nabla(a_\varepsilon \dot{W}), \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $\partial_x^j = \frac{\partial^j}{\partial x^j}$ ($j \in \mathbb{N}$), $V(u)$ as in (1.2), and $a_\varepsilon(x) = a(x\varepsilon^\beta)$, for $\beta > 2$ and a a C^∞ positive function with $\mathrm{supp}(a) \subset [-1,1]$, $a(0) = 1$, and $\|a\|_\infty = 1$. Finally, $\dot{W} = \dot{W}_{x,t}$ is a space–time white noise, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The initial condition u_0 is taken close to the centered interface $\overline{m}(x) = \tanh(x)$.

A precise sense for the above equation is given by the integral equation obtained in terms of the Green function $G(x, y, t)$ corresponding to the operator $\mathrm{e}^{-(1/2)t\Delta^2}$, where $\Delta^2 f = \partial_x^4 f$ when restricted to $f \in C^\infty$,

$$u(x,t) = \int \mathrm{d}y G(x,y,t) u_0(y) + \int_0^t \mathrm{d}s \int \mathrm{d}y \partial_y^2 G(x,y,t-s) V'\big(u(y,s)\big) + \sqrt{\varepsilon} Y(x,t). \tag{2.2}$$

The last term above is the Gaussian process obtained formally as

$$Y(x,t) = \int_0^t \mathrm{d}s \int \mathrm{d}y G(x,y,t-s) \partial_y \big(a_\varepsilon(y) \dot{W}_{y,s}\big).$$

Precisely, $Y(x,t)$ is the centred Gaussian process with covariance,

$$\mathbb{E}\big(Y(x,t)Y(x',t')\big) = \int_0^{t\wedge t'} \mathrm{d}s \int \mathrm{d}y \partial_y G(x,y,t-s) \partial_y G(x',y,t'-s) a_\varepsilon(y)^2. \tag{2.3}$$

As usual, for $g \in L_2(\mathbb{R} \times [0,\infty))$, we denote $\dot{W}(g) = \int g(x,t)\,\mathrm{d}W_{x,t}$, omitting the variables x and t in the integral when clear otherwise. The elements in the space Ω will be denoted by ω. We also consider the filtration,

$$\mathscr{F}_t = \sigma\big\{\dot{W}\big(A \times [0,t]\big); A \in \mathscr{B}(\mathbb{R})\big\}, \qquad \mathscr{B}(\mathbb{R}) \text{ the Borel sets in } \mathbb{R}.$$

In the sequel, we will denote by C a generic positive constant, whose numerical value may change from line to line and from one side to the other in an inequality. The notation $a \wedge b$ $(a \vee b)$ stands for the minimum (maximum) between the real numbers a and b. Given $p \in [1,\infty]$, we let $\|\cdot\|_p$ be the norm in $L_p(\mathbb{R}, \mathrm{d}x)$. We consider $C(\mathbb{R}_+)$ equipped with the (metrizable) topology of uniform convergence on compact sets, and denote by $\Longrightarrow$ weak convergence of processes in that space. Finally, to simplify the writing, we assume that the scaling parameter $\varepsilon \in (0,1)$.

Our main results are stated as follows.

Theorem 2.1. *Given $0 < \gamma < \frac{2}{3}$ and $T > 0$, there exists a set $B(\varepsilon,\gamma,T) \in \mathscr{F}$ such that $\mathbb{P}(B(\varepsilon,\gamma,T)) \to 1$ as $\varepsilon \to 0$, and, for $\omega \in B(\varepsilon,\gamma,T)$, the stochastic Cahn–Hilliard equation* (2.1) *with initial condition $\overline{m}$ has a unique bounded continuous solution $u(x,t)$, for $x \in \mathbb{R}$ and $t \le \varepsilon^{-\gamma} T$. Moreover, there exists a one dimensional $\mathscr{F}_t$-adapted process $\zeta_\varepsilon(t)$ such that*

(i) *For each $\eta > 0$,*

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\sup_{t \le \varepsilon^{-\gamma} T} \big\|u(\cdot,t) - \overline{m}_{\zeta_\varepsilon(t)}\big\|_\infty > \varepsilon^{(1-\gamma)\wedge 1/2 - \eta}\Big) = 0.$$

(ii) *Consider the real process* $X_\varepsilon(t) := \varepsilon^{-1/2+\gamma/4}\zeta_\varepsilon(\varepsilon^{-\gamma}t)$. *Then*,

$$X_\varepsilon(t) \Longrightarrow (8\pi)^{1/4} r(t) \qquad \text{as } \varepsilon \to 0,$$

where $r(t)$ *is the one dimensional centered Gaussian process with covariance*,

$$\mathbb{E}\big(r(t)r(t')\big) = \sqrt{t+t'} - \sqrt{t-t'}, \qquad t \geq t'. \tag{2.4}$$

Proposition 2.2. *The one dimensional Gaussian process* $r(t)$ *with covariance given by* (2.4) *is a continuous process, self-similar of order* $\frac{1}{4}$. *It admits the following three representations, in terms of well known processes*:

- *Let* $\nu^{(H)}(t)$ *be the usual two sided fractional Brownian motion with Hurst parameter* H, *and* $\nu_O^{(H)}$ *its odd part. Then*,
$$r(t) = 2\nu_O^{(1/4)}(t).$$
- *Let* $h(x,t)$ *be the solution to the heat equation in* $\mathbb{R}$ *with additive space–time white noise and initial condition* 0. *Then*,
$$r(t) = (2\pi)^{1/4} h(0,t).$$
- *Consider* $b(s)$ *a standard Brownian motion. Then*,
$$r(t) = c\int_0^t \frac{s^{1/4}}{(t^2 - s^2)^{1/4}}\, \mathrm{d}b(s),$$
where the constant $c = (\frac{1}{2}B(\frac{3}{4}, \frac{3}{4}))^{-1/2}$, *with* $B(\cdot,\cdot)$ *the usual Euler beta function*.

The paper is organized as follows. In the next section, we discuss the properties of the Gaussian process Y, obtaining some *sup*-norm estimates that allow to establish a local existence and uniqueness theorem for equation (2.2). In Section 4, we introduce a new integral equation, in terms of the kernel of the linearized equation around $\overline{m}$. With the aid of estimates for this kernel derived by Bricmont et al. (1999), we obtain estimates that are used in Section 5 to show stability of the front in a convenient time-scaling. In Section 6, we study the Gaussian process H appearing in the linearization about the front under proper time scaling, and prove Theorem 2.1. Finally, Proposition 2.2 is proved in Section 7. Some more technical proofs are reported in the Appendix.

3 Local existence and uniqueness of solutions

3.1 The process *Y*

We investigate first the properties of the Gaussian process Y, which are deduced from that of its covariance (2.3). In the next lemma, we provide some properties of the Green function G appearing in (2.2), that will be used to obtain the estimates on that covariance stated in Proposition 3.2, which in turn permit to estimate the *sup*-norm of Y in Proposition 3.3.

Lemma 3.1. *The Green function $G(x,y,t)$ associated to $\mathrm{e}^{-(1/2)t\Delta^2}$ is given by*

$$G(x,y,t)=\frac{1}{t^{1/4}}\phi\Big(\frac{x-y}{t^{1/4}}\Big) \qquad \text{where } \phi(x)=\frac{1}{2\pi}\int \mathrm{d}\omega\, \mathrm{e}^{-(1/2)\omega^4}\mathrm{e}^{\mathrm{i}\omega x}. \tag{3.1}$$

Therefore,

$$\partial_y G(x,y,t)=-\frac{1}{t^{1/2}}\phi'\Big(\frac{x-y}{t^{1/4}}\Big), \qquad \partial_y^2 G(x,y,t)=\frac{1}{t^{3/4}}\phi''\Big(\frac{x-y}{t^{1/4}}\Big). \tag{3.2}$$

Moreover, for each $\lambda>0$ and non-negative integer k,

$$\bar c=\bar c(\lambda,k):=\sup_{x\in\mathbb{R}}\Big|\mathrm{e}^{\lambda|x|}\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)\Big|<+\infty. \tag{3.3}$$

Also, for any $H>0$ and $A\ge 1$,

$$\Big|\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x+H)-\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)\Big|\le CH\Big(\mathrm{e}^{-|x|}\mathbb{1}_{\{|x|>2H\}}+\frac{1}{1+H}\mathbb{1}_{\{|x|\le 2H\}}\Big), \tag{3.4}$$

$$\Big|\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(Ax)-\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)\Big|\le C\frac{A-1}{1+(A-1)|x|}|x|\mathrm{e}^{-|x|}\le C\frac{A-1}{A}\mathrm{e}^{-|x|}. \tag{3.5}$$

For the second derivative, we have also that

$$\big|\phi''(x+H)-\phi''(x)\big|\le C\frac{H^2+H|x|}{1+H^2+H|x|}. \tag{3.6}$$

Proof. The expression (3.1) follows by standard Fourier analysis. To prove (3.3), we observe that, as the entire function $f(z)=\exp(-\frac12 z^4+\mathrm{i}xz)$ satisfies $\max_{0\le\eta\le\lambda}|f(\pm R+\mathrm{i}\eta)|\to 0$ as $R\to\infty$, by Cauchy's theorem we have

$$\phi(x)=\frac{1}{2\pi}\int \mathrm{d}\omega\, \mathrm{e}^{-(1/2)(\omega+\mathrm{i}\lambda)^4}\mathrm{e}^{\mathrm{i}(\omega+\mathrm{i}\lambda)x},$$

and hence

$$\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)=\frac{\mathrm{e}^{-\lambda x}}{2\pi}\int \mathrm{d}\omega\,(\mathrm{i}\omega-\lambda)^k\mathrm{e}^{-(1/2)(\omega+\mathrm{i}\lambda)^4}\mathrm{e}^{\mathrm{i}\omega x}.$$

As the integral on the right-hand side is uniformly bounded in x for each λ, the previous expression implies the bound (3.3) for $x\ge 0$, and then also for $x<0$ as ϕ is an even function.

The estimates (3.4) and (3.5) follow from (3.3) with $\lambda=2$, as

$$\Big|\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x+H)-\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)\Big|\le\bar c\int_x^{x+H}\mathrm{d}y\,\mathrm{e}^{-2|y|}\le\begin{cases}\bar c H\mathrm{e}^{-|x|} & \text{if } |x|>2H\,,\\ \bar c(H\wedge 1) & \text{if } |x|\le 2H,\end{cases}$$

and

$$\left|\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(Ax)-\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x)\right|\le \bar{c}\left|\int_x^{Ax}\mathrm{d}y\mathrm{e}^{-2|y|}\right|\le \bar{c}\big((A-1)|x|\wedge 1\big)\mathrm{e}^{-|x|}$$

(note that $a\wedge 1\le 2a(1+a)^{-1}\ \forall a\ge 0$).

Finally, to prove (3.6), we consider the integral expression for ϕ given in (3.1) and observe that

$$\big|\cos(\omega x)\big(\cos(\omega H)-1\big)\big|\le \frac{(\omega H)^2}{2}\wedge 2,\qquad \big|\sin(\omega x)\sin(\omega H)\big|\le \omega^2|x|H\wedge 1.$$

Therefore,

$$\begin{aligned}
&\big|\phi''(x+H)-\phi''(x)\big|\\
&\quad=\frac{1}{2\pi}\left|\int \mathrm{d}\omega\,\mathrm{e}^{-(1/2)\omega^4}\omega^2\mathrm{e}^{\mathrm{i}\omega x}\big(\mathrm{e}^{\mathrm{i}\omega H}-1\big)\right|\\
&\quad\le\frac{1}{2\pi}\int \mathrm{d}\omega\,\mathrm{e}^{-(1/2)\omega^4}\omega^2\big(\big|\cos(\omega x)\big(\cos(\omega H)-1\big)\big|+\big|\sin(\omega x)\sin(\omega H)\big|\big)\\
&\quad\le C\int \mathrm{d}\omega\,\mathrm{e}^{-(1/2)\omega^4}\omega^2\big[\big(H^2\omega^2+\omega^2|x|H\big)\wedge 1\big]\le C\big[\big(H^2+H|x|\big)\wedge 1\big],
\end{aligned}$$

which proves (3.6) (using, as before, that $a\wedge 1\le 2a(1+a)^{-1}\ \forall a\ge 0$). □

The following proposition, whose proof is given in the Appendix, is a consequence of the estimates detailed in Lemma 3.1.

Proposition 3.2. *Let $Y(x,t)$ be the Gaussian process with covariance (2.3). Then, for any $h>0$,*

$$\mathbb{E}Y(x,t)^2\le Ct^{1/4},\tag{3.7}$$

$$\mathbb{E}\big(Y(x+h,t)-Y(x,t)\big)^2\le Ch\log\big(1+h^{-1}t^{1/4}\big),\tag{3.8}$$

$$\mathbb{E}\big(Y(x,t+h)-Y(x,t)\big)^2\le Ch^{1/4}.\tag{3.9}$$

Using well-known results for Gaussian processes, (3.8) and (3.9) imply Hölder continuity of the paths of $Y(x,t)$ in both variables. From (3.7), it follows at once that $Y(x,t)\in L^2(\mathbb{R}\times[0,T],\mathrm{d}\mu)$ with probability one, for any $T>0$ given, and μ a finite measure on $\mathbb{R}\times[0,T]$. To establish uniqueness and existence of continuous solutions to (2.2), we need however estimates on the *sup*-norm of Y over convenient sets in the parameter space.

Let us define, for each positive ε, γ, and T, the set

$$\mathcal{T}_\varepsilon=\big\{(x,t):x\in\mathbb{R},t\in\big[0,T\varepsilon^{-\gamma}\big]\big\}.\tag{3.10}$$

In the next proposition we deduce, from Proposition 3.2, a bound for the entropy of $\mathcal{T}_\varepsilon$, from which the desired estimates follow, using the Gaussian structure of Y

(e.g., see Adler (1990)). Recall that, although not explicit in the notation, the process Y depends on ε.

Proposition 3.3. *Consider the set $\mathcal{T}_\varepsilon$ as defined in* (3.10). *Then the process $Y(x,t)$ satisfies the following properties.*

(i) *For each $\xi > 0$ there exists a constant $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\mathbb{E} \sup_{(x,t)\in\mathcal{T}_\varepsilon} |Y(x,t)| < C\varepsilon^{-\gamma/8-\xi}. \tag{3.11}$$

(ii) *Given $0 < \gamma < 4$,*

$$\mathbb{P}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \sqrt{\varepsilon}|Y(x,t)| < \infty\Big) = 1. \tag{3.12}$$

(iii) *Given $0 < \gamma < 4$, for any $0 < \xi < \frac{4-\gamma}{16}$ there exist $\zeta > 0$ and a constant $C > 0$ such that,*

$$\mathbb{P}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \sqrt{\varepsilon}|Y(x,t)| > \varepsilon^{\xi}\Big) \le C\mathrm{e}^{-\varepsilon^{-\zeta}}. \tag{3.13}$$

Proof. We first show that items (ii) and (iii) follow from item (i). Indeed, by (3.11) we have,

$$\mathbb{E}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \sqrt{\varepsilon}|Y(x,t)|\Big) < \varepsilon^{(4-\gamma)/16}, \tag{3.14}$$

which implies (3.12). Next, from (3.7),

$$\sigma^2_{\mathcal{T}_\varepsilon} := \sup_{(x,t)\in\mathcal{T}_\varepsilon} \mathrm{Var}\big(Y(x,t)\big) \le C\varepsilon^{-\gamma/4}, \tag{3.15}$$

so that, from (3.14), (3.15) and Borell's inequality [Adler (1990)],

$$\begin{aligned}\mathbb{P}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \sqrt{\varepsilon}|Y(x,t)| > \varepsilon^{\xi}\Big) &\le 4\exp\Big(-\frac{(\varepsilon^{-1/2+\xi} - \varepsilon^{-1/2+(4-\gamma)/16})^2}{2\sigma^2_{\mathcal{T}_\varepsilon}}\Big)\\ &\le 4\exp\big(-C\varepsilon^{-1+2\xi+\gamma/4}\big),\end{aligned} \tag{3.16}$$

which implies (3.13).

We are left with the proof of item (i). Without loss of generality, we can suppose $T = 1$ and $\varepsilon \in (0, \frac{1}{2})$, what we do. Consider the pseudo-metric d on $\mathcal{T}_\varepsilon$ defined by

$$d\big((x,t),(y,s)\big) = \big(\mathbb{E}\big(Y(x,t) - Y(y,s)\big)^2\big)^{1/2},$$

let $B_\delta(x,t)$ denote the d-ball of radius δ and center (x,t), and $N(\delta)$ the minimum number of d-balls of radius δ needed to cover $\mathcal{T}_\varepsilon$. Then $\log N(\delta)$ is the entropy of $\mathcal{T}_\varepsilon$, and it is known [Adler (1990)] that there is a universal positive constant κ such that

$$\mathbb{E} \sup_{(x,t)\in\mathcal{T}_\varepsilon} |Y(x,t)| < \kappa \int_0^{\mathrm{diam}\,\mathcal{T}_\varepsilon} \mathrm{d}\delta\big(\log N(\delta)\big)^{1/2}. \tag{3.17}$$

Recall that $Y(x,0)\equiv 0$. It follows from (3.7) that the diameter of $\mathcal{T}_\varepsilon$ satisfies $\operatorname{diam}\mathcal{T}_\varepsilon \le C_0\varepsilon^{-\gamma/8}$. To estimate $N(\delta)$ we let R be a positive parameter, to be fixed later as a function of δ and ε. Proceeding as in (A.1), with the aid of (3.3) with $\lambda=1$ and after recalling the definition of a_ε, if $|x|>\varepsilon^{-\beta}+R$ and $t\le\varepsilon^{-\gamma}$, we have,

$$\begin{aligned}\mathbb{E}Y(x,t)^2 &\le \int_0^t \mathrm{d}s s^{-3/4}\int_{s^{-1/4}(x-\varepsilon^{-\beta})}^{s^{-1/4}(x+\varepsilon^{-\beta})}\mathrm{d}y\phi'(y)^2 \le \frac{\overline{c}}{2}\int_0^{\varepsilon^{-\gamma}}\mathrm{d}s s^{-3/4}\mathrm{e}^{-2s^{-1/4}R}\\ &\le \frac{\overline{c}R}{2}\mathrm{e}^{-\varepsilon^{\gamma/4}R}\int_0^{R^{-4}\varepsilon^{-\gamma}}\mathrm{d}\tau\tau^{-3/4}\mathrm{e}^{-\tau^{-1/4}} \le \frac{\overline{c}R}{2}\mathrm{e}^{-\varepsilon^{\gamma/4}R}C\big(R^{-1}\varepsilon^{-\gamma/4}\big)\\ &= C_1\mathrm{e}^{-\varepsilon^{\gamma/4}R}\varepsilon^{-\gamma/4},\end{aligned}$$

with $C_1>0$.

Let R be the unique non-negative solution to $C_1\mathrm{e}^{-\varepsilon^{\gamma/4}R}\varepsilon^{-\gamma/4}=\delta^2$, which clearly exists if $\delta^2\le C_1\varepsilon^{-\gamma/4}$, and it is

$$R=\varepsilon^{-\gamma/4}\left|\log\frac{\delta^2\varepsilon^{\gamma/4}}{C_1}\right|. \tag{3.18}$$

In particular, for each δ such that $\delta^2<C_1\varepsilon^{-\gamma/4}$ and R as in (3.18), the set $\mathcal{R}_1=\mathcal{T}_\varepsilon\cap\{x:|x|>\varepsilon^{-\beta}+R\}$ is contained in $B_\delta(0,0)$, so it is covered with one ball of radius δ. If $\delta^2>C_1\varepsilon^{-\gamma/4}$, the corresponding set with $R=0$ is covered by one ball of radius δ. Let us consider the rest of the parameter set, $\mathcal{R}_2=\mathcal{T}_\varepsilon\cap\{x:|x|\le\varepsilon^{-\beta}+R\}$. Let $(x_0,t_0)\in\mathcal{R}_2$, and denote by $Q(x_0,t_0)=\{(x,t):|x-x_0|\le b, |t-t_0|\le b^4\}$ the rectangle of sides b and b^4 in the usual metric, for b that will be conveniently chosen. If $(x,t)\in Q(x_0,t_0)\cap\mathcal{T}_\varepsilon$, from (3.8) and (3.9) we get that,

$$\begin{aligned}\mathbb{E}\big(Y(x,t)-Y(x_0,t_0)\big)^2 &\le C\big(|x-x_0|\log\big(1+|x-x_0|^{-1}t^{1/4}\big)+|t-t_0|^{1/4}\big)\\ &\le C\big(|x-x_0|\log\big(1+|x-x_0|^{-1}\big)+|x-x_0|\log\varepsilon^{-1}+b\big)\\ &\le C_2\log\varepsilon^{-1}\big(\sqrt{b}\mathbb{1}_{\{b\le1\}}+b\mathbb{1}_{\{b>1\}}\big),\end{aligned}$$

with $C_2>0$ (recall we are assuming $\varepsilon^{-1}\ge 2$). Then, choosing

$$b=\left(\frac{\delta^2}{C_2\log\varepsilon^{-1}}\right)^2\mathbb{1}_{\{\delta^2\le C_2\log\varepsilon^{-1}\}}+\frac{\delta^2}{C_2\log\varepsilon^{-1}}\mathbb{1}_{\{\delta^2>C_2\log\varepsilon^{-1}\}}, \tag{3.19}$$

we obtain $Q(x_0,t_0)\subset B_\delta(x_0,t_0)$. It is now clear that $\mathcal{R}_2$ is covered by $b^{-5}\varepsilon^{-\gamma}\times(\varepsilon^{-\beta}+R)$ rectangles (convenient translations of $Q(x_0,t_0)$). We conclude that $N(\delta)\le 1+b^{-5}\varepsilon^{-\gamma}(\varepsilon^{-\beta}+R)$. Then, noticing that (3.18) implies $R\le C\varepsilon^{-\gamma/4}\times(|\log\varepsilon|+|\log\delta|)$, by (3.19) we get,

$$\begin{aligned}N(\delta)&\le 1+C\varepsilon^{-\gamma}\big(\varepsilon^{-\beta}+\varepsilon^{-\gamma/4}\big(|\log\varepsilon|+|\log\delta|\big)\big)\\ &\quad\times\left(\frac{(C_2\log\varepsilon^{-1})^{10}}{\delta^{20}}\mathbb{1}_{\{\delta^2\le C_2\log\varepsilon^{-1}\}}+\frac{(C_2\log\varepsilon^{-1})^5}{\delta^{10}}\mathbb{1}_{\{\delta^2>C_2\log\varepsilon^{-1}\}}\right).\end{aligned}$$

For $\delta \le \operatorname{diam} \mathcal{T}_\varepsilon \le C\varepsilon^{-\gamma/8}$, the last estimate gives

$$N(\delta) \le 1 + C\varepsilon^{-A}\delta^{-21},$$

with $A > 0$ sufficiently large. Substitution of this in (3.17) yields

$$\begin{aligned}
&\int_0^{\operatorname{diam} \mathcal{T}_\varepsilon} \mathrm{d}\delta \big(\log N(\delta)\big)^{1/2} \\
&\quad \le \int_0^{C_0\varepsilon^{-\gamma/8}} \mathrm{d}\delta \big(\log\big(1 + C\varepsilon^{-A}\delta^{-21}\big)\big)^{1/2} \\
&\quad = \frac{C^{1/21}\varepsilon^{-A/21}}{21} \int_{CC_0^{-21}\varepsilon^{21\gamma/8-A}}^{\infty} \mathrm{d}u \big(\log(1+u)\big)^{1/2} u^{-22/21} \\
&\quad \le C\varepsilon^{-\gamma/8 - A\rho},
\end{aligned} \tag{3.20}$$

where in the last inequality we estimated the integrand by $u^{-22/21+\rho}$ with $\rho \in (0, \frac{1}{21})$. The estimate (3.11) now follows from (3.17) and (3.20) by choosing $\rho < A^{-1}\xi$ for the given ξ. □

3.2 Existence and uniqueness of solutions

From the previous results, it is not difficult to prove the existence of a unique continuous solution to the integral equation (2.2) for $t \le T_0$ if T_0 is small enough. To that end, denote by g and $\mathcal{G}$ the following operators, defined in terms of the Green function G,

$$\begin{aligned}
g u_0(x,t) &= \int \mathrm{d}y\, G(x,y,t) u_0(y), \\
\mathcal{G}F(x,t) &= \int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_y^2 G(x,y,t-s) F(y,s),
\end{aligned} \tag{3.21}$$

so that equation (2.2) reads,

$$u = g u_0 + \mathcal{G}\big(V'(u)\big) + \sqrt{\varepsilon} Y. \tag{3.22}$$

Proposition 3.4. *Given u_0 continuous, $\|u_0\|_\infty = M < \infty$, there exists a time T_0 (depending on $\|Y\|_\infty$ and on M) such that the equation (3.22) has a unique continuous bounded solution on $\mathbb{R} \times [0, T_0]$.*

Proof. Denote $q = g u_0 + \sqrt{\varepsilon} Y$ and consider, for each $T > 0$ fixed, the set

$$\mathcal{C} = \big\{ v \in C\big(\mathbb{R} \times [0,T]\big) : \|v\|_{\infty,T} < 2\|q\|_{\infty,T} \big\},$$

where, for each $t > 0$, $\|v\|_{\infty,t} := \sup\{|v(x,s)| : (x,s) \in \mathbb{R} \times [0,t]\}$. Consider on $\mathcal{C}$ the function $v \mapsto F(v) = \mathcal{G}(V'(v)) + q$, and observe that if $v \in \mathcal{C}$ then

$\|V'(v)\|_{\infty,T} \leq 8\|q\|_{\infty,T}(\|q\|^2_{\infty,T}+1)$. Therefore, from (3.2) and (3.3), for any $t \in [0,T]$,

$$\begin{aligned} \big|\mathcal{G}\big(V'(v)\big)(x,t)\big| &\leq \int_0^t \mathrm{d}s \int \mathrm{d}y \big|\partial_y^2 G(x,y,t-s)V'\big(u(y,s)\big)\big| \\ &\leq C\|q\|_{\infty,T}\big(\|q\|^2_{\infty,T}+1\big)t^{1/2}. \end{aligned} \tag{3.23}$$

Then, choosing $T = T_0$ small enough, from (3.23) we see that $F(\mathcal{C}) \subseteq \mathcal{C}$. Moreover, the Picard iterates given by $v_0 = q$, $v_{n+1} = F(v_n)$, form a Cauchy sequence on $\mathcal{C}$ (with sup norm) if T_0 is small, that converges to a limit u, which is a solution to (3.22). To prove uniqueness, fix a realization of Y and suppose that $u, \tilde{u}$ are continuous bounded solutions on $\mathbb{R} \times [0,T]$ with the same initial condition u_0 and same realization of Y. By (3.22), (3.3), and Hölder inequality, for any $t \in [0,T]$,

$$\begin{aligned} \|u-\tilde{u}\|_{\infty,t} &\leq C\int_0^t \mathrm{d}s \frac{1}{\sqrt{t-s}}\|u-\tilde{u}\|_{\infty,s} \\ &\leq C\bigg(\int_0^t \mathrm{d}s (t-s)^{-2/3}\bigg)^{3/4}\bigg(\int_0^t \mathrm{d}s \|u-\tilde{u}\|^4_{\infty,s}\bigg)^{1/4} \end{aligned}$$

and therefore,

$$\|u-\tilde{u}\|^4_{\infty,t} \leq CT\int_0^t \mathrm{d}s \|u-\tilde{u}\|^4_{\infty,s},$$

which implies that $u = \tilde{u}$ on $\mathbb{R} \times [0,T]$. □

4 Another integral equation

We introduce next a different integral equation equivalent to (2.1), which is more convenient to analyze the stability of $\overline{m}$. In particular, we need to go beyond local existence. Following Bricmont et al. (1999), we consider the kernel arising from a convenient linearization of equation (2.1) around $\overline{m}$. Precisely, let

$$u_0 = \overline{m} + h(x) \qquad \text{with } h = \partial_x f \text{ for some } f \text{ satisfying } f(\pm\infty) = 0, \tag{4.1}$$

and denote by

$$Lu = \frac{1}{2}\partial_x^2 u - V''(\overline{m})u,$$

the linearization around $\overline{m}$ of the non-linear operator $u \mapsto \frac{1}{2}\partial_x^2 u - V'(u)$. If u solves (2.1) for u_0 satisfying (4.1), then $v = u - \overline{m}$ satisfies,

$$\begin{cases} \partial_t v = -\partial_x^2 Lv + \partial_x^2(3\overline{m}v^2 + v^3) + \sqrt{\varepsilon}\nabla(a_\varepsilon \dot{W}), \\ v(x,0) = h(x). \end{cases}$$

Again, the previous equation is to be understood as the following integral equation for v in terms of the Green function $K(x, y, t)$ corresponding to the operator $e^{-t\partial_x L\partial_x}$,

$$
\begin{aligned}
v(x,t) = &\int dy \partial_x K(x,y,t) f(y) \\
&- \int_0^t ds \int dy \partial_x \partial_y K(x,y,t-s)\big(3\overline{m} v^2 + v^3\big)(y,s) \\
&+ \sqrt{\varepsilon}\int_0^t \int \partial_x K(x,y,t-s) a_\varepsilon(y)\, dW_{y,s}.
\end{aligned} \tag{4.2}
$$

Let us now consider the right-hand side above. We remark that although K is not explicitly computed, we have the following expression, that is a simple adaptation of Bricmont et al. (1999), Propositions 3.1 and 3.2, where the authors consider different numerical coefficients for L. We keep their notation.

Proposition 4.1. *There exists $t_0 > 0$ such that the kernel $K(x, y, t)$ satisfies,*

$$
K(x,y,t) = \begin{cases} K_\infty(x-y,t) + \tilde{K}(x,y,t) & \text{if } t \in (0,t_0), \\ K^*(x,y,t) + k(x,y,t) & \text{if } t \geq t_0, \end{cases} \tag{4.3}
$$

where $K_\infty(x - y, t)$ is the kernel associated with $e^{-t\partial_x((1/2)\partial_x^2 - 2)\partial_x}$ and the following estimates hold. For $i, j \in \{0, 1\}$, there exists a constant $C > 0$ such that

$$
\big|\partial_x^i \partial_y^j K_\infty(x-y,t)\big| \leq C t^{-(1+i+j)/4} \exp\big(-2^{1/4} t^{-1/4}|x-y|\big), \tag{4.4}
$$

$$
\big|\partial_x^i \partial_y^j \tilde{K}(x,y,t)\big| \leq C t^{-(i+j)/4} \exp\big(-2^{1/4} t^{-1/4}|x-y|\big) \tag{4.5}
$$

and

$$
\begin{aligned}
K^*(x,y,t) = \frac{1}{\sqrt{2\pi t}}\bigg\{ &-\frac{1}{2}\varphi(x)\varphi(y)\,\mathrm{sign}(xy) + \frac{1}{2} e^{-y^2/(8t)}\varphi(x)\,\mathrm{sign}(xy) \\
&+ \frac{1}{2} e^{-x^2/(8t)}\varphi(y)\,\mathrm{sign}(xy) \\
&- e^{-(x+y)^2/(8t)} \mathbb{1}_{\{\mathrm{sign}(xy)=1\}}\bigg\},
\end{aligned} \tag{4.6}
$$

where

$$
\varphi(x) = \begin{cases} 1 - \overline{m}(x) & \text{if } x \geq 0, \\ 1 + \overline{m}(x) & \text{if } x \leq 0 \end{cases} \tag{4.7}
$$

while, concerning the kernel $k(x, y, t)$, for $i, j \in \{0, 1\}$ there exist $\mu > 0$ and $C > 0$ such that,

$$
\big|\partial_x^i \partial_y^j k(x,y,t)\big| \leq \frac{C}{t} \exp\big(-\mu t^{-1/2}|x-y|\big). \tag{4.8}
$$

For future reference, we compute $\partial_x K^*(x,y,t)$ and $\partial_y\partial_x K^*(x,y,t)$,

$$
\begin{aligned}
&\partial_x K^*(x,y,t)\\
&\quad= \frac{1}{\sqrt{2\pi t}}\Big\{\frac{1}{2}\overline{m}'(x)\varphi(y)\,\mathrm{sign}(y) - \frac{1}{2}\overline{m}'(x)\mathrm{e}^{-y^2/(8t)}\,\mathrm{sign}(y)\\
&\qquad\qquad - \frac{x}{8t}\mathrm{e}^{-x^2/(8t)}\varphi(y)\,\mathrm{sign}(x)\,\mathrm{sign}(y)\\
&\qquad\qquad + \mathrm{e}^{-(x+y)^2/(8t)}\frac{(x+y)}{4t}\mathbb{1}_{\{\mathrm{sign}(xy)=1\}}\Big\},
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&\partial_x\partial_y K^*(x,y,t)\\
&\quad= \frac{1}{\sqrt{2\pi t}}\Big\{-\frac{1}{2}\overline{m}'(x)\overline{m}'(y) + \overline{m}'(x)\frac{y}{8t}\mathrm{e}^{-y^2/(8t)}\,\mathrm{sign}(y)\\
&\qquad\qquad + \frac{x}{8t}\mathrm{e}^{-x^2/(8t)}\overline{m}'(y)\,\mathrm{sign}(x) + \frac{1}{4t}\mathrm{e}^{-(x+y)^2/(8t)}\mathbb{1}_{\{\mathrm{sign}(xy)=1\}}\\
&\qquad\qquad - \frac{(x+y)^2}{16t^2}\mathrm{e}^{-(x+y)^2/(8t)}\mathbb{1}_{\{\mathrm{sign}(xy)=1\}}\Big\}.
\end{aligned}
\tag{4.10}
$$

Our aim is now to obtain estimates for $v = u - \overline{m}$ from the integral equation (4.2), valid for times of the order of $\varepsilon^{-\gamma}$, for $\gamma > 0$ convenient. They clearly rely on the properties of K presented above. We start by examining the stochastic integral on the last line, that we denote by H,

$$
H(x,t) = \int_0^t \int \partial_x K(x,y,t-s)a_\varepsilon(y)\,\mathrm{d}W_{y,s}. \tag{4.11}
$$

More precisely, $H(x,t)$ is the Gaussian process with covariance,

$$
\begin{aligned}
&\mathbb{E}\big(H(x,t)H(x',t')\big)\\
&\quad= \int_0^{t\wedge t'} \mathrm{d}s \int \mathrm{d}y\,\partial_x K(x,y,t-s)\partial_{x'}K\big(x',y,t'-s\big)a_\varepsilon(y)^2.
\end{aligned}
\tag{4.12}
$$

An expression for H in terms of Y also holds. Indeed, from (2.1),

$$
\partial_t(H-Y) = -\partial_x^2\Big(\frac{1}{2}\partial_x^2(H-Y) - V''(\overline{m})H\Big). \tag{4.13}
$$

Recalling (3.21), and solving in terms of the Green functions G and K, respectively we obtain

$$
\begin{aligned}
H &= Y + G\partial_x^2\big(V''(\overline{m})H\big) = Y + \mathcal{G}\big(V''(\overline{m})H\big),\\
H &= Y + \nabla K\nabla\big(V''(\overline{m})Y\big).
\end{aligned}
\tag{4.14}
$$

It is not difficult to see that the process H is bounded and continuous as long as Y is bounded and continuous. Moreover, u is a solution to (2.2) with initial condition as in (4.1) if and only if $v = u - \overline{m}$ is a solution to (4.2). The next proposition gives estimates on the increments of H. As the corresponding ones for Y, they will be useful to estimate the *sup*-norm of H on $\mathcal{T}_\varepsilon$, and consequently the *sup*-norm of v. The proof is rather technical due to the long formulas for K given in Proposition 4.1, and is given in the Appendix.

Proposition 4.2. *Let $H(x,t)$ be as in (4.11). Then, for any $h > 0$,*

$$\mathbb{E}H(x,t)^2 \le C\big(t^{1/4}\mathbb{1}_{\{t\le 1\}} + \big(1+\overline{m}'(x)^2 t^{1/2}\big)\mathbb{1}_{\{t>1\}}\big), \tag{4.15}$$

$$\mathbb{E}\big(H(x+h,t)-H(x,t)\big)^2 \le Ch\big(ht^{1/2}\big(1+h^2+|x|^2\big)+\log\big(1+h^{-1}t^{1/4}\big)\big), \tag{4.16}$$

$$\mathbb{E}\big(H(x,t+h)-H(x,t)\big)^2 \le C\big(h^{1/4}+h^{3/2}+\big(h+h^{1/2}\big)t^{1/2}\big). \tag{4.17}$$

The estimate (4.15) for the variance of the process $H(x,t)$ is uniform in x. However, we will need more precise estimates for x large, which are considered in the next lemma. Recall that, as was already observed for the process Y, the process H depends on ε through a_ε, see (4.11), although the dependence is not explicit in the notation.

Lemma 4.3. *For each $\delta > 0$, $T > 0$, $R > \varepsilon^{-11\gamma/10}\delta^{-2} + 1$, and any ε sufficiently small, the process H satisfies*

$$\sup_{|x|\ge R+\varepsilon^{-\beta},\, t\le \varepsilon^{-\gamma}T} \mathbb{E}H(x,t)^2 \le \delta^2. \tag{4.18}$$

Proof. Recalling (4.11) and (4.3),

$$\begin{aligned}\mathbb{E}H(x,t)^2 = {} & \int_0^{t\wedge t_0}\mathrm{d}s\int \mathrm{d}y\big(\partial_x K_\infty(x,y,s)+\partial_x\tilde{K}(x,y,s)\big)^2 a_\varepsilon(y)^2 \\ & + \int_{t\wedge t_0}^{t}\mathrm{d}s\int \mathrm{d}y\big(\partial_x K^*(x,y,s)+\partial_x k(x,y,s)\big)^2 a_\varepsilon(y)^2.\end{aligned} \tag{4.19}$$

From (4.4), for $R > 1$ and $|x| > R + \varepsilon^{-\beta}$,

$$\begin{aligned}\int_0^{t\wedge t_0}\mathrm{d}s\int \mathrm{d}y\big(\partial_x K_\infty(x,y,s)\big)^2 a_\varepsilon(y)^2 &\le C\int_0^{t\wedge t_0}\mathrm{d}s\frac{1}{s}\int_{-\varepsilon^{-\beta}}^{\varepsilon^{-\beta}}\mathrm{d}y\mathrm{e}^{-s^{-1/4}|x-y|} \\ &\le C\int_0^{t\wedge t_0}\mathrm{d}s\frac{1}{s^{3/4}}\mathrm{e}^{-Rs^{-1/4}} \\ &\le C\frac{(t\wedge t_0)^{1/2}}{R}.\end{aligned} \tag{4.20}$$

Similar computations yield, from (4.5) and (4.8) respectively,

$$\int_0^{t\wedge t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_x \tilde{K}(x,y,s)\big)^2 a_\varepsilon(y)^2 \le C\frac{t\wedge t_0}{R}, \tag{4.21}$$

$$\int_{t\wedge t_0}^{t} \mathrm{d}s \int \mathrm{d}y \big(\partial_x k(x,y,s)\big)^2 a_\varepsilon(y)^2 \le \frac{C}{R}. \tag{4.22}$$

With the aid of (4.9) we estimate the term involving $\partial_x K^*$ in (4.19). Observe that $\overline{m}'(x) \le \mathrm{e}^{-|x|}$, and then

$$\begin{aligned}\int_{t\wedge t_0}^{t} \mathrm{d}s \frac{\overline{m}'(x)^2}{8\pi s} \int \mathrm{d}y \big(\varphi(y)\,\mathrm{sign}(y) - \mathrm{e}^{-y^2/(8s)}\,\mathrm{sign}(y)\big)^2 a_\varepsilon(y)^2 \\ \le C\mathrm{e}^{-2|x|}\big(\log t - \log(t\wedge t_0) + t^{1/2}\big).\end{aligned} \tag{4.23}$$

Finally,

$$\begin{aligned}\int_{t\wedge t_0}^{t} \mathrm{d}s \int \mathrm{d}y \Big(\frac{\varphi(y)}{\sqrt{2\pi s}}\frac{x}{4s}\mathrm{e}^{-x^2/(8s)}\Big)^2 a_\varepsilon(y)^2 &\le C\int_{t\wedge t_0}^{t} \mathrm{d}s \frac{x^2}{s^3}\mathrm{e}^{-x^2/(4s)} \\ &\le C\frac{1}{x^2},\end{aligned} \tag{4.24}$$

$$\int_{t\wedge t_0}^{t} \mathrm{d}s \int \mathrm{d}y \Big(\frac{(x+y)}{4s\sqrt{2\pi s}}\mathrm{e}^{-(x+y)^2/(8s)}\Big)^2 \mathbb{1}_{\{\mathrm{sign}(xy)=1\}} a_\varepsilon(y)^2 \le C\frac{1}{|x|}. \tag{4.25}$$

From (4.20)–(4.25), (4.18) follows for $R > \varepsilon^{-11\gamma/10}\delta^{-2} + 1$. □

The continuity of the process $H(x,t)$ in both variables follows from Proposition 4.2. We can also obtain estimates for the *sup*-norm of H on $\mathcal{T}_\varepsilon$.

Proposition 4.4. *Consider the set $\mathcal{T}_\varepsilon$ as defined in (3.10) with $T > 0$ and $\gamma > 0$. Then the process $H(x,t)$ satisfies the following properties.*

(i) *For each $\xi > 0$ there exists a constant $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\mathbb{E} \sup_{(x,t)\in\mathcal{T}_\varepsilon} \big|H(x,t)\big| < C\varepsilon^{-\gamma/4-\xi}. \tag{4.26}$$

(ii)

$$\mathbb{P}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \varepsilon^{\gamma/4+\xi}\big|H(x,t)\big| < \infty\Big) = 1. \tag{4.27}$$

(iii) *There exist $\zeta > 0$ and a constant $C > 0$ such that,*

$$\mathbb{P}\Big(\sup_{(x,t)\in\mathcal{T}_\varepsilon} \big|H(x,t)\big| > \varepsilon^{-\gamma/4-\xi}\Big) \le C\mathrm{e}^{-\varepsilon^{-\zeta}}. \tag{4.28}$$

Proof. We omit the details of the proof: item (i) follows from Proposition 4.2 and Lemma 4.3, adapting the proof of Proposition 3.3. Items (ii) and (iii) follow from (4.15) and (4.26), proceeding again as in the demonstration of Proposition 3.3. □

5 Stability of $\overline{m}$

In this section, we prove the stability of the front $\overline{m}$ up to times of the order $\varepsilon^{-\gamma}$, with $\gamma < \frac{2}{3}$. A precise statement is given in Proposition 5.2.

Lemma 5.1. *There exists a constant $M > 0$ such that for any $T \geq 0$,*

$$\sup_{x\in\mathbb{R}} \int_0^T \mathrm{d}t \int \mathrm{d}y \big|\partial_x \partial_y K(x, y, t)\big| \leq M\sqrt{T}. \tag{5.1}$$

Proof. It follows simply by integration of the expression (4.3) for the kernel K, and estimation of each term with the aid of (4.4), (4.5), (4.8), and (4.10). Recall the definition of t_0 in Proposition 4.1, and observe indeed that, from (4.3), (4.4), and (4.5),

$$\begin{aligned}\int_0^{T\wedge t_0} \mathrm{d}t \int \mathrm{d}y \big|\partial_x \partial_y K(x, y, t)\big| &\leq C \int_0^{T\wedge t_0} \mathrm{d}t \int \mathrm{d}y \big(t^{-3/4} + t^{-1/2}\big)\mathrm{e}^{-t^{-1/4}|x-y|} \\ &\leq C\big((T\wedge t_0)^{1/2} + (T\wedge t_0)^{3/4}\big).\end{aligned}$$

If $T > t_0$, from (4.3) and (4.8),

$$\int_{T\wedge t_0}^{T} \mathrm{d}t \int \mathrm{d}y \big|\partial_x \partial_y k(x, y, t)\big| \leq C \int_{T\wedge t_0}^{T} \mathrm{d}t \frac{1}{t} \int \mathrm{d}y \mathrm{e}^{-\mu t^{-1/2}|x-y|} \leq C T^{1/2},$$

and we are left with the estimation of the term containing K^*, which can be done by integration of the five terms in (4.10). Call $I_1, \ldots, I_5$ the resulting integrals. To conclude the proof, it is easy to see that

$$\begin{aligned}|I_1 + I_2| &\leq C \int_{T\wedge t_0}^{T} \mathrm{d}t\, t^{-1/2} \leq C T^{1/2}, \\ |I_3| &\leq C \int_{T\wedge t_0}^{T} \mathrm{d}t\, |x| t^{-3/2} \mathrm{e}^{-x^2/(8t)} \leq C T^{1/2}, \\ |I_4 + I_5| &\leq C \int_{T\wedge t_0}^{T} \mathrm{d}t \frac{1}{t} \leq C T^{1/2}.\end{aligned}$$

The lemma is thus proved. □

Proposition 5.2. *There exists a time $T_\varepsilon(\omega)$ such that the equation (2.1) with initial condition $\overline{m}$ has a unique continuous bounded solution $u(x,t)$ for $t \leq T_\varepsilon$. Moreover, given $T > 0$, $\gamma < \frac{2}{3}$, and $\xi \in (0, \frac{1}{2} - \frac{3\gamma}{4})$, there exists a set $B_\varepsilon(T,\xi) \in \mathscr{F}$ such that*

$$\mathbb{P}\big(B_\varepsilon(T,\xi)\big) \to 1 \qquad \text{as } \varepsilon \to 0,$$

and for $\omega \in B_\varepsilon(T,\xi)$ and ε sufficiently small, $u(x,t)$ satisfies,

$$\sup_{(x,t)\in\mathcal{T}_\varepsilon} \big|u(x,t)-\overline{m}(x)\big| < \varepsilon^{1/2-\gamma/4-\xi/3}, \tag{5.2}$$

$$\sup_{(x,t)\in\mathcal{T}_\varepsilon} \big|u(x,t)-\overline{m}(x)-\sqrt{\varepsilon}H(x,t)\big| < \varepsilon^{1-\gamma-\xi}, \tag{5.3}$$

where the set $\mathcal{T}_\varepsilon$ is defined in (3.10)*. In particular,* $\mathbb{P}(T_\varepsilon > \varepsilon^{-\gamma}T) \to 1$ *as* $\varepsilon \to 0$.

Proof. Recall that the equation (2.1) has to be understood in the sense of its integral version (2.2). We already know from Proposition 3.4 that, for each $\omega \in \Omega$, there exists a positive time $T_\varepsilon(\omega)$ such that the equation (2.2) admits a unique continuous solution u for $t \le T_\varepsilon(\omega)$. From (3.12), we also know that $\|\sqrt{\varepsilon}Y\|_{\infty,t} < \infty$ if $t \le \varepsilon^{-4}T$. We will exhibit a set $B_\varepsilon(T,\xi) \in \mathscr{F}$ such that, if $\omega \in B_\varepsilon(T,\xi)$, then the corresponding solution u with initial condition $\overline{m}$ satisfies (5.2), which in particular implies (from the proof of Proposition 3.4 for this particular initial condition and ω) that we can take $T_\varepsilon(\omega) > \varepsilon^{-\gamma}T$, for $\gamma < \frac{2}{3}$. To complete the proof, we need then to show the existence of the set $B_\varepsilon(T,\xi)$, whose probability goes to 1 as $\varepsilon \to 0$, and such that (5.2) and (5.3) hold on $B_\varepsilon(T,\xi)$. This will be done for the solution to the integral equation (4.2) with $f=0$, which is equivalent to (2.2) with $u_0=\overline{m}$.

In the sequel, we shall use the rescaled variables $(x,\varepsilon^{-\gamma}t)$, with $(x,t)\in\mathcal{T}=\{(x,t): x\in\mathbb{R}, t\in[0,T]\}$, to parametrize the elements in $\mathcal{T}_\varepsilon$. Define the set

$$B_\varepsilon(T,\xi) = \Big\{\omega : \sup_{(x,t)\in\mathcal{T}} \varepsilon^{\gamma/4+\xi/3}\big|H(x,\varepsilon^{-\gamma}t)\big| \le \frac{1}{2}\Big\}.$$

From (4.28), $\mathbb{P}(B_\varepsilon(T,\xi)) \to 1$ as $\varepsilon \to 0$. To prove that (5.2) holds on this set, observe that a simple time scaling in equation (4.2) gives, after (4.11),

$$\begin{aligned} v(x,\varepsilon^{-\gamma}t) = {} & -\varepsilon^{-\gamma}\int_0^t \mathrm{d}s \int \mathrm{d}y\,\partial_x\partial_y K\big(x,y,\varepsilon^{-\gamma}(t-s)\big)\big(3\overline{m}v^2+v^3\big)\big(y,\varepsilon^{-\gamma}s\big) \\ & + \sqrt{\varepsilon}H\big(x,\varepsilon^{-\gamma}t\big). \end{aligned} \tag{5.4}$$

In terms of $D_\varepsilon(x,t) = \varepsilon^{-1/2+\gamma/4+\xi/3}v(x,\varepsilon^{-\gamma}t)$, the previous equation reads,

$$\begin{aligned} D_\varepsilon(x,t) = {} & \varepsilon^{1/2-5\gamma/4-\xi/3}\int_0^t \mathrm{d}s \int \mathrm{d}y\,\partial_x\partial_y K\big(x,y,\varepsilon^{-\gamma}(t-s)\big) \\ & \times \big(3\overline{m}D_\varepsilon^2 + \varepsilon^{1/2-\gamma/4-\xi/3}D_\varepsilon^3\big)(y,s) + \varepsilon^{\gamma/4+\xi/3}H\big(x,\varepsilon^{-\gamma}t\big). \end{aligned}$$

Recall that $D_\varepsilon(x,0)=0$ and define next the stopping time,

$$T^* = \inf\big\{t>0 : \big\|D_\varepsilon(\cdot,t)\big\|_\infty \ge 1\big\}.$$

From (5.1), we obtain then that for any $\tau \le T^*$,

$$\begin{aligned} \big\|D_\varepsilon(\cdot,\tau)\big\|_\infty \le {} & \varepsilon^{1/2-3\gamma/4-\xi/3}C\tau^{1/2}\big(3+\varepsilon^{1/2-\gamma/4-\xi/3}\big) \\ & + \sup_{x\in\mathbb{R},t\le\tau} \varepsilon^{\gamma/4+\xi/3}\big|H\big(x,\varepsilon^{-\gamma}t\big)\big|. \end{aligned}$$

This inequality implies that $T^* > T$ for $\omega \in B_\varepsilon(T,\xi)$, for otherwise, evaluating at T^* we would get,

$$1 \le C T^{1/2} \varepsilon^{1/2-3\gamma/4-\xi/3} + \frac{1}{2},$$

which cannot be true for sufficiently small ε, under the assumptions on γ and ξ. But $T^* > T$ is precisely (5.2).

To prove (5.3), we notice that on the set $B_\varepsilon(T,\xi)$, from (5.4) and since (5.2) holds on that set, from (5.1)

$$\begin{aligned}
&\sup_{(x,t)\in\mathcal{T}} \big|v(x,\varepsilon^{-\gamma}t) - \sqrt{\varepsilon}H(x,\varepsilon^{-\gamma}t)\big| \\
&\quad \le 4\varepsilon^{-\gamma+1-\gamma/2-2\xi/3} \sup_{(x,t)\in\mathcal{T}} \left|\int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_x\partial_y K\big(x,y,\varepsilon^{-\gamma}(t-s)\big)\right| \\
&\quad \le C T^{1/2} \varepsilon^{1-\gamma-2\xi/3},
\end{aligned}$$

which implies (5.3), for ε sufficiently small. □

The previous result shows that, with probability going to 1, the solution u to the stochastic Cahn–Hilliard equation with initial condition $\overline{m}$ remains close to it for times of the order of $\varepsilon^{-\gamma}$ if $\gamma < \frac{2}{3}$. It also gives an idea about the fluctuations. To make it precise, we recall the notion of center of a front, already considered by Brassesco et al. (1995), Brassesco et al. (1998) and Bertini et al. (2008) to study the front fluctuations for the Allen–Cahn and phase field equations. Recall the definition (1.4) of $\mathcal{M}$, and consider

$$\mathcal{M}_\delta = \Big\{ m \in C^0(\mathbb{R}) : \mathrm{dist}(m,\mathcal{M}) = \inf_{x_0\in\mathbb{R}} \|m - \overline{m}_{x_0}\| \le \delta \Big\},$$

and for $m \in \mathcal{M}_\delta$ define the *center* of m as the real number ξ such that

$$\langle m - \overline{m}_\xi, \overline{m}'_\xi \rangle = 0.$$

Motivation and properties can be found in those articles and some of the references therein. Let us just recall that it is a convenient manner to assign a location to each configuration in $\mathcal{M}_\delta$. The following result, proved in Brassesco et al. (1995) guarantees the existence of a unique center, and provides an expansion.

Lemma 5.3. *There exists $\delta_0 > 0$ such that, if $\delta \le \delta_0$ and $m \in \mathcal{M}_\delta$, then m has a unique center $\zeta \in \mathbb{R}$. If x_0 is such that $\|m - \overline{m}_{x_0}\|_\infty < \delta$, then there exists a constant C depending only on δ such that*

(i) $|x - \zeta| \le C\|m - \overline{m}_{x_0}\|_\infty$,
(ii) $\zeta = x_0 - \frac{3}{4}\langle m - \overline{m}_{x_0}, \overline{m}'_{x_0}\rangle - \frac{9}{16}\langle m - \overline{m}_{x_0}, \overline{m}'_{x_0}\rangle\langle m - \overline{m}_{x_0}, \overline{m}''_{x_0}\rangle + R$,

where the remainder $R \le C\|m - \overline{m}_{x_0}\|_\infty^3$.

We can now prove the following lemma.

Lemma 5.4. *Let u be the solution to (2.1) with initial condition $\overline{m}$, $T > 0$, and $\gamma \in (0, \frac{2}{3})$. Then, on a set whose probability goes to 1 as $\varepsilon \to 0$, $u(\cdot, t)$ has a unique center $z_\varepsilon(t)$ for any $t \le \varepsilon^{-\gamma} T$ and any ε sufficiently small. It satisfies,*

$$z_\varepsilon(t) = -\frac{3}{4}\langle u(\cdot, t) - \overline{m}, \overline{m}'\rangle + R_\varepsilon, \tag{5.5}$$

where, for any given $\xi > 0$, $\sup_{t \le \varepsilon^{-\gamma} T} |R_\varepsilon| \le C\varepsilon^{1-\gamma/2-\xi}$.

Proof. From (5.2), in the set $B_\varepsilon(T, \xi)$ the solution $u(\cdot, t)$ has a unique center $z_\varepsilon(t)$ for any $t \le \varepsilon^{-\gamma} T$. Moreover, by item (ii) of Lemma 5.3 with $x_0 = 0$ and (5.2), the center $z_\varepsilon(t)$ satisfies (5.5). □

6 The process *H*

We proceed now to establish some properties of the process H defined in (4.11), that will be useful to study the fluctuations of the center $z_\varepsilon(t)$ of u, as suggested by (5.5) and (5.3).

Lemma 6.1. *The process $H(x, t)$ may be decomposed as*

$$H(x, t) = H_1(x, t) + H_2(x, t), \tag{6.1}$$

where

(i) $H_2(x, t) = -\frac{1}{2}\overline{m}'(x) \int_0^t \int \frac{e^{-y^2/(8(t-s))}}{\sqrt{2\pi(t-s)}} \operatorname{sign}(y) a_\varepsilon(y)\, dW_{y,s}$.

(ii) *For each $\xi > 0$, $T > 0$, and $\gamma < \frac{2}{3}$, the process $H_1(x, t)$ satisfies,*

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\sup_{(x,t) \in \mathcal{T}_\varepsilon} |H_1(x, t)| > \varepsilon^{-\xi} \Big) = 0. \tag{6.2}$$

Proof. From (4.11) and (4.3) we may write,

$$\begin{aligned} H(x, t) = \mathbb{1}_{\{t \le t_0\}} \int_0^t \int \partial_x (K_\infty + \tilde{K})(x, y, t-s) a_\varepsilon(y)\, dW_{y,s} \\ + \mathbb{1}_{\{t > t_0\}} \Big(\int_0^{t-t_0} \int (T_1 + T_3 + T_4)(x, y, t-s) a_\varepsilon(y)\, dW_{y,s} \\ + \int_0^{t-t_0} \int \partial_x k(x, y, t-s) a_\varepsilon(y)\, dW_{y,s} \\ + \int_{t-t_0}^{t} \int \partial_x (K_\infty + \tilde{K})(x, y, t-s) a_\varepsilon(y)\, dW_{y,s} \end{aligned}$$

$$+\,\overline{m}'(x)\int_{t-t_0}^{t}\frac{1}{2\sqrt{2\pi(t-s)}}\times \operatorname{sign}(y)\mathrm{e}^{-y^2/(8(t-s))}a_\varepsilon(y)\,\mathrm{d}W_{y,s}\Big)$$
$$+\,H_2(x,t).$$

The term T_j above ($j=1,3,4$) is defined as the first, third and fourth term in the kernel $\partial_x K^*$ in (4.9), respectively. Call $H_1(x,t)$ the process given by all the terms but the last one on the right-hand side above and let us show that H_1 so defined satisfies (ii), thus concluding the proof.

From the proof of (4.15) in the Appendix, one can see that the term of order $t^{1/2}$ in the right-hand side of (A.11) is precisely the one coming from that part of $\partial_x K^*$ which is now in $H_2(x,t)$. The remaining terms in the right-hand side of (A.11) are bounded by $C(1+\log(1+t))$, while the terms in the right-hand side of (A.8), (A.9), and (A.10) are uniformly bounded in time. Therefore,

$$\mathbb{E}H_1(x,t)^2\le C\big(\log(1+t)+1\big), \tag{6.3}$$

for some constant C independent of x and t. Also, from (4.20)–(4.25), it is clear that the estimate (4.18) is valid for H_1 as well: given $\delta>0$, for any ε sufficiently small,

$$R>\varepsilon^{-11\gamma/10}\delta^{-2}+1 \quad \text{implies} \quad \sup_{|x|\ge R+\varepsilon^{-\beta},\,t\le\varepsilon^{-\gamma}T}\mathbb{E}H_1(x,t)^2\le\delta^2. \tag{6.4}$$

On the other hand, given $h>0$,

$$\begin{aligned}&\mathbb{E}\big(H_2(x+h,t)-H_2(x,t)\big)^2\\&\quad=\frac14\big(\overline{m}'(x+h)-\overline{m}'(x)\big)^2\int_0^t\mathrm{d}s\int\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(4s)}}{2\pi s}a_\varepsilon(y)^2\le Cht^{1/2},\end{aligned} \tag{6.5}$$

$$\begin{aligned}&\mathbb{E}\big(H_2(x,t+h)-H_2(x,t)\big)^2\\&\quad=\frac{\overline{m}'(x)^2}{4}\int_t^{t+h}\mathrm{d}s\int\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(4(t+h-s))}}{2\pi(t+h-s)}a_\varepsilon(y)^2\\&\qquad+\frac{\overline{m}'(x)^2}{4}\int_0^t\mathrm{d}s\int\mathrm{d}y\Big(\frac{\mathrm{e}^{-y^2/(8(t+h-s))}}{\sqrt{2\pi(t+h-s)}}-\frac{\mathrm{e}^{-y^2/(8(t-s))}}{\sqrt{2\pi(t-s)}}\Big)^2a_\varepsilon(y)^2\\&\quad\le C\Big(\int_0^h\frac{\mathrm{d}s}{\sqrt{s}}+\int_0^t\mathrm{d}s\,\frac{1}{\sqrt{2(s+h)}}+\frac{1}{\sqrt{2s}}-\frac{2}{\sqrt{2s+h}}\Big)\le Ch^{1/2}.\end{aligned} \tag{6.6}$$

From (6.1), (4.16), and (6.5) we have,

$$\mathbb{E}\big(H_1(x+h,t)-H_1(x,t)\big)^2\le Ch\big(t^{1/2}(1+h^3+h|x|^2)+\log(1+h^{-1}t^{1/4})\big). \tag{6.7}$$

Analogously, from (4.17) and (6.6),

$$\mathbb{E}\big(H_1(x,t+h)-H_1(x,t)\big)^2 \le C\big(h^{1/4}+h^{3/2}+\big(h+h^{1/2}\big)t^{1/2}\big). \tag{6.8}$$

With the aid of (6.7), (6.8), and (6.4), proceeding as in the proof of (4.26), it follows that, for any $\xi>0$ and sufficiently small ε,

$$\mathbb{E}\Big(\sup_{x\in\mathcal{T}_\varepsilon}\big|H_1(x,t)\big|\Big) < \varepsilon^{-\xi/5}.$$

Finally, from (6.3) and Borell's inequality (as in (3.16)), (ii) follows. □

We consider next the asymptotics for the scaled process $\varepsilon^{\gamma/4}H(x,\varepsilon^{-\gamma}t)$, which yields the leading term for $u(x,\varepsilon^{-\gamma}t)-\overline{m}(x)$ (see Proposition 5.2).

Proposition 6.2. *For any $x,x'\in\mathbb{R}$ and $t>t'\ge 0$,*

$$\lim_{\varepsilon\to 0}\varepsilon^{\gamma/2}\mathbb{E}\big(H\big(x,\varepsilon^{-\gamma}t\big)H\big(x',\varepsilon^{-\gamma}t'\big)\big)=\frac{\overline{m}'(x)\overline{m}'(x')}{2\sqrt{2\pi}}\big(\sqrt{t+t'}-\sqrt{t-t'}\big). \tag{6.9}$$

Proof. Changing variables in (4.12) we obtain,

$$\begin{aligned}&\mathbb{E}\big(H\big(x,\varepsilon^{-\gamma}t\big)H\big(x',\varepsilon^{-\gamma}t'\big)\big)\\&\quad=\int_0^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\,\partial_x K(x,y,s)\partial_{x'}K\big(x',y,s+\varepsilon^{-\gamma}\big(t-t'\big)\big)a_\varepsilon(y)^2.\end{aligned} \tag{6.10}$$

Recall formula (4.3) for K, observe that, as $\varepsilon\to 0$, we only need to consider $\varepsilon^{-\gamma}t'>t_0$, split the time integral above according to $s\le t_0$ or not, and call $\mathcal{E}_1$ and $\mathcal{E}_2$ the resulting terms. Let us consider first the latter. From (4.3),

$$\begin{aligned}\mathcal{E}_2=&\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\big(\partial_x K^*(x,y,s)+\partial_x k(x,y,s)\big)a_\varepsilon(y)^2\\&\times\big(\partial_{x'}K^*\big(x',y,s+\big(t-t'\big)\varepsilon^{-\gamma}\big)+\partial_{x'}k\big(x',y,s+\big(t-t'\big)\varepsilon^{-\gamma}\big)\big).\end{aligned} \tag{6.11}$$

Substituting $\partial_x K^*$ and $\partial_{x'}K^*$ by the expressions in (4.9), we write $\mathcal{E}_2$ as a sum of integrals. Let us single out the integral corresponding to the product of each second term in the right-hand side of (4.9), and denote it by $\mathcal{E}_{22}$,

$$\begin{aligned}\mathcal{E}_{22}=&\frac{\overline{m}'(x)\overline{m}'(x')}{4}\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\\&\times\int\mathrm{d}y\frac{\mathrm{e}^{-y^2/(8s)}\mathrm{e}^{-y^2/(8(s+\varepsilon^{-\gamma}(t-t')))}a_\varepsilon(y)^2}{\sqrt{2\pi s}\sqrt{2\pi(s+(t-t')\varepsilon^{-\gamma})}}.\end{aligned} \tag{6.12}$$

To conclude the proof, we will show that $\mathcal{E}_{22}$ is the only term that contributes to the limit in (6.9), that is,

(i) $\lim_{\varepsilon\to 0}\varepsilon^{\gamma/2}\mathcal{E}_{22}=\frac{1}{2\sqrt{2\pi}}\overline{m}'(x)\overline{m}'(x')(\sqrt{t+t'}-\sqrt{t-t'})$;
(ii) $\lim_{\varepsilon\to 0}\varepsilon^{\gamma/2}(\mathcal{E}_2-\mathcal{E}_{22})=0$;
(iii) $\lim_{\varepsilon\to 0}\varepsilon^{\gamma/2}\mathcal{E}_1=0$.

Proof of (i). Let us compute the integral in (6.12), but taking $a_\varepsilon=1$,

$$\begin{aligned}&\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(8s)}\mathrm{e}^{-y^2/(8(s+\varepsilon^{-\gamma}(t-t')))}}{\sqrt{2\pi s}\sqrt{2\pi(s+\varepsilon^{-\gamma}(t-t'))}}\\&\quad=2\int_{t_0}^{\varepsilon^{-\gamma}t'}\frac{\mathrm{d}s}{\sqrt{2\pi(2s+\varepsilon^{-\gamma}(t-t'))}}\\&\quad=\frac{2}{\sqrt{2\pi}}\Big(\sqrt{\varepsilon^{-\gamma}(t+t')}-\sqrt{2t_0+\varepsilon^{-\gamma}(t-t')}\Big).\end{aligned}\tag{6.13}$$

Then, (i) follows once we show that

$$\lim_{\varepsilon\to 0}\varepsilon^{\gamma/2}\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(8s)}\mathrm{e}^{-y^2/(8(s+\varepsilon^{-\gamma}(t-t')))}(1-a_\varepsilon(y)^2)}{\sqrt{2\pi s}\sqrt{2\pi(s+(t-t')\varepsilon^{-\gamma})}}=0.\tag{6.14}$$

To do this, we split the spatial integral in (6.14) according to $|y|\le\varepsilon^{-\beta/2}$ or not. For the first case, from the properties of a, we know that, given $\eta>0$, $|1-a_\varepsilon(y)^2|<\eta$ for ε sufficiently small. Computing the integral as above we get,

$$\begin{aligned}&\varepsilon^{\gamma/2}\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int_{|y|\le\varepsilon^{-\beta/2}}\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(8s)}\mathrm{e}^{-y^2/(8(s+\varepsilon^{-\gamma}(t-t')))}(1-a_\varepsilon(y)^2)}{\sqrt{2\pi s}\sqrt{2\pi(s+(t-t')\varepsilon^{-\gamma})}}\\&\quad\le C\eta.\end{aligned}\tag{6.15}$$

In the other case, we have,

$$\begin{aligned}&\varepsilon^{\gamma/2}\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int_{|y|>\varepsilon^{-\beta/2}}\mathrm{d}y\,\frac{\mathrm{e}^{-y^2/(8s)}\mathrm{e}^{-y^2/(8(s+\varepsilon^{-\gamma}(t-t')))}(1-a_\varepsilon(y)^2)}{\sqrt{2\pi s}\sqrt{2\pi(s+(t-t')\varepsilon^{-\gamma})}}\\&\quad\le C\varepsilon^{\gamma/2}\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\,\frac{\mathrm{e}^{-\varepsilon^{-\beta}/(8s)}}{\sqrt{2\pi s}}\le C\mathrm{e}^{-\varepsilon^{(-\beta+\gamma)/8}}\to 0\qquad\text{as }\varepsilon\to 0,\end{aligned}\tag{6.16}$$

since $\gamma<\beta$, and (i) follows.

Proof of (ii). We need to consider all the terms in (6.11) except $\mathcal{E}_{22}$. From Hölder's inequality and (A.10) we have,

$$\begin{aligned}&\bigg|\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\,\partial_x k(x,y,s)\partial_{x'}k\big(x',y,s+\varepsilon^{-\gamma}(t-t')\big)a_\varepsilon(y)^2\bigg|\\&\quad\le\bigg(\int_{t_0}^{\varepsilon^{-\gamma}t'}\mathrm{d}s\int\mathrm{d}y\big(\partial_x k(x,y,s)\big)^2\bigg)^{1/2}\\&\qquad\times\bigg(\int_{t_0+\varepsilon^{-\gamma}(t-t')}^{\varepsilon^{-\gamma}t}\mathrm{d}s\int\mathrm{d}y\,a_\varepsilon(y)^2\big|\partial_{x'}k(x',y,s)\big|^2\bigg)^{1/2}\\&\quad\le C.\end{aligned}\tag{6.17}$$

Analogously, from (A.10) and (A.11),

$$\begin{aligned}
&\Big|\int_{t_0}^{\varepsilon^{-\gamma}t'} \mathrm{d}s \int \mathrm{d}y \partial_x K^*(x,y,s)\partial_{x'}k\big(x',y,s+\varepsilon^{-\gamma}(t-t')\big)a_\varepsilon(y)^2\Big| \\
&\quad \le \Big(\int_{t_0}^{\varepsilon^{-\gamma}t'} \mathrm{d}s \int \mathrm{d}y \big(\partial_x K^*(x,y,s)\big)^2 a_\varepsilon(y)^2\Big)^{1/2} \\
&\quad\quad \times \Big(\int_{t_0+\varepsilon^{-\gamma}(t-t')}^{\varepsilon^{-\gamma}t} \mathrm{d}s \int \mathrm{d}y \big(\partial_{x'}k(x',y,s)\big)^2 a_\varepsilon(y)^2\Big)^{1/2} \\
&\quad \le C\big(\log(\varepsilon^{-\gamma}t)+\sqrt{\varepsilon^{-\gamma}t}\big)^{1/2}.
\end{aligned} \tag{6.18}$$

Finally, to estimate the terms coming from $\partial_x K^*\partial_{x'}K^*$ in (6.11), we observe that, from (4.9), Hölder's inequality and (A.11),

$$\begin{aligned}
&\Big|\int_{t_0}^{\varepsilon^{-\gamma}t'} \mathrm{d}s \int \mathrm{d}y \partial_x K^*(x,y,s)\frac{\overline{m}'(x')\varphi(y)a_\varepsilon(y)^2}{2\sqrt{2\pi(s+\varepsilon^{-\gamma}(t-t'))}}\Big| \\
&\quad \le C\big(\log(t'\varepsilon^{-\gamma})+(\varepsilon^{-\gamma}t')^{1/2}\big)^{1/2}\big(\log(\varepsilon^{-\gamma}t)\big)^{1/2},
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
&\Big|\int_{t_0}^{\varepsilon^{-\gamma}t'} \mathrm{d}s \int \mathrm{d}y \partial_x K^*(x,y,s)\frac{x'\varphi(y)\mathrm{e}^{-x'^2/(8(s+\varepsilon^{-\gamma}(t-t')))}a_\varepsilon(y)^2}{8\sqrt{2\pi}(s+\varepsilon^{-\gamma}(t-t'))^{3/2}}\Big| \\
&\quad \le C\big(\log(\varepsilon^{-\gamma}t')+(\varepsilon^{-\gamma}t')^{1/2}\big)^{1/2},
\end{aligned} \tag{6.20}$$

$$\begin{aligned}
&\Big|\int_{t_0}^{\varepsilon^{-\gamma}t'} \mathrm{d}s \int \mathrm{d}y \partial_x K^*(x,y,s)\frac{(x'+y)\mathrm{e}^{-(x'+y)^2/(8(s+\varepsilon^{-\gamma}(t-t')))}a_\varepsilon(y)^2}{4\sqrt{2\pi}(s+\varepsilon^{-\gamma}(t-t'))^{3/2}}\Big| \\
&\quad \le C\big(\log(\varepsilon^{-\gamma}t)+(\varepsilon^{-\gamma}t)^{1/2}\big)^{1/2}.
\end{aligned} \tag{6.21}$$

Observe that (6.17), (6.18), (6.19), (6.20), and (6.21) go to zero when multiplied by $\varepsilon^{\gamma/2}$, as well as the remaining terms (which are similarly estimated), and (ii) follows.

Proof of (iii). From Hölder's inequality,

$$\begin{aligned}
|\mathcal{E}_1| &= \Big|\int_0^{t_0} \mathrm{d}s \int \mathrm{d}y \partial_x K(x,y,s)\partial_{x'}K\big(x',y,s+\varepsilon^{-\gamma}(t-t')\big)a_\varepsilon(y)^2\Big| \\
&\le \Big(\int_0^{t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_x K(x,y,s)\big)^2 a_\varepsilon(y)^2\Big)^{1/2} \\
&\quad \times \Big(\int_0^{t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_{x'}K\big(x',y,s+\varepsilon^{-\gamma}(t-t')\big)\big)^2 a_\varepsilon(y)^2\Big)^{1/2}.
\end{aligned}$$

The first factor on the right-hand side above is bounded by a constant, as can be seen from (4.3), (A.8) and (A.9). For the last factor, from (4.3) we have that, if $s \ge t_0$,

$$\big|\partial_{x'}K(x',y,s)\big|^2 \le 2\big|\partial_{x'}K^*(x',y,s)\big|^2 + 2\big|\partial_{x'}k(x',y,s)\big|^2.$$

From (A.10), we obtain,

$$\int_{\varepsilon^{-\gamma}(t-t')}^{\varepsilon^{-\gamma}(t-t')+t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_{x'} k(x', y, s)\big)^2 a_\varepsilon(y)^2 \le C.$$

Finally, we need to integrate K^*. As in (A.11), from (4.9),

$$\int_{\varepsilon^{-\gamma}(t-t')}^{\varepsilon^{-\gamma}(t-t')+t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_{x'} K^*(x', y, s)\big)^2 a_\varepsilon(y)^2 \le C \int_{\varepsilon^{-\gamma}(t-t')}^{\varepsilon^{-\gamma}(t-t')+t_0} \mathrm{d}s \left(\frac{1}{s} + \frac{1}{s^{1/2}} + \frac{1}{s^{3/2}}\right) \le C.$$

From the previous estimates, $|\mathcal{E}_1| \le C$, which implies (ii), and concludes the proof. □

The previous proposition implies that, as $\varepsilon \to 0$, the scaled process $\varepsilon^{\gamma/4} H(x, t\varepsilon^{-\gamma})$ converges in the sense of finite dimensional distributions to the process $(8\pi)^{-1/4}\overline{m}'(x) r(t)$, where $r(t)$ is the one dimensional Gaussian process with covariance (2.4), which is clearly continuous. Recall now the decomposition of H given in Lemma 6.1, and denote by h_2 the temporal part of H_2, that is,

$$h_2(t) = \int_0^t \int \frac{\mathrm{e}^{-y^2/(8(t-s))}}{2\sqrt{2\pi(t-s)}} \operatorname{sign}(y)\, \mathrm{d}W_{y,s}. \tag{6.22}$$

We show next that h_2, when suitably scaled, converges weakly in $C(\mathbb{R}_+)$ (equipped with the topology of uniform convergence in compacts) to the process r.

Lemma 6.3. *As $\varepsilon \to 0$, the real process*

$$h^{(\varepsilon)}(t) = \varepsilon^{\gamma/4} h_2(\varepsilon^{-\gamma} t) \tag{6.23}$$

converges weakly in $C(\mathbb{R}_+)$ to $(8\pi)^{1/4} r(t)$.

Proof. From (6.13) and (6.14) with $t_0 = 0$, it follows that

$$\lim_{\varepsilon \to 0} \mathbb{E}\big(h^{(\varepsilon)}(t) h^{(\varepsilon)}(t')\big) = (8\pi)^{-1/2}\big(\sqrt{t+t'} - \sqrt{t-t'}\big),$$

which implies $h^{(\varepsilon)}(t) \to r(t)$, in the sense of finite dimensional distributions. From (6.6), we also know that

$$\mathbb{E}\big(h^{(\varepsilon)}(t+h) - h^{(\varepsilon)}(t)\big)^2 = \varepsilon^{\gamma/2} \mathbb{E}\big(h_2(\varepsilon^{-\gamma}(t+h)) - h_2(\varepsilon^{-\gamma} t)\big)^2 \le C\varepsilon^{\gamma/2}\big(h\varepsilon^{-\gamma}\big)^{1/2} = Ch^{1/2},$$

which, together with the fact $h^{(\varepsilon)}(0) = 0$ implies that the corresponding family of laws is tight, and then the weak limit has to be r. □

Proof of Theorem 2.1. The statement concerning the uniqueness and existence of a solution u to the Cahn–Hilliard equation (2.1), or, equivalently (as already discussed), to the integral equation (4.2) follows from Proposition 5.2, just by taking $B(\varepsilon,\gamma,T)=B_\varepsilon(T,\xi)$ with any ξ small enough.

To prove item (i), given $\eta>0$ we fix $\xi\in(0,\frac{1}{2}-\frac{3\gamma}{4})$ such that $2\xi<\eta$ and we let $B_\varepsilon(T,\xi)$ be as in Proposition 5.2. Consider δ_0 as in Lemma 5.3 and define the process $\zeta_\varepsilon(t)=z_\varepsilon(t\wedge\tau)$, where

$$\tau=\inf\{t>0:u(\cdot,t)\notin\mathcal{M}_{\delta_0}\}$$

and $z_\varepsilon(t)$ is the center of $u(\cdot,t)$ (which is well defined, as follows from Lemma 5.4). The process $\zeta_\varepsilon(t)$ is clearly continuous and adapted to $\mathscr{F}_t$; note also that $B_\varepsilon(T,\xi)\subset\{\tau>\varepsilon^{-\gamma}T\}$ for any ε sufficiently small (recall (5.2)). Let now,

$$\nu_\varepsilon(t)=-\frac{3}{4}\langle u(\cdot,t)-\overline{m},\overline{m}'\rangle,$$

which is an approximation to $\zeta_\varepsilon(t)$. Indeed, from (5.5), on the set $B_\varepsilon(T,\xi)$, $\sup_{t\le\varepsilon^{-\gamma}T}|\zeta_\varepsilon(t)-\nu_\varepsilon(t)|\le\varepsilon^{1-\gamma/2-\xi}$, which implies, for any ε small enough,

$$\sup_{t\le\varepsilon^{-\gamma}T}\|\overline{m}_{\zeta_\varepsilon(t)}-\overline{m}_{\nu_\varepsilon(t)}\|_\infty\le\varepsilon^{1-\gamma/2-2\xi}. \tag{6.24}$$

Observe next that, from (6.1) and (6.22), after recalling that $\langle\overline{m}',\overline{m}'\rangle=\frac{4}{3}$,

$$\nu_\varepsilon(t)=-\frac{3}{4}\langle u(\cdot,t)-\overline{m}-\sqrt{\varepsilon}H,\overline{m}'\rangle-\frac{3}{4}\langle\sqrt{\varepsilon}H_1,\overline{m}'\rangle+\sqrt{\varepsilon}h_2(t).$$

Therefore, by (5.3) and (6.2), there is a set $\tilde{B}_\varepsilon(T,\xi)\subset B_\varepsilon(T,\xi)$, with $\mathbb{P}(\tilde{B}_\varepsilon(T,\xi))\to1$ as $\varepsilon\to0$, such that

$$\sup_{t\le\varepsilon^{-\gamma}T}\left|\nu_\varepsilon(t)-\sqrt{\varepsilon}h_2(t)\right|\le\varepsilon^{1-\gamma-\xi}+\varepsilon^{1/2-\xi}\qquad\forall\omega\in\tilde{B}_\varepsilon(T,\xi),$$

and then, for any ε small enough,

$$\sup_{t\le\varepsilon^{-\gamma}T}\|\overline{m}_{\nu_\varepsilon(t)}-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty\le\varepsilon^{1-\gamma-2\xi}+\varepsilon^{1/2-2\xi}\qquad\forall\omega\in\tilde{B}_\varepsilon(T,\xi). \tag{6.25}$$

Now, by the triangle inequality we may write,

$$\begin{aligned}\|u(\cdot,t)-\overline{m}_{\zeta_\varepsilon(t)}\|_\infty&\le\|\overline{m}_{\zeta_\varepsilon(t)}-\overline{m}_{\nu_\varepsilon(t)}\|_\infty+\|\overline{m}_{\nu_\varepsilon(t)}-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty\\&\quad+\|u(\cdot,t)-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty.\end{aligned} \tag{6.26}$$

For this last term, from (6.1) and (6.22) we have in turn,

$$\begin{aligned}&\|u(\cdot,t)-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty\\&\quad\le\|u(\cdot,t)-\overline{m}-\sqrt{\varepsilon}H\|_\infty+\|\overline{m}+\sqrt{\varepsilon}H-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty\\&\quad\le\|u(\cdot,t)-\overline{m}-\sqrt{\varepsilon}H\|_\infty\\&\qquad+\|\overline{m}-\sqrt{\varepsilon}\overline{m}'h_2-\overline{m}_{\sqrt{\varepsilon}h_2(t)}\|_\infty+\sqrt{\varepsilon}\|H_1\|_\infty.\end{aligned} \tag{6.27}$$

The first and last terms on this last inequality are bounded with the aid of (5.3) and (6.2), while for the middle one we have, for some $\theta \in \mathbb{R}$,

$$\left|\overline{m} - \overline{m}_{\sqrt{\varepsilon}h_2(t)} - \overline{m}'\sqrt{\varepsilon}h_2(t)\right| = \frac{\varepsilon}{2}\left|\overline{m}''(\theta)\right|h_2^2(t), \tag{6.28}$$

and, from Lemma 6.3, we know that

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\le\varepsilon^{-\gamma}T} \varepsilon\left|h_2^2(t)\right| > \varepsilon^{1-\gamma/2-2\xi}\Big) = 0. \tag{6.29}$$

Item (i) follows now from (6.26), (6.24), (6.25), (6.27), (6.28), and (6.29).

The convergence of $X_\varepsilon(t) = \varepsilon^{-1/2+\gamma/4}\zeta_\varepsilon(\varepsilon^{-\gamma}t)$ follows from the above estimates and Lemma 6.3 after writing

$$\zeta_\varepsilon(t) = \big(\zeta_\varepsilon(t) - \nu_\varepsilon(t)\big) + \big(\nu_\varepsilon - \sqrt{\varepsilon}h_2(t)\big) + \sqrt{\varepsilon}h_2(t),$$

which proves (ii). □

7 The one dimensional process *r*

In the next proposition, we summarize some properties of r, that follow at once from the form of the covariance function.

Proposition 7.1. *The one dimensional Gaussian process with covariance given by* (2.4) *satisfies*,

(i) *It has* (*a modification with*) *continuous paths*.
(ii) *It is a self similar process of order* $\frac{1}{4}$, *that is*, *for any given* $a > 0$,

$$\{r(at)\}_{t\ge 0} \stackrel{\text{law}}{=} \{a^{1/4}r(t)\}_{t\ge 0}.$$

We also obtain several representations of the process r, in terms of fractional Brownian motion, the solution to the one dimensional stochastic heat equation and a stochastic integral with respect to Brownian motion, that may be of independent interest.

Let us recall that a two sided fractional-Brownian motion with Hurst parameter H, is a one dimensional Gaussian process $\nu^{(H)}(t)$ characterized by its covariance function,

$$\mathbb{E}\big(\nu^{(H)}(t)\nu^{(H)}(s)\big) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).$$

Define the odd part of $\nu^{(H)}(t)$ as usual,

$$\nu_O^{(H)}(t) = \frac{1}{2}\big(\nu^{(H)}(t) - \nu^{(H)}(-t)\big). \tag{7.1}$$

We refer to Mandelbrot and Van Ness (1968) for an introduction and properties of fractional Brownian motion.

Next, consider $h(x,t)$ the solution of the stochastic heat equation for $x \in \mathbb{R}$, with zero initial condition,

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \dot{W}, \\ h(x,0) = 0, \end{cases}$$

where $\dot{W} = \dot{W}_{x,t}$ is a space–time white noise.

Proposition 7.2. *The process $r(t)$ satisfies,*

$$\{r(t)\}_{t\geq 0} \overset{\text{law}}{=} \{2\nu_O^{(1/4)}(t)\}_{t\geq 0},$$

$$\{r(t)\}_{t\geq 0} \overset{\text{law}}{=} \{(2\pi)^{1/4} h(0,t)\}_{t\geq 0}.$$

Proof. Both statements follow by computing covariances, since all the processes involved are Gaussian. Indeed, the process $h(x,t)$ is given by

$$h(x,t) = \int_0^t \frac{\mathrm{e}^{-(x-y)^2/(2(t-s))}}{\sqrt{2\pi(t-s))}}\,\mathrm{d}W_{y,s},$$

and the covariance is easily computed. □

We also obtain a representation of r as an integral with respect to Brownian motion.

Proposition 7.3. *The process r can be represented as the following integral with respect to a Brownian motion b,*

$$r(t) = c\int_0^t \frac{u^{1/4}}{(t^2-u^2)^{1/4}}\,\mathrm{d}b(u), \tag{7.2}$$

where $c = (\frac{1}{2}B(\frac{3}{4},\frac{3}{4}))^{-1/2}$, with $B(\cdot,\cdot)$ the usual Euler beta function.

Proof. Suppose that $t \geq t' \geq 0$. Then, Formula 9.121(4), page 1040 in Gradshteyn and Ryzhik (1980) reads,

$$(t+t')^\rho - (t-t')^\rho = 2\rho t' t^{\rho-1} F\left(-\frac{\rho-1}{2}, -\frac{\rho-2}{2}; \frac{3}{2}, \frac{t'^2}{t^2}\right), \tag{7.3}$$

where F is the hypergeometric Gauss function. Taking $\rho = \frac{1}{2}$, and using for F the Integral Formula 9.11.1, page 1040 in Gradshteyn and Ryzhik (1980) we obtain,

$$F\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}, \frac{t'^2}{t^2}\right) = \frac{1}{B(3/4,3/4)}\int_0^1 \mathrm{d}s\, s^{-1/4}(1-s)^{-1/4}\left(1-\frac{st'^2}{t^2}\right)^{-1/4}. \tag{7.4}$$

The change of variables $u = t's^{1/2}$ in this last integral, together with (7.3) yields,

$$\sqrt{t+t'} - \sqrt{t-t'} = \frac{2}{B(3/4,3/4)} \int_0^{t'} \mathrm{d}u \frac{u^{1/2}}{(t^2-u^2)^{1/4}(t'^2-u^2)^{1/4}}, \tag{7.5}$$

which proves (7.2), since the integral on the right-hand side above is just the covariance of the process given by the stochastic integral in (7.2). □

Appendix

A.1 Proof of Proposition 3.2

From (2.3) and (3.2)

$$\begin{aligned} \mathbb{E}Y(x,t)^2 &= \int_0^t \mathrm{d}s \int \mathrm{d}y a_\varepsilon(y)^2 \frac{1}{t-s} \phi'\Big(\frac{x-y}{(t-s)^{1/4}}\Big)^2 \\ &= \int_0^t \mathrm{d}s \frac{1}{s^{3/4}} \int \mathrm{d}z a_\varepsilon^2(x - s^{1/4}z)\phi'(z)^2 \le t^{1/4}\|\phi'\|_2^2, \end{aligned} \tag{A.1}$$

and then (3.7) follows from (3.3).

Before proving the remaining estimates, we remark that (3.4) implies that

$$\big|\phi'(x+H) - \phi'(x)\big| \le C\frac{H}{1+H}. \tag{A.2}$$

We then have,

$$\begin{aligned} &\mathbb{E}\big(Y(x+h,t) - Y(x,t)\big)^2 \\ &\quad = \int_0^t \mathrm{d}s \frac{1}{s^{3/4}} \int \mathrm{d}z a_\varepsilon^2(x - s^{1/4}z)\big[\phi'(z + s^{-1/4}h) - \phi'(z)\big]^2 \\ &\quad \le \int_0^t \mathrm{d}s \frac{1}{s^{3/4}} \int \mathrm{d}z \big[\phi'(z + s^{-1/4}h) - \phi'(z)\big]^2 \\ &\quad = 2\int_0^t \mathrm{d}s \frac{1}{s^{3/4}} \int \mathrm{d}z \phi'(z)\big[\phi'(z) - \phi'(z + s^{-1/4}h)\big] \\ &\quad \le C\int_0^t \mathrm{d}s \frac{1}{s^{3/4}} \frac{h}{s^{1/4}+h} = 4Ch\log\big(1 + h^{-1}t^{1/4}\big), \end{aligned} \tag{A.3}$$

where the last inequality follows from (3.3) and (A.2) with $H = s^{-1/4}h$. This proves the estimate (3.8).

We next have, after changing variables in the time integrals,

$$\begin{aligned} &\mathbb{E}\big(Y(x,t+h) - Y(x,t)\big)^2 \\ &\quad = \int_0^h \mathrm{d}s \int \mathrm{d}y \frac{a_\varepsilon(y)^2}{s} \phi'\Big(\frac{x-y}{s^{1/4}}\Big)^2 \\ &\qquad + \int_0^t \mathrm{d}s \int \mathrm{d}y a_\varepsilon^2(y) \Big[\frac{1}{(s+h)^{1/2}} \phi'\Big(\frac{x-y}{(s+h)^{1/4}}\Big) - \frac{1}{s^{1/2}}\phi'\Big(\frac{x-y}{s^{1/4}}\Big)\Big]^2. \end{aligned}$$

Let us denote by I_1 and I_2 the integrals on the right-hand side. The first integral can be bounded as in (A.1) to obtain

$$I_1 \le Ch^{1/4}. \tag{A.4}$$

For the second one, we observe that $I_2 \le I_{21} + I_{22}$ with

$$I_{21} = 2\int_0^t \mathrm{d}s \int \mathrm{d}y \Big(\frac{1}{s^{1/2}} - \frac{1}{(s+h)^{1/2}}\Big)^2 \phi'\Big(\frac{x-y}{s^{1/4}}\Big)^2,$$
$$I_{22} = 2\int_0^t \mathrm{d}s \int \mathrm{d}y \frac{1}{s+h}\Big[\phi'\Big(\frac{x-y}{(s+h)^{1/4}}\Big) - \phi'\Big(\frac{x-y}{s^{1/4}}\Big)\Big]^2.$$

Now, after the change of variables $z = s^{-1/4}(x-y)$ and then $s = h\tau$ we get,

$$I_{21} = 2\|\phi'\|_2^2 h^{1/4} \int_0^{h^{-1}t} \mathrm{d}\tau \frac{1}{\tau^{3/4}}\Big(1 - \frac{\tau^{1/2}}{(\tau+1)^{1/2}}\Big)^2 \le Ch^{1/4}, \tag{A.5}$$

where we used that the last integral in the right-hand side is bounded above by $\int_0^\infty \mathrm{d}\tau (1+\tau)^{-1}\tau^{-3/4} < +\infty$. Next, denoting $A = s^{-1/4}(s+h)^{1/4}$ and using (3.5),

$$\begin{aligned} I_{22} &= 2\int_0^t \mathrm{d}s \int \mathrm{d}z \frac{1}{(s+h)^{3/4}}[\phi'(z) - \phi'(Az)]^2 \\ &\le C\int_0^t \mathrm{d}s \frac{(A-1)^2}{(s+h)^{3/4}} \\ &= Ch^{1/4}\int_0^{h^{-1}t} \mathrm{d}\tau \frac{[(1+\tau)^{1/4} - \tau^{1/4}]^2}{\tau^{1/2}(1+\tau)^{3/4}} \le Ch^{1/4}, \end{aligned} \tag{A.6}$$

having used that the last integral is bounded by $\int_0^\infty \mathrm{d}\tau (1+\tau)^{-3/4}\tau^{-1/2} < +\infty$. The estimate (3.9) follows from (A.4), (A.5), and (A.6). □

A.2 Proof of Proposition 4.2

From (4.3) and recalling $\|a_\varepsilon\|_\infty \le 1$, we have that

$$\begin{aligned} \mathbb{E}H(x,t)^2 &= \int_0^t \mathrm{d}s \int \mathrm{d}y \big[\partial_x K(x,y,s) a_\varepsilon(y)\big]^2 \\ &\le 2\int_0^{t\wedge t_0} \mathrm{d}s \int \mathrm{d}y \big[\big(\partial_x K_\infty(x,y,s)\big)^2 + \big(\partial_x \tilde{K}(x,y,s)\big)^2\big] \\ &\quad + 2\int_{t\wedge t_0}^t \mathrm{d}s \int \mathrm{d}y \big[\big(\partial_x K^*(x,y,s)\big)^2 + \big(\partial_x k(x,y,s)\big)^2\big]. \end{aligned} \tag{A.7}$$

(Note that the last integral in the right-hand side is present only if $t > t_0$.)

Now, from (4.4) we get that

$$\begin{aligned}\int_0^{t\wedge t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_x K_\infty(x,y,s)\big)^2 &\le C\int_0^{t\wedge t_0} \mathrm{d}s \frac{1}{s}\int \mathrm{d}y \mathrm{e}^{-(2/s)^{1/4}|x-y|} \\ &\le C\int_0^{t\wedge t_0} \mathrm{d}s \frac{1}{s^{3/4}}\int \mathrm{d}z \mathrm{e}^{-|z|} \\ &= C(t\wedge t_0)^{1/4}.\end{aligned} \tag{A.8}$$

Analogously, from (4.5) and (4.8),

$$\int_0^{t\wedge t_0} \mathrm{d}s \int \mathrm{d}y \big(\partial_x \tilde{K}(x,y,s)\big)^2 \le C(t\wedge t_0)^{3/4}, \tag{A.9}$$

$$\int_{t\wedge t_0}^{t} \mathrm{d}s \int \mathrm{d}y \big(\partial_x k(x,y,s)\big)^2 \le C\big((t\wedge t_0)^{-1/2} - t^{-1/2}\big). \tag{A.10}$$

We are left with the estimation of the integral of $(\partial_x K^*)^2$, that gives the leading term (as $t\to\infty$). By formula (4.9), taking the square and using the bound $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$ we have,

$$\begin{aligned}\big(\partial_x K^*(x,y,s)\big)^2 &\le \bigg(\frac{\varphi(y)^2}{2\pi s} + \frac{\mathrm{e}^{-y^2/(4s)}}{2\pi s}\bigg)\overline{m}'(x)^2 \\ &\quad + \frac{x^2}{32\pi s^3}\mathrm{e}^{-x^2/(4s)}\varphi(y)^2 + \frac{(x+y)^2}{8\pi s^3}\mathrm{e}^{-(x+y)^2/(4s)},\end{aligned}$$

whence, as φ is square integrable,

$$\begin{aligned}\int \mathrm{d}y \big(\partial_x K^*(x,y,s)\big)^2 &\le \bigg(\frac{\|\varphi\|_2^2}{2\pi s} + \frac{1}{\pi s^{1/2}}\int \mathrm{d}z \mathrm{e}^{-z^2}\bigg)\overline{m}'(x)^2 \\ &\quad + \frac{\|\varphi\|_2^2}{8\pi s^2}\sup_z z^2 \mathrm{e}^{-z^2} + \frac{1}{\pi s^{3/2}}\int \mathrm{d}z z^2 \mathrm{e}^{-z^2} \\ &\le C\big((s^{-1}+s^{-1/2})\overline{m}'(x)^2 + s^{-2} + s^{-3/2}\big),\end{aligned}$$

so that (noticing $s^{-2} \le t_0^{-1/2} s^{-3/2}$ for $s \ge t_0$),

$$\begin{aligned}&\int_{t\wedge t_0}^{t} \mathrm{d}s \int \mathrm{d}y \big(\partial_x K^*(x,y,s)\big)^2 \\ &\quad \le C\bigg(\overline{m}'(x)^2\bigg(\log\frac{t}{t\wedge t_0} + t^{1/2} - (t_0\wedge t)^{1/2}\bigg) + (t\wedge t_0)^{-1/2} - t^{-1/2}\bigg).\end{aligned} \tag{A.11}$$

We now observe that $(t\wedge t_0)^{1/4} + (t\wedge t_0)^{3/4} \le (1+\sqrt{t_0})(t\wedge t_0)^{1/4}$ and that the terms in the right-hand side of (A.10) and (A.11) are zero for $t \le t_0$ and bounded by $C(1+\overline{m}'(x)^2 t^{1/2})$. Therefore, by (A.7), (A.8), (A.9), (A.10), and (A.11) we conclude that

$$\mathbb{E}H(x,t)^2 \le C\big((t\wedge t_0)^{1/4} + \big(1+\overline{m}'(x)^2 t^{1/2}\big)\mathbb{1}_{\{t>t_0\}}\big).$$

The estimate (4.15) now follows by noticing that if $t \le 1$ the right-hand side in the last display is bounded by

$$Ct^{1/4}\mathbb{1}_{\{t\le t_0\wedge 1\}} + C\big(1+\big\|(\overline{m}')^2\big\|_\infty\big)\mathbb{1}_{\{t>t_0\wedge 1\}} \le C\bigg(1+\frac{1+\|(\overline{m}')^2\|_\infty}{(t_0\wedge 1)^{1/4}}\bigg)t^{1/4},$$

while, if $t > 1$, it is bounded by $C(t_0^{1/4}+1+\overline{m}'(x)^2 t^{1/2})$.

To prove the rest of the proposition, let us use formula (4.14) for H, to write

$$\begin{aligned}\mathbb{E}\big(H(x+h,t)-H(x,t)\big)^2 & \\ \le 2\mathbb{E}\big(Y(x+h,t)-Y(x,t)\big)^2 & \qquad (A.12)\\ +2\mathbb{E}\big(\mathcal{G}\big(V''(\overline{m})H\big)(x+h,t)-\mathcal{G}\big(V''(\overline{m})H\big)(x,t)\big)^2. &\end{aligned}$$

Recalling (3.21) (with the substitution $s\to t-s$) and using $V''(\overline{m}) \le 3\overline{m}^2+1\le 4$, we estimate the last expectation above by Cauchy–Schwarz inequality,

$$\mathbb{E}\big(\mathcal{G}\big(V''(\overline{m})H\big)(x+h,t)-\mathcal{G}\big(V''(\overline{m})H\big)(x,t)\big)^2 \le J(x,t)^2, \qquad (A.13)$$

where

$$J(x,t) = 4\int_0^t \mathrm{d}s\int \mathrm{d}y\big|\partial_y^2 G(x+h,y,s)-\partial_y^2 G(x,y,s)\big|\big(\mathbb{E}H(y,t-s)^2\big)^{1/2}.$$

From (4.15), we have $(\mathbb{E}H(y,t-s)^2)^{1/2}\le C(1+\overline{m}'(y)(t-s)^{1/4}\mathbb{1}_{\{t-s>1\}})$. Then, by formula (3.2) for $\partial_y^2 G$, after changing variable $y = x - s^{1/4}z$ we get,

$$\begin{aligned}J(x,t) \le{}& C\int_0^t \mathrm{d}s\frac{1}{s^{1/2}}\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big|\\ &+Ct^{1/4}\int_0^{(t-1)_+}\mathrm{d}s\frac{1}{s^{1/2}}\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big|\overline{m}'\big(x-s^{1/4}z\big),\end{aligned}$$

which implies, as $\overline{m}'$ is bounded,

$$\begin{aligned}J(x,t) \le{}& C\int_0^t \mathrm{d}s\frac{1}{s^{1/2}}\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big|\\ &+Ct^{1/4}\int_0^{1\wedge(t-1)_+}\mathrm{d}s\frac{1}{s^{1/2}}\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big| \qquad (A.14)\\ &+Ct^{1/4}\int_{1\wedge(t-1)_+}^{(t-1)_+}\mathrm{d}s\frac{1}{s^{1/2}}\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big|\overline{m}'\big(x-s^{1/4}z\big).\end{aligned}$$

Now, by (3.4) with $H = s^{-1/4}h$ and $k=2$,

$$\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big| \le Cs^{-1/4}h,$$

while, by (3.6) and using that $\overline{m}'$ vanishes exponentially fast at $\pm\infty$,

$$\int \mathrm{d}z\big|\phi''\big(z+s^{-1/4}h\big)-\phi''(z)\big|\overline{m}'\big(x-s^{1/4}z\big) \le Cs^{-3/4}\big(h^2+\big(1+|x|\big)h\big).$$

Using the above estimates in the right-hand side of (A.14), we obtain

$$J(x,t) \le Ct^{1/4}\big(h^2 + \big(1+|x|\big)h\big). \tag{A.15}$$

Estimate (4.16) follows now from (A.12), (3.8), (A.13), and (A.15).

A similar reasoning proves (4.17). Indeed, by formula (4.14) for H we have in this case,

$$\begin{aligned} &\mathbb{E}\big(H(x,t+h) - H(x,t)\big)^2 \\ &\quad \le 2\mathbb{E}\big(Y(x,t+h) - Y(x,t)\big)^2 \\ &\qquad + 2\mathbb{E}\big(\mathcal{G}\big(V''(\overline{m})H\big)(x,t+h) - \mathcal{G}\big(V''(\overline{m})H\big)(x,t)\big)^2, \end{aligned} \tag{A.16}$$

with now (as before, recalling (3.21), that $V''(\overline{m}) \le 3\overline{m}^2 + 1 \le 4$, changing $s \to t-s$, and using Cauchy Schwarz inequality),

$$\begin{aligned} &\mathbb{E}\big(\mathcal{G}\big(V''(\overline{m})H\big)(x,t+h) - \mathcal{G}\big(V''(\overline{m})H\big)(x,t)\big)^2 \\ &\quad \le 2J_1(x,t)^2 + 2J_2(x,t)^2, \end{aligned} \tag{A.17}$$

where

$$J_1(x,t) = 4\int_0^h \mathrm{d}s \int \mathrm{d}y \big|\partial_y^2 G(x,y,s)\big| \big(\mathbb{E}H(y,t+h-s)^2\big)^{1/2}$$

and

$$J_2(x,t) = 4\int_0^t \mathrm{d}s \int \mathrm{d}y \big|\partial_y^2 G(x,y,s+h) - \partial_y^2 G(x,y,s)\big| \big(\mathbb{E}H(y,t-s)^2\big)^{1/2}.$$

Proceeding as in (A.13), from (3.2), (3.3) and (4.15) we obtain

$$\begin{aligned} J_1(x,t) &\le 4 \sup_{z\in\mathbb{R},\tau\le t+h} \big(\mathbb{E}H(y,\tau)^2\big)^{1/2} \int_0^h \mathrm{d}s \frac{1}{s^{1/2}} \int \mathrm{d}z \big|\phi''(z)\big| \\ &\le Ch^{1/2}\big(1 + h^{1/4} + t^{1/4}\big). \end{aligned} \tag{A.18}$$

Recalling (4.15) implies $(\mathbb{E}H(y,t-s)^2)^{1/2} \le 1 + \overline{m}'(y)(t-s)^{1/4}\mathbb{1}_{\{t-s>1\}}$, we now have,

$$\begin{aligned} &J_2(x,t) \\ &\quad \le 4\int_0^t \mathrm{d}s \frac{1}{(s+h)^{3/4}} \int \mathrm{d}y \left|\phi''\left(\frac{x-y}{(s+h)^{1/4}}\right) - \phi''\left(\frac{x-y}{s^{1/4}}\right)\right| \\ &\qquad + 4\int_0^t \mathrm{d}s \left(\frac{1}{s^{3/4}} - \frac{1}{(s+h)^{3/4}}\right) \int \mathrm{d}y \left|\phi''\left(\frac{x-y}{s^{1/4}}\right)\right| \\ &\qquad + 4t^{1/4}\int_0^{(t-1)_+} \mathrm{d}s \frac{1}{(s+h)^{3/4}} \int \mathrm{d}y \left|\phi''\left(\frac{x-y}{(s+h)^{1/4}}\right) - \phi''\left(\frac{x-y}{s^{1/4}}\right)\right| \overline{m}'(y) \\ &\qquad + 4t^{1/4}\int_0^{(t-1)_+} \mathrm{d}s \left(\frac{1}{s^{3/4}} - \frac{1}{(s+h)^{3/4}}\right) \int \mathrm{d}y \left|\phi''\left(\frac{x-y}{s^{1/4}}\right)\right| \overline{m}'(y). \end{aligned}$$

We change variables $y = x - (s+h)^{1/4} z$ in the first and third integral and $y = x - s^{1/4} z$ in the second one, and shorthand $A = (\frac{s+h}{s})^{1/4}$ to obtain,

$$
\begin{aligned}
&J_2(x,t) \\
&\quad \le 4 \int_0^t \mathrm{d}s \frac{1}{(s+h)^{1/2}} \int \mathrm{d}z \big|\phi''(z) - \phi''(Az)\big| \\
&\quad + 4 \int_0^t \mathrm{d}s \frac{1}{s^{1/2}} \biggl(1 - \frac{s^{3/4}}{(s+h)^{3/4}}\biggr) \int \mathrm{d}z \big|\phi''(z)\big| \\
&\quad + 4t^{1/4} \int_0^{(t-1)_+} \mathrm{d}s \frac{1}{(s+h)^{1/2}} \int \mathrm{d}z \big|\phi''(z) - \phi''(Az)\big| \overline{m}'\bigl(x - (s+h)^{1/4} z\bigr) \\
&\quad + 4t^{1/4} \int_0^{(t-1)_+} \mathrm{d}s \biggl(\frac{1}{s^{3/4}} - \frac{1}{(s+h)^{3/4}}\biggr) \int \mathrm{d}y \biggl|\phi''\biggl(\frac{x-y}{s^{1/4}}\biggr)\biggr| \overline{m}'(y).
\end{aligned}
$$

We denote by J_{21}, J_{22}, J_{23}, J_{24} the four integrals on the right-hand side. By (3.5) and noticing

$$
\begin{aligned}
A - 1 &= \frac{(s+h)^{1/4} - s^{1/4}}{s^{1/4}} = \frac{(s+h)^{1/2} - s^{1/2}}{s^{1/4}((s+h)^{1/4} + s^{1/4})} \\
&= \frac{h}{s^{1/4}((s+h)^{1/4} + s^{1/4})((s+h)^{1/2} + s^{1/2})} \le \biggl(\frac{h}{s}\biggr)^{3/4},
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
J_{21} &\le C \int_0^t \mathrm{d}s \frac{1}{(s+h)^{1/2}} \biggl(\frac{h}{s}\biggr)^{1/4} \\
&= Ch^{1/2} \int_0^{h^{-1}t} \mathrm{d}\tau \frac{1}{\tau^{3/4}(1+\tau)^{1/2}} \le Ch^{1/2},
\end{aligned}
$$

while, as ϕ'' is integrable,

$$
\begin{aligned}
J_{22} &\le C \int_0^t \mathrm{d}s \frac{1}{s^{1/2}} \biggl(1 - \frac{s^{3/4}}{(s+h)^{3/4}}\biggr) \\
&= Ch^{1/2} \int_0^{h^{-1}t} \mathrm{d}\tau \frac{(1+\tau)^{3/4} - \tau^{3/4}}{\tau^{1/2}(1+\tau)^{3/4}} \le Ch^{1/2}.
\end{aligned}
$$

To estimate J_{23} we observe that (3.5) implies $|\phi''(z) - \phi''(Az)| \le C\frac{A-1}{A}$, and d$z$-integration of $\overline{m}'(x + (s+h)^{1/4} z)$ gives an extra factor $(s+h)^{-1/4}$. Therefore,

$$
\begin{aligned}
J_{23} &\le Ct^{1/4} \int_0^t \mathrm{d}s \frac{1}{(s+h)^{3/4}} \biggl(\frac{h}{s}\biggr)^{3/4} \\
&= C(ht)^{1/4} \int_0^{h^{-1}t} \mathrm{d}\tau \frac{1}{\tau^{3/4}(1+\tau)^{3/4}} \le C(ht)^{1/4}.
\end{aligned}
$$

Analogously, as ϕ'' is bounded and $\overline{m}'$ is integrable we finally have,

$$J_{24} \le C(ht)^{1/4} \int_0^{h^{-1}t} \mathrm{d}\tau \frac{(1+\tau)^{3/4} - \tau^{3/4}}{\tau^{3/4}(1+\tau)^{3/4}} \le C(ht)^{1/4}.$$

We conclude that

$$J_2(x,t) \le C\bigl(h^{1/2} + (ht)^{1/4}\bigr). \tag{A.20}$$

The estimate (4.17) follows from (A.16), (3.9), (A.17), (A.18), and (A.20). □

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