FUBINI'S THEOREM IN CODIMENSION TWO

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Abstract. We classify codimension two analytic submanifolds of projective space $X^n \subset \mathbb{CP}^{n+2}$ having the property that any line through a general point $x \in X$ having contact to order two with X at x automatically has contact to order three. We give applications to the study of the Debarre-de Jong conjecture and of varieties whose Fano variety of lines has dimension $2n-4$.

1. Introduction

1.1. **Statement of the main result.** Let V be a complex vector space, and $X \subset \mathbb{P}V$ be a complex submanifold or algebraic variety and let $x \in X$ be a smooth point. Define $\mathcal{C}_{k,x} \subset \mathbb{P}T_xX$ to be the set of tangent directions at x for which there exists a line $l \simeq \mathbb{P}^1$ in $\mathbb{P}V$ having contact to order k with X at x, or, in the language of algebraic geometry, $mult(l \cap X)_x \geq k+1$. Let $\mathcal{C}_x = \mathcal{C}_{\infty,x} \subset \mathbb{P}T_xX$ denote the tangent directions to lines on X through x.

One way to state the classical Fubini theorem [2] is as follows:

Theorem 1.1 (Fubini)**.** *Let $X^n \subset \mathbb{CP}^{n+1}$ be a complex analytic hypersurface with $n > 1$ and at least a two dimensional Gauss image. Let $x \in X$ be a general point. If*

$$\mathcal{C}_{2,x} = \mathcal{C}_{3,x} \quad \text{(Fubini hypothesis)}$$

then X is (an open subset of) a quadric hypersurface.

We stated the redundant hypotheses $n > 1$ for emphasis. When $n = 1$ the Fubini hypothesis is vacuous. If X is a hypersurface whose Gauss image has dimension one, then X is locally ruled by $\mathbb{P}^{n-1}$'s [14, 12]. (I.e., if X is variety, it is a scroll of $\mathbb{P}^{n-1}$'s.) So, all hypersurfaces satisfying the Fubini hypothesis are classified.

In this paper we present a generalization of Fubini's theorem to codimension two. There are several formulations of the Fubini hypothesis, all of which are equivalent for hypersurfaces but do not coincide in codimension two. Thus our first task is to come up with proper hypotheses. Let $X \subset \mathbb{P}V$ be a variety or analytic submanifold and let $x \in X$ be a general point. There is a well defined sequence of ideals defined on the tangent space T_xX given by the *relative differential invariants* $F_k \in S^kT^*_xX \otimes N_xX$, where F_k is an equivalence class of vector spaces of homogeneous polynomials of degree k on T_xX parametrized by the conormal space N^*_xX. A coordinate definition of these invariants is as follows: Take adapted local coordinates (w^α, z^μ), $1 \leq \alpha \leq n$, $n+1 \leq \mu \leq \dim \mathbb{P}V$, on $\mathbb{P}V$ such that $[x] = (0,0)$ and $T_{[x]}X$ is spanned by the first n coordinates $(1 \leq \alpha \leq n)$. Then locally X is given by equations

$$z^\mu = f^\mu(w^\alpha) \tag{1.1}$$

and, at $(0,0)$,

$$F_k\left(\frac{\partial}{\partial w^{i_1}}, ..., \frac{\partial}{\partial w^{i_k}}\right) = (-1)^k \sum_\mu \frac{\partial^k f^\mu}{\partial w^{i_1}, ..., \partial w^{i_k}} \frac{\partial}{\partial z^\mu}.$$

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The invariant $II = F_2$ is called the *projective second fundamental form* and for it there is no equivalence to mod out by. For the other invariants, different choices (e.g., of a complement to T_xX in $T_x\mathbb{P}V$) will yield different systems of polynomials, but the new higher degree polynomials will be the old plus polynomials in the ideal generated by the lower degree forms (see [6], §3.5). So, letting $|F_k| = F_k(N_x^*X) \subseteq S^kT_x^*X$, the ideals in $Sym(T_x^*X)$ generated by $\{|F_2|, ..., |F_k|\}$ are well defined.

The set $\mathcal{C}_{k,x}$ is the zero set of $\{|F_2|, ..., |F_k|\}$. Because points can and do occur with multiplicities, it is more precise to work with the ideals $I_{\mathcal{C}_{k,x}}$ which we define to be the ideals generated by $\{|F_2|, ..., |F_k|\}$. So we will consider the Fubini hypothesis in the form

$$I_{\mathcal{C}_{3,x}} = I_{\mathcal{C}_{2,x}} \quad \text{(Fubini hypothesis)}$$

Now let $X^n \subset \mathbb{CP}^{n+2}$ be a submanifold of codimension two and satisfy the Fubini hypothesis. What can we say about X?

Evident examples for X satisfying the Fubini hypothesis are: the intersection of two quadric hypersurfaces, the product of a curve with an $(n-1)$-fold having an $n-3$ dimensional family of lines through a general point (i.e., a quadric of dimension $n-1$) or a variety that is a one parameter family of $\mathbb{P}^{n-2}$'s. Note that to have a meaningful result we should assume $n > 2$.

A less evident class of examples are certain products of two curves with a $\mathbb{P}^{n-2}$, more precisely the product of a curve with a variety with a one-dimensional Gauss image. Note that one could not have three curves as we only have two independent quadrics in the second fundamental form.

We prove

Theorem 1.2 (Codimension two Fubini)**.** *Let $X^n \subset \mathbb{CP}^{n+2}$ be an analytic submanifold with $n > 2$. Let $x \in X$ be a general point. If*

$$I_{\mathcal{C}_{2,x}} = I_{\mathcal{C}_{3,x}} \quad \text{(Fubini hypothesis)}$$

Then X is one of:

(1) *a complete intersection of two quadric hypersurfaces.*
(2) *locally the product of a curve with a quadric hypersurface $Q^{n-1} \subset \mathbb{P}^n$. (I.e., a general point of X is contained in a $Q^{n-1} \subset X$.)*
(3) *A cone over $Seg(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^5$.*
(4) *Locally the product of a curve with a variety with a one dimensional Gauss image. In particular, X is locally the product of two curves with a $\mathbb{P}^{n-2}$.*
(5) *Locally the product of a curve with a $\mathbb{P}^{n-1}$,i.e, a scroll of $\mathbb{P}^{n-1}$'s.*
(6) *A quadric hypersurface in $\mathbb{P}^{n+1}$.*
(7) *A linear $\mathbb{P}^n$.*

Remarks:

(a) Under the hypotheses of the theorem $\mathcal{C}_{2,x} = \mathcal{C}_x$ by [8], Theorem 2. (We do not use this result in the proof of Theorem 1.2.) In particular, $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ is the intersection of (at most) two quadric hypersurfaces.

(b) The dual variety of X is degenerate if and only if none of the quadrics in the ideal $I_{\mathcal{C}_{2,x}}$ are smooth (see [3]). This occurs in cases 3-5.

(c) Our results are valid over $\mathbb{R}$ in the sense that if one assumes the same normalizations, the same results hold. However over $\mathbb{R}$, there are more cases (e.g., due to the signature of a quadratic form), although each individual case should be solvable by the methods of this paper.

(d) The meaning of a general point in our hypotheses can be made more precise: we assume that $|II|_x$ and $|II + F_3|_x$ have base loci having the same number of components and dimension of singular sets as all points in some open neighborhood of x.

1.2. Related work and problems.

1.2.1. *Rogora's theorem.* As remarked above, $\mathcal{C}_{2,x} = \mathcal{C}_x$. Thus a generalization of the problem would be to classify the codimension two submanifolds containing (at least) an $(n-3)$-dimensional family of lines passing through a general point, or equivalently, the codimension two linearly nondegenerate varieties whose Fano variety of lines $\mathbb{F}(X) = \{l \in \mathbb{G}(\mathbb{P}^1, \mathbb{P}V) \mid l \subset X\}$ has dimension $2n-4$. This is a generalization because $\mathcal{C}_{2,x}$ may have several components of dimension $(n-3)$ and the Fubini problem only addresses the case when all components are also in $\mathcal{C}_x$. Now max $\dim \mathbb{F}(X) = 2n-2$, with equality if and only if $X = \mathbb{P}^n$. The classical Fubini Theorem classifies the varieties with $\dim \mathbb{F}(X) = 2n-3$, namely quadric hypersurfaces and curves of $\mathbb{P}^{n-1}$'s. The next case, where $\dim \mathbb{F}(X) = 2n-4$, was solved when codim $(X) > 2$ by Rogora [11]. The only possibilities are one parameter families of quadrics, two parameter families of $\mathbb{P}^{n-2}$'s or linear sections of $G(2,5)$, the Grassmannian of 2-planes in $\mathbb{C}^5$. The codimension two case is partially addressed in this paper:

Corollary 1.3. *Let $X^n \subset \mathbb{P}V$ be a projective variety such that $dim \mathbb{F}_1(X) = 2n-4$ and $\mathcal{C}_{2,x}$ has one component (or such that $\mathcal{C}_{2,x} = \mathcal{C}_{3,x}$). Then unless X is a hypersurface,it is one of the varieties 1,2,3,4 in the conclusion of theorem 1.2.*

It is interesting to consider the near counter-example of a linear projection of $G(2,5) \subset \mathbb{P}^9$ to a $\mathbb{P}^8$. In this case $\mathcal{C}_{2,x}$ is the union of $\mathbb{P}^1 \times \mathbb{P}^2$ and a $\mathbb{P}^3$, but $\mathcal{C}_{3,x} = \mathcal{C}_x = \mathbb{P}^1 \times \mathbb{P}^2$. This reflects the general principle that under linear projection from a point, $|II|_x$ loses a quadric but that quadric shows up (multiplied by linear forms) in F_3.

Similarly, for a two parameter family of $\mathbb{P}^{n-2}$'s, $\mathcal{C}_{2,x}$ always has multiple components.

The case of $\mathcal{C}_{2,x}$ having multiple components would be in principle treatable by the methods of this paper, but one would have to do a separate calculation for each individual case. One could study the hypersurface case using the methods of this paper but it appears one would have to take at least twelve derivatives using the moving frame to get an answer.

1.2.2. *The Debarre–de Jong conjecture.* Both Debarre and de Jong have conjectured that a smooth hypersurface $Z^{m-1} \subset \mathbb{P}^m$ of degree d has $\dim \mathbb{F}(Z) = 2m-3-d$ for $m \geq d$ (the expected dimension). They observed that by taking linear sections, it would be sufficient to prove the conjecture for $d = m$, and moreover proved that any potential counter-example Z with a larger space of lines would have to contain a hypersurface X (a variety of codimension two in $\mathbb{P}^m$), with the property that $\mathbb{F}_1(X) = 2m-3-d$, see [1].

As an application of our theorem, in Section §3 we give a new proof of this conjecture when $m = 6$ (the largest m for which the conjecture is known to be true), a problem recently solved by R. Beheshti [1].

1.2.3. *Other generalizations of Fubini.* In codimension one, the Fubini hypothesis implies that there exists a choice of F_3 such that $F_3 = 0$. In [9], it was shown that a n-fold in $\mathbb{P}^{\binom{n+1}{2}-1}$ having the expected second fundamental form (i.e. $|F_2| = S^2T^*_xX$) and admitting a choice of F_3 that is identically zero, must be the quadratic Veronese embedding of projective space. For minimally embedded compact Hermitian symmetric spaces (CHSS), something much stronger is true: in [9, 10] Fubini's theorem was generalized to all rank two CHSS in the stronger form that if $\mathcal{C}_{2,x}$ is the same as that of a rank two CHSS, then X must be (an open subset of) the corresponding CHSS. It was then generalized further in [5] to arbitrary CHSS by requiring that the base loci of the fundamental forms coincide. (Roughly speaking, the k-th fundamental form is a component of F_k that is well defined independent of adapted coordinates.)

1.2.4. *An analogue for multi-secant lines?* Tangent lines are limits of secant lines, and directions in $\mathcal{C}_{k,x}$ are limits of k-secant lines. Are there natural analogues of these results related to k-secant lines? For example, much easier than Fubini's theorem is the fact that a variety X having the property that any trisecant line is contained in X is either a quadric or a linear space. (However in general it may be much more difficult to prove analogous statements for multi-secant lines.)

1.3. **Outline of the proof.** We know of two proofs of Fubini's result (Theorem 1.1). One can either reduce to the surface case by taking a general $\mathbb{P}^3$-section and then prove the theorem for surfaces (which follows because a surface having two distinct lines through a general point is necessarily a quadric) or by reducing the frame bundle of an unknown variety satisfying the Fubini hypothesis to the reduced frame bundle of a quadric hypersurface. Any proof of the codimension two Fubini theorem must necessarily be more complicated because for quadric hypersurfaces (the codimension 1 case), there is only a discrete invariant (the rank), but for pencils of quadrics (the codimension 2 case) there are moduli. Thus a moving frames proof would have to reduce to a Frobenius system on the frame bundle (i.e., one whose solutions were parametrized by a fixed number of constants). For a linear section argument, one needs to be sure that the sections cannot be coming from a more complicated variety (since the sections will not necessarily be isomorphic).

Moreover, not only do the expected answers have moduli, the possible second fundamental forms do as well (as they too are pencils of quadrics), whereas in the original Fubini theorem there was only the discrete invariant of rank. Our proof combines methods of both proofs of Fubini's theorem.

If a variety satisfies Fubini's hypothesis, then so will any general linear section. For most cases we prove Theorem 1.2 for $n = 3$ and then use the fact that any general $\mathbb{P}^5$ section of X^n is of the type found in the $n = 3$ analysis to characterize these varieties. In other cases we just argue directly in n dimensions. For the generic $\mathcal{C}_{2,x}$ (Case 1 of Theorem 1.2) both methods work equally well. The various cases are treated as follows:

(1) Whenever a general linear section of a variety is a complete intersection cut out by varieties of degrees $d_1, ..., d_s$, then the original variety must also be a complete intersection cut out by varieties of degrees $d_1, ..., d_s$.
(2) Here we prove the result directly for arbitrary n.
(3) The only variety whose general $\mathbb{P}^5$ section is $\mathbb{P}^1 \times \mathbb{P}^2$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^2$.
(4) If a general linear section of X is locally the product of a curve with a variety with a one dimensional Gauss image, then X will have that property as well.
(5-7) These cases are degenerate and covered by the original (codimension one) Fubini theorem remarks.

Since we are working in codimension 2, $\dim|II_{X,x}| \leq 2$. If the dimension is zero, X is a $\mathbb{P}^4$. When the dimension is one, $|II_{X,x}|$ consists of a single quadric. If the quadric has rank 1, then X is a curve of $\mathbb{P}^{n-1}$'s. Otherwise X is a quadric hypersurface in $\mathbb{P}^{n+1}$.

Now suppose that $\dim|II_{X,x}| = 2$, so we have a pencil of quadrics. Restrict to $n = 3$. As explained in [4], there are seven possibilities for the pencil as characterized by the base loci: if the pencil contains a smooth conic, then the base locus $\mathcal{C}_{2,x}$ consists of four points (counted with multiplicity) in $\mathbb{P}^2 = \mathbb{P}T_xX$. The cases are: (i) four distinct points; (ii) two double points; (iii) a double point and two distinct points; (iv) a single four-fold point; (v) a triple point and a distinct point. These cases are addressed in Subsections §2.1-2.5. The sixth and seventh cases arise when the pencil contains no smooth quadrics. Equivalently, the dual is degenerate. For these cases $|II_{X,x}|$ respectively has the normal forms $\{\omega^1_0\omega^2_0, \omega^1_0\omega^3_0\}$ and $\{(\omega^1_0)^2, (\omega^2_0)^2\}$. They are analyzed in Subsections §2.6–2.7.

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2. Moving frames

We use notation for the moving frame and differential invariants as in [6]. We use index ranges

$$1 \le a, b, c, d, e \le n$$
$$n+1 \le u, v \le n+2$$
$$0 \le A, B \le n+2.$$

NOTE: In calculations we will use the convention that indices a, b are *not* to be summed over unless explicitly specified but use the summation convention for all other indices.

We work on the open subset of a codimension 2 submanifold $X^n \subset \mathbb{CP}^{n+2}$ consisting of general points and slightly abuse notation by calling it X.

The bundle of first order adapted frames $\mathcal{F}^1_X$ for a submanifold $X^n \subset \mathbb{P}^{n+2} = \mathbb{P}V$ is the set of ordered bases $g = (e_0, ..., e_{n+1})$ of V such that $[e_0] \in X$ and the affine tangent space $\hat{T}_x X$ is the span of $e_0, ..., e_n$. It is a bundle over X and the Maurer-Cartan form $\omega = (\omega^A_B) = g^{-1}dg$ of $GL(V)$ pulls back to give forms on $\mathcal{F}^1_X$. We write $g = (g^A_B) \in GL(V)$.

The first order adaption forces

$$\omega^u_0 = 0\,.$$

Differentiating these equations produces

$$\omega^u_a = q^u_{ae}\omega^e_0\,, \tag{2.1}$$

for symmetric functions $q^u_{ab} = q^u_{ba}$. A moving frame definition of the second fundamental form $F_2 = II_X \in \Gamma(X, S^2T^*X \otimes NX)$ is obtained by pushing down $\omega^u_e \otimes \omega^e_0 \otimes e_u \in \Gamma(\mathcal{F}^1_X, \pi^*(S^2T^*X \otimes NX))$ down to X. We denote the Fubini cubic by $F_3 = r^u_{cde}\omega^c_0\omega^d_0\omega^e_0 \otimes e_u \in \Gamma(\mathcal{F}^1_X, \pi^*(S^3T^*X \otimes NX))$ where the coefficients r^u_{abc} of F_3 are defined by

$$r^u_{abc}\omega^c_0 = -dq^u_{ab} - q^u_{ab}\omega^0_0 - q^v_{ab}\omega^u_v + q^u_{ae}\omega^e_b + q^u_{be}\omega^e_a\,. \tag{2.2}$$

See [6], Chapter 3 for details.

We now add the Fubini hypothesis that $|F_3| \subset |II \circ T^*|$ on the coefficients of F_3:

$$r^u_{abc} = \mathfrak{S}_{abc}\, \rho^u_{av} q^v_{bc}\,.$$

The notation $\mathfrak{S}$ denotes cyclic summation on the indices.

2.1. **Case (i): $n = 3$, and $\mathcal{C}_{2,x}$ is linearly nondegenerate and smooth.** Here we begin our seven part analysis of the case that $n = 3$ and $|II|_{X,x}$ contains a pencil of quadrics. When the base locus $\mathcal{C}_{2,x}$ contains four distinct points we may normalize the q^u_{ab} so that

$$q^{n+1}_{ab} = \delta_{ab} \qquad \text{and} \qquad q^{n+2}_{ab} = \lambda_a\delta_{ab} \tag{2.3}$$

for pairwise distinct functions λ_a. We will see that X is the intersection of two quadrics. To do so it is sufficient, by Theorem 4.28 of [7], to show that the coefficients of F_4 and F_5 satisfy

$$r^u_{abcd} \;=\; \mathfrak{S}_{abc}\, \sigma^u_{vw}\, q^v_{ab}\, q^w_{cd} \;+\; \mathfrak{S}_{abcd}\, \rho^u_{av}\, r^v_{bcd}\,, \tag{2.4}$$

$$r^u_{abcde} \;=\; \mathfrak{S}_{abcde}\, \left(\rho^u_{av}\, r^v_{bcde} + \sigma^u_{vw}\, (q^v_{ab}\, r^w_{cde} + q^v_{ac}\, r^w_{ebd})\right)\,. \tag{2.5}$$

Here $\sigma^u_{vw} = \sigma^u_{wv}$. (Although we will not need to use this in our calculations, it will serve as a useful guide.) Recall that these coefficients are defined by

$$\begin{aligned}
r^u_{abcd}\omega^d_0 &= -dr^u_{abc} - 2r^u_{abc}\omega^0_0 - r^v_{abc}\omega^u_v \\
&\quad +\mathfrak{S}_{abc}\left(r^u_{abe}\omega^e_c + q^u_{ab}w^0_c - q^u_{ae}q^v_{bc}\omega^e_v\right), \qquad (2.6)\\
r^u_{abcde}\omega^e_0 &= -dr^u_{abcd} - 3r^u_{abcd}\omega^0_0 - r^v_{abcd}\omega^u_v \\
&\quad -\mathfrak{S}_{abc}\left\{(r^u_{abe}q^v_{cd} + r^u_{ade}q^v_{bc})\omega^e_v + (q^u_{ab}q^v_{cd} + q^u_{ad}q^v_{bc})\omega^0_v\right\} \\
&\quad +\mathfrak{S}_{abde}\left\{r^u_{abce}\omega^e_d + 2r^u_{abc}\omega^0_d - q^u_{ae}r^v_{bcd}\omega^e_v\right\}. \qquad (2.7)
\end{aligned}$$

We will use the notation

$$\lambda_{ab} := \lambda_a - \lambda_b \neq 0\,.$$

Note that (2.1) gives us

$$\omega^{n+1}_a = \omega^a_0 \qquad \text{and} \qquad \omega^{n+2}_a = \lambda_a\omega^a_0\,. \tag{2.8}$$

Recall our convention that there is no sum on a in the last equation.

Assume Fubini's hypothesis holds. For a suitable choice of g^0_a, g^a_v, the transformation $e_u \mapsto e_u + g^a_u e_a$ and $e_a \mapsto e_a + g^0_a e_0$ further refines the frames so that $\rho^4_{av} = 0 = \rho^5_{a5}$ and $\rho^5_{a4} = \rho_a$. Now (2.2) implies Fubini's hypothesis holds on our reduced frame bundle if and only if

$$\begin{aligned}
0 &= \omega^a_b + \omega^b_a & (a \neq b) \qquad (2.9)\\
0 &= -\omega^0_0 - w^4_4 - \lambda_a\omega^4_5 + 2\omega^a_a & (2.10)\\
\rho_a\omega^b_0 + \rho_b\omega^a_0 &= \lambda_a\omega^a_b + \lambda_b\omega^b_a & (a \neq b) \qquad (2.11)\\
2\rho_a\omega^a_0 + \rho_e\omega^e_0 &= -d\lambda_a - \lambda_a\omega^0_0 - \omega^5_4 - \lambda_a\omega^5_5 + 2\lambda_a\omega^a_a\,. & (2.12)
\end{aligned}$$

(The first two equations come from $u = n+1 = 4$, and the last two from $u = n+2 = 5$.)

2.1.1. *Determination of F_4.* Differentiating (2.9) produces functions C^a_e and E^a_e so that

$$\begin{aligned}
\omega^a_5 &= C^a_e\omega^e_0 \\
\omega^0_a - \omega^a_4 - \rho_a\omega^4_5 &= E^a_e\omega^e_0\,.
\end{aligned}$$

The functions E^a_e satisfy the relations

$$(E^a_e) = \begin{pmatrix} E^1_1 & \lambda_3 C^1_2 & \lambda_2 C^1_3 \\ \lambda_3 C^2_1 & E^2_2 & \lambda_1 C^2_3 \\ \lambda_2 C^3_1 & \lambda_1 C^3_2 & E^3_3 \end{pmatrix},$$

and

$$C^a_a\lambda_b - C^b_b\lambda_a = E^a_a - E^b_b\,.$$

The last is a set of $\binom{n}{2} = 3$ linear equations for the $2n = 6$ unknowns C^a_a, E^b_b The system has rank $2n-4 = 2$ so there is a 4-dimensional space of solutions. We may parameterize the solutions as follows by introducing new variables R, S, T:

$$C^a_a = R\lambda_a + S \qquad \text{and} \qquad E^a_a = -S\lambda_a + T\,. \tag{2.13}$$

The derivative of (2.10) forces the off-diagonal terms of C, and therefore E as well, to vanish. Whence

$$\begin{aligned}
\omega^a_5 &= C^a_a\omega^a_0 & (2.14)\\
\omega^0_a - \omega^a_4 - \rho_a\omega^4_5 &= E^a_a\omega^a_0\,. & (2.15)
\end{aligned}$$

We may use g^0_4, g^0_5 to normalize $S, T = 0 \implies E^a_a = 0$.

Making use of the identities derived thus far, a computation of (2.6) in the $u = n+1 = 4$ case yields

$$r^4_{abcd} = \mathfrak{S}_{abc}\sigma^4_{uv}q^u_{ab}q^v_{cd}\,,$$

with $\sigma^4_{4,4} = \sigma^4_{4,5} = \sigma^4_{5,4} = 0$ and $\sigma^4_{5,5} = -R$. Taking into account the normalizations of ρ, this gives us the $u = n+1 = 4$ half of (2.4).

Next we differentiate (2.11) and obtain functions F^a_e, G^a_e such that

$$\begin{aligned} w^a_4 + 2\rho_a w^4_5 &= F^a_e\omega^e_0 \\ d\rho_a - \rho_a(2w^4_4 + 3\lambda_a\omega^4_5 - w^5_5) + \sum_e \rho_e\omega^a_e &= G^a_e\omega^e_0\,. \end{aligned}$$

The functions G^a_e satisfy

$$(G^a_e) = \begin{pmatrix} G^1_1 & \lambda_{31}F^1_2 & \lambda_{21}F^1_3 \\ \lambda_{32}F^2_1 & G^2_2 & \lambda_{12}F^2_3 \\ \lambda_{23}F^3_1 & \lambda_{13}F^3_2 & G^3_3 \end{pmatrix},$$

and

$$(F^a_a - {\lambda_a}^2R)\lambda_b - (F^b_b - {\lambda_b}^2R)\lambda_a = \left(G^a_a + \lambda_aF^a_a\right) - \left(G^b_b + \lambda_bF^b_b\right).$$

As above for (2.13), this is a corank three system and, introducing new variables U, V, W gives,

$$\begin{aligned} F^a_a - {\lambda_a}^2R &= U\lambda_a + V \\ G^a_a + \lambda_aF^a_a &= -V\lambda_a + W\,. \end{aligned} \tag{2.16}$$

The derivative of (2.12) forces the off-diagonal entries of F (and therefore G, as well) to vanish. With an application of (2.15) we have

$$\omega^0_a + \rho_a\omega^4_5 = F^a_a\omega^a_0\,, \tag{2.17}$$

$$d\rho_a - \rho_a(2w^4_4 + 3\lambda_a\omega^4_5 - w^5_5) + \sum_e \rho_e\omega^a_e = G^a_aw^a_0\,. \tag{2.18}$$

Now a computation of (2.6) in the $u = n+2 = 5$ case yields

$$r^5_{abcd} = \mathfrak{S}_{abc}\sigma^5_{uv}q^u_{ab}q^v_{cd}\,,$$

with $\sigma^5_{4,4} = -W$, $\sigma^5_{4,5} = V = \sigma^5_{5,4}$ and $\sigma^5_{5,5} = U$. In particular, (2.4) holds.

2.1.2. *Determination of* F_5. It remains to verify (2.5). These coefficients are given by (2.7) which requires that we compute $-dr^u_{abcd}$. In particular, we need expressions for dR, dU, dV and dW. We obtain information on the first three differentials by differentiating the expressions

$$\omega^a_5 - R\lambda_a\omega^a_0 = 0 \quad \text{and} \tag{2.19}$$

$$\omega^0_a - \omega^a_4 - \rho_aw^4_5 = 0\,, \tag{2.20}$$

which are consequences of (2.13,2.14,2.15). In particular, we find

$$\begin{aligned} dR &= R(2\omega^5_5 - \omega^0_0 - \omega^4_4) + U\omega^4_5 \\ 0 &= \omega^0_5 + +V\omega^4_5 + R\omega^5_4 \\ 0 &= -2\omega^0_4 + W\omega^4_5\,. \end{aligned} \tag{2.21}$$

The first and second expressions are derived from the derivative of (2.19), and the third from (2.20).

Next, (2.16,2.17,2.18) give us

$$
\begin{aligned}
&(2.22) & 0 &= \omega^0_a + \rho_a\omega^4_4 - (R{\lambda_a}^2 + U\lambda_a + V)\omega^a_0 \\
&(2.23) & 0 &= \left(R{\lambda_a}^3 + U{\lambda_a}^2 + 2V\lambda_a - W\right)\omega^a_0 \\
& & &\quad + d\rho_a - \rho_a(2w^4_4 + 3\lambda_a\omega^4_5 - \omega^5_5) + \sum_e \rho_e\omega^a_e\,.
\end{aligned}
$$

Differentiating (2.22) provides expressions for dU and dV; dW is given by (2.23). Summing over the index $e \in \{1,\dots,n\}$, we have

$$
(2.24)\qquad
\begin{aligned}
dU &= -\omega^0_5 + R\rho_e\omega^e_0 + U(\omega^5_5 - \omega^0_0) + 3V\omega^4_5 + 2R\omega^5_4 \\
dV &= -\omega^0_4 - R\lambda_e\rho_e\omega^e_0 + V(w^4_4 - \omega^0_0) - W\omega^4_5 + U\omega^5_4 \\
dW &= (2R{\lambda_e}^2 + 2U\lambda_e + 2V - T)\rho_e\omega^e_0 - 4{\rho_e}^2\omega^4_5 \\
&\quad + W(2\omega^4_4 - \omega^0_0 - \omega^5_5) - 2V\omega^5_4\,.
\end{aligned}
$$

Now a computation of (2.7) reveals that the coefficients of F_5 are indeed of the form (2.5), and X must be a complete intersection.

Note that one can avoid the use of [7], Theorem 4.28 as follows: Differentiating (2.21,2.24) yields no additional relations and we may make the following observation. Let $\mathbb{C}^3_\lambda$ and $\mathbb{C}^3_\rho$ denote two copies of $\mathbb{C}^3$ with coordinates $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ and $\rho = (\rho_1,\rho_2,\rho_3)$, respectively. Denote the coordinates of $\mathbb{C}^4_\sigma$ by (R,U,V,W). Let $M = \{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3_\lambda \ :\ \lambda_a \neq \lambda_b \text{ whenever } a \neq b\}$, and $\Sigma = GL_{n+3}\mathbb{C} \times M \times \mathbb{C}^3_\rho \times \mathbb{C}^4_\sigma$ (here $n=3$). Then the system given by the equations $\{\omega^u_0 = 0\}$ and (2.8,2.9,2.10,2.11,2.12,2.19,2.20,2.21,2.22,2.23,2.24) is Frobenius. Note that $\dim_{\mathbb{C}}\Sigma = 46$, and that the system consists of 36 independent equations. So the maximal integral submanifolds are of dimension 10 and may be identified with the graphs of those the natural maps $\mathcal{F} \to M \times \mathbb{C}^3_\rho \times \mathbb{C}^4_\sigma$, where $\mathcal{F} \subset GL_{n+3}\mathbb{C}$ is a sub-bundle of the adapted frame bundle over a smooth variety of codimension 2 which satisfies Fubini's hypothesis (and with distinct eigenvalues λ_a). In fact the resulting integral manifolds have ideal generated by

$$
\begin{aligned}
& x^0x^4 - \sum_a (x^a)^2 + R(x^5)^2, \\
& x^0x^5 - \sum_a \lambda_a(x^a)^2 - \sum_a \rho_a x^a x^4 + W(x^4)^2 - Vx^4x^5 - U(x^5)^2.
\end{aligned}
$$

Remark. This computation is easily generalized to arbitrary n. In particular, suppose the second quadric in II may be normalized as in (2.3). Additionally assume that there exists as least two distinct eigenvalues λ_a, and that no eigenvalue occurs with multiplicity $n-1$. (In the case $n=3$ this is equivalent to hypothesis of three distinct eigenvalues.) The analogous calculation for $n>3$ shows that each numbered equation in this section holds when the indices $(4,5)$ are replaced with $(n+1,n+2)$. Again we have a Frobenius system whose integral manifolds have ideal generated by

$$
\begin{aligned}
& x^0x^{n+1} - \sum_a (x^a)^2 + R(x^{n+2})^2, \\
& x^0x^{n+2} - \sum_a \lambda_a(x^a)^2 - \sum_a \rho_a x^a x^{n+1} + W(x^{n+1})^2 - Vx^{n+1}x^{n+2} - U(x^{n+2})^2\,.
\end{aligned}
$$

This places us in Case 2 of Theorem 1.2.

2.2. **Case (ii).** Fix $n \geq 3$ and assume we are in Case (ii). As in Case (i) the q^u_{ab} may be normalized so that

$$q^{n+1}_{ab} = \delta_{ab} \qquad \text{and} \qquad q^{n+2}_{ab} = \lambda_a \delta_{ab} \,.$$

Additionally, assume a $1, n-1$ split of the eigenvalues: $\lambda_1 \neq \lambda_2 = \cdots = \lambda_n$. (When $n = 3$, this is the case that $\mathcal{C}_{2,x}$ contains two points, each counted with multiplicity 2.)

A second normalization puts the coefficients in the form $\lambda_1 = 0$ and $\lambda_\alpha = \lambda \neq 0$ for $2 \leq \alpha \leq n$. Then (2.9),(2.11) imply

$$\omega^\alpha_1 = -\omega^1_\alpha = \frac{\rho_1}{\lambda}\omega^\alpha_0$$

which in turn implies the hyperplane distribution $\{\omega^1_0\}^\perp$ is integrable. Since any line field is integrable as well, we see that X is locally a product $C \times Y$. But now II_Y consists of a single quadric of rank greater than one, so by e.g. [6], Cor. 3.5.7, Y is a hypersurface in some $\mathbb{P}^{n-1}$. Fubini's hypothesis also holds for Y, and it must be a quadric hypersurface (Theorem 1.1). This places us in Case (2) of Theorem 1.2.

2.3. **Case (iii).** This is the case that $\mathcal{C}_{2,x}$ consists of 3 points, one with multiplicity 2. We may normalize the second fundamental form as follows

$$\left(q^4_{ab}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad \left(q^5_{ab}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some function $\lambda \neq 0$. In particular, (2.1) gives us

$$\begin{array}{rclrclrcl} \omega^4_1 & = & 0 & \omega^4_2 & = & \omega^2_0 & \omega^4_3 & = & \lambda\omega^3_0 \\ \omega^5_1 & = & \omega^2_0 & \omega^5_2 & = & \omega^1_0 & \omega^5_3 & = & \omega^3_0 \,. \end{array} \tag{2.25}$$

Assume Fubini's hypothesis holds. As in §2.1 a suitable choice of g^0_a, g^a_v allows us to normalize ρ. In this case we may refine the framing so that $\rho^5_{av} = 0 = \rho^4_{a4}$ and $\rho^4_{a5} = \rho_a$. (Contrast with §2.1,2.2 where $\rho_a = \rho^5_{a4}$.) Computations with (2.2) produce

$$\begin{aligned} (u,a,b) = (5,1,1) &\implies \omega^2_1 = 0 \\ (u,a,b) = (4,3,1) &\implies \lambda\omega^3_1 = \rho_3\omega^2 \\ (u,a,b) = (5,3,1) &\implies \omega^2_3 + \omega^3_1 = 0 \,. \end{aligned}$$

The last two equations tell us that $\omega^2_3 \equiv 0 \bmod \omega^2_0$. Along with the first equation above, this implies the hyperplane distribution $\{\omega^2_0\}^\perp$ is integrable. As in §2.2 X is locally the product of a curve and surface Y. In this case

$$II_Y = (\omega^4_1\omega^1_0 + \omega^4_3\omega^3_0) \otimes e_4 + (\omega^5_1\omega^1_0 + \omega^5_3\omega^3_0) \otimes e_5 + (\omega^2_1\omega^1_0 + \omega^2_3\omega^3_0) \otimes e_2 \bmod \omega^2_0,$$

so $|II_Y| = \{(\omega^3_0)^2\}$. Hence the Gauss map of Y is degenerate and we are in Case (4) of Theorem 1.2. (Note that when $n = 3$, we can say more as Y is either a cone over a curve, or the tangential variety of a curve. Cf. [13], p.105; or [6], Thm.3.4.6.)

2.4. **Case (iv).** Here $\mathcal{C}_{2,x}$ contains a single point of multiplicity 4. The second fundamental form may be normalized as in §2.3, but with $\lambda = 0$. Again,

$$\begin{aligned} (u,a,b) = (4,3,1) &\implies \rho_3 = 0 \\ (u,a,b) = (5,1,1) &\implies \omega^2_1 = 0 \\ (u,a,b) = (4,1,2) &\implies -\omega^4_5 = 2\rho_2\omega^2_0 \\ (u,a,b) = (4,3,3) &\implies -\omega^4_5 = \rho_2\omega^2_0 \end{aligned}$$

Those last two equations imply $\rho_2 = 0$. Now

$$(u,a,b) = (4,2,3) \implies \omega^2_3 = 0\,,$$

and $\{\omega^2_0\}^\perp$ is again integrable and again $|II_Y| = \{(\omega^3_0)^2\}$ and we are in Case (4) as above.

2.5. **Case (v).** Here $\mathcal{C}_{2,x}$ consists a triple point, and a singleton. We may normalize the second fundamental form as follows

$$\left(q^4_{ab}\right) = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & \lambda & 1 \\ \lambda & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \left(q^5_{ab}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

for some function $\lambda \neq 0$. The Fubini cubic may be normalized so that $\rho^4_{av} = 0 = \rho^5_{a5}$ and $\rho_a := \rho^5_{a4}$. Now computations of (2.1) yield

$$\begin{aligned} (u,a,b) = (4,1,1) &\implies \omega^3_1 = 0 \\ (u,a,b) = (5,1,1) &\implies \omega^3_1 = \lambda\rho_1\omega^3_0 \\ (u,a,b) = (5,1,2) &\implies \omega^3_2 = \lambda\rho_1\omega^2_0 + (\rho_1 + \lambda\rho_2)\omega^3_0\,. \end{aligned}$$

In particular, $\rho_1 = 0$, and $\omega^3_2 \equiv 0 \mod \omega^3_0$. It follows that the hyperplane distribution $\{\omega^3_0\}^\perp$ is integrable and $|II_Y| = \{(\omega^2_0)^2\}$. Again we are in Case (4).

2.6. **Degenerate dual and nondegenerate Gauss map case.** Here there is a unique pencil of quadrics up to equivalence satisfying the hypotheses: we may normalize $|II_X| = \{\omega^1_0\omega^3_0, \omega^2_0\omega^3_0\}$. Now the hypothesis on F_3 allows us to reduce the frame bundle on X to a sub-bundle upon which the Maurer-Cartan forms pull-back to satisfy the same relations as those satisfied by the Maurer-Cartan forms on the frame bundle of $\mathbb{P}^1 \times \mathbb{P}^2$ in its Segre embedding. In particular, both bundles are integral manifolds of a Frobenius system defined by left-invariant 1-forms on $GL_6\mathbb{C}$. Hence, X is (projectively equivalent to an open subset of) $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^2)$.

2.7. **Degenerate dual and rank two Gauss map case.** Here we may normalize $|II_X| = \{(\omega^1_0)^2, (\omega^2_0)^2\}$. This case also reduces to Case (4) and the calculation is even easier than the above cases.

3. Proof of the Debarre-de Jong conjecture for degree six hypersurfaces

Our proof will use general results of [1] but avoid the case by case study in section 4.2 of [1]. The argument may be useful in either proving the degree seven case, or as a guide to counter-examples for all higher degrees.

Suppose $Z^5 \subset \mathbb{P}^6$ is a counter example, i.e., some component of $\mathbb{F}_1(Z)$ has dimension at least 4. Let $X \subseteq Z$ be the variety swept out by the lines of this component. It is known that no lines pass through a general point of Z, so X must have codimension 1 or 2 in Z. If X has codimension 2 then it is a $\mathbb{P}^3$. Since no smooth Z^5 can contain a $\mathbb{P}^3$, X must have codimension 1 in Z (and 2 in $\mathbb{P}^6$). Thus there is a 1-parameter family of lines passing through a general point $x \in X$, i.e., $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ is a curve.

Since X is of codimension 2 in $\mathbb{P}^6$, $\dim |II_{X,x}| \leq 2$. If the dimension is 0, then X is a $\mathbb{P}^4$. This is a contradiction as the smooth $Z^5 \subset \mathbb{P}^6$ can not contain even a $\mathbb{P}^3$. Next suppose that $\dim |II_{X,x}| = 1$, so $|II_{X,x}|$ contains a single quadric. If the quadric has rank greater than 1, then X is a hypersurface in a $\mathbb{P}^5$. We postpone the argument for this case to the end of the section. If the quadric has rank one, then X is a curve of $\mathbb{P}^3$'s, again a contradiction.

Now suppose that $\dim |II_{X,x}| = 2$. Then $\mathcal{C}_{2,x}$ is a curve in $\mathbb{P}(T_xX) = \mathbb{P}^3$. First suppose that $\mathcal{C}_{2,x}$ consists of several components (counted with multiplicity). They must be curves of degrees one, two or three. Any degree three curve in this setting will be non-planar; see [4], page

307, Case (x). In particular, the components of $\mathcal{C}_{2,x}$ are rational. Since $\mathcal{C}_x \subset \mathcal{C}_{2,x}$, this implies $\mathbb{F}_1(Z) = \cup_{x\in X}\mathcal{C}_x$ is uniruled by rational curves contradicting Theorem 2.1 of [1].

Thus we are reduced to the case $I_{\mathcal{C}_{2,x}}$ is reduced and irreducible. Since $\mathcal{C}_x$ is also a curve, it follows that $I_{\mathcal{C}_{2,x}} = I_{\mathcal{C}_x}$. Theorem 1.2 classifies the 4-folds in $\mathbb{P}^6$ satisfying Fubini's hypothesis. Let's look at this list: (1) and (6) are eliminated by degree reasons using the Lefschetz hyperplane theorem. (2-5) and (7) are ruled out by Beheshti's Theorem 2.1 [1] (and for more elementary reasons in some of the cases).

It remains to deal with the case when X^4 is contained in a $\mathbb{P}^5$, ie. X is a hyperplane section of Z. By Zak's theorem on tangencies [15], X can have at most isolated singularities. Note also that X cannot be a cone as then it would support at most a 3 dimensional family of lines (unless it were a $\mathbb{P}^4$). To finish we appeal to a result supplied to us by I. Coskun (personal communication) which he believes to be "known to the experts":

Theorem 3.1. *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree at least $n+1$. Suppose X has only isolated singularities and X is not a cone. Then X is not covered by lines.*

Proof. Since X is not a cone, a general line on X has a well defined normal bundle $N_{L/X}$ over L. This bundle if of rank $n-1$ and fits into the exact sequence

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^{n+1}} \to N_{X/\mathbb{P}^{n+1}}\,|_L \ \to 0$$

Note that the second term is just $\mathcal{O}(1)^{\oplus n}$ and the last is $\mathcal{O}(d)$ where $d \geq n+1$ is the degree of X. Write $N_{L/X} = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-1})$ (by the Segre-Grothendieck splitting theorem), with $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. Then we see $a_1 < 0$ which means the deformations of L cannot cover X, a contradiction. $\square$

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