Full and maximal squashed flat antichains of minimum weight

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June 22, 2012

Abstract

A squashed full flat antichain (SFFA) in the Boolean lattice B_n is a family $\mathcal{A}\cup\mathcal{B}$ of subsets of $[n]=\{1,2,\ldots,n\}$ such that, for some $k\in[n]$ and $0\le m\le\binom{n}{k}$, $\mathcal{A}$ is the family of the first m k-sets in squashed (reverse-lexicographic) order and $\mathcal{B}$ contains exactly those $(k-1)$-subsets of $[n]$ that are not contained in some $A\in\mathcal{A}$. If, in addition, every k-subset of $[n]$ which is not in $\mathcal{A}$ contains some $B\in\mathcal{B}$, then $\mathcal{A}\cup\mathcal{B}$ is a squashed maximal flat antichain (SMFA). For any n,k and positive real numbers α,β, we determine all SFFA and all SMFA of minimum weight $\alpha\cdot|\mathcal{A}|+\beta\cdot|\mathcal{B}|$. Based on this, asymptotic results on SMFA with minimum size and minimum BLYM value, respectively, are derived.

Keywords: Antichain, Sperner family, Flat Antichain Theorem, Kruskal-Katona Theorem, BLYM inequality
MSC (2010): 05D05 (primary), 06A07
Proposed Running Head: Flat Antichains

1 Introduction

An *antichain* in the Boolean lattice B_n is a family of subsets of $[n] := \{1, 2, \ldots, n\}$ such that none of the subsets is properly contained in another. An antichain $\mathcal{F} \subseteq B_n$ is *flat* if $|F| \in \{k-1, k\}$ for all $F \in \mathcal{F}$ and some $1 \leq k \leq n$. In this paper, we study flat antichains with the property that no $(k-1)$-set can be added without destroying the antichain property and such that the k-sets form an initial segment in squashed (or colexicographic) order. Such *squashed full flat antichains* (SFFA) are known to generate ideals of minimum size among all antichains of the same size in B_n. This fact and the Flat Antichain Theorem are the main motivations for this research. More detailed explanations are given later in this introductory section and in the beginning of the next section. Our main result is a characterization of those SFFA that attain minimum weight with respect to certain weight functions. In particular, we determine all SFFA of minimum weight, minimum volume, and minimum BLYM value, respectively.

Throughout, let n be a positive integer. We use $2^{[n]}$ or B_n to denote the power set of $[n]$ and $\binom{[n]}{i}$ for the family of all i-subsets of $[n]$. The *volume* $V(\mathcal{F})$ of $\mathcal{F} \subseteq 2^{[n]}$ is defined as $V(\mathcal{F}) := \sum_{F \in \mathcal{F}} |F|$. The results of Kisvölcsey [5] and Lieby [7, 8] perfectly complement each other to give the following important theorem.

Theorem 1 (Flat Antichain Theorem (FLAT))**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. Then there is a flat antichain $\mathcal{F}' \subseteq 2^{[n]}$ with $|\mathcal{F}'| = |\mathcal{F}|$ and $V(\mathcal{F}') = V(\mathcal{F})$.*

The FLAT naturally induces an equivalence relation on the class of all antichains in B_n. We say that two antichains are equivalent if they have the same size and the same volume. By Theorem 1, each of the equivalence classes with respect to this relation contains some flat antichain. Proposition 2 below illustrates that flat antichains are in some sense the extremal representatives of their equivalence classes. Let $\mathbb{R}^+$ denote the set of nonnegative real numbers, and consider a weight function $w : 2^{[n]} \mapsto \mathbb{R}^+$ such that each i-set $F \subseteq [n]$ has the same weight $w(F) = w_i$. The weight of $\mathcal{F} \subseteq 2^{[n]}$ is defined to be $w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F)$. The sequence $\{w_i\}_{i=0}^n$ is *convex* if $\{w_i - w_{i-1}\}_{i=1}^n$ is increasing and *concave* if $\{w_i - w_{i-1}\}_{i=1}^n$ is decreasing.

Proposition 2. *Let $w : 2^{[n]} \mapsto \mathbb{R}^+$ be a weight function as above. Furthermore, let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain and $\mathcal{F} \subseteq 2^{[n]}$ a flat antichain such that $|\mathcal{F}| = |\mathcal{A}|$ and $V(\mathcal{F}) = V(\mathcal{A})$.*

(i) If the sequence $\{w_i\}_{i=0}^n$ is convex, then $w(\mathcal{F}) \leq w(\mathcal{A})$.

(ii) If the sequence $\{w_i\}_{i=0}^n$ is concave, then $w(\mathcal{F}) \geq w(\mathcal{A})$.

Proof: We only prove part (i) here. The proof of (ii) is analogous.

Assume that $\{w_i\}_{i=0}^n$ is convex. Let $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ be the profile vector of $\mathcal{A}$, i.e., $a_i = |\{A \in \mathcal{A} : |A| = i\}|$. Furthermore, let $\ell = \min\{i : a_i \neq 0\}$ and $u = \max\{i : a_i \neq 0\}$. The weight of $\mathcal{A}$ is determined by $\mathbf{a}$, and one has $w(\mathcal{A}) = \sum_{i=\ell}^{u} a_i w_i =: w(\mathbf{a})$. If $u - \ell \leq 1$, then $\mathcal{A}$ is flat. As $\mathcal{A}$ and $\mathcal{F}$ have the same size and the same volume, their profile vectors must be equal then, and it follows that $w(\mathcal{A}) = w(\mathcal{F})$. Hence, without loss of generality we can assume that $u - \ell \geq 2$.

Consider the vector $\mathbf{a}'$ obtained from $\mathbf{a}$ replacing a_ℓ by $a_\ell - 1$, $a_{\ell+1}$ by $a_{\ell+1} + 1$, a_u by $a_u - 1$, and a_{u-1} by $a_{u-1} + 1$. (That is, if $u - \ell = 2$, then $a_{\ell+1} = a_{u-1}$ will be increased by 2.) Note that $\sum a_i' = \sum a_i$, $\sum a_i' i = \sum a_i i$, and

$$w(\mathbf{a}) - w(\mathbf{a}') = (w_u - w_{u-1}) - (w_{\ell+1} - w_\ell).$$

As $\{w_i\}$ is convex, it follows that $w(\mathbf{a}') \leq w(\mathbf{a})$. (It should be pointed out that we do not claim or need that $\mathbf{a}'$ is the profile vector of some antichain in B_n.) Iterating this process, we transform $\mathbf{a}$ into the profile vector $\mathbf{f}$ of $\mathcal{F}$ as $\mathcal{A}$ and $\mathcal{F}$ agree in size and volume. This implies $w(\mathcal{F}) = w(\mathbf{f}) \leq w(\mathbf{a}) = w(\mathcal{A})$. ∎

The well-known BLYM inequality (see [1] for instance) states that BLYM value of any antichain in B_n is at most 1, where the BLYM value of $\mathcal{F} \subseteq 2^{[n]}$ is defined to be $\sum_{F \in \mathcal{F}} |F| / \binom{n}{|F|}$. In this context, Proposition 2 implies an interesting observation about flat antichains.

Corollary 3. *Flat antichains have minimum BLYM values within their equivalence classes.*

Proof: The claim follows from Proposition 2 and the fact that the sequence $\left\{1/\binom{n}{i}\right\}_{i=0}^n$ is convex which is straight forward to verify. ∎

For a family $\mathcal{G} \subseteq \binom{[n]}{i}$ the *shadow* and the *shade* (or *upper shadow*) of $\mathcal{G}$ are the families $\Delta\mathcal{G} := \{H \in \binom{[n]}{i-1} : H \subset G \text{ for some } G \in \mathcal{G}\}$ and $\nabla\mathcal{G} := \{H \in \binom{[n]}{i+1} : H \supset G \text{ for some } G \in \mathcal{G}\}$, respectively. Full flat antichains (FFA) $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ for some $1 \leq k \leq n$ are characterized by $\mathcal{B} = \binom{[n]}{k-1} \setminus \Delta\mathcal{A}$. Maximal flat antichains (MFA) are the ones that in addition satisfy $\mathcal{A} = \binom{[n]}{k} \setminus \nabla\mathcal{B}$.

For example, $\mathcal{F} = \big\{\{1,2\}, \{1,3\}, \{4\}\big\}$ is an FFA in B_4 as all singletons other than $\{4\}$ are covered by the 2-sets in $\mathcal{F}$. On the other hand, $\mathcal{F}$ is not an MFA since $\{2,3\}$ could be added in without destroying the antichain property.

It is easy to see that $\mathcal{F}$ is an MFA if and only if its complementary antichain $\overline{\mathcal{F}} := \{[n] \setminus F : F \in \mathcal{F}\}$ is an MFA. If $\mathcal{F}$ is an FFA, then $\overline{\mathcal{F}}$ is an FFA only if $\mathcal{F}$ is an MFA.

In [3], it is shown that the minimum size of an MFA with $k = 3$ (that is an MFA consisting of 2-sets and 3-sets) is $\binom{n}{2} - \lfloor (n+1)^2/8 \rfloor$, and all such MFA of minimum size are determined. Some further results are obtained for the more general problem of minimizing the total weight of an MFA with $k = 3$ when each 3-sets has some weight α and each 2-set has weight β, where $\alpha, \beta \in \mathbb{R}^+$.

In the next section, we solve a similar problem for squashed FFA and squashed MFA for any n and k.

2 SFFA and SMFA of minimum weight

We say that $F \subseteq [n]$ precedes $G \subseteq [n]$, $G \neq F$, in *squashed* (or *colexicographic*) order and write $F <_S G$ whenever $\max\big((F \cup G) \setminus (F \cap G)\big) \in G$.

A squashed FFA (SFFA) in B_n is an FFA of the form $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ for some $1 \leq k \leq n$ and such that $\mathcal{A}$ consists of the first m elements of $\binom{[n]}{k}$ with respect to squashed order for some $m \leq \binom{n}{k}$. Clearly, an SFFA $\mathcal{F}$ is completely determined by n, k, and m. If an SFFA $\mathcal{F}$ is an MFA, then we call it a squashed MFA (SMFA).

By Sperner's Theorem [9], any antichain in B_n has size at most $\binom{n}{\lfloor n/2 \rfloor}$. It is a remarkable fact that for any positive $s \leq \binom{n}{\lfloor n/2 \rfloor}$ there is an SFFA of size s in B_n. Moreover, among all antichains of size s a uniquely determined SFFA generates an ideal of minimum weight, where each i-set in B_n has the same weight w_i and $0 \leq w_0 \leq w_1 \leq \cdots \leq w_n$. For details, see Theorem 8.3.5 in Engel's book [1]. The last statement was generalized to Macaulay posets P with the the property that P and its dual are weakly shadow increasing in [2].

For $1 \leq k \leq n$ and $0 \leq m \leq \binom{n}{k}$, the *k-cascade representation* of m is a representation of m in the form

$$m = \sum_{i=1}^{k} \binom{a_i}{i} \quad \text{with} \quad a_k > a_{k-1} > \cdots a_t \geq t > 0 = a_{t-1} = \cdots = a_1. \tag{1}$$

The summands $\binom{a_i}{i}$ with $a_i = 0$ could clearly be removed from the above representation of m. Their only purpose here is that they will allow us a more compact formulation of the main result (Theorem 5). It is easy to see (cf. [4]) that for given k and m there is a unique k-cascade representation of m. Moreover, if $\mathcal{A}$ is the family of the first m k-sets in squashed order and (1) is the k-cascade representation of m, then

$$|\Delta \mathcal{A}| = \sum_{i=t}^{k} \binom{a_i}{i-1}. \tag{2}$$

By the Kruskal-Katona Theorem [6, 4], the family $\mathcal{A}$ of the first m k-sets in squashed order has a shadow of smallest size among all m-element subsets of $\binom{[n]}{k}$.

The following is folklore and easy to check:

Proposition 4. *Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ be a SFFA with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$, and let $m := |\mathcal{A}|$ be represented as in* (1)*. $\mathcal{F}$ is an SMFA if and only if $a_1 = 0$.*

Our main result is the following characterization of all SFFA of minimum weight. To avoid certain technicalities, the trivial cases $k = 1$ and $k = n$ are excluded.

Theorem 5. *Let $1 < k < n$ be integers, α, β positive real numbers and $\lambda := \beta/\alpha$. Furthermore, let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ be a SFFA, and let* (1) *be the k-cascade representation of $m := |\mathcal{A}|$. $\mathcal{F}$ has minimum weight $w(\mathcal{F}) = \alpha \cdot |\mathcal{A}| + \beta \cdot |\mathcal{B}|$ among all SFFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ if and only if*

$$a_i = \begin{cases} n-k-1+i & \text{if } i > 1 + (n-k)/\lambda, \\ \lceil (i-1)(\lambda+1) - 1) \rceil \text{ or } \lfloor (i-1)(\lambda+1) \rfloor & \text{if } 1 + (n-k)/\lambda \geq i \geq 1 + 2/\lambda, \\ i & \text{if } 1 + 2/\lambda > i > 1/\lambda, \\ 0 \text{ or } i & \text{if } 1/\lambda = i, \\ 0 & \text{if } 1/\lambda > i. \end{cases}$$

Proof: First, observe that with $g(m) := m - \lambda|\Delta\mathcal{A}|$ we have

$$w(\mathcal{F}) = \alpha \cdot g(m) + \binom{n}{k-1}.$$

Hence, our problem of minimizing $w(\mathcal{F})$ is equivalent to minimizing $g(m)$ over all $m \in \left\{0, 1, \ldots, \binom{n}{k}\right\}$.

If $m \in \left\{\binom{n}{k} - 1, \binom{n}{k}\right\}$, then $\Delta\mathcal{A} = \binom{[n]}{k-1}$ holds. Consequently, $m = \binom{n}{k}$ does not minimize $g(m)$, and we can assume that $m < \binom{n}{k}$, i.e. that $a_k \leq n-1$. As $a_k > a_{k-1} > \cdots > a_t$, this implies

$$a_i = 0 \quad \text{or} \quad i \leq a_i \leq n-1-k+i \qquad \text{for } \; i \in [k]. \tag{3}$$

By (2), we have

$$g(m) = \sum_{i=t}^{k} h_i(a_i), \tag{4}$$

where, for $i \in [k]$, the polynomial $h_i : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$h_i(x) := \binom{x}{i} - \lambda\binom{x}{i-1} = \begin{cases} x - \lambda & \text{if } i = 1, \\ \dfrac{x+1-i(\lambda+1)}{i!} \displaystyle\prod_{j=0}^{i-2}(x-j) & \text{if } i \geq 2. \end{cases}$$

Our strategy is as follows: For each $i \in [k]$, we determine those $x \in [i, n-1-k+i] \cap \mathbb{Z}$ for which $h_i(x)$ is smallest possible. For such x, we choose $a_i = x$ or $a_i = 0$

if $h_i(x)$ is negative or positive, respectively. If $h_i(x) = 0$, we choose $a_i \in \{0, x\}$. Eventually, we will verify that, with the a_i's chosen as described, we obtain a proper k-cascade representation (1), i.e. that the following implication is true:

$$(i \in [k-1]) \ \wedge \ (a_i > 0) \quad \Longrightarrow \quad (a_i < a_{i+1}). \tag{5}$$

To begin with, note that $h_1(x) = x - \lambda$ achieves its global minimum with respect to the interval $[1, n-k]$ at $x = 1$, and we have have $h_1(1) = 1 - \lambda$.

Let $i \in [k] \setminus \{1\}$. Then h_i is a polynomial of degree i with leading coefficient 1 and zeros $0, 1, \ldots, i-1$ and $i(\lambda+1)-1$. That means, $h_i(x)$ is positive and strictly increasing for $x > i(\lambda+1)-1$, and $h_i(x) < 0$ for $i-2 < x < i(\lambda+1)-1$. Moreover, h_i is strictly convex on $I := \big(i-2,\, i(\lambda+1)-1\big)$. The numbers $u := (i-1)(\lambda+1)$ and $u-1$ both lie in I, and one can easily check that $h_i(u-1) = h_i(u)$.

Based on the above discussion, we distinguish three cases to find the global minimum of $h_i(x)$ over all $x \in [i,\, n-1-k+i] \cap \mathbb{Z}$.

Case 1: Assume that $u - 1 < i$. Note that this is equivalent to $i < 1 + 2/\lambda$. In this case, $h_i(x)$ is a minimum only at $x = i$, and $h_i(i)$ is positive if $i < 1/\lambda$, equals 0 if $i = 1/\lambda$ and is negative if $i > 1/\lambda$.

Case 2: Assume that $i \le u - 1$ and that $u \le n - 1 - k + i$. Note that this is equivalent to $1 + 2/\lambda \le i \le 1 + (n-k)/\lambda$. In this case, $h_i(x)$ attains its minimum exactly for $x \in \{\lceil u-1 \rceil, \lfloor u \rfloor\}$, and this minimum is negative.

Case 3: Assume that $n - 1 - k + i < u$. Note that this is equivalent to $1 + (n-k)/\lambda < i$. In this case, $h_i(x)$ is a minimum only at $x = n-1-k+i$ and $h_i(n-1-k+i) < 0$.

By the results of the above case study and (4), $g(m)$ becomes a minimum when the a_i's are chosen as in the theorem, where the minimization is over all choices satisfying (3). Finally, a straight forward calculation shows that (5) holds for the a_i's as in the theorem. ∎

Note that, by Proposition 4, for $\lambda < 1$ the optimal SFFA in Theorem 5 are also SMFA. In general, Theorem 5 and its proof yield the following characterization of minimum weight SMFA.

Corollary 6. *Let $1 < k < n$ be integers, α, β positive real numbers and $\lambda := \beta/\alpha$. Furthermore, let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ be a SMFA, and let (1) be the k-cascade representation of $m := |\mathcal{A}|$. $\mathcal{F}$ has minimum weight $w(\mathcal{F}) = \alpha \cdot |\mathcal{A}| + \beta \cdot |\mathcal{B}|$ among all MFSA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ if and only if*

(a) $\lambda \le n-k+1$ *and*

$$a_i = \begin{cases} n-k-1+i & \textit{if } i > 1+(n-k)/\lambda, \\ \lceil (i-1)(\lambda+1)-1 \rceil \textit{ or } \lfloor (i-1)(\lambda+1) \rfloor & \textit{if } 1+(n-k)/\lambda \ge i \ge 1+2/\lambda, \\ i & \textit{if } 1+2/\lambda > i > \max\{1/\lambda, 1\}, \\ 0 \textit{ or } i & \textit{if } 1/\lambda = i > 1, \\ 0 & \textit{otherwise}, \end{cases}$$

or

(b) $\lambda \ge n-k+1$, $a_i = 0$ *for* $i = 1, \dots, k-1$ *and* $a_k = n$.

Proof: In the beginning of the proof of Theorem 5 we ruled out the case that $\mathcal{F} = \binom{[n]}{k}$ when looking for SFFA of minimum weight. For $\lambda < n-k+1$, the SMFA $\binom{[n]}{k}$ can not be an SMFA of minimum weight either. This follows from the simple observation that in this case, the SMFA

$$\left(\binom{[n]}{k} \setminus \nabla\{\{n-k+2, n-k+3, \dots, n\}\}\right) \cup \{\{n-k+2, n-k+3, \dots, n\}\}$$

has a smaller weight. Now the a_i's are determined as in the proof of Theorem 5, with the exception that a_1 must be 0 by Proposition 4. This proves the claim for $\lambda < n-k+1$.

If $\lambda > n-k+1$, then $\binom{[n]}{k}$ is the unique SMFA of minimum weight. To see this, assume that $\mathcal{B} \ne \emptyset$, and use $|\nabla\mathcal{B}| \le (n-k+1)|\mathcal{B}|$ which implies that

$$\binom{[n]}{k} = (\mathcal{F} \setminus \mathcal{B}) \cup \nabla\mathcal{B}$$

has a smaller weight that $\mathcal{F}$.

Finally, if $\lambda = n-k+1$, then choosing $a_1 = 0$ and the other a_i's as in Theorem 5 (i.e., $a_i = n-k-1+i$ for $i = 2, \dots, k$) gives an SMFA that has the same weight as $\binom{[n]}{k}$. ■

3 Cases of special interest

Theorem 5 and Corollary 6 together with (2) and $\mathcal{B} = \binom{[n]}{k-1} \setminus \Delta\mathcal{A}$ give the formula

$$w(\mathcal{F}) = \beta\tbinom{n}{k-1} + \textstyle\sum_{i=1}^{k}\left(\alpha\tbinom{a_i}{i} - \beta\tbinom{a_i}{i-1}\right) \tag{6}$$

for the smallest weight of an SFFA and an SMFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$, where the a_i's are chosen as in the theorem and the corollary, respectively. (Note that for our formula to be accurate we have to adopt the somewhat unusual convention that $\binom{0}{0}$ is 0.)

3.1 SFFA and SMFA of minimum size

Let $s(n,k)$ denote the minimum size of an SFFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. By Theorem 5 with $\alpha = \beta = 1$, $s(n,k)$ is equal to the right-hand side of (6) for

$$a_i = \begin{cases} n-k-1+i & \text{if } i > n-k+2, \\ 2i-3 \text{ or } 2i-2 & \text{if } 3 \le i \le n-k+1, \\ 2 & \text{if } i = 2, \\ 0 \text{ or } 1 & \text{if } i = 1. \end{cases}$$

For the minimum size of an SMFA we have to choose $a_1 = 0$ and the other a_i's as above by Corollary 6. Consequently, $s(n,k)$ is also the minimum size of an SMFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$.

Using the above values for the a_i's in (6) gives the following formula for $s(n,k)$.

Corollary 7. *Let $1 \le k \le (n+1)/2$. Then*

$$s(n,k) = s(n, n-k+1) = \binom{n}{k-1} - \sum_{i=1}^{k-1} \frac{1}{i+1}\binom{2i}{i}.$$

Corollary 7 implies that as n gets larger for fixed k the optimal SFFA look more and more like the $(k-1)$-st level of B_n.

Corollary 8. *For fixed $k \ge 1$ one has* $\lim_{n\to\infty} \frac{s(n,k)}{\binom{n}{k-1}} = 1$.

3.2 SFFA and SMFA of minimum volume

Using, $\alpha = k$ and $\beta = k-1$ in Theorem 5 gives a characterization of all SFFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ of minimum volume. As $\lambda = (k-1)/k < 1$ in this case, these SFFA are all also SMFA.

3.3 SFFA and SMFA of minimum BLYM value

To find the minimum BLYM value of an SFFA or SMFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$, we can use (6) with $\alpha = 1/\binom{n}{k}$ and $\beta = 1/\binom{n}{k=1}$ which means that $\lambda = (n-k+1)/k$. Note that the optimal SFFA given by Theorem 5 are also SMFA if $k > (n+1)/2$. For $k = (n+1)/2$ there is an optimal SFFA which also is an SMFA, but not for $k \le n/2$.

Let $\mathrm{BLYM}(n,k)$ be the minimum BLYM value of an SMFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. As the optimal SFFA and SMFA differ only marginally, it is easy to verify that the following asymptotic result still holds for SFFA. For brevity, we only look at SMFA here.

Corollary 9. *For fixed $k \geq 1$ one has* $\lim_{n\to\infty} \mathrm{BLYM}(n,k) = 1 - \dfrac{(k-1)^{k-1}}{k^k}$.

Proof: For the asymptotic to be shown, we can assume that $\lambda = (n-k+1)/k$ is large. Considering this, Corollary 6 implies that for an optimal SMFA we can choose $a_1 = 0$ and $a_i = \lfloor (i-1)(\lambda+1) \rfloor$.

For $2 \leq i \leq k-1$ we have $a_i = \lfloor (i-1)(\lambda+1) \rfloor = \lfloor \frac{i-1}{k}(n+1) \rfloor \leq \frac{i}{k} n < n$. Consequently, for $i \neq k$ the summands

$$\alpha\binom{a_i}{i} - \beta\binom{a_i}{i-1} = \frac{\binom{a_i}{i}}{\binom{n}{k}} - \frac{\binom{a_i}{i-1}}{\binom{n}{k-1}}$$

on the right-hand side of (6) all tend to 0 as $n \to \infty$.

The claim follows by $\beta\binom{n}{k-1} = 1$ and the fact that

$$\begin{aligned} \alpha\binom{a_k}{k} - \beta\binom{a_k}{k-1} &= \frac{\binom{\lfloor \frac{k-1}{k}(n+1) \rfloor}{k}}{\binom{n}{k}} - \frac{\binom{\lfloor \frac{k-1}{k}(n+1) \rfloor}{k-1}}{\binom{n}{k-1}} \\ &= \left(\frac{\lfloor \frac{k-1}{k}(n+1) \rfloor - k + 1}{n-k+1} - 1 \right) \prod_{j=0}^{k-2} \frac{\lfloor \frac{k-1}{k}(n+1) \rfloor - j}{n-j} \end{aligned}$$

tends to

$$\left(\frac{k-1}{k} - 1\right)\left(\frac{k-1}{k}\right)^{k-1} = -\frac{(k-1)^{k-1}}{k^k}$$

as $n \to \infty$. ∎

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