Full and Reduced-order Synchronization of Chaos in Josephson Junction

B. A. Idowu[1], A. Uçar[2] and U. E. Vincent[3,4,*]

[1]*Department of Physics, Lagos State University, Ojo, Festac, Festac Town, Lagos, Nigeria*

[2]*Department of Electrical and Electronic Engineering, Firat University, Elazig, Turkey*

[3]*Institute of Theoretical Physics, Technical University of Clausthal, Clausthal-Zellerfeld, Germany*

[4] *Department Physics, Olabisi Onabanjo University,Ago-Iwoye, Nigeria*

In this paper, the synchronization between two different Josephson junction models is investigated based on variable back-stepping technique. Firstly, we examined the synchronization of identical RCL-shunted Josephson junction (RCLSJ) systems emanating from different initial conditions by means of a proposed variable back-stepping technique. Secondly, by utilizing a reduced-order synchronization technique, we realized synchronization between the third order RCLSJ model and the parametrically modulated Josephson junction (PMJJ) system (second order). In both cases, the designed controllers are singular in nature and thus easier to construct and implement in practice. They ensure that the states of the controlled chaotic slave system exponentially synchronize with the state of the master system. Numerical simulations are illustrated to verify the proposed methods.

1. Introduction

Over the last three decades, researchers have been intensively interested in nonlinear systems that exhibit complex dynamical behavior due to their universality in nature. Among these nonlinear systems, the Josephson junction is a fascinating one. Most of the earlier studies on Josephson junction (JJ) dynamics have used an RC-shunted junction model (RCSJ) [1-6], in which a resistor and a capacitor are shunted parallel to the ideal JJ flown through by the super-currents. It appears in many applications as in JJ arrays and ladders [4-6], in superconducting quantum interference devices (SQUIDs) [1,7], as a millimeter and sub-millimeter wave oscillator [1,2] in digital systems [3], measurement of small magnetic flux, detection of electromagnetic waves, construction of logical gates using flux quanta and so on. Also, from the viewpoint of fundamental physics, Josephson devices have provided ideal physical systems studying nonlinear equations including soliton solutions and chaos [8,9].

The RCSJ model fails to describe the current-voltage characteristics of the junction if the resistive shunt due to its wiring contains a non-negligible inductance and for higher temperatures, anomalies in the characteristics have been found. Thus, in order to remedy this lack, a resistive-capacitive-inductive junction (RCLSJ) model has been suggested to describe these junction [10,11]. Consideration of this model leads to a better agreement of the numerical simulation with the experimentally obtained data [10,12] and thus it is a consideration in this paper. The junction shows chaotic behaviour in a selected parameter space even in absence of any external forcing [13,14]. The RCLSJ model is found more in high frequency applications [13].

Synchronization between coupled chaotic systems [15] is an interesting area of study for understanding the collective behaviour of nonlinear systems [16] and according to Pikovsky [17], studies on the synchronization of chaotic systems are in general important in science. Synchronization of the superconducting junction arrays is also important for the purpose of generating reasonably large output power [18]. However, synchronization of chaotic systems are related to the observer problem in control theory and many recent synchronization methods employ control techniques to achieve synchronization goals. In earlier works, the control of chaos on JJ was investigated, for instance in [19], where control was achieved using weak periodic forcing. Olsen et al. [20] used the external rf signal for controlling chaotic patterns in an RC-shunted junction. Likewise, the taming of chaos in a single RC-shunted junction using weak perturbation was reported by Hsu et al. [21], while the control of RCLSJ using the back-stepping design was presented by Vincent et al. [22]. Since Pecora and Carroll [24] introduced a method to synchronize two identical chaotic systems with different initial conditions, many methods and techniques have

*u.vincent@tu-clausthal.de

been developed. Most of the methods are used to synchronize two identical chaotic systems. However, it is difficult to find identical chaotic systems in reality, such as in physics, auto-control, information and secure communications. Therefore, it is important to synchronize two different chaotic systems, especially with different orders.

Recently, Dana et al. [18] proposed a method for synchronizing two identical junctions using a negative pulse forcing and the robustness of the technique to white noise was also established. Ucar et al. [23] employed the active control technique to synchronize two RCLSJ models with slightly different parameters. In Vincent et al. [22], the synchronization of RCLSJ based on back-stepping nonlinear control was achieved, whereas the synchronization of a parametrically modulated Josephson junction model (PMJJ) based on Lyapunov stability theory and Routh-Hurwitz criteria was examined in [24].

In this paper, we are employing the recursive back-stepping technique to achieve synchronization between two identical RCLSJ systems as well as reduced-order synchronization [25] between the RCLSJ and PMJJ. In all other works, the control functions are numerically equal to the dimension of the system and could be very complex. In control theory applications, it is important that the control should be as simple as possible. Thus, the controller complexity is a fundamental problem that should be addressed. Here we will be utilizing only one controller and this will reduce the complexity of the controller and in practical applications, the less the number of controllers designed in control process the better the control method [26].

The main feature of the reduced-order synchronization is that the order of the slave system is less than the master system. Reduced-order synchronization has received interesting attention due to its application in real life such as in the movement of neurons [27,28], where the output from higher-order neurons always drives the neurons with lower-order in the subsystem; circulatory and respiratory systems were observed to behave in a synchronous way and the master system is strictly different from the slave system where the difference may involve different orders. The reduced-order synchronization problem is pertinent in the study of neural networks [27-29]. According to Ho et al. [30], studying such problems can help us elucidate the coherent behaviour of complex systems; notwithstanding the inherent interest in the problem itself.

Recently Ho et al. [30] investigated the reduced-order synchronization of uncertain chaotic systems, which was based upon the parameters modulation and the adaptive control techniques. This was applied to generalized Lorenz system (fourth order) and Lu system (third order) as well as Rossler hyperchaotic system (fourth order) and Rossler system (third order). Likewise Lu et al. [31] designed a controller to realize synchronization between the Rossler and hypechaotic Chen system. In Li et al. [32], they proposed a new synchronization manner, reduced-order generalized synchronization, which has the characteristics of having a functional relation between the slave and the partial master systems.

In view of the above considerations, the motive of this paper is two-fold; firstly to show that the synchronization between two identical RCLSJ system could be achieved using a single control and secondly to achieve the reduced-order synchronization between the RCLSJ system (third order) with the PMJJ system (second order). In both cases, the required controllers are singular in nature, thus, easier to construct and achieve in practice. This is a deviation from the earlier works on RCLSJ and PMJJ [22-24].

The rest of this paper is organized as follows: Sec. 2 presents the Josephson junction models under consideration. Sec. 3 and Sec. 4 present the recursive back-stepping technique and reduced-order technique to achieve synchronization, respectively. Conclusions are finally drawn in Sec. 5.

2. The Josephson junction models

2.1 The RCL-shunted Josephson junction

In this paper, we study the synchronization of two different Josephson junction models – RCL-shunted Josephson Junction (RCLSJ) and the Periodically Modulated Josephson Junction (PMJJ). The RCLSJ model of JJ in dimensionless form is described by the following set of first order differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{\beta_L}(z - y), \\ \dot{z} &= \frac{1}{\beta_C}\left[i_0 - g(z)z - \sin(x) - y\right], \end{aligned} \tag{1}$$

Where, the nonlinear damping function $g(z)$ is approximated by a current-voltage relation between the two junctions and is defined by

$$g(z)=\begin{cases}0.366 & if \quad |z| \; > \; 2.9\\ 0.061 & if \quad |z| \; \le \; 2.9\end{cases}$$

x, y, and z represent the phase difference, the voltage at the junction, and the inductive current, respectively. β_C and β_L are constants representing the capacitive and inductive values, respectively, and i_0 is the *dc* external current.

Fig. 1 is a chaotic attractor for the following set of parameters: $i_0 = 1.15$, $\beta_C = 2.6$ and $\beta_L = 0.707$ with the initial conditions $(x(0), y(0), z(0)) = (0,0,0)$. The RCLSJ model exhibits chaotic dynamics for the *dc* external current in the region $1 < i_0 < 1.3$ [10-13].

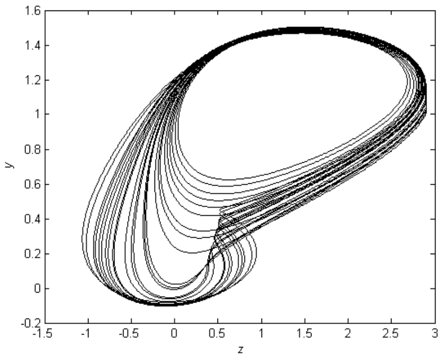

Fig. 1: Phase portrait of *y* and *z* variables of the system (1) for $i_0 = 1.15$.

2.2 The periodically modulated Josephson junction

The periodically modulated Josephson Junction (PMJJ) model is a second order dynamical system which was recently analyzed by Wu and Li [33] and is given by

$$\ddot{\phi} = -[1+\xi\cos(\Omega t+\theta)]\sin\phi + \rho_0 - \delta\dot{\phi} + \gamma\cos(\omega t), \quad (2)$$

where ϕ is the phase difference between quantum-mechanical wave function of the two separated superconductor of the junction; $\xi\cos(\Omega t+\theta)$ is the modulating term with amplitude ξ, phase θ and frequency Ω; ρ_0 is the dc bias, δ is the damping parameter; and γ and ω are the amplitude and frequency of the rf current, respectively. When the parameters are fixed as follows: $\Omega = 4, \theta = 0, \rho_0 = 0.1, \delta = 0.1, \gamma = 2.0$, and $\omega = 3.0$, one can observe then a sequence of period-doubling route to chaos as ξ is progressively increased. Fig. 2 shows a chaotic attractor for $\xi = 2.462$ arising from a tori-doubling bifurcation. The plot in Fig. 2 is essentially the same with those obtained by Wu and Li [33].

3. Synchronization for the RCLSJ

In this section, we consider a one controller scheme based on variable back-stepping control, by letting

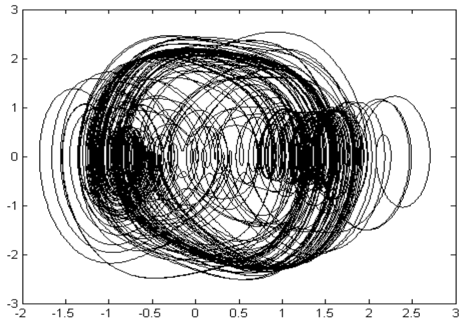

Fig. 2: Phase portrait of *y* and *z* variables of the system (2) for $\xi = 2.462$.

$$\begin{aligned} x &= x_1; \quad \dot{x} = \dot{x}_1\\ z &= y_1; \quad \dot{z} = \dot{y}_1\\ y &= z_1; \quad \dot{y} = \dot{z}_1 \end{aligned} \qquad (2)$$

We can recast the RCLSJ system as a driver system given by

$$\begin{aligned} \dot{x}_1 &= z_1,\\ \dot{y}_1 &= \frac{1}{\beta_L}(z_1 - y_1)\\ \dot{z}_1 &= \frac{1}{\beta_C}[i_0 - g(z_1)z_1 - \sin(x_1) - y_1] \end{aligned} \qquad (3)$$

The corresponding response system is thus,

$$\begin{aligned} \dot{x}_2 &= z_2\\ \dot{y}_2 &= \frac{1}{\beta_L}(z_2 - y_2)\\ \dot{z}_2 &= \frac{1}{\beta_C}[i_0 - g(z_2)z_2 - \sin(x_2) - y_2] + u \end{aligned} \qquad (4)$$

Where, *u* is the undetermined controller.

Let the error variables be

$$\begin{aligned} e_1 &= x_2 - x_1\\ e_2 &= y_2 - y_1\\ e_3 &= z_2 - z_1 \end{aligned} \qquad (5)$$

Thus, the time derivative of above gives

$$\begin{aligned}\dot{e}_1 &= \dot{x}_2 - \dot{x}_1\\ \dot{e}_2 &= \dot{y}_2 - \dot{y}_1\\ \dot{e}_3 &= \dot{z}_2 - \dot{z}_1\end{aligned} \tag{6}$$

Substituting (3) and (4) into (6), gives the error dynamics equation

$$\begin{aligned}\dot{e}_1 &= e_3\\ \dot{e}_2 &= \frac{1}{\beta_L}(e_3 - e_2)\\ \dot{e}_3 &= -\frac{1}{\beta_c}\{g(z_2)z_2 - g(z_1)z_1 + \sin(x_2) - \sin(x_1)\}\\ &\quad -\frac{1}{\beta_c}e_2 + u\end{aligned} \tag{7}$$

Considering the subsystem

$$\dot{e}_1 = e_3 \tag{8}$$

of equation (7), and assume the following Lyapunov function,

$$V_1 = \frac{1}{2}e_1^2 \tag{9}$$

Thus,

$$\dot{V}_1 = e_1\dot{e}_1 \tag{10}$$

Suppose $e_{3d} = -e_1$ (e_{3d} is desired error state for e_3). Then, $\dot{e}_1 = e_3$, which implies that $e_3 = -e_1$, so that equation (10) becomes $\dot{V}_1 = -e_1^2$, which is negative definite, ensuring the stability of system (8).

Suppose the error between the desired states e_{3d} and e_3 is *L*, that is,

$$L = e_3 - e_{3d}\ .$$

Assume a Lyapunov function of the form

$$V_2 = V_1 + \frac{1}{2}L^2\ ,$$

for the error L, then,

$$\dot{V}_2 = \dot{V}_1 + L\dot{L}\ , \tag{11}$$

Therefore,

$$\dot{V}_2 = \dot{V}_1 + L\left\{\begin{aligned}&-\frac{1}{\beta_c}[g(z_2)z_2 - g(z_1)z_1 + \sin(x_2) - \sin(x_1)]\\ &-\frac{1}{\beta_c}e_2 + e_3 + u\end{aligned}\right\} \tag{12}$$

Thus, if

$$u = \frac{1}{\beta_c}\{e_2 + g(z_2)z_2 - g(z_1)z_1 + \sin(x_2) - \sin(x_1)\} - e_3 \tag{13}$$

Then, $\dot{V}_2 = \dot{V}_1 = -e_1^2$, is negative definite and systems (3) and (4) is synchronized.

Proof:

Substituting (13) in error dynamics (7) leads to

$$\begin{aligned}\dot{e}_1 &= e_3\\ \dot{e}_2 &= \frac{1}{\beta_L}(e_3 - e_2)\\ \dot{e}_3 &= -e_3\end{aligned} \tag{14}$$

The eigenvalues of the characteristic equations of system (14) are $\lambda_1 = 0$, $\lambda_2 = -1.41$ and $\lambda_3 = -1$. Thus, the synchronization is achieved.

The time response of the system (3) for $i_0 = 1.15$ and initial state conditions $(x_1(0), y_1(0), z_1(0)) = (0,0,0)$ is displayed in Fig. 3 and that for system (4) for initial state conditions $(x_2(0), y_2(0), z_2(0)) = (1,2,1)$ is displayed in Fig. 4.

In Fig. 5, we display the time response of the error signals of the coupled RCLSJ chaotic system with the controller applied at a time $t \geq 250$ *sec* for same system parameters and external input $i_0 = 1.15$. The state initial conditions are as for the systems (3) and (4).

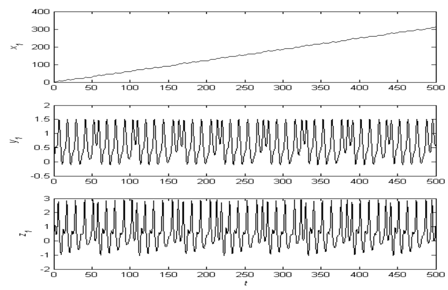

Fig. 3: The time response of the system (3) for $i_0 = 1.15$ and initial conditions $(x_1(0), y_1(0), z_1(0)) = (0,0,0)$.

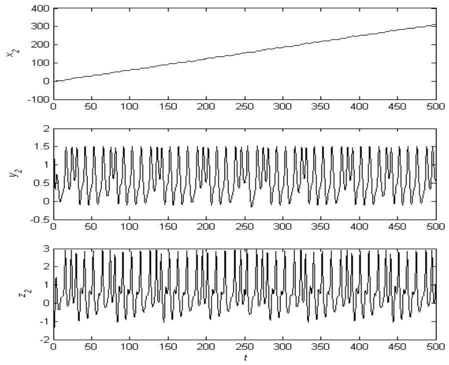

Fig. 4: The time response of the system (4) for $i_0 = 1.15$ and initial state conditions $(x_2(0), y_2(0), z_2(0)) = (1,2,1)$.

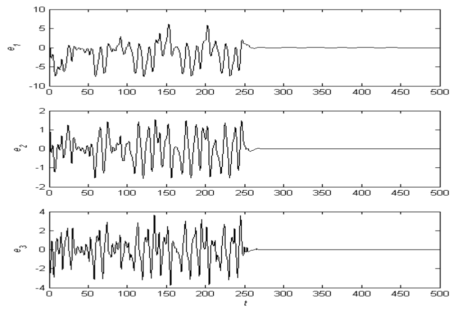

Fig. 5: The time response of the error signals of the coupled RCLSJ chaotic system with the controller applied at time $t \geq 250$ *sec* for same system parameters and the external input, $i_0 = 1.15$. The state initial conditions for the systems (3) and (4) were $(x_1(0), y_1(0), z_1(0)) = (0,0,0)$ and $(x_2(0), y_2(0), z_2(0)) = (1,2,1)$, respectively.

4. Reduced-order synchronization for RCLSJ and PMJJ

This section will address the problem of synchronization between two different Josephson junction models with different orders, where the drive system is given in (3) and expressed as

$$\begin{aligned} \dot{x}_1 &= z_1, \\ \dot{y}_1 &= \frac{1}{\beta_L}(z_1 - y_1) \\ \dot{z}_1 &= \frac{1}{\beta_C}\left[i_0 - g(z_1)z_1 - \sin(x_1) - y_1\right] \end{aligned} \tag{15}$$

with the following system parameters $\beta_C = 2.6$ and $\beta_L = 0.707$ and the external input is $i_0 = 1.15$. The slave system is the second-order PMJJ given in (2) and expressed as:

$$\begin{aligned} \dot{y}_2 &= z_2 \\ \dot{z}_2 &= -[1+\xi\cos(\Omega t+\theta)]\sin y_2 - \delta z_2 + \rho_0 + \gamma\cos(\omega t) + u \end{aligned} \tag{16}$$

Where, u is the undetermined controller and system parameters are: $\Omega = 4$, $\theta = 0$, $\rho_0 = 0.1$, $\delta = 0.1$, $\gamma = 2.0$, $\omega = 3.0$ and $\xi = 2.462$.

In the following, a systematic way of designing the controller u is presented. Let the first error variable be

$$e_1 = y_2 - y_1 \tag{17}$$

Then, its time derivative along the solutions of systems (15) and (16) is

$$\begin{aligned} \dot{e}_1 &= \dot{y}_2 - \dot{y}_1 \\ \dot{e}_1 &= z_2 - \frac{1}{\beta_L}(z_1 - y_1) \end{aligned} \quad . \tag{18}$$

Defining the second error variable as

$$e_2 = \dot{e}_1 . \tag{19}$$

Then, its time derivative

$$\dot{e}_2 = \dot{z}_2 - \frac{1}{\beta_L}(\dot{z}_1 - \dot{y}_1)$$

along the solutions of systems (15) and (16) is

$$\begin{aligned} \dot{e}_2 = &-[1+\xi\cos(\Omega t+\theta)]\sin y_2 - \delta z_2 + \rho_0 \\ &+ \gamma\cos(\omega t) - \frac{1}{\beta_L^2}[\frac{\beta_L}{\beta_C}(i_0 - g(z_1)z_1 \\ &- \sin(x_1) - y_1) - (z_1 - y_1)] + u \end{aligned} \tag{20}$$

If the controller u is taken as

$$\begin{aligned} u = &[1+\xi\cos(\Omega t+\theta)]\sin y_2 + \delta z_2 - \rho_0 \\ &- \gamma\cos(\omega t) - k_1 e_1 - k_2 e_2 \\ &+ \frac{1}{\beta_L^2}[\frac{\beta_L}{\beta_C}(i_0 - g(z_1)z_1 - \sin(x_1) - y_1) - (z_1 - y_1)] \end{aligned} \tag{21}$$

and the feedback coefficients satisfy the following conditions:

$$k_1 > 0, k_2 > 0 ; \tag{22}$$

then, the reduced-order synchronization between systems (15) and (16) can be achieved.

Proof:
With the choice of controller (21), the error dynamics of the system is given by

$$\begin{aligned}\dot{e}_1 &= e_2 \\ \dot{e}_2 &= -k_1 e_1 - k_2 e_2\end{aligned} \tag{23}$$

It is obvious that system (23) has only one equilibrium point at (0,0). The corresponding characteristic equation is

$$\lambda^2 + k_2\lambda + k_1 = 0 \tag{24}$$

Where, λ is the eigenvalue.

According to the Routh-Hurwitz criterion, the real eigenvalues or all the real parts of complex conjugate eigenvalues are negatives when k_i meets the conditions (22). Therefore, the origin is asymptotically stable.

In order to achieve synchronization within $t \geq t_s = 0.4$ sec after control signal is activated, $k_1 = 100$, $k_2 = 20$ are chosen. The numerical results for the closed loop system are depicted in Fig. 6 and Fig. 7 when the control signals are activated at 250 s*ec*.

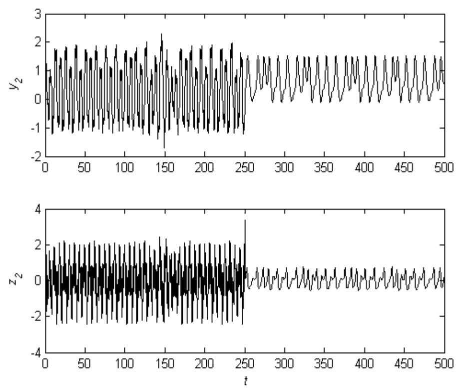

Fig. 6: The time response of the system (16) for $\xi = 2.462$ and the initial conditions $(y_2(0), z_2(0)) = (0{,}0)$ with the controller applied at a time $t \geq 250$ *sec*.

Fig. 6 displays **t**he time response of the system (16) for $\xi = 2.462$ and the initial conditions $(y_2(0), z_2(0)) = (0{,}0)$ with the controller applied at a time $t \geq 250$ *sec*. Fig. 7 shows the time response of the error signals defined in (17) and (19) for the reduced order synchronization of RCLSJ chaotic system with the controller applied at a time $t \geq 250$ *sec* and this shows that reduced order synchronization have been achieved.

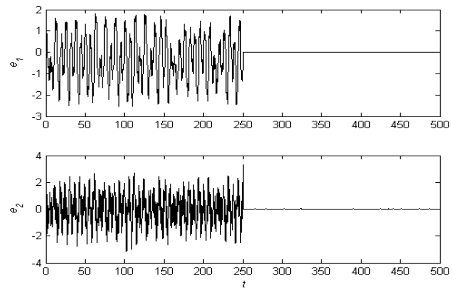

Fig. 7: The time response of the error signals defined in (17) and (19) for the reduced order synchronization of RCLSI chaotic system coupled with PMJJ and the controller applied at a time $t \geq 250$ *sec*.

5. Conclusions

In this paper, we have investigated the chaos synchronization and reduced-order synchronization between two different Josephson junctions with different orders by utilizing a proposed recursive back-stepping technique based on Lyapunov stability theory and Routh-Hurwitz criterion, which is aimed at designing a single controller in contrast to others that have control functions numerically equal to the dimension of the system. The approach makes practical application much easier and provides answer to the question of controller complexity.

The simulation results show that the controllers in both cases are effective and the states of the systems are asymptotically synchronized.

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