# Full-Rate Full-Diversity Space–Frequency Codes With Optimum Coding Advantage

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***Abstract*—In this paper, a general space–frequency (SF) block code structure is proposed that can guarantee full-rate (one channel symbol per subcarrier) and full-diversity transmission in multiple-input multiple-output–orthogonal frequency-division multiplexing (MIMO-OFDM) systems. The proposed method can be used to construct SF codes for an arbitrary number of transmit antennas, any memoryless modulation and arbitrary power-delay profiles. Moreover, assuming that the power-delay profile is known at the transmitter, we devise an interleaving method to maximize the overall performance of the code. We show that the diversity product can be decomposed as the product of the "intrinsic" diversity product, which depends only on the used signal constellation and the code design, and the "extrinsic" diversity product, which depends only on the applied interleaving method and the power delay profile of the channel. Based on this decomposition, we propose an interleaving strategy to maximize the "extrinsic" diversity product. Extensive simulation results show that the proposed SF codes outperform the previously existing codes by about 3–5 dB, and that the proposed interleaving method results in about 1–3-dB performance improvement compared to random interleaving.**

***Index Terms*—Frequency-selective fading channels, full diversity, multiple-input multiple-output–orthogonal frequency-division multiplexing (MIMO-OFDM) systems, permutation, space–frequency (SF) coding.**

## I. Introduction

THE idea of using multiple transmit and receive antennas in wireless communication systems to accommodate high data rates has attracted considerable attention recently. It has been shown that multiple-input multiple-output (MIMO) systems offer considerable performance improvement over single-antenna systems. This performance improvement has been characterized by the achievable diversity order, which describes the available degrees of freedom present in the MIMO channel. In order to take advantage of the spatial and temporal diversity, a large number of *space–time* (ST) coding and modulation methods have been proposed, for example, [1]–[18].

In case of frequency-selective MIMO channels, there is an additional source of diversity, frequency diversity, due to the existence of multiple propagation paths between each transmit and receive antenna pair. By combining the orthogonal frequency-division multiplexing (OFDM) modulation [19], [20] with MIMO systems, *space–frequency* (SF) codes have been proposed[1] to exploit the spatial and frequency diversity present in frequency-selective MIMO channels [21]–[29]. The strategy of SF coding is to distribute the channel symbols over different transmit antennas and OFDM tones within one OFDM block. If longer decoding delay and higher decoding complexity are allowable, one may consider coding over several OFDM block periods, resulting in space–time–frequency codes [30], [31].

The first SF coding scheme was proposed in [21], in which previously existing ST codes were used by replacing the time domain with the frequency domain. The resulting SF codes could achieve only spatial diversity and were not guaranteed to achieve full (spatial and frequency) diversity. Later, similar schemes were described in [22]–[25]. The performance criteria for SF-coded MIMO-OFDM systems were derived in [26], [27]. The maximum achievable diversity order was found to be the product of the number of transmit antennas, the number of receive antennas, and the number of delay paths. The authors of [27] showed that, in general, existing ST codes cannot exploit the frequency diversity available in the frequency-selective MIMO channels, and it was suggested that a completely new code design procedure will have to be developed for MIMO-OFDM systems. Later in [28], they provided a construction method for a class of SF codes by multiplying a part of the discrete Fourier transform (DFT) matrix with the input symbol vectors. The obtained SF codes achieve full spatial and frequency diversity at the expense of bandwidth efficiency. Moreover, this approach relied on the assumption that all of the path delays are located exactly at the sampling instances of the receiver, and the power is distributed uniformly across the paths. Recently, in [29], a systematic design method to obtain full-diversity SF codes was proposed for arbitrary power delay profiles and any number of transmit antennas. It was shown that *any* ST code (block or trellis) achieving full (spatial) diversity in quasi-static flat-fading environment can be used to construct full-diversity SF codes via a simple mapping. The resulting SF codes provide higher data rates than the approach described in [28], but they still cannot achieve *full rate* (one channel symbol per subcarrier) transmission. Therefore, it is of interest to devise new SF code design methods that can guarantee both performance (full diversity) and high data rate (full symbol rate).

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[1]Another coding approach is to consider ST coding directly for single-carrier frequency-selective MIMO systems (see [32], [33], and the references therein). In this paper, we follow the SF coding approach for MIMO-OFDM systems.

In MIMO-OFDM systems, the DFT operation introduces correlation into the channel frequency response at different subcarriers, even if the individual delay paths are independent of each other [34], [35]. A natural idea to decrease the correlation of the channel frequency response is to interleave, or permute, the subcarriers. If the power delay profile of the channel is not known *a priori*, random interleaving may offer desirable performance. Assuming that the delay paths are equally spaced and fall onto the sampling instances of the receiver, an optimum subcarrier grouping method was proposed in [36]. However, the proposed grouping method was not guaranteed to be optimum for arbitrary power delay profiles.

In this paper, we consider the problem of *systematic* SF block code design for MIMO-OFDM systems. We propose an SF code design approach that offers full symbol rate and guarantees full diversity for an arbitrary number of transmit antennas, any memoryless modulation method, and arbitrary power delay profiles. First, we describe a general SF code structure and show that the combination of this code structure and the algebraically rotated signal constellations [37]–[42] or the diagonal ST signal constellations [8] can guarantee full-rate full-diversity transmission. Second, assuming that the statistics of the channel (the power delay profile) is known at the transmitter, we devise a permutation (or interleaving) method to maximize the overall performance of the code. We show that the diversity product can be decomposed as the product of the "intrinsic" and the "extrinsic" diversity products. The "intrinsic" diversity product depends only on the used signal constellations and the SF code design, while the "extrinsic" diversity product depends only on the applied permutation and the power delay profile of the channel. We also obtain some upper bounds on the "extrinsic" diversity product for any permutation and any power delay profile. Based on this decomposition, we propose a permutation strategy and determine the optimum permutation to maximize the "extrinsic" diversity product.

The rest of the paper is organized as follows. In Section II, we introduce the system model and briefly review the SF code design criteria. In Section III, we describe the general code structure, and discuss two approaches to obtain full-rate full-diversity SF codes. In Section IV, we investigate the effect of permutations on the proposed SF codes, and determine the optimum permutation for different power delay profiles. The simulation results are presented in Section V, and some conclusions are drawn in Section VI.

## II. Channel Model and SF Code Design Criteria

We consider an SF-coded MIMO-OFDM system with $M_t$ transmit antennas, $M_r$ receive antennas, and $N$ subcarriers. The MIMO channel is assumed to be constant over each OFDM block period. The frequency-selective fading channels between different transmit and receive antenna pairs are assumed to have $L$ independent paths and the same power delay profile. The channel impulse response from transmit antenna $i$ to receive antenna $j$ is modeled as

$$h_{i,j}(\tau) = \sum_{l=0}^{L-1} \alpha_{i,j}(l)\delta(\tau - \tau_l) \tag{2.1}$$

where $\tau_l$ is the delay of the $l$th path, and $\alpha_{i,j}(l)$ is the complex amplitude of the $l$th path. The $\alpha_{i,j}(l)$'s are modeled as zero-mean, complex Gaussian random variables with variances $E|\alpha_{i,j}(l)|^2 = \delta_l^2$, where $E$ stands for the expectation. The powers of the $L$ paths are normalized such that $\sum_{l=0}^{L-1}\delta_l^2 = 1$. From (2.1), the frequency response of the channel is given by

$$H_{i,j}(f) = \sum_{l=0}^{L-1} \alpha_{i,j}(l) e^{-\boldsymbol{j}2\pi f \tau_l} \tag{2.2}$$

where $\boldsymbol{j} = \sqrt{-1}$ is the imaginary unit. We assume that the MIMO channel is spatially uncorrelated, i.e., the channel taps $\alpha_{i,j}(l)$ are independent for different indices $(i,j)$.

The input bit stream is divided into $b$ bit long segments, and each segment is mapped onto an SF codeword. Each SF codeword can be represented as an $N \times M_t$ matrix

$$C = \begin{bmatrix} c_1(0) & c_2(0) & \cdots & c_{M_t}(0) \\ c_1(1) & c_2(1) & \cdots & c_{M_t}(1) \\ \vdots & \vdots & \ddots & \vdots \\ c_1(N-1) & c_2(N-1) & \cdots & c_{M_t}(N-1) \end{bmatrix} \tag{2.3}$$

where $c_i(n)$ denotes the channel symbol transmitted over the $n$th subcarrier by transmit antenna $i$. The SF code is assumed to satisfy the energy constraint $E\|C\|_F^2 = NM_t$, where $\|C\|_F$ is the Frobenius norm[2] of $C$. The OFDM transmitter applies an $N$-point inverse fast Fourier transform (IFFT) to each column of the matrix $C$, and after appending the cyclic prefix, the OFDM symbol corresponding to the $i$th $(i = 1, 2, \ldots, M_t)$ column of $C$ is transmitted by transmit antenna $i$. Note that all of the $M_t$ OFDM symbols are transmitted simultaneously from different transmit antennas.

At the receiver, after matched filtering, removing the cyclic prefix, and applying the fast Fourier transform (FFT), the received signal at the $n$th subcarrier at receive antenna $j$ is given by

$$y_j(n) = \sqrt{\frac{\rho}{M_t}} \sum_{i=1}^{M_t} c_i(n) H_{i,j}(n) + z_j(n) \tag{2.4}$$

where

$$H_{i,j}(n) = \sum_{l=0}^{L-1} \alpha_{i,j}(l) e^{-\boldsymbol{j}2\pi n \Delta f \tau_l} \tag{2.5}$$

is the channel frequency response at the $n$th subcarrier between the transmit antenna $i$ and the receive antenna $j$, $\Delta f = 1/T$ is the subcarrier separation in the frequency domain, and $T$ is the OFDM symbol period. We assume that the channel state information $H_{i,j}(n)$ is known at the receiver, but not at the transmitter. In (2.4), $z_j(n)$ denotes the additive complex Gaussian noise with zero mean and unit variance at the $n$th subcarrier at receive antenna $j$. The noise samples $z_j(n)$ are assumed to be uncorrelated for different $j$'s and $n$'s. The factor $\sqrt{\rho/M_t}$ in (2.4) ensures that $\rho$ is the average signal-to-noise ratio (SNR) at

[2]The Frobenius norm of $C$ is defined as

$$\|C\|_F^2 = \mathrm{tr}(C^{\mathcal{H}}C) = \mathrm{tr}(CC^{\mathcal{H}}) = \sum_{n=0}^{N-1}\sum_{i=1}^{M_t} |c_i(n)|^2.$$

each receive antenna, independently of the number of transmit antennas.

The channel frequency response vector between transmit antenna $i$ and receive antenna $j$ will be denoted by

$$H_{i,j} = [H_{i,j}(0)\ H_{i,j}(1)\ \cdots\ H_{i,j}(N-1)]^{\mathcal{T}}. \tag{2.6}$$

Using the notation $w = e^{-\boldsymbol{j}2\pi\Delta f}$, $H_{i,j}$ can be decomposed as

$$H_{i,j} = W \cdot A_{i,j} \tag{2.7}$$

where

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ w^{\tau_0} & w^{\tau_1} & \cdots & w^{\tau_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(N-1)\tau_0} & w^{(N-1)\tau_1} & \cdots & w^{(N-1)\tau_{L-1}} \end{bmatrix}_{N\times L}$$

which is related to the delay distribution, and

$$A_{i,j} = [\alpha_{i,j}(0)\ \alpha_{i,j}(1)\ \ldots\ \alpha_{i,j}(L-1)]^{\mathcal{T}}$$

which is related to the power distribution of the channel impulse response. In general, $W$ is not a unitary matrix. If all of the $L$ delay paths fall at the sampling instances of the receiver, $W$ is part of the DFT matrix, which is unitary. From (2.7), the correlation matrix of the channel frequency response vector between transmit antenna $i$ and receive antenna $j$ can be calculated as

$$\begin{aligned} R_{i,j} &= E\left\{H_{i,j}H_{i,j}^{\mathcal{H}}\right\} \\ &= WE\left\{A_{i,j}A_{i,j}^{\mathcal{H}}\right\}W^{\mathcal{H}} \\ &= W\mathrm{diag}(\delta_0^2, \delta_1^2, \ldots, \delta_{L-1}^2)W^{\mathcal{H}} \triangleq R \end{aligned} \tag{2.8}$$

where the superscript $\mathcal{H}$ stands for the complex conjugate and transpose of a matrix. The third equality follows from the assumption that the path gains $\alpha_{i,j}(l)$ are independent for different paths. Note that the correlation matrix $R$ is independent of the transmit and receive antenna indices $i$ and $j$.

For two distinct SF codewords $C$ and $\tilde{C}$, we use the notation

$$\Delta = (C-\tilde{C})(C-\tilde{C})^{\mathcal{H}}. \tag{2.9}$$

Then, assuming spatially uncorrelated MIMO channel, the pairwise error probability between $C$ and $\tilde{C}$ can be upper-bounded as [29], [13], [14]

$$P(C\to\tilde{C}) \le \binom{2\nu M_r - 1}{\nu M_r}\left(\prod_{i=1}^{\nu}\lambda_i\right)^{-M_r}\left(\frac{\rho}{M_t}\right)^{-\nu M_r} \tag{2.10}$$

where $\nu$ is the rank of $\Delta\circ R$, $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ are the nonzero eigenvalues of $\Delta\circ R$, and $\circ$ denotes the Hadamard product.[3] Based on the upper bound (2.10), two SF code design criteria were proposed in [29].

[3]Suppose that $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$ are two matrices of size $m\times n$. The *Hadamard product* of $A$ and $B$ is defined as

$$A\circ B = \begin{bmatrix} a_{1,1}b_{1,1} & \cdots & a_{1,n}b_{1,n} \\ \cdots & \cdots & \cdots \\ a_{m,1}b_{m,1} & \cdots & a_{m,n}b_{m,n} \end{bmatrix}.$$

- *Diversity (rank) criterion:* The minimum rank of $\Delta\circ R$ over all pairs of distinct codewords $C$ and $\tilde{C}$ should be as large as possible.
- *Product criterion:* The minimum value of the product $\prod_{i=1}^{\nu}\lambda_i$ over all pairs of distinct codewords $C$ and $\tilde{C}$ should also be maximized.

If the minimum rank of $\Delta\circ R$ is $\nu_0$ for any pair of distinct codewords $C$ and $\tilde{C}$, we say that the SF code achieves a diversity order of $\nu_0 M_r$. According to a rank inequality on Hadamard products ([50, p. 307]), we know that

$$\mathrm{rank}(\Delta\circ R) \le \mathrm{rank}(\Delta)\mathrm{rank}(R).$$

Since the rank of $\Delta$ is at most $M_t$, the rank of $R$ is at most $L$, and the rank of $\Delta\circ R$ is at most $N$, the maximum achievable diversity (or full diversity) is at most $\min\{LM_tM_r, NM_r\}$ [26], [27], [29], [30]. Clearly, in order to achieve a diversity order of $\nu_0 M_r$, the number of nonzero rows of $C-\tilde{C}$ cannot be less than $\nu_0$ for any pair of distinct SF codewords $C$ and $\tilde{C}$. If an SF code achieves full diversity, the *diversity product*, which is the normalized coding advantage, is given by [29], [8], [9]

$$\zeta = \frac{1}{2\sqrt{M_t}}\min_{C\ne\tilde{C}}\left|\prod_{i=1}^{\nu}\lambda_i\right|^{\frac{1}{2\nu}} \tag{2.11}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ are the nonzero eigenvalues of $\Delta\circ R$ for any pair of distinct SF codewords $C$ and $\tilde{C}$.

## III. Full-Rate and Full-Diversity Code Design

In this section, we describe a systematic method to obtain full-rate SF codes achieving full diversity. Specifically, we will design a class of SF codes that can achieve a diversity order of $\Gamma M_t M_r$ for any fixed integer $\Gamma$ $(1\le\Gamma\le L)$.

### A. Code Structure

We consider a coding strategy where each SF codeword $C$ is a concatenation of some matrices $G_p$

$$C = \left[G_1^{\mathcal{T}}\ G_2^{\mathcal{T}}\ \ldots\ G_P^{\mathcal{T}}\ \mathbf{0}_{N-P\Gamma M_t}^{\mathcal{T}}\right]^{\mathcal{T}} \tag{3.1}$$

where $P = \lfloor N/(\Gamma M_t)\rfloor$, and each matrix $G_p$, $p = 1, 2, \ldots, P$, is of size $\Gamma M_t$ by $M_t$. The zero padding in (3.1) is used if the number of subcarriers $N$ is not an integer multiple of $\Gamma M_t$. Each matrix $G_p$ $(1\le p\le P)$ has the same structure given by

$$G = \sqrt{M_t}\,\mathrm{diag}\left(X_1, X_2, \ldots, X_{M_t}\right) \tag{3.2}$$

where $\mathrm{diag}(X_1, X_2, \ldots, X_{M_t})$ is a block diagonal matrix,

$$X_i = [x_{(i-1)\Gamma+1}\ x_{(i-1)\Gamma+2}\ \ldots\ x_{i\Gamma}]^{\mathcal{T}}, \qquad i = 1, 2, \ldots, M_t$$

and all $x_k$, $k = 1, 2, \ldots, \Gamma M_t$ are complex symbols and will be specified later. The energy constraint is $E(\sum_{k=1}^{\Gamma M_t}|x_k|^2) = \Gamma M_t$. For a fixed $p$, the symbols in $G_p$ are designed jointly, but the design of $G_{p_1}$ and $G_{p_2}$, $p_1\ne p_2$, is independent of each other. The symbol rate of the code is $P\Gamma M_t/N$, ignoring the cyclic prefix. If $N$ is a multiple of $\Gamma M_t$, the symbol rate is 1. If not, the rate is less than 1, but since usually $N$ is much greater than $\Gamma M_t$, the symbol rate is very close to 1.

Now we derive sufficient conditions for the previouslu described SF codes to achieve a diversity order of $\Gamma M_t M_r$. Suppose that $C$ and $\tilde{C}$ are two distinct SF codewords which are constructed from $G_1, G_2, \ldots, G_P$ and $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_P$, respectively. We would like to determine the rank of $\Delta \circ R$, where $\Delta$ is defined in (2.9) and $R$ is the correlation matrix defined in (2.8). For two distinct codewords $C$ and $\tilde{C}$, there exists at least one index $p_0$ $(1 \le p_0 \le P)$ such that $G_{p_0} \neq \tilde{G}_{p_0}$. We may further assume that $G_p = \tilde{G}_p$ for any $p \neq p_0$ since the rank of $\Delta \circ R$ does not decrease if $G_p = \tilde{G}_p$ for some $p \neq p_0$ ([50, Corollary 3.1.3, p. 149]).

From (2.8), we know that the correlation matrix $R \triangleq \{r_{i,j}\}_{1 \le i,j \le N}$ is a Toeplitz matrix. The entries of $R$ are given by

$$r_{i,j} = \sum_{l=0}^{L-1} \delta_l^2 w^{(i-j)\tau_l}, \qquad 1 \le i,j \le N. \tag{3.3}$$

Under the assumption that $G_p = \tilde{G}_p$ for any $p \neq p_0$, we observe that the nonzero eigenvalues of $\Delta \circ R$ are the same as those of $[(G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}}] \circ Q$, where $Q = \{q_{i,j}\}_{1 \le i,j \le \Gamma M_t}$ is also a Toeplitz matrix whose entries are

$$q_{i,j} = \sum_{l=0}^{L-1} \delta_l^2 w^{(i-j)\tau_l}, \qquad 1 \le i,j \le \Gamma M_t. \tag{3.4}$$

Note that $Q$ is independent of the index $p_0$, i.e., it is independent of the position of $G_{p_0} - \tilde{G}_{p_0}$ in $C - \tilde{C}$. Suppose that $G_{p_0}$ and $\tilde{G}_{p_0}$ have symbols $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_{\Gamma M_t}]$ and $\tilde{\boldsymbol{X}} = [\tilde{x}_1\ \tilde{x}_2\ \ldots\ \tilde{x}_{\Gamma M_t}]$, respectively. Then, the difference matrix between $G_{p_0}$ and $\tilde{G}_{p_0}$ is

$$\begin{aligned} G_{p_0} - \tilde{G}_{p_0} &= \sqrt{M_t}\mathrm{diag}(X_1 - \tilde{X}_1, X_2 - \tilde{X}_2, \ldots, X_{M_t} - \tilde{X}_{M_t}) \\ &= \sqrt{M_t}\,\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})\,(I_{M_t} \otimes \mathbf{1}_{\Gamma \times 1}) \end{aligned} \tag{3.5}$$

where

$$\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}}) \triangleq \mathrm{diag}(x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \ldots, x_{\Gamma M_t} - \tilde{x}_{\Gamma M_t})$$

$I_{M_t}$ is the identity matrix of size $M_t \times M_t$, $\mathbf{1}_{\Gamma \times 1}$ is an all one matrix of size $\Gamma \times 1$, and $\otimes$ stands for the tensor product. Thus, we have

$$\begin{aligned} &\left[(G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}}\right] \circ Q \\ &= M_t\left[\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})(I_{M_t} \otimes \mathbf{1}_{\Gamma \times 1})(I_{M_t} \otimes \mathbf{1}_{\Gamma \times 1})^{\mathcal{H}}\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})^{\mathcal{H}}\right] \circ Q \\ &= M_t\left[\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})\,(I_{M_t} \otimes \mathbf{1}_{\Gamma \times \Gamma})\,\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})^{\mathcal{H}}\right] \circ Q \\ &= M_t\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})\,[(I_{M_t} \otimes \mathbf{1}_{\Gamma \times \Gamma}) \circ Q]\,\mathrm{diag}(\boldsymbol{X} - \tilde{\boldsymbol{X}})^{\mathcal{H}}. \end{aligned} \tag{3.6}$$

In the preceding derivation, the second equality follows from the identities

$$[I_{M_t} \otimes \mathbf{1}_{\Gamma \times 1}]^{\mathcal{H}} = I_{M_t} \otimes \mathbf{1}_{1 \times \Gamma}$$

and

$$(A_1 \otimes B_1)(A_2 \otimes B_2)(A_3 \otimes B_3) = (A_1 A_2 A_3) \otimes (B_1 B_2 B_3)$$

([50, p. 251]), and the last equality follows from a property of the Hadamard product ([50, p. 304]). If all of the eigenvalues of $[(G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}}] \circ Q$ are nonzero, the product of the eigenvalues is

$$\begin{aligned} &\det\left(\left[(G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}}\right] \circ Q\right) \\ &= M_t^{\Gamma M_t} \prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k|^2 \cdot \det\left((I_{M_t} \otimes \mathbf{1}_{\Gamma \times \Gamma}) \circ Q\right) \\ &= M_t^{\Gamma M_t} \prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k|^2 \cdot (\det(Q_0))^{M_t} \end{aligned} \tag{3.7}$$

where $Q_0 = \{q_{i,j}\}_{1 \le i,j \le \Gamma}$ and $q_{i,j}$ is specified in (3.4). Similar to the correlation matrix $R$ in (2.8), $Q_0$ can also be expressed as

$$Q_0 = W_0 \mathrm{diag}(\delta_0^2, \delta_1^2, \ldots, \delta_{L-1}^2) W_0^{\mathcal{H}} \tag{3.8}$$

where

$$W_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ w^{\tau_0} & w^{\tau_1} & \cdots & w^{\tau_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(\Gamma-1)\tau_0} & w^{(\Gamma-1)\tau_1} & \cdots & w^{(\Gamma-1)\tau_{L-1}} \end{bmatrix}_{\Gamma \times L}.$$

Clearly, with $\tau_0 < \tau_1 < \cdots < \tau_{L-1}$, $Q_0$ is nonsingular. Therefore, from (3.7) we observe that if $\prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k| \neq 0$, the determinant of $[(G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}}] \circ Q$ is nonzero. This implies that the SF code achieves a diversity order of $\Gamma M_t M_r$.

The assumption that $G_p = \tilde{G}_p$ for any $p \neq p_0$ is also sufficient to calculate the diversity product. If the rank of $\Delta \circ R$ is $\Gamma M_t$ and $G_p \neq \tilde{G}_p$ for some $p \neq p_0$, the product of the nonzero eigenvalues of $\Delta \circ R$ cannot be less than that with the assumption that $G_p = \tilde{G}_p$ for any $p \neq p_0$ ([50, Corollary 3.1.3, p. 149]). Specifically, the diversity product can be calculated as in (3.9) at the top of the following page, and

$$\zeta_{\mathrm{in}} = \frac{1}{2} \min_{\boldsymbol{X} \neq \tilde{\boldsymbol{X}}} \left(\prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k|\right)^{\frac{1}{\Gamma M_t}} \tag{3.10}$$

is termed as the "intrinsic" diversity product of the SF code. The "intrinsic" diversity product $\zeta_{\mathrm{in}}$ does not depend on the power delay profile of the channel. Thus, we have the following theorem.

*Theorem 3.1:* For any SF code constructed by (3.1) and (3.2), if $\prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k| \neq 0$ for any pair of distinct sets of symbols $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_{\Gamma M_t}]$ and $\tilde{\boldsymbol{X}} = [\tilde{x}_1\ \tilde{x}_2\ \ldots\ \tilde{x}_{\Gamma M_t}]$, the SF code achieves a diversity order of $\Gamma M_t M_r$, and the diversity product is

$$\zeta = \zeta_{\mathrm{in}}\,|\det(Q_0)|^{\frac{1}{2\Gamma}} \tag{3.11}$$

where $Q_0$ is defined in (3.8), and $\zeta_{\mathrm{in}}$ is the "intrinsic" diversity product defined in (3.10).

$$\begin{aligned}\zeta &= \frac{1}{2\sqrt{M_t}} \min_{G_{p0} \neq \tilde{G}_{p0}} \left| \det\left( \left[ (G_{p_0} - \tilde{G}_{p_0})(G_{p_0} - \tilde{G}_{p_0})^{\mathcal{H}} \right] \circ Q \right) \right|^{\frac{1}{2\Gamma M_t}} \\ &= \frac{1}{2} \min_{\boldsymbol{X} \neq \tilde{\boldsymbol{X}}} \left( \prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k| \right)^{\frac{1}{\Gamma M_t}} |\det(Q_0)|^{\frac{1}{2\Gamma}} \\ &= \zeta_{\text{in}} \cdot |\det(Q_0)|^{\frac{1}{2\Gamma}} \end{aligned} \tag{3.9}$$

From Theorem 3.1, we observe that $|\det(Q_0)|$ depends only on the power delay profile of the channel, and the "intrinsic" diversity product $\zeta_{\text{in}}$ depends only on

$$\min_{\boldsymbol{X} \neq \tilde{\boldsymbol{X}}} \left( \prod_{k=1}^{\Gamma M_t} |x_k - \tilde{x}_k| \right)^{1/(\Gamma M_t)}$$

which is called the minimum product distance of the set of symbols $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_{\Gamma M_t}]$ [37], [38]. Therefore, given the code structure (3.2), it is desirable to design the set of symbols $\boldsymbol{X}$ such that the minimum product distance is as large as possible.

### *B. Maximizing the "Intrinsic" Diversity Product $\zeta_{\text{in}}$: Two Approaches*

The problem of maximizing the minimum product distance of a set of signal points has arisen previously as the problem of constructing signal constellations for Rayleigh-fading channels [39], [40], [42]. In this subsection, we will discuss two approaches to design the set of variables $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_{\Gamma M_t}]$. For simplicity, we will use the notation $K = \Gamma M_t$.

One approach to designing the signal points $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_K]$ is to apply a transform over a $K$-dimensional signal set. Specifically, assume that $\Omega$ is a set of signal points (a constellation such as quadrature amplitude modulation (QAM), pulse amplitude modulation (PAM), and so on). For any signal vector $S = [s_1\ s_2\ \ldots\ s_K] \in \Omega^K$, let

$$\boldsymbol{X} = S\mathcal{M}_K \tag{3.12}$$

where $\mathcal{M}_K$ is a $K \times K$ matrix. For a given signal constellation $\Omega$, the transform $\mathcal{M}_K$ should be optimized such that the minimum product distance of the set of $\boldsymbol{X}$ vectors is as large as possible. Both Hadamard transforms and Vandermonde matrices have been proposed for constructing $\mathcal{M}_K$ [39], [40], [42]. The results have been used recently to design ST block codes with full diversity [41], [42]. Note that the transforms $\mathcal{M}_K$ based on Vandermonde matrices result in larger minimum product distance than those based on Hadamard transforms. Here we summarize only some best known transforms $\mathcal{M}_K$ based on Vandermonde matrices. A Vandermonde matrix with variables $\theta_1, \theta_2, \ldots, \theta_K$ is a $K \times K$ matrix

$$V(\theta_1, \theta_2, \ldots, \theta_K) \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \cdots & \theta_K \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{K-1} & \theta_2^{K-1} & \cdots & \theta_K^{K-1} \end{bmatrix}. \tag{3.13}$$

First, two classes of optimum transforms $\mathcal{M}_K$ were proposed in [39] as follows.

i) If $K = 2^s\,(s \geq 1)$, the optimum transform $\mathcal{M}_K$ for a signal constellation $\Omega$ from $\boldsymbol{Z}[\boldsymbol{j}] \triangleq \{a + b\boldsymbol{j}$ : both $a$ and $b$ are integers, $\boldsymbol{j} = \sqrt{-1}\}$ is given by

$$\mathcal{M}_K = \frac{1}{\sqrt{K}} V(\theta_1, \theta_2, \ldots, \theta_K) \tag{3.14}$$

where $\theta_1, \theta_2, \ldots, \theta_K$ are the roots of the polynomial $\theta^K - \boldsymbol{j}$ over field $\boldsymbol{Q}[\boldsymbol{j}] \triangleq \{c + d\boldsymbol{j}$ : both $c$ and $d$ are rational numbers$\}$, and they can be determined as

$$\theta_k = e^{\boldsymbol{j}\frac{4k-3}{2K}\pi}, \qquad k = 1, 2, \ldots, K. \tag{3.15}$$

ii) If $K = 3 \cdot 2^s\,(s \geq 0)$, the optimum transform $\mathcal{M}_K$ for a signal constellation $\Omega$ from

$$\boldsymbol{Z}[\boldsymbol{\omega}] \triangleq \{a + b\boldsymbol{\omega} : \text{both } a \text{ and } b \text{ are integers}, \quad \boldsymbol{\omega} = (-1 + \boldsymbol{j}\sqrt{3})/2\}$$

is given by

$$\mathcal{M}_K = \frac{1}{\sqrt{K}} V(\theta_1, \theta_2, \ldots, \theta_K) \tag{3.16}$$

where $\theta_1, \theta_2, \ldots, \theta_K$ are the roots of the polynomial $\theta^K + \boldsymbol{\omega}$ over field $\boldsymbol{Q}[\boldsymbol{\omega}] \triangleq \{c + d\boldsymbol{\omega}$ : both $c$ and $d$ are rational numbers$\}$, and they can be specified as

$$\theta_k = e^{\boldsymbol{j}\frac{6k-1}{3K}\pi}, \qquad k = 1, 2, \ldots, K. \tag{3.17}$$

The signal constellations $\Omega$ from $\boldsymbol{Z}[\boldsymbol{j}]$ such as QAM and PAM constellations are of practical interest. In case of $K = 2^s (s \geq 1)$, the optimum transforms (3.14) and (3.15) were also described in [40]. Moreover, in [40], some transforms $\mathcal{M}_K$ (not optimum) were proposed in the case when $K$ is not a power of two. If $K = 2^s \cdot 3^t\,(s \geq 1, t \geq 1)$, a class of transforms $\mathcal{M}_K$ for signal constellations $\Omega$ from $\boldsymbol{Z}[\boldsymbol{j}]$ was given in [40] as $\mathcal{M}_K = \frac{1}{\sqrt{K}} V(\theta_1, \theta_2, \ldots, \theta_K)$, where

$$\theta_k = e^{\boldsymbol{j}\frac{6k-5}{3K}\pi}, \qquad k = 1, 2, \ldots, K. \tag{3.18}$$

Recently, in [42], some optimum transforms $\mathcal{M}_K$ were introduced for the case of $K$ not being a power of two. Specifically, if $K$ is not a power of two, but $K = \varphi(J)$ for some $J$ with $J \not\equiv 0 \pmod 4$, where $\varphi(\cdot)$ is the Euler function,[4] the optimum transform $\mathcal{M}_K$ for a signal constellation $\Omega$ from $\boldsymbol{Z}[\boldsymbol{j}]$ can be expressed as [42] $\mathcal{M}_K = \frac{1}{\sqrt{K}} V(\theta_1, \theta_2, \ldots, \theta_K)$, where

$$\theta_k = e^{\boldsymbol{j}\frac{2m_k}{J}\pi}, \qquad \gcd(m_k, J) = 1,\ 1 \leq m_k < J. \tag{3.19}$$

For example, when $K$ is $6, 10, 12, 18$, the corresponding $J$ is $7, 11, 13, 19$, respectively. In these cases, $\theta_k = e^{\boldsymbol{j}\frac{2k}{J}\pi}$, $k =$

[4] $\varphi(J)$ denotes the number of integers $m\,(1 \leq m < J)$ such that $m$ is relatively prime to $J$, i.e., $\gcd(m, J) = 1$.

$1, 2, \ldots, J-1$. In case of some odd $K$, for example $K = 5, 7, 9$, and so on, an experimental result was given in [42] as

$$\mathcal{M}_K = \frac{1}{\sqrt{\gamma}} V(\theta_1, \theta_2, \ldots, \theta_K) \qquad (3.20)$$

where $\theta_1, \theta_2, \ldots, \theta_K$ are the roots of polynomial $\theta^K - (1 + \boldsymbol{j})$ over the field $\boldsymbol{Q}[\boldsymbol{j}]$, and they can be calculated as

$$\theta_k = 2^{\frac{1}{2K}} e^{\boldsymbol{j}\frac{8k-7}{4K}\pi}, \qquad k = 1, 2, \ldots, K. \qquad (3.21)$$

In (3.20), the factor $\gamma$ is equal to $\sqrt{(2^{1/K} - 1)/K}$ for the energy normalization. Note that although the transforms given in (3.18) and (3.21) are not optimum, they do provide large minimum product distance. For more details, we refer the reader to [39], [40], [42].

The other approach to designing the signal set $\boldsymbol{X}$ is to exploit the structure of the diagonal ST block codes. Suppose that the spectral efficiency of the SF code is $r$ bits per second per hertz (bits/s/Hz). We may consider designing the set of $L_0 = 2^{rK}$ variables directly under the energy constraint $E\|\boldsymbol{X}\|_F^2 = K$. We can take advantage of the results from [8], in which diagonal ST block codes were constructed as follows:

$$C_l = \mathrm{diag}(e^{\boldsymbol{j}u_1\theta_l},\ e^{\boldsymbol{j}u_2\theta_l},\ \ldots,\ e^{\boldsymbol{j}u_K\theta_l}), \quad l = 0, 1, \ldots, L_0 - 1 \qquad (3.22)$$

where $\theta_l = \frac{l}{L_0}2\pi$, $0 \le l \le L_0 - 1$, and $u_1, u_2, \ldots, u_K \in \{0, 1, \ldots, L_0 - 1\}$. The parameters $u_1, u_2, \ldots, u_K$ need to be optimized such that the metric

$$\min_{l \ne l'} \prod_{k=1}^{K} \left| e^{\boldsymbol{j}u_k\theta_l} - e^{\boldsymbol{j}u_k\theta_{l'}} \right| = \min_{1 \le l \le L_0 - 1} \prod_{k=1}^{K} \left| 2 \sin\left(\frac{u_k l}{L_0}\pi\right) \right| \qquad (3.23)$$

is maximized. Then, we can design a set of variables $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_K]$ as follows. For any $l = 0, 1, \ldots, L_0 - 1$, let

$$x_k = e^{\boldsymbol{j}u_k\theta_l}, \qquad k = 1, 2, \ldots, K. \qquad (3.24)$$

As a consequence, the minimum product distance of the set of the resulting signal vectors $\boldsymbol{X}$ is determined by the metric in (3.23). The optimum parameters $\boldsymbol{u} = [u_1\ u_2\ \ldots\ u_K]$ can be obtained via computer search. For example [8]

$K = 4,\quad L_0 = 16,\quad \boldsymbol{u} = [1\ 3\ 5\ 7]$
$K = 4,\quad L_0 = 256,\quad \boldsymbol{u} = [1\ 25\ 97\ 107]$
$K = 6,\quad L_0 = 64,\quad \boldsymbol{u} = [1\ 9\ 15\ 17\ 23\ 25]$
$K = 6,\quad L_0 = 1024,\quad \boldsymbol{u} = [1\ 55\ 149\ 327\ 395\ 417]; \cdots.$

## IV. Maximizing the Coding Advantage by Permutations

In the previous section, we obtained a class of SF codes with full rate and full diversity assuming that the transmitter has no *a priori* knowledge about the channel. In this case, the performance of the SF codes can be improved by random interleaving, as it can reduce the correlation between adjacent subcarriers. However, if the power delay profile of the channel is available at the transmitter side, further improvement can be achieved by developing a permutation (or interleaving) method that explicitly takes the power delay profile into account. This possibility will be explored in this section.

In [36], an optimum subcarrier grouping method was proposed under the assumption that the path delays are equally spaced and fall onto the sampling instances of the receiver, i.e., $\tau_l = lT/N$ for $l = 0, 1, \ldots, L-1$. Here we will consider the optimum permutation for any arbitrary power delay profile.

### A. Diversity Product of the SF Codes With Permutations

Suppose that the path delays $\tau_0, \tau_1, \ldots, \tau_{L-1}$ and powers $\delta_0^2, \delta_1^2, \ldots, \delta_{L-1}^2$ are available at the transmitter. Our objective is to develop an optimum permutation (or interleaving) method for the SF codes defined by (3.1) and (3.2) such that the resulting coding advantage is maximized. By permuting the rows of a SF codeword $C$, we obtain an interleaved codeword $\sigma(C)$. We know that for two distinct SF codewords $C$ and $\tilde{C}$ constructed from $G_1, G_2, \ldots, G_P$ and $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_P$, respectively, there exists at least one index $p_0$ ($1 \le p_0 \le P$) such that $G_{p_0} \ne \tilde{G}_{p_0}$. In order to determine the minimum rank of $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R$, we may further assume that $G_p = \tilde{G}_p$ for any $p \ne p_0$ for the same reason as stated in the previous section.

Suppose that $G_{p_0}$ and $\tilde{G}_{p_0}$ consist of symbols $\boldsymbol{X} = [x_1\ x_2\ \ldots\ x_{\Gamma M_t}]$ and $\tilde{\boldsymbol{X}} = [\tilde{x}_1\ \tilde{x}_2\ \ldots\ \tilde{x}_{\Gamma M_t}]$, respectively, with $x_k \ne \tilde{x}_k$ for all $1 \le k \le \Gamma M_t$. For simplicity, we use the notation $\Delta x_k = x_k - \tilde{x}_k$ for $k = 1, 2, \ldots, \Gamma M_t$. After row permutation, we assume that the $k$th ($1 \le k \le \Gamma M_t$) row of $G_{p_0} - \tilde{G}_{p_0}$ is located at the $n_k$th ($0 \le n_k \le N - 1$) row of $\sigma(C - \tilde{C})$, i.e., the $k$th row of $G_{p_0}$ will be transmitted at the $n_k$th subcarrier. Then, all the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th, $1 \le i, j \le \Gamma$ and $1 \le m \le M_t$, entries of $\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}$ are nonzero, and the other entries are zero. Thus, all the entries of $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R$ are zeros except the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th entries for $1 \le i, j \le \Gamma$ and $1 \le m \le M_t$. For convenience, we define the matrices $A_m$, $m = 1, 2, \ldots, M_t$, such that the $(i, j)$th ($1 \le i, j \le \Gamma$) entry of $A_m$ is the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th entry of $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R$. Since the correlation matrix $R$ is a Toeplitz matrix (see (3.3)), the $(i, j)$th, $1 \le i, j \le \Gamma$, entry of $A_m$ can be expressed as

$$M_t \Delta x_q \Delta x_r^* \sum_{l=0}^{L-1} \delta_l^2 \omega^{(n_q - n_r)\tau_l} \qquad (4.1)$$

where $q = (m-1)\Gamma + i$ and $r = (m-1)\Gamma + j$. Note that the nonzero eigenvalues of $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R$ are determined by the matrices $A_m, m = 1, 2, \ldots, M_t$. It is shown in Appendix A that the product of the nonzero eigenvalues of $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R, \lambda_1, \lambda_2, \ldots, \lambda_{\Gamma M_t}$, can be calculated as

$$\prod_{k=1}^{\Gamma M_t} \lambda_k = \prod_{m=1}^{M_t} \left|\det(A_m)\right|. \qquad (4.2)$$

From (4.1), for each $m = 1, 2, \ldots, M_t$, the $\Gamma \times \Gamma$ matrix $A_m$ can be decomposed as follows:

$$A_m = D_m W_m \Lambda W_m^{\mathcal{H}} D_m^{\mathcal{H}} \qquad (4.3)$$

where

$$\Lambda=\mathrm{diag}(\delta_0^2,\ \delta_1^2,\ \ldots,\ \delta_{L-1}^2),$$
$$D_m=\sqrt{M_t}\,\mathrm{diag}(\Delta x_{(m-1)\Gamma+1}, \Delta x_{(m-1)\Gamma+2}, \ldots, \Delta x_{m\Gamma}),$$
$$W_m=\begin{bmatrix} w^{n_{(m-1)\Gamma+1}\tau_0} & w^{n_{(m-1)\Gamma+1}\tau_1} & \cdots & w^{n_{(m-1)\Gamma+1}\tau_{L-1}} \\ w^{n_{(m-1)\Gamma+2}\tau_0} & w^{n_{(m-1)\Gamma+2}\tau_1} & \cdots & w^{n_{(m-1)\Gamma+2}\tau_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w^{n_{m\Gamma}\tau_0} & w^{n_{m\Gamma}\tau_1} & \cdots & w^{n_{m\Gamma}\tau_{L-1}} \end{bmatrix}. \tag{4.4}$$

As a consequence, the determinant of $A_m$ is given by

$$\det(A_m) = M_t^\Gamma \prod_{i=1}^{\Gamma} \left|\Delta x_{(m-1)\Gamma+i}\right|^2 \det(W_m \Lambda W_m^{\mathcal{H}}). \tag{4.5}$$

Substituting (4.5) into (4.2), the expression for the product of the nonzero eigenvalues of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}]\circ R$ takes the form

$$\prod_{k=1}^{\Gamma M_t} \lambda_k = M_t^{\Gamma M_t} \prod_{k=1}^{\Gamma M_t} |\Delta x_k|^2 \prod_{m=1}^{M_t} \left|\det(W_m \Lambda W_m^{\mathcal{H}})\right|. \tag{4.6}$$

Therefore, the diversity product of the permuted SF code can be calculated as

$$\zeta=\frac{1}{2}\min_{\boldsymbol{X}\neq\tilde{\boldsymbol{X}}}\left(\prod_{k=1}^{\Gamma M_t}|\Delta x_k|\right)^{\frac{1}{\Gamma M_t}}\left(\prod_{m=1}^{M_t}\left|\det(W_m\Lambda W_m^{\mathcal{H}})\right|\right)^{\frac{1}{2\Gamma M_t}}$$
$$=\zeta_{\rm in}\cdot\zeta_{\rm ex} \tag{4.7}$$

where $\zeta_{\rm in}$ is the "intrinsic" diversity product defined in (3.10), and $\zeta_{\rm ex}$, the "extrinsic" diversity product, is defined by

$$\zeta_{\rm ex} = \left(\prod_{m=1}^{M_t}\left|\det(W_m\Lambda W_m^{\mathcal{H}})\right|\right)^{\frac{1}{2\Gamma M_t}}. \tag{4.8}$$

The "extrinsic" diversity product $\zeta_{\rm ex}$ depends only on the permutation and the power delay profile of the channel. The permutation does not effect the "intrinsic" diversity product $\zeta_{\rm in}$.

From (4.4), for each $m = 1, 2, \ldots, M_t$, $W_m$ can be written as

$$W_m = V_m \cdot \mathrm{diag}(w^{n_{(m-1)\Gamma+1}\tau_0}, w^{n_{(m-1)\Gamma+1}\tau_1}, \ldots, w^{n_{(m-1)\Gamma+1}\tau_{L-1}}) \tag{4.9}$$

where $V_m$ is given in (4.10) at the bottom of the page. Thus, $\det(W_m\Lambda W_m^{\mathcal{H}}) = \det(V_m\Lambda V_m^{\mathcal{H}})$. We observe that the determinant of $W_m\Lambda W_m^{\mathcal{H}}$ depends only on the relative positions of the permuted rows with respect to the position $n_{(m-1)\Gamma+1}$, not on their absolute positions.

In the following theorem, we summarize the above results, and obtain upper bounds on the "extrinsic" diversity product $\zeta_{\rm ex}$ for arbitrary permutations.

*Theorem 4.1:* For any subcarrier permutation, the diversity product of the resulting SF code is

$$\zeta = \zeta_{\rm in}\cdot\zeta_{\rm ex} \tag{4.11}$$

where $\zeta_{\rm in}$ and $\zeta_{\rm ex}$ are the "intrinsic" and "extrinsic" diversity products defined in (3.10) and (4.8), respectively. Moreover, the "extrinsic" diversity product $\zeta_{\rm ex}$ is upper-bounded as

i) $\zeta_{\rm ex} \le 1$; and more precisely,
ii) if we sort the power profile $\delta_0, \delta_1, \ldots, \delta_{L-1}$ in a nonincreasing order as $\delta_{l_1} \ge \delta_{l_2} \ge \cdots \ge \delta_{l_L}$, then

$$\zeta_{\rm ex} \le \left(\prod_{i=1}^{\Gamma}\delta_{l_i}\right)^{\frac{1}{\Gamma}} \left|\prod_{m=1}^{M_t}\det(V_m V_m^{\mathcal{H}})\right|^{\frac{1}{2\Gamma M_t}} \tag{4.12}$$

where equality holds when $\Gamma = L$. As a consequence,

$$\zeta_{\rm ex} \le \sqrt{L}\left(\prod_{i=1}^{\Gamma}\delta_{l_i}\right)^{\frac{1}{\Gamma}}. \tag{4.13}$$

The proof of Theorem 4.1 can be found in Appendix B. We observe from Theorem 4.1 ii) that the "extrinsic" diversity product $\zeta_{\rm ex}$ depends on the power delay profile in two ways. First, it depends on the power distribution through the square root of the geometric average of the largest $\Gamma$ path powers, i.e., $(\prod_{i=1}^{\Gamma}\delta_{l_i})^{1/\Gamma}$. In case of $\Gamma = L$, the best performance is expected if the power distribution is uniform (i.e., $\delta_l^2 = 1/L$) since the sum of the path powers is unity. Second, the "extrinsic" diversity product $\zeta_{\rm ex}$ also depends on the delay distribution and the applied subcarrier permutation. On the other hand, the "intrinsic" diversity product, $\zeta_{\rm in}$, is not affected by the power delay profile or the permutation method. It only depends on the signal constellation and the SF code design via the achieved minimum product distance.

### B. Maximizing the "Extrinsic" Diversity Product $\zeta_{\rm ex}$

By carefully choosing the applied permutation method, the overall performance of the SF code can be improved by increasing the value of the "extrinsic" diversity product $\zeta_{\rm ex}$. Toward this end, we consider a specific permutation strategy.

We decompose any integer $n$ $(0 \le n \le N-1)$ as

$$n = e_1\Gamma + e_0 \tag{4.14}$$

where $0 \le e_0 \le \Gamma - 1$, $e_1 = \lfloor \frac{n}{\Gamma} \rfloor$, and $\lfloor x \rfloor$ denotes the largest integer not greater than $x$. For a fixed integer $\mu\,(\mu \ge 1)$, we further decompose $e_1$ in (4.14) as

$$e_1 = v_1\mu + v_0 \tag{4.15}$$

where $0 \le v_0 \le \mu - 1$ and $v_1 = \lfloor \frac{e_1}{\mu} \rfloor$.

We permute the rows of the $N \times M_t$ SF codeword constructed from (3.1) and (3.2) in such a way that the $n$th $(0 \le n \le N-1)$ row of $C$ is moved to the $\sigma(n)$th row, where

$$\sigma(n) = v_1\mu\Gamma + e_0\mu + v_0 \tag{4.16}$$

where $e_0, v_0, v_1$ come from (4.14) and (4.15). We call the integer $\mu$ as the *separation factor*. The separation factor $\mu$ should be chosen such that $\sigma(n) \le N$ for any $0 \le n \le N-1$, or

$$V_m = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ w^{[n_{(m-1)\Gamma+2}-n_{(m-1)\Gamma+1}]\tau_0} & w^{[n_{(m-1)\Gamma+2}-n_{(m-1)\Gamma+1}]\tau_1} & \cdots & w^{[n_{(m-1)\Gamma+2}-n_{(m-1)\Gamma+1}]\tau_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w^{[n_{m\Gamma}-n_{(m-1)\Gamma+1}]\tau_0} & w^{[n_{m\Gamma}-n_{(m-1)\Gamma+1}]\tau_1} & \cdots & w^{[n_{m\Gamma}-n_{(m-1)\Gamma+1}]\tau_{L-1}} \end{bmatrix}. \tag{4.10}$$

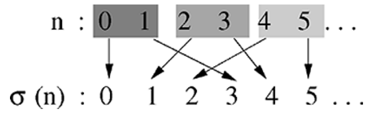



Fig. 1. An illustration of the permutation with $\Gamma = 2$ and separation factor $\mu = 3$.

equivalently, $\mu \leq \lfloor \frac{N}{\Gamma} \rfloor$. Moreover, in order to guarantee that the mapping (4.16) is one-to-one over the set $\{0, 1, \ldots, N-1\}$ (i.e., it defines a permutation), $\mu$ must be a factor of $N$. The role of the permutation specified in (4.16) is to separate two neighboring rows of $C$ by $\mu$ subcarriers. An example of this permutation method is depicted in Fig. 1.

The following result characterizes the extrinsic diversity product of the SF code that is permuted with the above described method. The proof can be found in Appendix C.

*Theorem 4.2:* For the permutation specified in (4.16) with a separation factor $\mu$, the "extrinsic" diversity product of the permuted SF code is

$$\zeta_{\text{ex}} = |\det(V_0 \Lambda V_0^{\mathcal{H}})|^{\frac{1}{2\Gamma}} \tag{4.17}$$

where

$$V_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ w^{\mu\tau_0} & w^{\mu\tau_1} & \cdots & w^{\mu\tau_{L-1}} \\ w^{2\mu\tau_0} & w^{2\mu\tau_1} & \cdots & w^{2\mu\tau_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(\Gamma-1)\mu\tau_0} & w^{(\Gamma-1)\mu\tau_1} & \cdots & w^{(\Gamma-1)\mu\tau_{L-1}} \end{bmatrix}_{\Gamma \times L}. \tag{4.18}$$

Moreover, if $\Gamma = L$, the "extrinsic" diversity product $\zeta_{\text{ex}}$ can be calculated as

$$\zeta_{\text{ex}} = \left(\prod_{l=0}^{L-1} \delta_l\right)^{\frac{1}{L}} \left(\prod_{0 \leq l_1 < l_2 \leq L-1} \left| 2\sin\left(\frac{\mu(\tau_{l_2} - \tau_{l_1})\pi}{T}\right)\right|\right)^{\frac{1}{L}}. \tag{4.19}$$

The permutation (4.16) is determined by the separation factor $\mu$. Our objective is to find a separation factor $\mu_{\text{op}}$ that maximizes the "extrinsic" diversity product $\zeta_{\text{ex}}$

$$\mu_{\text{op}} = \arg \max_{1 \leq \mu \leq \lfloor N/\Gamma \rfloor} |\det(V_0 \Lambda V_0^{\mathcal{H}})|. \tag{4.20}$$

If $\Gamma = L$, the optimum separation factor $\mu_{\text{op}}$ can be expressed as

$$\mu_{\text{op}} = \arg \max_{1 \leq \mu \leq \lfloor N/\Gamma \rfloor} \prod_{0 \leq l_1 < l_2 \leq L-1} \left| \sin\left(\frac{\mu(\tau_{l_2} - \tau_{l_1})\pi}{T}\right)\right| \tag{4.21}$$

which is independent of the path powers. The optimum separation factor can be easily found via low-complexity computer search. However, in some cases, closed-form solutions can also be obtained.

- If $\Gamma = L = 2$, the "extrinsic" diversity product $\zeta_{\text{ex}}$ is

$$\zeta_{\text{ex}} = \sqrt{\delta_0 \delta_1} \left| 2\sin\left(\frac{\mu(\tau_1 - \tau_0)\pi}{T}\right)\right|^{\frac{1}{2}}. \tag{4.22}$$

TABLE I
TYPICAL URBAN (TU) SIX-RAY POWER DELAY PROFILE

| Delay profile ($\mu s$) | 0.0 | 0.2 | 0.5 | 1.6 | 2.3 | 5.0 |
|---|---|---|---|---|---|---|
| Power profile | 0.189 | 0.379 | 0.239 | 0.095 | 0.061 | 0.037 |

TABLE II
HILLY TERRAIN (HT) SIX-RAY POWER DELAY PROFILE

| Delay profile ($\mu s$) | 0.0 | 0.1 | 0.3 | 0.5 | 15.0 | 17.2 |
|---|---|---|---|---|---|---|
| Power profile | 0.413 | 0.293 | 0.145 | 0.074 | 0.066 | 0.008 |

Suppose that the system has $N = 128$ subcarriers, and the total bandwidth is $BW = 1$ MHz. Then, the OFDM block duration is $T = 128$ $\mu$s without the cyclic prefix. If $\tau_1 - \tau_0 = 5$ $\mu$s, then $\mu_{\text{op}} = 64$ and $\zeta_{\text{ex}} = \sqrt{2\delta_0\delta_1}$. If $\tau_1 - \tau_0 = 20$ $\mu$s, then $\mu_{\text{op}} = 16$ and $\zeta_{\text{ex}} = \sqrt{2\delta_0\delta_1}$. In general, if $\tau_1 - \tau_0 = 2^a b$ microseconds, where $a$ is an nonnegative integer and $b$ is an odd integer, $\mu_{\text{op}} = 128/2^{a+1}$. In all of these cases, the "extrinsic" diversity product is $\zeta_{\text{ex}} = \sqrt{2\delta_0\delta_1}$, which achieves the upper bound (4.13) of Theorem 4.1.

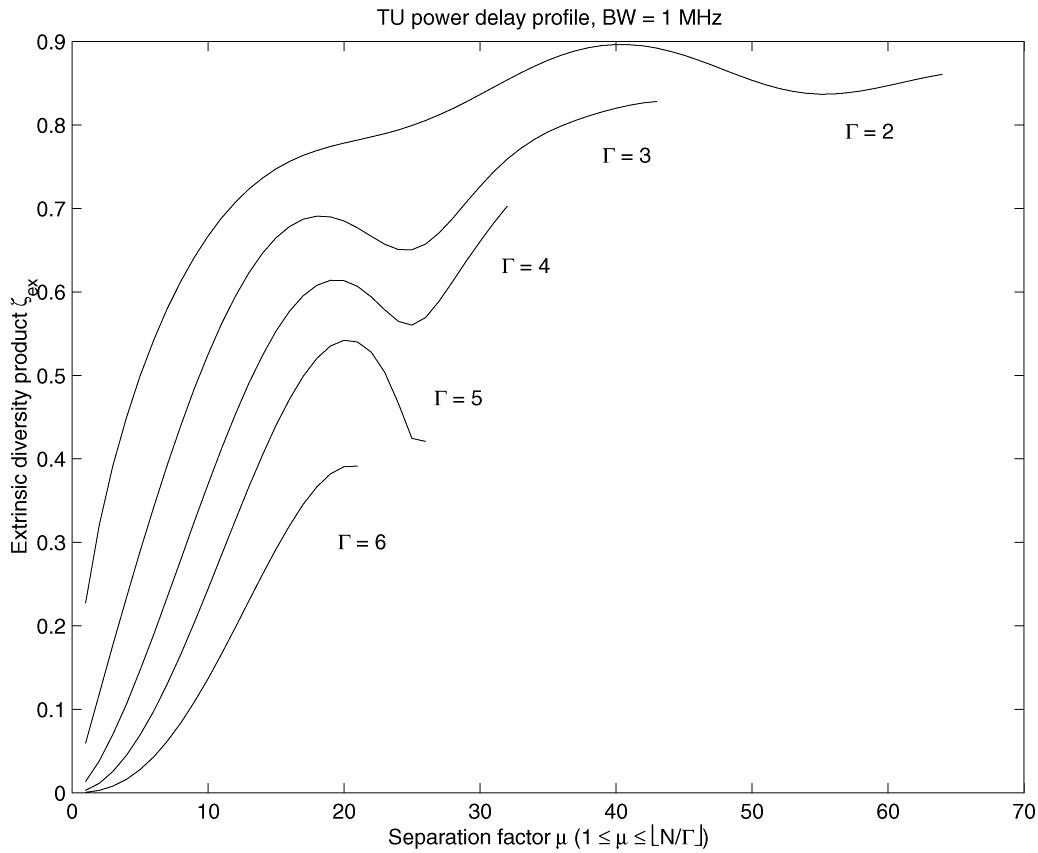
- Assume that $\tau_l - \tau_0 = lN_0T/N$, $l = 1, 2, \ldots, L-1$, and $N$ is an integer multiple of $LN_0$, where $N_0$ is a constant and not necessarily an integer. If $\Gamma = L$ or $\delta_0^2 = \delta_1^2 = \cdots = \delta_{L-1}^2 = 1/L$, the optimum separation factor is

$$\mu_{\text{op}} = \frac{N}{LN_0} \tag{4.23}$$

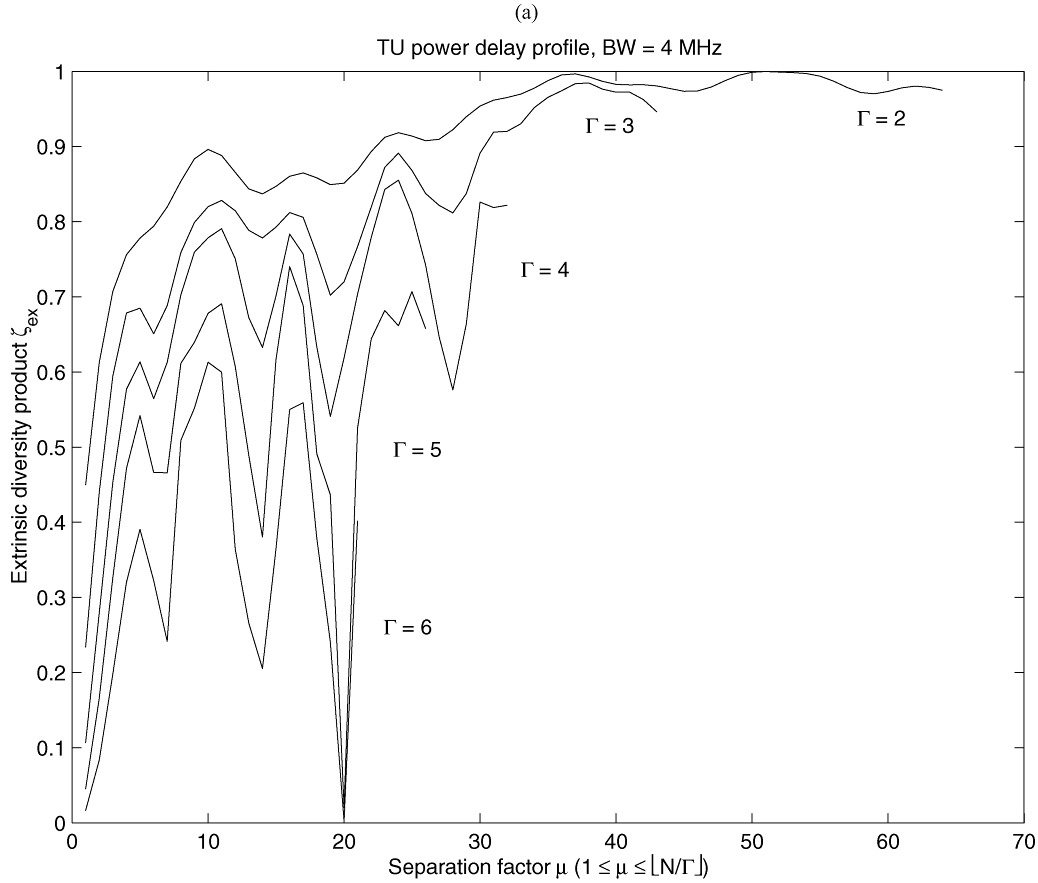
and the corresponding "extrinsic" diversity product is $\zeta_{\text{ex}} = \sqrt{L}(\prod_{l=0}^{L-1} \delta_l)^{1/L}$ (see Appendix D for the proof). In particular, in case of $\delta_0^2 = \delta_1^2 = \cdots = \delta_{L-1}^2 = 1/L$, $\zeta_{\text{ex}} = 1$. In both cases, the "extrinsic" diversity products achieve the upper bounds of Theorem 4.1. Note that if $\tau_l = lT/N$ for $l = 0, 1, \ldots, L-1$, $\Gamma = L$, and $N$ is an integer multiple of $L$, the permutation with the optimum separation factor $\mu_{\text{op}} = N/L$ is similar to the optimum subcarrier grouping method proposed in [36], which is not optimal for arbitrary power delay profiles.

We now determine the optimum separation factors for two commonly used multipath fading models. The COST 207 six-ray power delay profiles for *typical urban* (TU) and *hilly terrain* (HT) environment [48] are described in Tables I and II, respectively. We consider two different bandwidths: a) BW $=$ 1 MHz, and b) BW $=$ 4 MHz. Suppose that the OFDM has $N = 128$ subcarriers. The plots of the "extrinsic" diversity product $\zeta_{\text{ex}}$ as the function of the separation factor $\mu$ for the TU and HT channel models are shown in Figs. 2 and 3, respectively. In each figure, the curves of the "extrinsic" diversity product are depicted for different $\Gamma\,(2 \leq \Gamma \leq L)$ values. Note that for a fixed $\Gamma, \Gamma = 2, 3, \ldots, L$, the separation factor $\mu$ cannot be greater than $\lfloor N/\Gamma \rfloor$.

Let us focus on the $\Gamma = 2$ case. For the TU channel model with BW $=$ 1 MHz (Fig. 2(a)), the maximum "extrinsic" diversity product is $\zeta_{\text{ex}} = 0.8963$. The corresponding separation factor is $\mu_{\text{op}} = 40$. However, to ensure one-to-one mapping,




(a)




(b)

Fig. 2. Extrinsic diversity product $\zeta_{\rm ex}$ versus separation factor $\mu$ for different $\Gamma$ $(2 \leq \Gamma \leq 6)$, TU channel model. (a) BW $=$ 1 MHz. (b) BW $=$ 4 MHz.

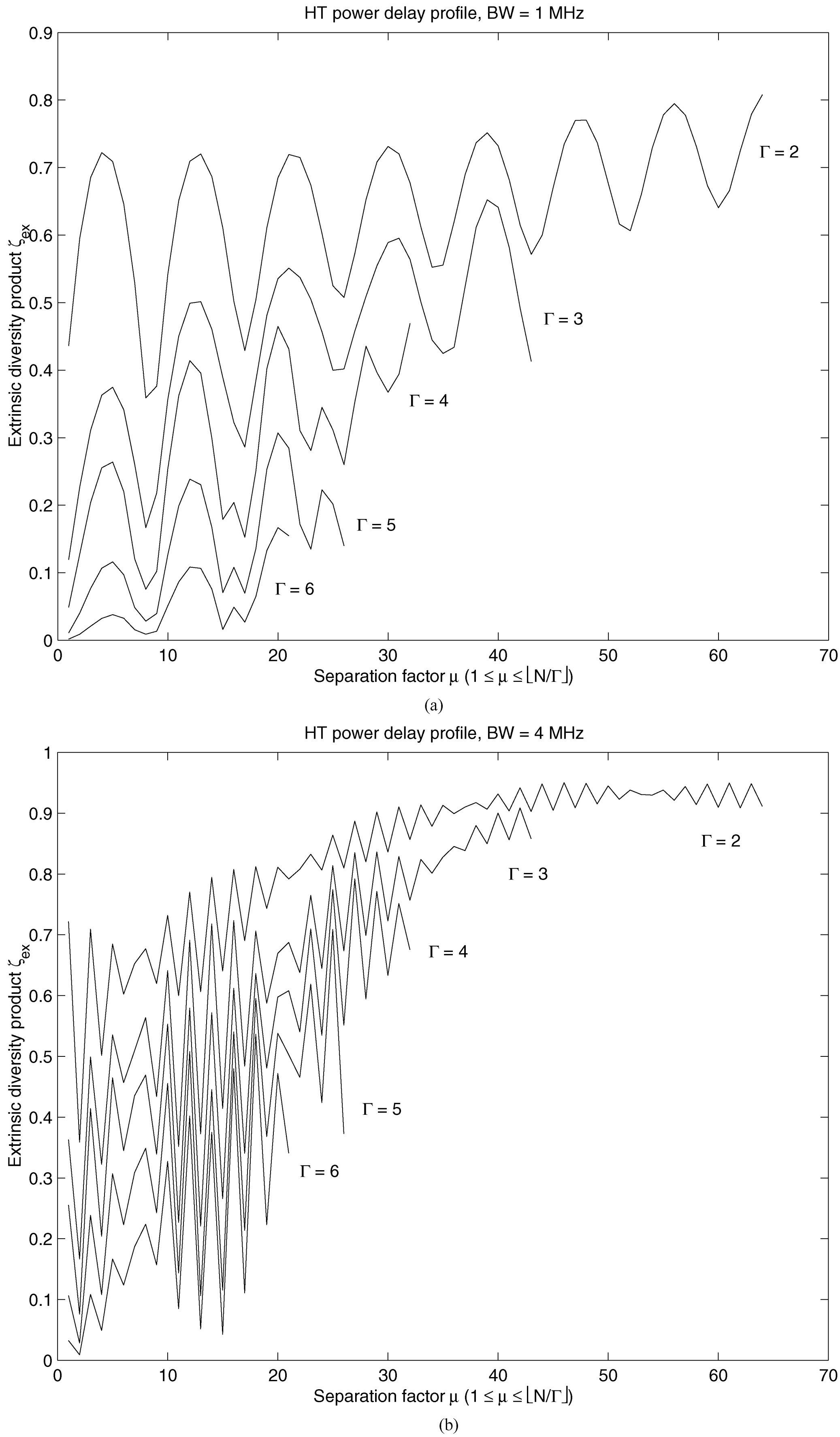



Fig. 3. Extrinsic diversity product $\zeta_{\rm ex}$ versus separation factor $\mu$ for different $\Gamma$ $(2 \leq \Gamma \leq 6)$, HT channel model. (a) BW $= 1$ MHz. (b) BW $= 4$ MHz.

we choose $\mu = 64$, which results in an "extrinsic" diversity product $\zeta_{\rm ex} = 0.8606$. For the TU channel model with BW = 4 MHz (Fig. 2(b)), the maximum "extrinsic" diversity product is $\zeta_{\rm ex} = 0.9998$, which approaches the upper bound 1 stated in Theorem 4.1. The corresponding separation factor is $\mu_{\rm op} = 51$. Similarly, we choose $\mu = 64$ to generate a permutation. The resulting "extrinsic" diversity product is $\zeta_{\rm ex} = 0.9751$, which is a slight performance loss compared to the maximum value $\zeta_{\rm ex} = 0.9998$. Finally, in case of the six-ray HT channel model with BW = 1 MHz (Fig. 3(a)), the maximum "extrinsic" diversity product is $\zeta_{\rm ex} = 0.8078$. The corresponding separation factor is $\mu_{\rm op} = 64$, which is desirable. For the HT channel model with BW = 4 MHz (Fig. 3(b)), the maximum "extrinsic" diversity product is $\zeta_{\rm ex} = 0.9505$, and the corresponding separation factor is $\mu_{\rm op} = 46$. To ensure one-to-one mapping, we choose $\mu = 64$, which results in an "extrinsic" diversity product $\zeta_{\rm ex} = 0.9114$.

## V. Simulation Results

To illustrate the preceding analytical results, we performed some computer simulations. The MIMO-OFDM system had $M_t = 2$ transmit antennas, $M_r = 1$ receive antenna, and $N = 128$ subcarriers. The simulated full-rate full-diversity SF codes were constructed according to (3.1) and (3.2) with $\Gamma = 2$, yielding the code block structure

$$G = \sqrt{2}\begin{bmatrix} x_1 & 0 \\ x_2 & 0 \\ 0 & x_3 \\ 0 & x_4 \end{bmatrix}. \tag{5.1}$$

The symbols $x_1, x_2, x_3, x_4$ were obtained as

$$[x_1\, x_2\, x_3\, x_4] = [s_1\, s_2\, s_3\, s_4] \cdot \frac{1}{2} V(\theta, -\theta, \boldsymbol{j}\theta, -\boldsymbol{j}\theta) \tag{5.2}$$

where $s_1, s_2, s_3, s_4$ were chosen from a binary phase-shift keying (BPSK) constellation ($s_i \in \{1, -1\}$) or a quaternary phase-shift keyong (QPSK) constellation ($s_i \in \{\pm 1, \pm \boldsymbol{j}\}$), $V(\cdot)$ is the Vandermonde matrix defined in (3.13), and $\theta = e^{\boldsymbol{j}\pi/8}$. This code targets a frequency diversity order of $\Gamma = 2$, thus, it achieves full diversity only if the number of delay paths is $L \leq 2$.

We simulated the proposed SF codes with three permutation schemes: no permutation, random permutation, and the proposed optimum permutation. The random permutation was generated by the Takeshita–Constello method [43], which is given by

$$\sigma(n) = \left(\frac{n(n+1)}{2}\right) \bmod N, \quad n = 0, 1, \ldots, N-1. \tag{5.3}$$

We present average bit-error rate (BER) curves as functions of the average SNR. In all simulation results, the curves with squares ("□"), pluses ("+"), and stars ("$*$") show the performance of the proposed full-rate full-diversity SF codes without permutation, with the random permutation (5.3) and with the proposed optimum permutation, respectively.

### A. *Code Performance With Different Permutation Schemes*

The first set of experiments were conducted to compare the performance of the proposed full-rate full-diversity SF codes using different permutation schemes. We simulated the proposed code (5.1) with the channel symbols $s_1, s_2, s_3, s_4$ chosen from BPSK constellation. The symbol rate of this code is 1, and its spectral efficiency is 1 bit/s/Hz, ignoring the cyclic prefix.

First, we assumed a simple two-ray, equal-power delay profile, with a delay $\tau$ microseconds between the two rays. We simulated two cases: a) $\tau = 5$ $\mu$s, and b) $\tau = 20$ $\mu$s with OFDM bandwidth BW = 1 MHz. From the BER curves, shown in Fig. 4(a) and (b), we observe that the performance of the proposed SF code with the random permutation is better than that without permutation. In case of $\tau = 5$ $\mu$s, the performance improvement is more significant. With the optimum permutation, the performance is further improved. In case of $\tau = 5$ $\mu$s, there is a 3-dB gain between the optimum permutation ($\mu_{\rm op} = 64$) and the random permutation at a BER of $10^{-5}$. In case of $\tau = 20$ $\mu$s, the performance improvement of the optimum permutation ($\mu_{\rm op} = 16$) over the random permutation is about 2 dB at a BER of $10^{-5}$. If no permutation is used, the performance of the code in the $\tau = 5$ $\mu$s case ($\zeta_{\rm ex} = 0.3499$) is worse than that in the $\tau = 20$ $\mu$s case ($\zeta_{\rm ex} = 0.6866$). However, if we apply the proposed optimum permutation, the performance of the SF code in both the $\tau = 5$ $\mu$s case ($\mu_{\rm op} = 64, \zeta_{\rm ex} = 1$) and the $\tau = 20$ $\mu$s case ($\mu_{\rm op} = 16, \zeta_{\rm ex} = 1$) is approximately the same. This confirms that by careful interleaver design, the performance of the SF codes can be significantly improved. Moreover, the consistency between the theoretical diversity product values and the simulation results suggest that the "extrinsic" diversity product $\zeta_{\rm ex}$ is a good indicator of the code performance.

We also simulated the code (5.1) with the TU and HT channel models. We considered two situations: a) BW = 1 MHz, and b) BW = 4 MHz. Fig. 5 provides the performance results of the code with different permutations for the TU channel model. From Fig. 5(a) and (b), we observe that in both cases, the code with random permutation has a significant improvement over the nonpermuted code. Using the proposed permutation with a separation factor $\mu = 64$, there is an additional gain of 1.5 and 1 dB at a BER of $10^{-5}$ in case of BW = 1 MHz and BW = 4 MHz, respectively.

Fig. 6 depicts the simulation results for the HT channel model. We can see that in the BW = 1 MHz case, the performance gain of the permuted codes is larger than in the BW = 4 MHz case. In both cases, the code with the proposed permutation ($\mu = 64$) outperforms the code with the random permutation. There is a 1.5-dB performance improvement at a BER of $10^{-5}$ in case of BW = 1 MHz, and a 1-dB improvement in case of BW = 4 MHz.

### B. *Comparison With Existing SF Codes*

We also compared the performance of the proposed full-rate full-diversity SF codes with that of the full-diversity SF codes described in [29]. We simulated the proposed code (5.1) with symbols $s_1, s_2, s_3, s_4$ chosen from a QPSK constellation. The symbol rate of the code is 1, and the spectral efficiency is 2 bits/s/Hz, ignoring the cyclic prefix. The full-diversity SF

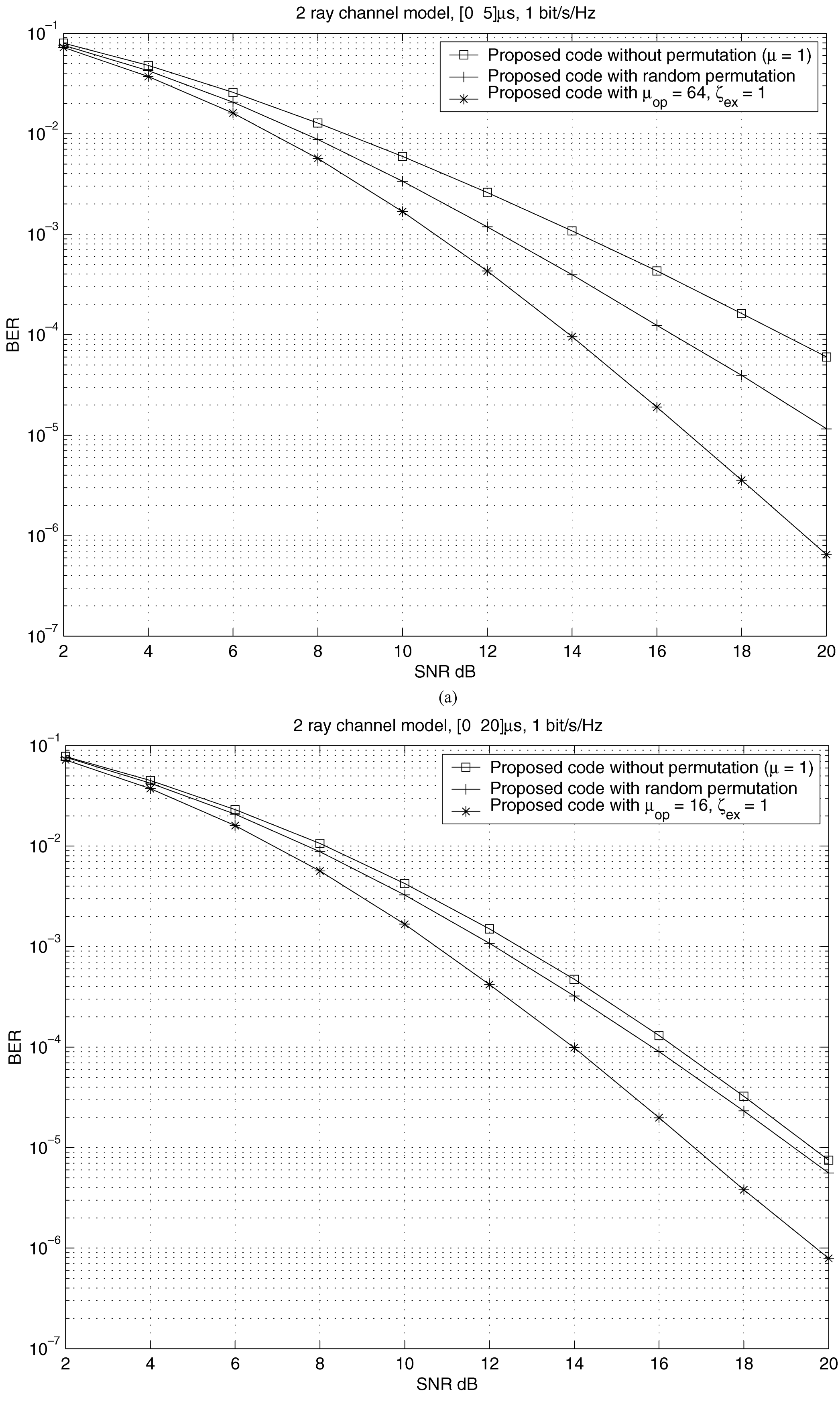
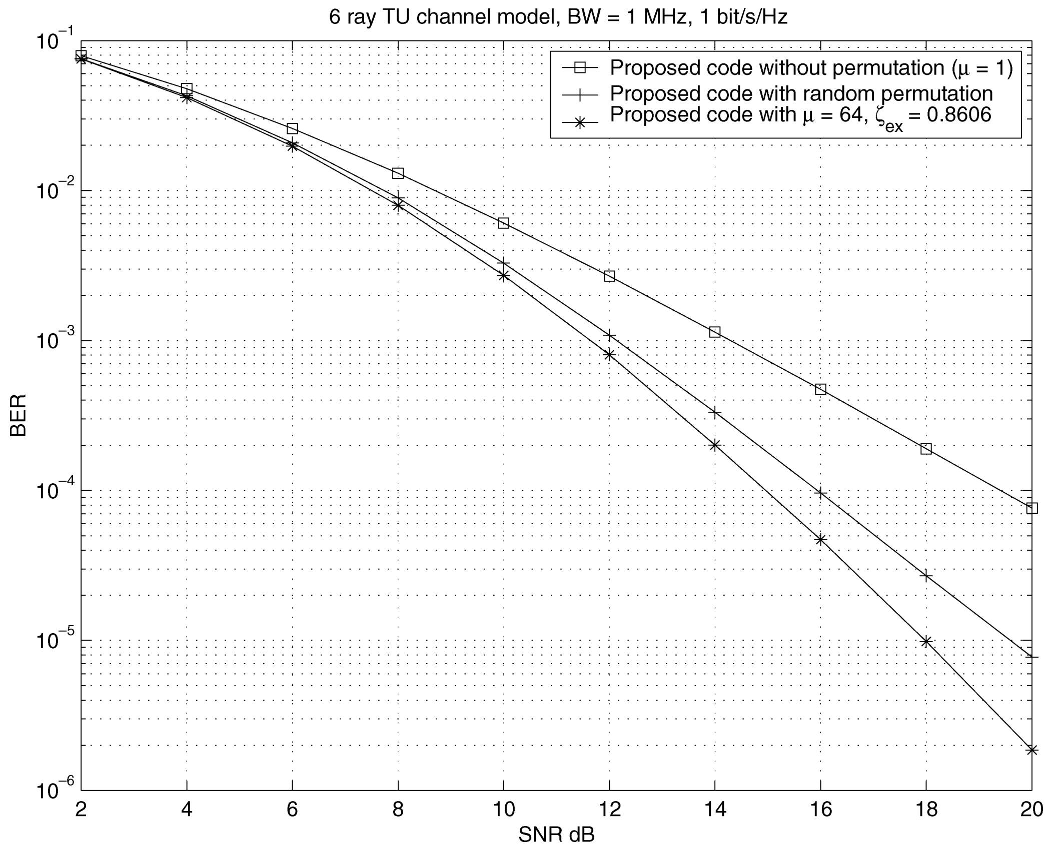
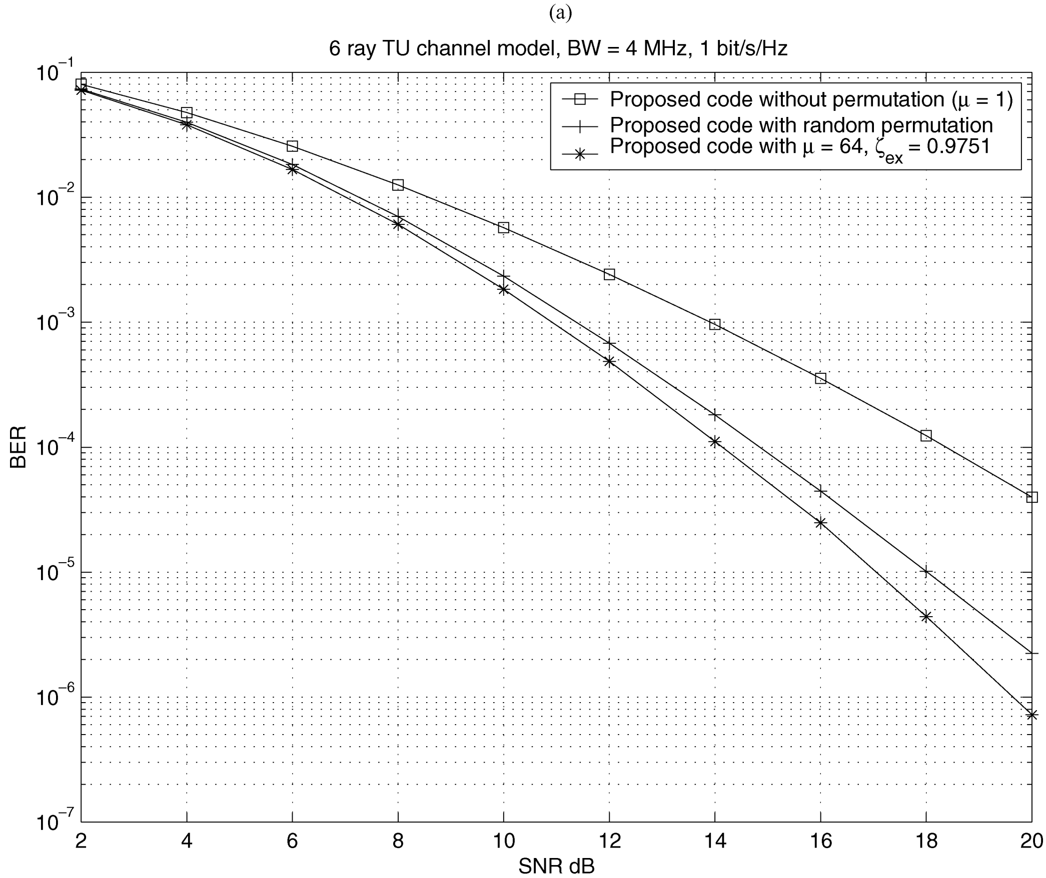



Fig. 4. Performance of the proposed SF code with different permutations, two-ray channel model. (a) Two rays at 0 and 5 $\mu$s. (b) Two rays at 0 and 20 $\mu$s.

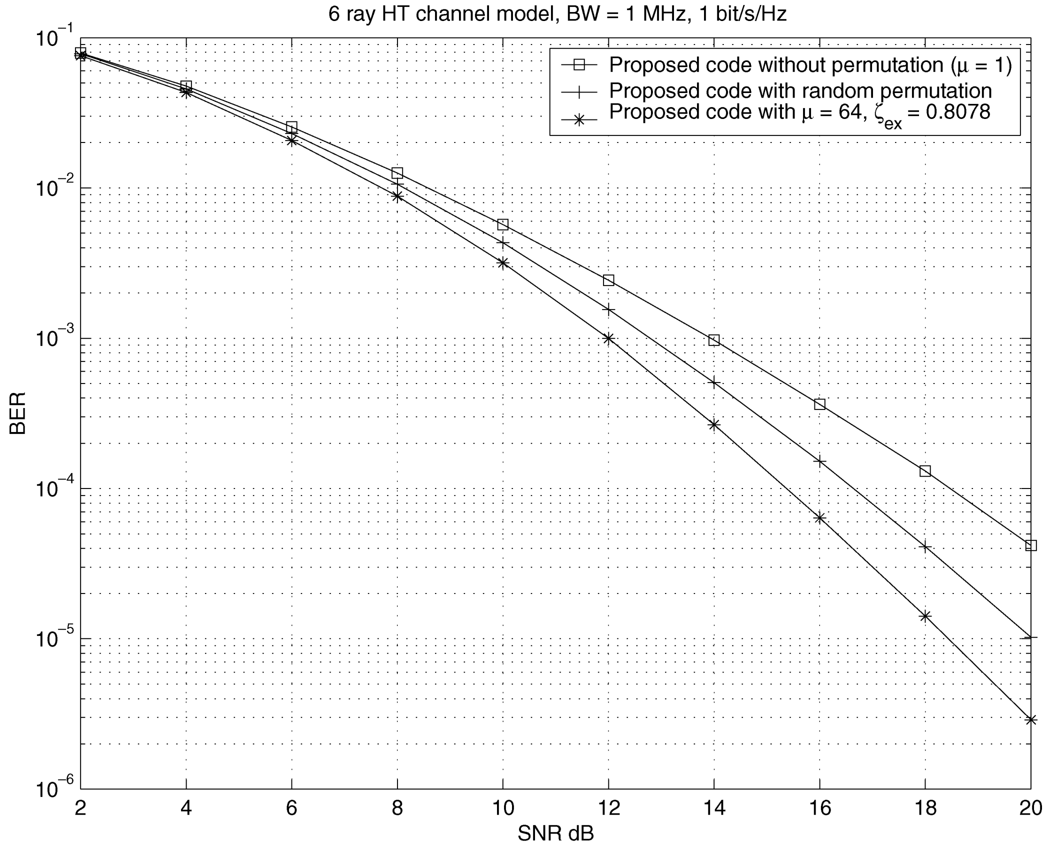



(a)

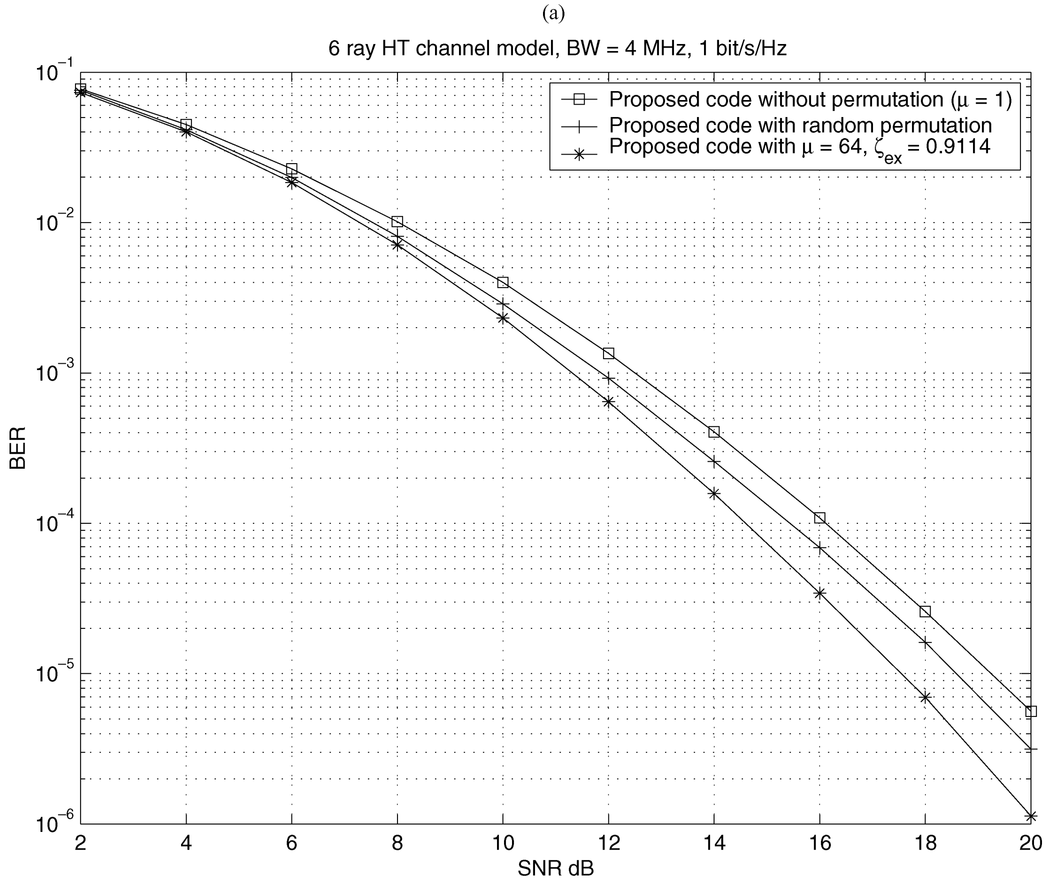



(b)

Fig. 5. Performance of the proposed SF code with different permutations, TU channel model. (a) BW $=$ 1 MHz. (b) BW $=$ 4 MHz.




(a)




(b)

Fig. 6. Performance of the proposed SF code with different permutations, HT channel model. (a) BW = 1 MHz. (b) BW = 4 MHz.

code [29] is a repetition of the Alamouti scheme [3] two times as follows:

$$G = \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \\ -x_2^* & x_1^* \\ -x_2^* & x_1^* \end{bmatrix} \tag{5.4}$$

where the channel symbols $x_1$ and $x_2$ were chosen from 16-QAM in order to maintain the same spectral efficiency. In all figures, the curves with diamonds ("$\diamond$") and circles ("$\circ$") show the performance of the SF code from orthogonal design (5.4) without permutation and with the random permutation (5.3), respectively.

First, we used the two-ray, equal-power profile, with a) $\tau = 5$ $\mu$s, and b) $\tau = 20$ $\mu$s. The total bandwidth was BW $= 1$ MHz. From the BER curve of the $\tau = 5$ $\mu$s case, depicted in Fig. 7(a), we observe that without permutation, the proposed SF code outperforms the SF code from orthogonal design by about 3 dB at a BER of $10^{-4}$. With the random permutation (5.3), the proposed code outperforms the code from orthogonal design by about 2 dB at a BER of $10^{-4}$. With the optimum permutation ($\mu_{\rm op} = 64$), the proposed code has an additional gain of 3 dB at a BER of $10^{-4}$. Compared to the code from orthogonal design with the random permutation, the proposed code with the optimum permutation has a total gain of 5 dB at a BER of $10^{-4}$. Fig. 7(b) shows the performance of the SF codes in the $\tau = 20$ $\mu$s case. It can be seen that without permutation, the proposed code outperforms the code (5.4) by about 2 dB at a BER of $10^{-4}$. With the random permutation (5.3), the performance of the proposed code is better than that of the code (5.4) by about 2 dB at a BER of $10^{-4}$. With the optimum permutation ($\mu_{\rm op} = 16$), an additional improvement of 2 dB at a BER of $10^{-4}$ is achieved by the proposed code.

We also simulated the two SF codes using the TU and HT channel models. We considered two situations: a) BW$=1$ MHz, and b) BW$=4$ MHz. Fig. 8 depicts the simulation results for the TU channel model. In case of BW$=1$ MHz, from Fig. 8(a), we can see that without permutation, the proposed SF code outperforms the SF code (5.4) by about 2 dB at a BER of $10^{-4}$. With the random permutation (5.3), the performance of the proposed code is better than that of the code from orthogonal design by about 2.5 dB at a BER of $10^{-4}$. With the proposed permutation ($\mu = 64$), an additional improvement of 1 dB at a BER of $10^{-4}$ is achieved by the proposed SF code. In case of BW$=4$ MHz, from Fig. 8(b), we observe that without permutation, the performance of the proposed code is better than that of the code from orthogonal design by about 3 dB at a BER of $10^{-4}$. With the random permutation, the proposed SF code outperforms the SF code (5.4) by about 2 dB at a BER of $10^{-4}$. With the proposed permutation ($\mu$=64), there is an additional gain of about 1 dB at a BER of $10^{-4}$. Compared to the SF code from orthogonal design with the random permutation, the proposed SF code with the proposed permutation has a total gain of 3 dB at a BER of $10^{-4}$.

Fig. 9 provides the simulation results for the HT channel model. In case of BW $= 1$ MHz, from Fig. 9(a), we observe that without permutation, the proposed SF code outperforms the SF code (5.4) by about 3 dB at a BER of $10^{-4}$. With the random permutation (5.3), the performance of the proposed code is better than that of the code (5.4) by about 2 dB at a BER of $10^{-4}$. With the proposed permutation ($\mu = 64$), an additional improvement of more than 1 dB is observed for the proposed SF code at a BER of $10^{-4}$. In case of BW $= 4$ MHz, Fig. 9(b) shows that without permutation, the performance of the proposed code is better than that of the code (5.4) by about 1.5 dB at a BER of $10^{-4}$. With the random permutation, the proposed SF code outperforms the SF code from orthogonal design by about 2 dB at a BER of $10^{-4}$. With the proposed permutation ($\mu = 64$), there is an additional gain of about 1 dB at a BER of $10^{-4}$. Compared to the SF code from orthogonal design with the random permutation, the proposed SF code with the proposed permutation has a total gain of 3 dB at a BER of $10^{-4}$.

## VI. Conclusion

In this paper, we proposed a general SF code structure that can guarantee full-rate and full-diversity transmission in MIMO-OFDM systems for an arbitrary number of transmit antennas, any memoryless modulation method and arbitrary power delay profiles. In addition, assuming that the power delay profile of the channel is available at the transmitter, we proposed an optimum interleaving scheme to further improve the performance. Based on the theoretical diversity product values and the simulation results, we can draw the following conclusions.

First, the proposed SF codes offer considerable performance improvement over previously existing approaches. We observed 3–5-dB gain over the full-diversity SF codes constructed from orthogonal design. Second, the applied interleaving method can have a significant effect on the overall performance of the SF code. Compared to the random permutation, the proposed optimum permutation resulted in 1–3-dB performance improvement. Finally, the maximum-likelihood decoding complexity of the proposed scheme increases exponentially with the number of transmit antennas and the targeted frequency diversity order, but sphere decoding methods [44]–[46] can be used to reduce the complexity.

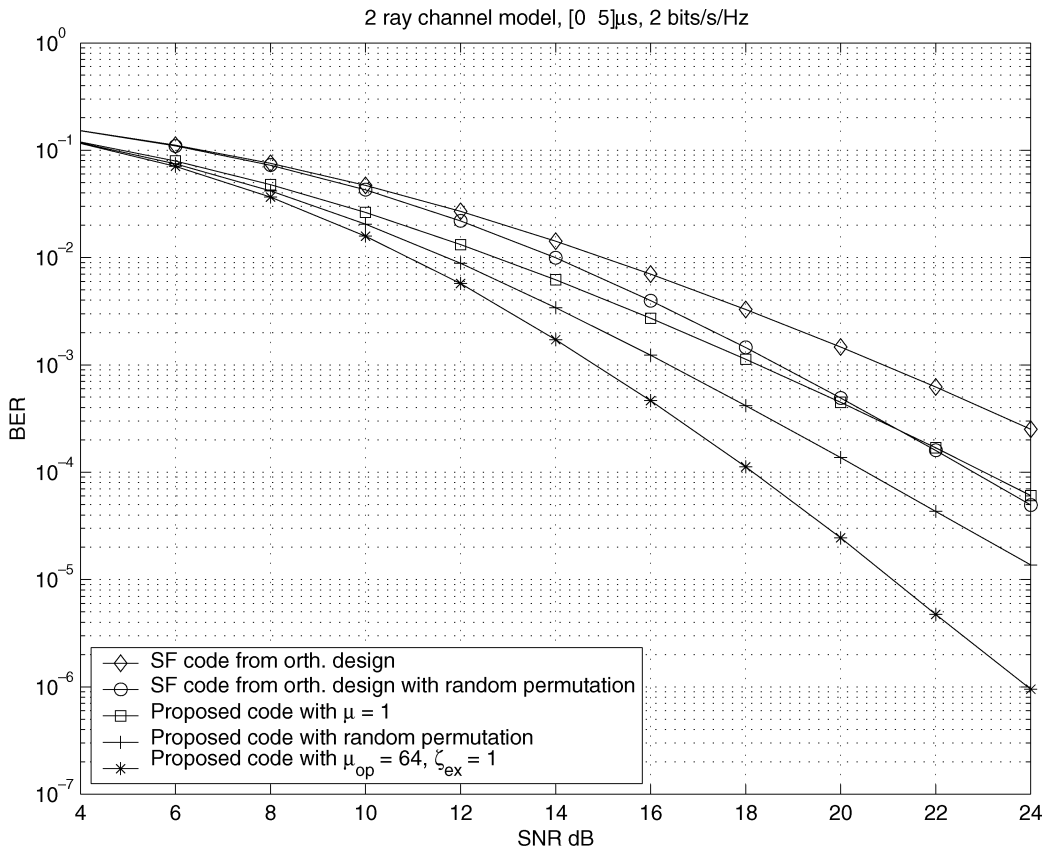
## Appendix A
## Proof of Equation (4.2)

By applying some row and column permutations to $[\sigma(C - \tilde{C})\sigma(C - \tilde{C})^{\mathcal{H}}] \circ R$, we move the $n_{(m-1)\Gamma+i}$th row and column to the $((m-1)\Gamma + i)$th row and column, respectively, for $1 \leq i \leq \Gamma$ and $1 \leq m \leq M_t$. The resulting matrix is denoted as $\underline{\Delta \circ R}$.

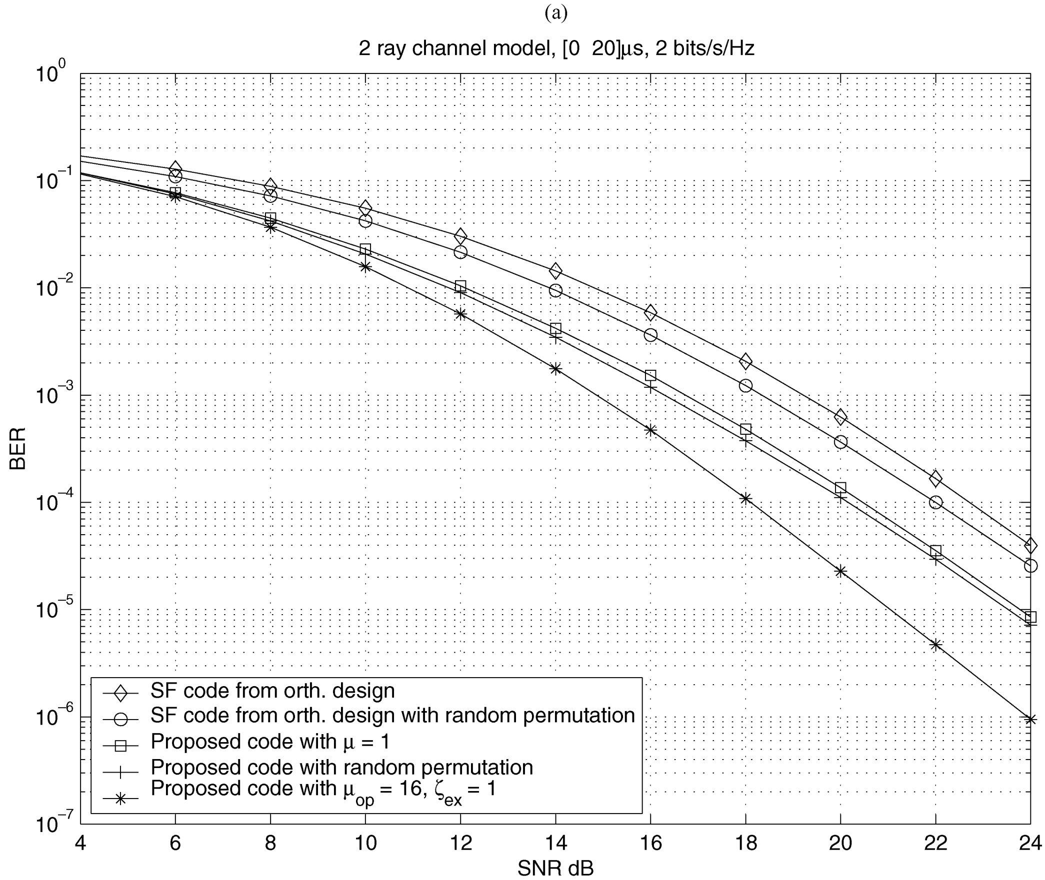
First, we show that $\underline{\Delta \circ R}$ is a block-diagonal matrix given by

$$\underline{\Delta \circ R} = \mathrm{diag}(A_1, A_2, \ldots, A_{M_t}, \mathbf{0}_{(N-\Gamma M_t)\times(N-\Gamma M_t)}). \tag{A1}$$

We observe that after the row and column permutations, the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th entry of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}] \circ R$ is the $((m-1)\Gamma + i, (m-1)\Gamma + j)$th entry of $\underline{\Delta \circ R}$ for $1 \leq i,j \leq \Gamma$, and $1 \leq m \leq M_t$. On the other hand, we recall that all the entries of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}] \circ R$ are zeros except the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th entry for $1 \leq i,j \leq \Gamma$ and $1 \leq m \leq M_t$. Thus, all the entries of $\underline{\Delta \circ R}$ are zeros except the $((m-1)\Gamma + i, (m-1)\Gamma + j)$th entry for $1 \leq i,j \leq \Gamma$ and




(a)




(b)

Fig. 7. Comparison of the proposed SF code and the code from orthogonal design, two-ray channel model. (a) Two rays at 0 and 5 $\mu$s. (b) Two rays at 0 and 20 $\mu$s.

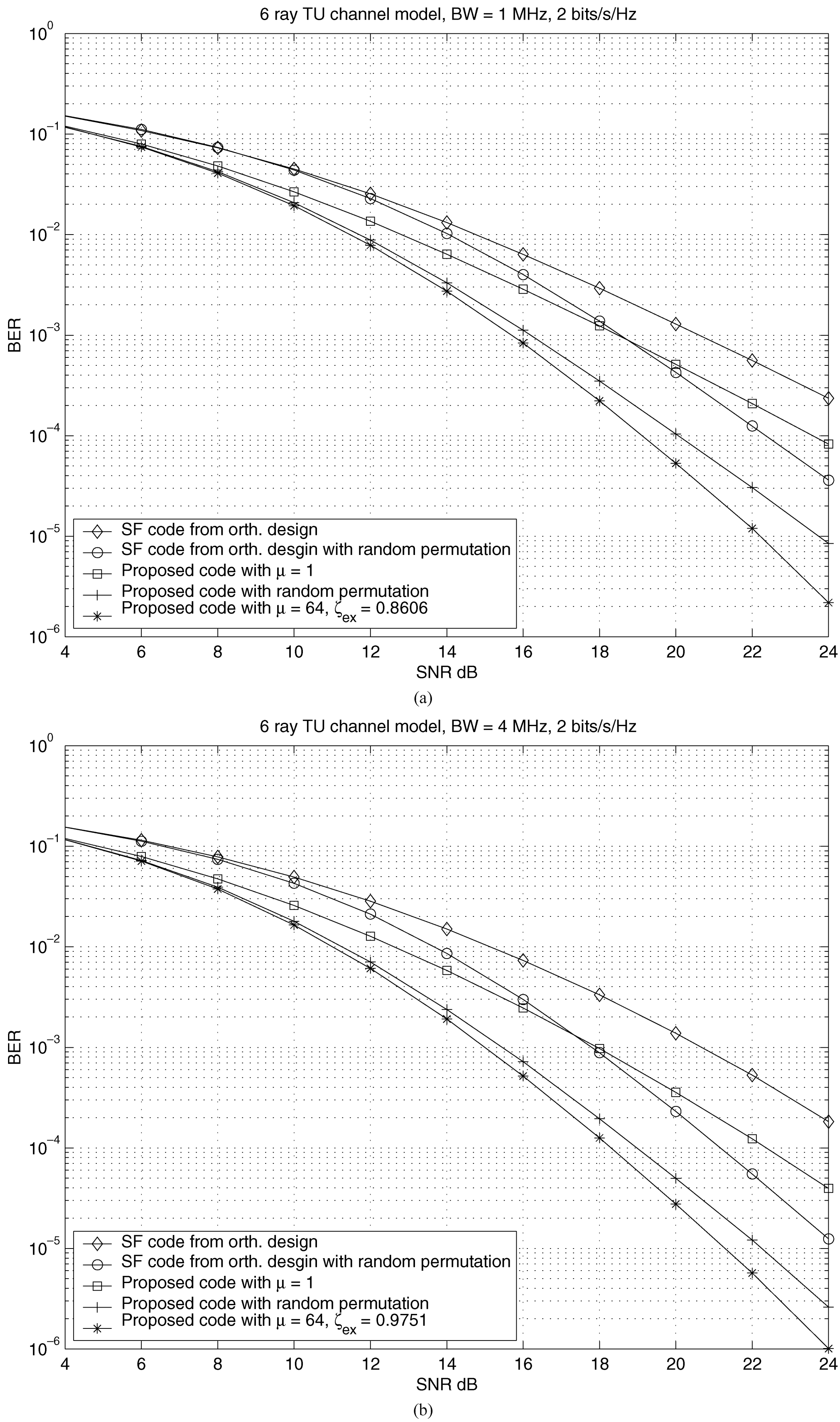



Fig. 8. Comparison of the proposed SF code and the code from orthogonal design, TU channel model. (a) BW = 1 MHz. (b) BW = 4 MHz.

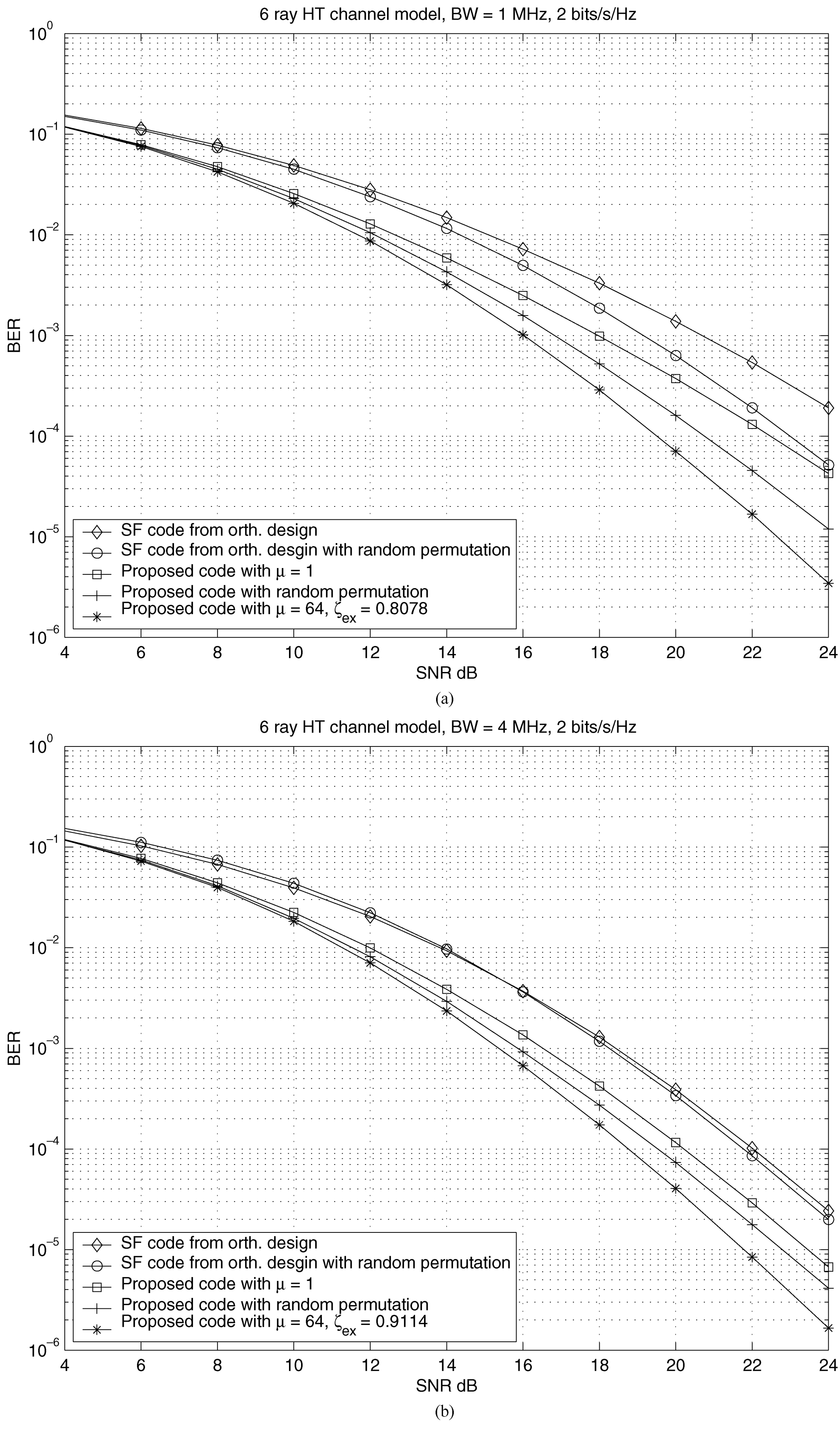



Fig. 9. Comparison of the proposed SF code and the code from orthogonal design, HT channel model. (a) BW = 1 MHz. (b) BW = 4 MHz.

$1 \le m \le M_t$. For a fixed $m\,(1 \le m \le M_t)$, since the $(i,j)$th entry of $A_m$ is the same as the $(n_{(m-1)\Gamma+i}, n_{(m-1)\Gamma+j})$th entry of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}] \circ R$, the $((m-1)\Gamma+i, (m-1)\Gamma+j)$th entry of $\underline{\Delta \circ R}$ is the same as the $(i,j)$th entry of $A_m$. Therefore, we have the expression of $\underline{\Delta \circ R}$ in (A1).

Since row and column permutations are unitary operations, $\lambda_1, \lambda_2, \ldots, \lambda_{\Gamma M_t}$, the nonzero eigenvalues of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}] \circ R$ are the same as those of $\underline{\Delta \circ R}$. Thus, according to (A1), the product of the nonzero eigenvalues of $[\sigma(C-\tilde{C})\sigma(C-\tilde{C})^{\mathcal{H}}] \circ R$ is

$$\prod_{k=1}^{\Gamma M_t} \lambda_k = \prod_{k=1}^{M_t} |\det(A_m)|$$

which is the result in (4.2). □

## Appendix B
## Proof of Theorem 4.1

For each $m = 1, 2, \ldots, M_t$, $W_m \Lambda W_m^{\mathcal{H}}$ is nonnegative definite, and all of its diagonal entries are $\sum_{l=0}^{L-1} \delta_l^2 = 1$. Thus, according to Hadamard's inequality ([49, p. 477]), the determinant of $W_m \Lambda W_m^{\mathcal{H}}$ is less than or equal to 1, so from (4.8), we have $\zeta_{\text{ex}} \le 1$, which is result i) of Theorem 4.1.

In order to prove the result in Theorem 4.1 ii), we define some additional notation. We denote the eigenvalues of an $n \times n$ nonnegative-definite matrix $A$ (in nondecreasing order) by

$$\text{eig}_1(A) \ge \text{eig}_2(A) \ge \cdots \ge \text{eig}_n(A).$$

Similarly, the singular values of an $m \times n$ matrix $B$ are denoted by

$$\text{sing}_1(B) \ge \text{sing}_2(B) \ge \cdots \ge \text{sing}_{\min(m,n)}(B).$$

For $m = 1, 2, \ldots, M_t$, we have

$$\begin{aligned} |\det(V_m \Lambda V_m^{\mathcal{H}})| &= \prod_{i=1}^{\Gamma} \text{sing}_i(V_m \Lambda V_m^{\mathcal{H}}) \\ &\le \prod_{i=1}^{\Gamma} \text{sing}_i(V_m)\, \text{sing}_i(\Lambda)\, \text{sing}_i(V_m^{\mathcal{H}}) \\ &= \prod_{i=1}^{\Gamma} \text{sing}_i(\Lambda)\, \text{eig}_i(V_m V_m^{\mathcal{H}}) \end{aligned} \quad \text{(B1)}$$

where the inequality follows from Horn's theorem on singular values ([50, pp. 171–172]), and the last equality follows from the fact that the singular values of $V_m$ are the square roots of the eigenvalues of $V_m V_m^{\mathcal{H}}$. The singular values of $\Lambda$ are $\delta_0^2, \delta_1^2, \ldots, \delta_{L-1}^2$. If we sort the power profile $\delta_0, \delta_1, \ldots, \delta_{L-1}$ in a nonincreasing order as $\delta_{l_1} \ge \delta_{l_2} \ge \cdots \ge \delta_{l_L}$, we obtain $\text{sing}_i(\Lambda) = \delta_{l_i}^2$ for $i = 1, 2, \ldots, L$, so

$$\prod_{i=1}^{\Gamma} \text{sing}_i(\Lambda) = \prod_{i=1}^{\Gamma} \delta_{l_i}^2. \quad \text{(B2)}$$

Since $\det(W_m \Lambda W_m^{\mathcal{H}}) = \det(V_m \Lambda V_m^{\mathcal{H}})$ for $m = 1, 2, \ldots, M_t$, from (4.8), (B1), and (B2), we have

$$\begin{aligned} \zeta_{\text{ex}} &= \left( \prod_{m=1}^{M_t} \left|\det(V_m \Lambda V_m^{\mathcal{H}})\right| \right)^{\frac{1}{2\Gamma M_t}} \\ &\le \left( \prod_{i=1}^{\Gamma} \delta_{l_i} \right)^{\frac{1}{\Gamma}} \left| \prod_{m=1}^{M_t} \det(V_m V_m^{\mathcal{H}}) \right|^{\frac{1}{2\Gamma M_t}} \end{aligned}$$

which is the upper bound (4.12) in Theorem 4.1 ii). Finally, since $V_m V_m^{\mathcal{H}}$ is nonnegative definite and all of the diagonal entries of $V_m V_m^{\mathcal{H}}$ are $L$, we have $\det(V_m V_m^{\mathcal{H}}) \le L^{\Gamma}$ for any $m = 1, 2, \ldots, M_t$ by Hadamard's inequality. Therefore,

$$\left| \prod_{m=1}^{M_t} \det(V_m V_m^{\mathcal{H}}) \right|^{\frac{1}{2\Gamma M_t}} \le \sqrt{L}$$

which implies the upper bound (4.13) in Theorem 4.1 ii). □

## Appendix C
## Proof of Theorem 4.2

In order to obtain (4.17) in Theorem 4.2, it is sufficient to show that

$$\det(W_m \Lambda W_m^{\mathcal{H}}) = \det(V_0 \Lambda V_0^{\mathcal{H}})$$

for each $1 \le m \le M_t$.

For $m = 1, 2, \ldots, M_t$, we know that $\det(W_m \Lambda W_m^{\mathcal{H}}) = \det(V_m \Lambda V_m^{\mathcal{H}})$, where $V_m$ is defined in (4.10). According to the permutation (4.16), the permuted locations $n_{(m-1)\Gamma+i}$, $i = 1, 2, \ldots, \Gamma$, can be expressed as

$$n_{(m-1)\Gamma+i} = \sigma\left((p_0-1)\Gamma M_t + (m-1)\Gamma + i\right), i = 1, 2, \ldots, \Gamma, \quad \text{(C1)}$$

for some $p_0\,(1 \le p_0 \le P)$. From (4.14), (4.15), and (4.16), we have

$$\begin{aligned} &\sigma\left((p_0-1)\Gamma M_t + (m-1)\Gamma + i\right) \\ &\qquad = v_1 \mu \Gamma + (i-1)\mu + v_0, \quad i = 1, 2, \ldots, \Gamma \end{aligned} \quad \text{(C2)}$$

where $v_0$ and $v_1$ are two integers which do not depend on $i$. Combining (C1) and (C2), we obtain

$$n_{(m-1)\Gamma+i} - n_{(m-1)\Gamma+1} = (i-1)\mu, \qquad i = 1, 2, \ldots, \Gamma. \quad \text{(C3)}$$

Substituting (C3) into (4.10), we conclude that $V_m = V_0$, i.e., $\det(W_m \Lambda W_m^{\mathcal{H}}) = \det(V_0 \Lambda V_0^{\mathcal{H}})$ for each $m = 1, 2, \ldots, M_t$.

Furthermore, if $\Gamma = L$, then $V_0$ in (4.18) is a Vandermonde matrix with variables $w^{\mu\tau_0}, w^{\mu\tau_1}, \ldots, w^{\mu\tau_{L-1}}$ ([50, p. 400]), so its determinant can be calculated as

$$\begin{aligned} |\det(V_0)| &= \prod_{0 \le l_1 < l_2 \le L-1} \left| w^{\mu(\tau_{l_2} - \tau_{l_1})} - 1 \right| \\ &= \prod_{0 \le l_1 < l_2 \le L-1} \left| 2 \sin\left( \frac{\mu(\tau_{l_2} - \tau_{l_1})\pi}{T} \right) \right|. \end{aligned} \quad \text{(C4)}$$

Finally, from (4.17) and (C4), the "extrinsic" diversity product can be further expressed as

$$\zeta_{\rm ex} = |\det(\Lambda)|^{\frac{1}{2L}} |\det(V_0)|^{\frac{1}{L}} = \left(\prod_{l=0}^{L-1} \delta_l\right)^{\frac{1}{L}} \left(\prod_{0\le l_1<l_2\le L-1} \left|2\sin\left(\frac{\mu(\tau_{l_2}-\tau_{l_1})\pi}{T}\right)\right|\right)^{\frac{1}{L}}$$

which is the desired result (4.19). □

## Appendix D
## Proof of Equation (4.21)

If $\Gamma = L$, the "extrinsic" diversity product is given by

$$\zeta_{\rm ex} = |\det(\Lambda)|^{\frac{1}{2L}} |\det(V_0 V_0^{\mathcal{H}})|^{\frac{1}{2L}}. \tag{D1}$$

With the assumption that $\tau_l - \tau_0 = lN_0T/N$ for $l = 1, 2, \ldots, L-1$, we observe that $\frac{1}{\sqrt{L}}V_0$ is a unitary matrix if $\mu = \frac{N}{LN_0}$. If we use the notation $V_0V_0^{\mathcal{H}} = \{v_{i,j}\}_{1\le i,j\le L}$, we have

$$v_{i,j} = \sum_{l=0}^{L-1} w^{(i-j)\mu(lN_0T/N+\tau_0)} = \begin{cases} L, & \text{if } i=j \\ e^{-\boldsymbol{j}2\pi(i-j)\mu\tau_0/T}\,\frac{1-e^{-\boldsymbol{j}2\pi(i-j)\mu LN_0/N}}{1-e^{-\boldsymbol{j}2\pi(i-j)\mu N_0/N}}, & \text{if } i\neq j. \end{cases} \tag{D2}$$

It follows that if $\mu = \frac{N}{LN_0}$, $V_0V_0^{\mathcal{H}} = LI_L$, where $I_L$ is the $L\times L$ identity matrix. Substituting $V_0V_0^{\mathcal{H}} = LI_L$ into (D1), we obtain

$$\zeta_{\rm ex} = \sqrt{L}\left(\prod_{l=0}^{L-1}\delta_l\right)^{\frac{1}{L}}$$

which achieves the upper bound (4.13) of Theorem 4.1. Therefore, $\mu = \frac{N}{LN_0}$ is an optimum separation factor.

If

$$\delta_0^2 = \delta_1^2 = \cdots = \delta_{L-1}^2 = 1/L$$

from Theorem 4.2, the "extrinsic" diversity product is

$$\zeta_{\rm ex} = \frac{1}{\sqrt{L}} |\det(V_0V_0^{\mathcal{H}})|^{\frac{1}{2\Gamma}}. \tag{D3}$$

Similarly to the derivation in (D2), it can be shown that $V_0V_0^{\mathcal{H}} = LI_\Gamma$ if $\mu = \frac{N}{LN_0}$. Note that $\Gamma$ and $L$ may be different integers. Thus, $\det(V_0V_0^{\mathcal{H}}) = L^\Gamma$. Substituting this into (D3), we arrive at $\zeta_{\rm ex} = 1$, which achieves the upper bound of Theorem 4.1 i). Therefore, $\mu = \frac{N}{LN_0}$ is also an optimum separation factor for this case. □

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