Full-Time Schooling in Life-Cycle Models of Human Capital Accumulation

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A reduced-form equation relating length of "formal schooling" to market, endowment, and ability parameters was derived for a life-cycle human capital accumulation model with alternative assumptions: (*a*) equal borrowing and lending rates and (*b*) no loans for human capital investment. Length of "formal schooling" increases when loans are unavailable. For both cases, length of "formal schooling" varies directly with length of work life, a Hicks-neutral "ability" index, and the ratio of the human capital rental rate to the price of associated inputs, and varies inversely with the discount rate, deterioration rate, and initial human capital stock.

I. Introduction

The Ben-Porath (1967) model of human capital accumulation is an imaginative attempt to give a neoclassical basis for inference about investments in human capital. He assumed that the individual was confronted with exogenous economic forces such as the price of educational inputs, a rental rate on human capital, and a market rate of interest for borrowing and lending for whatever purpose, and fixed retirement time. Further, he assumed internal constraints: (i) a fixed deterioration rate on the stock of human capital, (ii) an initial exogenous stock on which to build, (iii) an internalized production function whose parameters and form determine the ability to augment the stock of human capital, and (iv) a recognition that human capital as an input to produce future human capital resides in the individual.

The Ben-Porath solution yielded the result that early in life the individual specializes in the production of human capital. Thus, his solution broke into two phases, the first phase characterized by zero earnings. If one therefore directs attention to the shape of earnings streams implied by the model, the second phase deserves greater attention; Ben-Porath concentrated his attention on the second phase. In fact, Ben-Porath did not solve his model for some of the reduced forms implicit in it. In particular, he did not obtain a formal expression relating the end of the period of specialization to the parameters of the model. It is of some importance to focus attention on implications of a Ben-Porath-type model regarding the length of the period of specialization in the production of human capital. That is, much of the empirical work on earnings takes the length of formal schooling as a datum, not accounting for the possibility that the individual is free to decide how long to stay in school, depending on his environment, his ability, and his market alternatives. Elsewhere, time of labor force entry and length of time spent in school are taken as the variables to be explained, but often such studies are not guided by any assumed formal choice mechanism.

One of the objectives of the present paper is to give a solution for the path of human capital accumulation during the "formal schooling" phase of the Ben-Porath model and use that solution to characterize the end of the period of specialization in terms of the exogenous parameters. Another objective is to develop and examine a similar model given an alternative assumption that there are no opportunities for borrowing for the purpose of investing in human capital.

The main justification for our second objective is pragmatic. That is, the reduced form relating the length of formal schooling to the parameters of the model turns out to be more tractable in our case than in Ben-Porath's, and a numerical experiment shows that, over at least some range of the parameter space, directional effects of parameter changes on the length of formal schooling are invariant between the two models. Moreover, some general analytical results regarding the Ben-Porath period of specialization become apparent through a comparison of the reduced forms implicit in the two models. Thus, the work reported here provides a basis for inference and a guide to empirical work regarding the length of formal schooling.

Our second objective is partially justified by the common supposition running through the literature on human capital that loan markets for investment in education are imperfect.[1] Many such speculations can be cited, and we have chosen the following examples.

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[1] In private correspondence, Ben-Porath pointed out that his assumption about loan rates does not preclude the exogenous rates from varying for different individuals. However, in this paper, we define "imperfection" to be a divergence in the borrowing rate for financing educational investment and other rates.

Schultz (1961, p. 4) states: "It is indeed elementary to stress the greater imperfection of the capital market in providing funds for investment in human beings than for investment in physical capital." Concurring, Friedman (1962, p.102) states: "Investment in human capital cannot be financed on the same terms or with the same ease as investment in physical capital." Indeed, for some students it is held that the loan price is so high that one can in effect say that loan funds do not exist. Thus, Friedman and Kuznets (1954, p. vi) in an earlier work state: "The economic stratification of the population is important because capital invested in professional training, unlike capital invested in factories and machines, can rarely be obtained in the open market; it must be provided by the prospective practitioner himself, his parents, or a benefactor."

If no loans are available to the individual for educational investment, the separation theorem can be retained only by artifact. That is, to retain equivalence of maximizing present value of income and utility, one must now assume that borrowing rates for individual consumption (or other types of investment) are exogenous and equal to the lending rate. As Becker (1964) points out, such an assumption is a bit specious, because the borrower has some freedom to convert funds to alternative uses. However, much consumer borrowing is for durables (autos, houses, appliances) and the lender frequently retains title, thus precluding conversion. Whatever the value of these remarks, we proceed by maximizing the present value of the rental flow of human capital, and, as a consequence of our "no educational loans" assumption, we have less cause for invoking the separation theorem than did Ben-Porath.

We agree with a reviewer of an earlier draft of this paper that retention of the separation theorem with divergence of lending-borrowing rates is a luxury. And we agree with another reviewer that zero borrowing for schooling is not descriptive. The "truth" lies somewhere between, and choice thus depends on individual time preference. Since our assumptions lead to an implication that the length of schooling is greater for every case than the length of schooling in the Ben-Porath model, bias runs in the same direction for both models. That is, divergence in borrowing-lending rates that dictates choice via intertemporal utility leads to the possibility that some individuals would fail to invest, assuming, of course, that the borrowing rate exceeds the lending rate rather than vice versa.

The objectives we attempt to achieve in this paper should be viewed as intermediate steps en route to confronting these models with data. To date, most empirical effort guided by the Ben-Porath-type model has been directed toward earnings data.[2] The solutions given here and the resulting reduced form equations have implications for alternative data.

[2] See Ben-Porath (1970), Haley (1971), and Lillard (1972).

II. A Ben-Porath-Type Model with No Loans for Educational Investment

What follows is a presentation and analysis of a human capital accumulation model similar to Ben-Porath's except for the loan market assumption. For expository purposes, it is necessary to duplicate some of the Ben-Porath discussion.

An individual is assumed to arrive at a specified age, t_0, with an acquired and exogenous stock of human capital, E_0. The question of units of measurement for E_t, the stock of human capital at time t, is begged. It is a stock of "*Eds*" from which rental earnings may flow. Potential earnings are

$$Y_t^* = RE_t, \qquad t \geq t_0, \tag{1}$$

and it is assumed that the individual has acquired no assets at initial time other than his stock of human capital. The rental rate, R, is fixed.

The individual can add to his human capital stock, thereby augmenting potential earnings. Assume that the gross addition to "*Eds*" is technically constrained by a Cobb-Douglas production function,

$$Q_t = \beta_0 K_t^{\beta_1} D_t^{\beta_2}, \tag{2}$$

where β_0, β_1, and β_2 are positive and scale (i.e., $\beta_1 + \beta_2$) is less than unity.[3]

The symbol Q_t represents the gross addition to human capital in time t; K_t is that part of E_t used to augment human capital as opposed to earning an immediate cash flow; and D_t is other educational inputs taken as a bundle.

The price of educational inputs is taken to be exogenous and fixed through time. The price of a unit of D is denoted by the symbol P. The opportunity cost of K is the rental rate, R. Hence, net earnings is potential earnings net of investment in schooling. That is, net earnings, Y_t, is

$$Y_t = R(E_t - K_t) - PD_t, \qquad t \geq t_0. \tag{3}$$

We assume that educational inputs must be purchased out of current cash flow. That is, contrary to the Ben-Porath assumption that the individual can defray outlay for tuition, books, etc., by borrowing, we assume that net earnings are always nonnegative. Thus, a constraint that must be

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[3] In Ben-Porath's notation, our E_t is his K_t and our K_t is his $s_t K_t$; that is, Ben-Porath's s_t is our K_t/E_t. As noted by a reviewer, some of our results do not depend on this specific form for the production process. The nature of this invariance is clarified at the appropriate points.

met is

$$Y_t = g^2(t), \qquad t \geq t_0, \tag{4}$$

where $g(t)$ is any arbitrary differentiable real time function.[4]

Note that (4) implies $E_t \geq K_t$, since R, P, and D_t are nonnegative.[5] Assuming that human capital deteriorates at a constant rate in the absence of augmentation, the net addition to the human capital stock is governed by

$$\dot{E}_t = Q_t - \delta E_t, \qquad 0 < \delta < 1, \tag{5}$$

where the dot notation indicates a time derivative and δ is the rate of deterioration of the human capital stock.

It is assumed that there is a fixed endpoint in time, t_n, at which the individual retires. This implies that R, the rental rate on human capital, jumps to zero at time t_n.

Finally, it is assumed that the individual faces an exogenous market rate of interest, r, at which he can loan or borrow for consumption purposes, and the objective is assumed to be that of maximizing the present value of net earnings, given the restrictions as stated. That is, the objective is to maximize

$$J = \int_{t_0}^{t_n} \Lambda \, dt, \tag{6}$$

where

$$\Lambda = e^{-rt}[R(E_t - K_t) - PD_t] - \lambda_{1t}(\dot{E}_t - \beta_0 K_t^{\beta_1} D_t^{\beta_2} + \delta E_t) \\ - \lambda_{2t}[R(E_t - K_t) - PD_t - g^2(t)] \tag{7}$$

and λ_{1t} and λ_{2t} are Lagrangians.[6]

Before we present results, table 1 is given so that the reader may have a ready list of the variables of the model.

A. Necessary Conditions for Maximizing J

The reader is referred to Sage (1968) for a general discussion of necessary conditions for maximizing an integral over time and to Haley (1971) for

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[4] The arbitrary real function $g(t)$ enters the model as an endogenous variable. By making net earnings equal to its square, we ensure the nonnegativity of net earnings (see Sage 1968).

[5] The Ben-Porath assumption analogous to (4) is that human capital diverted to production cannot be greater than the total stock available.

[6] See Sage (1968) for methods of incorporating equality constraints into variational calculus problems.

TABLE 1

LIST OF VARIABLES: MODEL FOR HUMAN CAPITAL
INVESTMENT WITH NO LOANS

ENDOGENOUS		EXOGENOUS	
Variable	Description	Variable	Description
Y_t^*	Potential earnings	R	Rental rate on human capital
Y_t	Net earnings	δ	Rate of deterioration of E_t
E_t	Human capital stock	P	Price of educational inputs
K_t	Part of E_t being reinvested	$\beta_0, \beta_1, \beta_2$	Production function parameters
D_t	Purchased educational inputs	E_0	Initial stock of human capital
Q_t	Produced human capital		
$\dot{E}_t$	Net additions to E_t	t_0, t_n	Initial and ending times of
λ_{1t}	Shadow price of Q_t^*		planning horizon†
λ_{2t}	Part of shadow price of D_t, K_t^*	r	Market rate of interest‡
$g(t)$	Square root of net earnings		
t^*	Point in time at which net earnings turn positive*		

* Implicit in solution.
† t_0 is taken to be zero without loss in generality.
‡ At which loans or borrowing for consumption purposes can take place.

a complete discussion in a similar case.[7] In the present case, the general
necessary conditions are:

$$Re^{-rt} - \delta\lambda_{1t} - R\lambda_{2t} + \dot{\lambda}_{1t} = 0, \tag{8}$$

$$\lambda_{1t}\beta_0\beta_1 K_t^{\beta_1-1}D_t^{\beta_2} = R(e^{-rt} - \lambda_{2t}), \tag{9}$$

$$\lambda_{1t}\beta_0\beta_2 K_t^{\beta_1}D_t^{\beta_2-1} = P(e^{-rt} - \lambda_{2t}), \tag{10}$$

$$\dot{E}_t = \beta_0 K_t^{\beta_1}D_t^{\beta_2} - \delta E_t, \tag{11}$$

$$R(E_t - K_t) - PD_t = g^2(t), \tag{12}$$

$$2g(t)\lambda_{2t} = 0. \tag{13}$$

Note that equations (11) and (12) are simply restatements of the
constraints. Equations (9) and (10) are "value of marginal product
equals factor price" equations. Equation (8) is a differential equation
describing the path of the implicit price of gross additions to human

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[7] In general, if the objective is to maximize

$$\int_{t_0}^{t_n} \Lambda(\mathbf{x}, \dot{\mathbf{x}}, t)dt,$$

the Euler-Lagrange conditions are

$$\frac{\partial \Lambda}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{\mathbf{x}}} = 0,$$

where $\mathbf{x}$ is a vector of time variables, $\dot{\mathbf{x}}$ is the vector of time derivatives, etc. To these
must be added transversality conditions, but for the present problem it becomes clear
how to resolve constants of integration without appeal to transversality conditions.

capital (price of Q_t), and equation (13) says that if the shadow part of the input prices is zero, net earnings are not necessarily zero, or, conversely, if net earnings are zero, there can be a shadow component to input prices. This makes economic sense because when the individual specializes in educational investment (zero net cash flow), his human capital stock is a fixed and limiting input as of a moment in time.

Thus, the solution can break into two phases, and, where it exists, we denote the period of zero net earnings as Phase I, the period of "formal schooling."[8]

B. Phase I

Note that even when λ_{2t} is nonzero, as it is in Phase I, a ratio of the two *VMP* curves, equations (9) and (10), eliminates the shadow part of the input prices. This is in contrast to the Ben-Porath case and makes for a simpler solution.[9] Hence, the important relationships for characterizing the period of "formal schooling" are

$$\frac{\beta_1 D_t}{\beta_2 K_t} = \frac{R}{P}, \tag{14}$$

$$E_t = K_t + \frac{P}{R} D_t, \tag{15}$$

$$\dot{E}_t = \beta_0 K_t^{\beta_1} D_t^{\beta_2} - \delta E_t. \tag{16}$$

Equations (14) and (15) imply

$$K_t = \frac{\beta_1}{\beta_1 + \beta_2} E_t, \tag{17}$$

$$D_t = \frac{\beta_2}{\beta_1 + \beta_2} \left(\frac{R}{P} E_t \right). \tag{18}$$

Substituting (17) and (18) into (16), we get a differential equation in E_t alone:

$$\dot{E}_t - \mu E_t^{1-\Delta} + \delta E_t = 0, \tag{19}$$

[8] The conditions under which Phase I exists become clear in the derivation of the optimal path of human capital stock, E_t, which follows. Note that when loans are not available, the end of the period of specialization or "formal schooling" is not the time at which labor force entry occurs but instead is the time at which net earnings become positive, that is, completion of "full-time schooling." This result is consistent with observed behavior, for as one reviewer noted entry into the labor force often precedes completion of schooling.

[9] See Appendix.

where

$$\mu = \beta_0 \left(\frac{\beta_1}{1 - \Delta}\right)^{1 - \Delta} \left(\frac{R\beta_2}{P\beta_1}\right)^{\beta_2} \quad \text{and} \quad \Delta = 1 - \beta_1 - \beta_2.^{10}$$

An explicit solution for E_t from (19) is contained in the following equation:

$$-\frac{1}{\delta\Delta} \log (\mu - \delta E_t^\Delta) = t + c, \tag{20}$$

where c is a constant of integration.

The constant, c, can be found by setting $t = t_0$ in (20). Without loss of generality we take $t_0 = 0$, so

$$c = -\frac{1}{\delta\Delta} \log (\mu - \delta E_0^\Delta). \tag{21}$$

Resolution of the constant of integration instructs us that Phase I exists only if μ is larger than δE_0^Δ. Otherwise c is undefined.[11]

If we incorporate the constant of integration and rewrite (20) in altered form, the optimal path of human capital stock during specialization is described by

$$E_t^I = \left[\frac{\mu}{\delta} + \left(E_0^\Delta - \frac{\mu}{\delta}\right)e^{-\delta\Delta t}\right]^{1/\Delta}, \quad 0 \le t \le t^*, \tag{22}$$

[10] As pointed out by a reviewer, a similar differential equation governing accumulation of "Eds" in Phase I would be obtained from any homogeneous production function. The parameter μ would remain a function of relative prices—only the parametric representation would vary. For example, if one chose a constant elasticity of substitution (CES) function,

$$Q_t = (\alpha_1 K_t^{-\beta} + \alpha_2 D_t^{-\beta})^{-\nu/\beta},$$

where ν is scale (less than unity), then equation (19) would become

$$\dot{E}_t - \tilde{\mu}E_t^\nu + \delta E_t = 0,$$

where

$$\tilde{\mu} = \left[\alpha_1 + \alpha_2 \left(\frac{\alpha_2 R}{\alpha_1 P}\right)^{-\beta\sigma}\right]^{-(\nu/\beta)} \div \left[1 + \frac{P}{R}\left(\frac{\alpha_2 R}{\alpha_1 P}\right)^\sigma\right]^\nu$$

and where σ is the elasticity of substitution $1/(1 + \beta)$. A similar generality holds in Phase II, as noted by the same reviewer (see n. 14 below). The reasons for working with the Cobb-Douglas are to preserve manageable closed forms for relating the length of formal schooling to parameters and to facilitate comparison with similar closed forms implicit in the Ben-Porath model.

[11] Since the net accumulation of human capital is at the rate $Q_t - \delta E_t$, and optimal Q_t in Phase I is $\mu E_t^{1-\Delta}$, the requirement that μ be larger than δE_0^Δ is simply a requirement that the initial optimal gross rate of capital accumulation be larger than deterioration. Thus, Phase I, if it exists at all, is a period of positive net additions to the stock of human capital.

where the superscript on E_t^I indicates Phase I and where t^*, which marks the end of the period of specialization in the production of human capital, remains to be determined.

From (22), the general shape of human capital accumulation during Phase I can be discerned. The stock begins at E_0 and, given the existence of a specialization phase, monotonically increases to an upper asymptote, the upper asymptote being $(\mu/\delta)^{1/\Delta}$. The path of accumulation has the potential of passing through a point of inflection.

Via equations (17) and (18), similar paths can be immediately constructed for K_t and D_t in Phase I. Note that the optimal fraction of human capital devoted to production (Ben-Porath's s_t) is $\beta_1/(1 - \Delta)$ during Phase I: it is thus constant during specialization but smaller than unity.

C. Phase II

During the period of positive net earnings, the shadow price component of input prices, λ_{2t}, becomes zero. Hence, the relationships governing Phase II important to our purpose are

$$\lambda_{1t}\beta_0\beta_1 K_t^{\beta_1 - 1}D_t^{\beta_2} = Re^{-rt}, \tag{23}$$

$$\lambda_{1t}\beta_0\beta_2 K_t^{\beta_1}D_t^{\beta_2 - 1} = Pe^{-rt}, \tag{24}$$

$$Re^{-rt} - \delta\lambda_{1t} + \dot{\lambda}_{1t} = 0. \tag{25}$$

As previously noted, λ_{1t} is the shadow price of a unit of produced "Eds." Since at time t_n it is assumed that the rental rate on human capital goes to zero, it follows that the shadow price of produced "Eds" goes to zero at t_n.[12] This furnishes an integrating constant for the general solution to equation (25); hence, a specific solution to (25) is

$$\lambda_{1t} = \frac{R}{r + \delta} e^{-rt}[1 - e^{(r + \delta)(t - t_n)}], \qquad t^* \leq t \leq t_n.[13] \tag{26}$$

Using equation (26) in conjunction with the two VMP curves, (23) and (24), the optimal time path for the part of educational stock diverted

[12] In variational calculus language, educational stock is not constrained to be any particular value at final time; hence its "perturbation" is not zero at t_n, so the transversality condition implies that λ_{1t} evaluated at t_n must be zero (see Sage 1968).

[13] Equation (25) is a standard linear differential equation for which a general solution may be found in textbooks on differential equations.

as an input during Phase II is

$$K_t^{II} = \beta_1 (1 - \Delta) \frac{1 - \Delta}{\Delta} \mu^{1/\Delta} \left[\frac{1 - e^{(r+\delta)(t-t_n)}}{r + \delta} \right]^{1/\Delta},$$

(27)

$$t^* \leq t \leq t_n. [14]$$

An optimal path for D_t in Phase II can be similarly determined. The differential equation describing the optimal E_t^{II} path is difficult to integrate but can be handled.[15] However, it is not vital to our purposes and will be omitted from consideration here.

D. End of the Period of Specialization

At the point $t = t^*$, it must be true that the optimal stock of human capital, E_t^I, in Phase I coincides with E_t^{II} in Phase II. At the point $t = t^*$, equation (17) must also hold true. Thus there is a nondifferentiable (unsmooth) point in the E_t path at t^*, but there is no provision in the model for an instantaneous gain or loss in human capital at a single point in time. Therefore, at $t = t^*$,

$$K_{t^*} = \frac{\beta_1}{1 - \Delta} E_{t^*}.$$

(28)

Hence, to determine t^*, the relevant functions to equate are

$$\frac{1 - \Delta}{\beta_1} K_{t^*}^{II} = \left[\frac{(1 - \Delta)\mu}{r + \delta} \right]^{1/\Delta} [1 - e^{(r+\delta)(t^*-t_n)}]^{1/\Delta},$$

(29)

$$E_{t^*}^I = \left(\frac{\mu}{\delta} \right)^{1/\Delta} \left[1 - \left(1 - \frac{\delta}{\mu} E_0^\Delta \right) e^{-\delta \Delta t^*} \right]^{1/\Delta}.$$

(30)

Thus, a reduced-form equation relating the end of "formal schooling" to the parameters of the model is

$$1 - \left(1 - \frac{\delta}{\mu} E_0^\Delta \right) e^{-\delta \Delta t^*} = \frac{\delta(1 - \Delta)}{r + \delta} [1 - e^{(r+\delta)(t^*-t_n)}].$$

(31)

[14] With reference to n. 10, if the production function were CES, equation (27) would read

$$K_t = (\nu\alpha_1)^{1/(1-\nu)} \left[\alpha_1 + \alpha_2 \left(\frac{\alpha_2 R}{\alpha_1 P} \right)^\sigma \right]^{(\nu+\beta)/(\nu\beta-\beta)} \left[\frac{1 - e^{(r+\delta)(t-t_n)}}{r + \delta} \right]^{1/(1-\nu)}.$$

This is an example of the fact that, in general, the shapes of the optimal time paths involved are similar for any homogeneous production function, with scale being crucial. Only the constants change with variations in the function. Thus, anticipating the results given in Section III, we find that the partial effects of changes in r, δ, t_n, and E_0 the length of formal schooling are invariant to the form of the production function within the class of homogeneous functions.

[15] See Haley (1972).

E_t, K_t

$\left(\frac{\mu}{\delta}\right)^{\frac{1}{\Delta}}$

E_t^I

$\frac{(1-\Delta)}{\beta_1} K_t^{II}$

K_t^{II}

E_0

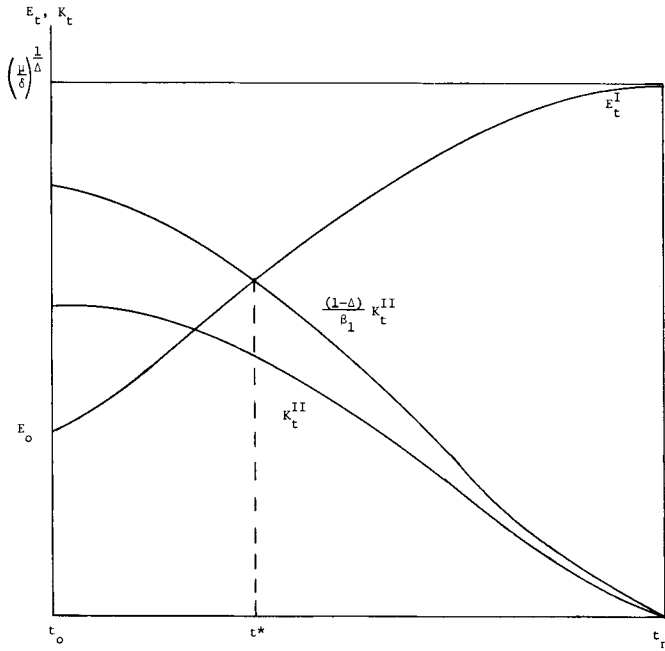t_0 t^* t_n

FIG. 1.—Determination of the end of "formal schooling"

The function on the left-hand side of (31) is an increasing function in t^*, and the right-hand side is a decreasing function in t^*.

Figure 1 pictures the crossing of the Phase I and Phase II functions that determine t^*.

III. Partial Effects of Parameter Changes on the Length of the Period of Specialization

Equation (31) is a reduced-form equation relating the length of the period of specialization (t^*) to the parameters of the model. Some of the partial effects of changes in the parameters on the length of specialization can be visualized with the aid of figure 1. Other effects are more difficult to visualize but can be derived analytically from the reduced form via an implicit-function theorem.[16]

An increase in the initial stock of human capital, E_0, shifts up the Phase I function of figure 1 without affecting the Phase II function. Hence, an increase in the initial stock of human capital, holding other

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[16] The useful implicit-function theorem is that if $F(x_1, x_2, \ldots, x_n) = 0$, $\partial x_i / \partial x_j = (-\partial F / \partial x_j) / (\partial F / \partial x_i)$. In the present case,

$$F = 1 - e^{-\delta \Delta t^*} - \frac{\delta(1 - \Delta)}{r + \delta} [1 - e^{(r+\delta)(t^*-t_n)}] + \frac{\delta}{\mu} E_0^\Delta e^{-\delta \Delta t^*}.$$

parameters constant, decreases the length of the period of specialization in the production of human capital.[17] This result may seem inconsistent with casual observation, for one would expect to find a positive correlation between, say, parental schooling or income and length of time spent in school by the offspring. There is an explanation for this apparent anomaly. First, the effect under discussion is partial; ability (as reflected by the parameters of the production function) and the price of schooling inputs are held constant. Students with higher ability also will tend to have a larger initial stock of human capital; thus, simple and partial correlations may have opposite signs. Similarly, parents with larger resources may subsidize their child's education for longer periods, thereby lowering the price of educational inputs. In addition, since institutions of higher education use nonprice rationing, the supply of schooling to an individual is likely to be influenced by both his ability and his achievement prior to enrollment. The reduced form under discussion here is a demand function. Finally, it should be reiterated that education in these models is not treated as a consumer good—for either the individual or his family.

Again, with reference to figure 1, a lengthening of work life, t_n, has the effect of shifting the Phase II function upward without affecting the path of human capital accumulation in Phase I. Thus, a postponing of retirement has the partial effect of lengthening the period of "formal schooling."

Other partial effects are more difficult to visualize with the aid of figure 1. For example, changes in the rate of deterioration, δ, the rental-price ratio, R/P, and the production function parameters affect the functions in both phases. Thus, an increase in the rental-price ratio, R/P, shifts both functions upward. This increase in R/P clearly increases discounted future earnings, but its effect on length of schooling is less clear. However, use of the implicit-function theorem establishes the result that the partial effect of increasing the rental-price ratio increases the period of specialization.

Other partial effects that can be established analytically from equation (31) are: (i) changing the interest rate, r, affects time in school in an inverse manner. Increasing r decreases t^* and conversely. (ii) Similarly, increasing the Hicks-neutral productivity of the individual in his capacity to augment human capital (increase β_0) increases the length of specialization in the production of human capital.

Changes in the remaining parameters, δ, β_1, and β_2, are ambiguous in their effects. However, in a simulation exercise some results were established (see Section V).

IV. End of the Period of Specialization in the Ben-Porath Case

The crucial difference in the Ben-Porath case is that there is no shadow price for the D_t input in the specialization phase, since the individual

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[17] As noted earlier, Phase I exists only if E_0 is less than $(\mu/\delta)^{1/\Delta}$.

is not constrained to pay as he goes for these inputs. The amount of "*Eds*" possessed by the individual is constraining during Phase I for both versions of the model; hence, the per unit cost of K_t as an input has an internal valuation. When the individual is specializing in the Ben-Porath sense, he wishes he could purchase additional "*Eds*" to use as an input but cannot, since they are internal to the individual. The end of the period of specialization is characterized by a coinciding of the shadow price of "*Eds*" and the opportunity cost of using "*Eds*" in production. Thus, in the Ben-Porath version, Phase I ends as λ_{2t} goes to zero and the individual is content to move along a usual Cobb-Douglas expansion path. In contrast, D_t is shadow priced in the same way as K_t in the no-borrowing situation; that is, all prices are internal. This difference makes the solution for Ben-Porath Phase I quite complex, and, in fact, no solution for Phase I was presented in his paper.

Presented below (eq. [32]) is an implicit function relating the end of the period of specialization in Ben-Porath's model to the parameters. Its complexity makes for analytical difficulty; however, some general results can be obtained by comparing the Ben-Porath model with the no-loans case. Both models have the same Phase II solution for K_t, the human capital input into the production process. The Phase I path for E_t lies above and to the left of the path of human capital accumulation during the zero net earnings phase for the no-loans case, since the Ben-Porath individual has more inputs at his disposal during specialization. Hence, the period of specialization in the Ben-Porath model is shorter than the length of "formal schooling" in our model. Also, initial human capital, E_0, only affects the Phase I function; therefore, the Ben-Porath length of specialization varies inversely with E_0. In his paper, Ben-Porath (1967) recognized the effect of E_0 on length of specialization.

Equation (32) is derived in the Appendix:

$$
1 - \delta e^{[\gamma\beta_2 - \delta(1-\beta_1)]\tau} - \frac{\delta}{\theta}\beta_0\beta_1\left[\frac{1 - e^{(r+\delta)(\tau-t_n)}}{r+\delta}\right]
$$

$$
+ (\beta_0\beta_1)^{-\beta_2/\Delta}\left(\frac{\beta_1 P}{\beta_2 R}\right)^{[\beta_2(1-\beta_1)]/\Delta}E_0^{1-\beta_1}e^{-\delta(1-\beta_1)\tau}
$$

$$
\times\left[\frac{1 - e^{(r+\delta)(\tau-t_n)}}{r+\delta}\right]^{-\beta_2/\Delta} = 0, \quad (32)
$$

where τ is the end of Ben-Porath's period of specialization and

$$
\gamma = \frac{\delta + r - \delta\beta_1}{1 - \beta_2}, \tag{33}
$$

$$
\theta = \frac{\beta_0(1-\beta_1)\delta}{\gamma\beta_2 + \delta(1-\beta_1)}. \tag{34}
$$

TABLE 2

PARAMETER VALUES USED FOR SOLUTIONS FOR t^* AND τ

β_0	β_1	β_2	E_0	δ	r	R/P	t_n
30	0.2	0.3	2,000	0.06	0.07	0.75	40
40	0.25	0.25	3,000	0.07	0.09	1.0	50
50	0.3	0.2	4,000	...	...	1.25	...

Equations (31) and (32) were solved for several specific parameter sets to obtain some partial comparisons; the numerical results are given in Section V.

V. Some Numerical Comparisons

Table 2 contains the values of parameters for which equations (31) and (32) were solved for t^* and τ.[18] The following considerations were a guide to choosing the values in table 2:

i) Since only the price ratio R/P is relevant, no generality is lost in fixing R at unity. Thus, "*Eds*" can be thought of as being identical to potential earnings. In particular, the reader can think of initial stock of "*Eds*," E_0, as potential annual earnings in dollars at initial time. Units of D_t are as arbitrary as units of K_t. However, varying the price ratio R/P around unity implies a D measure in annual units roughly the same as K. For example, in Phase I, when net income is zero and R/P and β_1/β_2 are unity, then $D = K = \frac{1}{2}E$ in the no-loans model.

ii) Values for human capital deterioration were picked close to those reported in Johnson (1970), and values for r were chosen in a range that seemed reasonable for real rates of return to nonhuman capital. Both r and δ are continuous discount rates; therefore, annual discrete equivalents would be slightly larger than the r and δ values in table 2.

iii) Values of β_1 and β_2 were chosen to sum to $\frac{1}{2}$. This allows a simple solution to the differential equation governing potential and net earnings during Phase II.[19]

iv) The parameter, t_n, represents expected years of productive life. That is, if the individual is assumed to make his choice at age 18, say, then $t_n = 50$ implies retirement at age 68.

v) The parameter β_0 is chosen to crudely represent "ability." The value 40 was chosen so that the individual would realize an initial gross

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[18] Richard Brook developed the solution programs.

[19] Two reviewers of an earlier draft suggested some discussion of earnings profiles implicit in the parameter sets.

internal rate of return of about 50 percent when $\beta_1 = \beta_2 = \frac{1}{4}$, $R/P = 1$, and $E_0 = 3,000$. (Initial production, Q_0, would be about 1,500 for this set of values.) Variations above and below 40 represent variations from a "norm" of a Hicks-neutral "ability" measure.

Results of the numerical comparisons were as follows:

i) Length of specialization in the Ben-Porath model, τ, was uniformly smaller than the period of zero net earnings in the no-loans case (t^*). As noted previously, this result has general validity, since with a no-loan restriction the individual can increase his capital stock at a faster rate during the specialization phase.

ii) Both τ and t^* varied directly with β_0, R/P, and t_n and inversely with δ, r, and E_0. At least for the initial human capital stock (E_0), generality can be claimed for the Ben-Porath model.

iii) For the parameter sets chosen, an increase in β_1 with scale fixed has ambiguous effects on t^*, depending on the price ratio, R/P. In contrast, increases in β_1 with offsetting decreases in β_2 to fix scale increased the Ben-Porath period of specialization, τ.

Maximum length of specialization in the no-loans case was approximately 6.3 years and occurred for the parameter set $\beta_0 = 50$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\delta = 0.06$, $r = 0.07$, $R/P = 1.25$, $E_0 = 2,000$, and $t_n = 50$. Maximum τ also was obtained for this parameter set and was about 2.7 years. If we assume the age of decision, t_0, to be 18, these numbers would correspond to job market entry at about age 21 for the Ben-Porath individual and a realization of income over and above schooling expenses at about age 24 for the individual with no access to loans for educational investment. Minimums were found for the set $\beta_0 = 30$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\delta = 0.07$, $r = 0.09$, $R/P = 0.75$, $E_0 = 4,000$, and $t_n = 40$. The minimums were, respectively, $t^* = -0.25$ and $\tau = -2.0$.[20]

Table 3 gives potential earnings (Y_t^*) and net earnings (Y_t) streams for maximum and minimum t^*'s and τ's and their corresponding parameter sets, assuming $t_0 = 18$.

At least for the parameter sets represented in table 3, peak earnings were rather insensitive to loans versus no-loans educational investment. In constrast, length of "full time schooling" is quite sensitive to the loan market assumption. Thus, the major difference in earnings streams for two individuals having equal opportunity other than access to loans during "formal schooling" lies in the early part of work life. Of course, discounted value of earnings at the time of decision favors the individual having access to educational loans, but earnings tend to converge later.

[20] The individual cannot, of course, undo the past. The negative figures indicate e.g., that the individual would have entered the labor force at, say, age 16 in the Ben-Porath case with an expected work life of 42 years and a smaller initial human capital stock.

TABLE 3

POTENTIAL AND NET EARNINGS ($1,000):
PATHS CORRESPONDING TO $t^* = 6.3$ AND -0.25, AND $\tau = 2.7$ AND -2.0

AGE ($t_0 = 18$)	$t^* = 6.3$		$\tau = 2.7$		AGE ($t_0 = 18$)	$t^* = -0.25$		$\tau = -2.0$	
	$Y^*(t)$	$Y(t)$	$Y^*(t)$	$Y(t)$		$Y^*(t)$	$Y(t)$	$Y^*(t)$	$Y(t)$
25	24.4	3.0	27.5	6.1	20	5.7	2.0	5.9	2.2
30	42.2	20.9	44.5	23.2	25	9.1	5.4	9.2	5.5
35	55.2	34.3	56.9	36.0	30	11.5	7.8	11.6	7.9
40	64.7	44.2	65.9	45.5	35	13.1	9.5	13.2	9.6
45	71.2	51.8	72.2	52.7	40	14.1	10.8	14.2	10.8
50	75.3	57.6	75.9	58.3	45	14.6	11.7	14.6	11.7
55	76.6	62.3	77.1	62.8	50	14.4	12.4	14.4	12.4
60	74.6	65.6	75.0	66.0	55	13.0	12.5	13.0	12.5
65	67.2	65.0	67.5	64.0	58	11.2	11.2	11.2	11.2
68	58.7	58.7	60.0	60.0					

VI. Summary

For a Ben-Porath-type model of human capital accumulation with the alternative assumption that the individual may not borrow for educational investment, a reduced-form equation was derived that relates the length of "formal schooling" to market, endowment, and ability parameters. The form of the relationship is invariant within the class of homogeneous production processes for augmenting human capital stock except for the manner in which the rental-price ratio, R/P, enters as a determining factor. Thus, the equation may serve as a guide to functional forms and to choosing determining variables in empirical investigation of time spent in school or age of labor force entry.

It was established analytically that the length of "formal schooling" varies directly with the length of work life and inversely with a market rate of discount, r, and the initial stock of human capital. For a Cobb-Douglas production function, it is also generally valid that the rental-price ratio, R/P, affects the length of "formal schooling" in a positive direction; a similar directional effect holds for a Hicks-neutral "ability" index. An experiment involving several specific parameter sets indicated that length of "formal schooling" varies inversely with rate of deterioration of human capital.

A similar reduced form was derived for the original Ben-Porath (1967) set of assumptions. The period of specialization is always shorter for individuals with access to educational loans but who otherwise face identical market, endowment, and technical data. It is generally true that a larger initial endowment of human capital shortens length of specialization in the Ben-Porath case. Other results based on a computational experiment indicate that the individual with access to educational loans specializes longer for larger rent-price ratios (R/P), smaller interest and deterioration rates $(r$ and $\delta)$, longer work life, and greater "ability"

in the Hicks-neutral sense. Thus, the directional effects of these parameter changes on length of "formal schooling" are invariant for the two models, at least over the range of parameter space considered.

The numerical experiment indicated that length of specialization is sensitive to the loan market assumption. However, earnings streams were mainly affected in early life, with peak earnings (both potential and net) being quite comparable for both assumptions, other things equal.

Appendix
Derivation of the Solution to Ben-Porath's Phase I

In the symbology of the present paper, the objective in the Ben-Porath model was to maximize

$$J = \int_{t_0}^{t_n} \Lambda \, dt, \tag{A.1}$$

where

$$\Lambda = e^{-rt}[R(E_t - K_t) - PD_t] - \lambda_{1t}[\dot{E}_t - \beta_0 K_t^{\beta_1} D_t^{\beta_2} + \delta E_t] \\ - \lambda_{2t}[E_t - K_t - g^2(t)].$$

The Euler-Lagrange conditions are

$$Re^{-rt} - \delta\lambda_{1t} - \lambda_{2t} + \dot{\lambda}_{1t} = 0, \tag{A.2}$$

$$\lambda_{1t}\beta_0\beta_1 K_t^{\beta_1 - 1} D_t^{\beta_2} = Re^{-rt} - \lambda_{2t}, \tag{A.3}$$

$$\lambda_{1t}\beta_0\beta_2 K_t^{\beta_1} D_t^{\beta_2 - 1} = Pe^{-rt}, \tag{A.4}$$

$$\dot{E}_t = \beta_0 K_t^{\beta_1} D_t^{\beta_2} - \delta E_t, \tag{A.5}$$

$$E_t - K_t = g^2(t), \tag{A.6}$$

$$2\lambda_{2t}g(t) = 0. \tag{A.7}$$

Phase I

During the specialization phase, $g(t) = 0$, $\lambda_{2t} \neq 0$, and

$$E_t = K_t. \tag{A.8}$$

From (A.4) and (A.8), it follows that

$$\frac{\dot{\lambda}_{1t}}{\lambda_{1t}} = -r + (1 - \beta_2)\frac{\dot{D}_t}{D_t} - \beta_1 \frac{\dot{E}_t}{E_t}. \tag{A.9}$$

If we substitute equation (A.3) into equation (A.2) to eliminate λ_{2t}, some rearranging results in

$$\frac{\dot{\lambda}_{1t}}{\lambda_{1t}} = \delta - \beta_0\beta_1 E_t^{\beta_1 - 1} D_t^{\beta_2}. \tag{A.10}$$

From equation (A.5), dividing by E and rearranging terms yield

$$\frac{\dot{E}_t}{E_t} = \beta_0 E_t^{\beta_1 - 1} D_t^{\beta_2} - \delta. \tag{A.11}$$

Combining (A.9), (A.10), and (A.11), we get

$$\frac{\dot{D}_t}{D_t} = \frac{\delta + r - \delta\beta_1}{1 - \beta_2}, \qquad t_0 \leq t \leq \tau. \tag{A.12}$$

Thus, during Phase I, the growth rate of the D input is constant.[21] Therefore, the optimal path for the D input during Phase I is

$$D_t = c_1 e^{\gamma t}, \qquad t_0 \leq t \leq \tau, \tag{A.13}$$

where γ is the positive constant on the right-hand side of (A.12) and c_1 is a constant of integration as yet undetermined.

Substituting (A.13) for D_t in (A.11) sets up a differential equation governing the optimal path for E_t (and K_t) during Phase I. The resulting equation is

$$\dot{E}_t - c_1^{\beta_2}\beta_0 E_t^{\beta_1} e^{\gamma\beta_2 t} + \delta E_t = 0, \tag{A.14}$$

which has the solution

$$\log\left[\frac{c_1^{\beta_2}\beta_0\delta(1 - \beta_1)}{\gamma\beta_2 + \delta(1 - \beta_1)} e^{\gamma\beta_2 t} - \delta E_t^{1-\beta_1}\right] = -\delta(1 - \beta_1)t + c_2, \tag{A.15}$$

where c_2 is another constant of integration. An alternative expression for (A.15) is

$$E_t^I = \left[c_1^{\beta_2}\frac{\theta}{\delta} e^{\gamma\beta_2 t} - \frac{c_2^*}{\delta} e^{-\delta(1-\beta_1)t}\right]^{1/(1-\beta_1)}, \qquad t_0 \leq t \leq \tau, \tag{A.16}$$

where

$$\theta = \frac{\beta_0(1 - \beta_1)\delta}{\gamma\beta_2 + \delta(1 - \beta_1)} \quad \text{and} \quad c_2^* = e^{c_2}.$$

One of the integration constants can be eliminated by the assumption that the individual begins the process with an exogenous stock of human capital E_0. By taking initial time to be $t_0 = 0$, (A.16) yields

$$c_2^* = c_1^{\beta_2}\theta - \delta E_0^{1-\beta_1}. \tag{A.17}$$

Phase II

Phase II is identical to both models, with the exception of its point of initiation. We symbolize the end of Ben-Porath's period of specialization by τ.

Determination of τ in the Ben-Porath Model

Equations (A.16) and (A.17) hold at the point τ, and, further, since $\gamma_{2\tau} = 0$,

$$\frac{\beta_1 D_\tau}{\beta_2 K_\tau} = \frac{R}{P}, \tag{A.18}$$

and since τ marks the end of the period of specialization, it is also true that

$$K_\tau = E_\tau. \tag{A.19}$$

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[21] The growth rate of the D input is constant, but the absolute level of its optimal path will be seen to be a function of input prices as well as other parameters.

Using equations (A.16), (A.17), (A.18), and (A.19), where K_τ is equation (27) evaluated at τ, we get an implicit function relating the length of specialization to the parameters of the model as follows:

$$1 - \delta e^{[\gamma\beta_2 - \delta(1-\beta_1)]\tau} - \frac{\delta}{\theta}\beta_0\beta_1\left[\frac{1 - e^{(r+\delta)(\tau-t_n)}}{r+\delta}\right]$$

$$+ (\beta_0\beta_1)^{-(\beta_2/\Delta)}\left(\frac{\beta_1 P}{\beta_2 R}\right)^{[\beta_2(1-\beta_1)]/\Delta} E_0^{1-\beta_1} e^{-\delta(1-\beta_1)\tau}\left[\frac{1 - e^{(r+\delta)(\tau-t_n)}}{r+\delta}\right]^{-\beta_2/\Delta} = 0.$$

$$(A.20)$$

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