Full Waveform Inversion via Matched Source Extension

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Outline

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Overview

Assume that the received pressure field $p(\mathbf{x}_r, t; \mathbf{x}_s)$ generated by a causal isotropic point radiator at source position $\mathbf{x} = \mathbf{x}_s$ satisfies the constant density acoustic wave equation,

$$\frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \delta(\mathbf{x} - \mathbf{x}_s) f(t) \quad \text{in } \mathbb{R}^2. \tag{1}$$

$$p|_{t=0} = \left.\frac{\partial p}{\partial t}\right|_{t=0} = 0 \tag{2}$$

Let's introduce the forward modeling operator $S[v]$ to relate the velocity $v(x, z)$ and wavelet function $f(t)$ to the scattering field at the receiver $\mathbf{x}_r$,

$$S[v, f](\mathbf{x}_r, t; \mathbf{x}_s) = p(\mathbf{x}_r, t; \mathbf{x}_s). \tag{3}$$

Full Waveform Inversion

Given recorded traces $d(\mathbf{x}_r, t; \mathbf{x}_s)$, find velocity v and wavelet function f such that $S[v, f] = d$.

- FWI via data fitting,

$$J_{\mathrm{FWI}}[v, f] = \frac{1}{2} \sum_{\mathbf{x}_r, \mathbf{x}_s} \int |S[v, f](\mathbf{x}_r, t; \mathbf{x}_s) - d(\mathbf{x}_r, t; \mathbf{x}_s)|^2 dt. \quad (4)$$

- FWI objective function is quadratic with respect to f, but highly nonlinear and nonconvex in velocity v (frequency dependent).
- Cycle skipping problem (eg. Symes, 1994).

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Extended Modeling & Null Space

Extended Modeling:

Let $\bar{f}(\mathbf{x}_r, \mathbf{x}_s, t)$ be the extended model of $f(t)$, define the extended modeling operator

$$\bar{S}[v, \bar{f}] = \bar{p}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where $\bar{p}$ is the solution of (1)-(2) with source function being $\bar{f}$.

Null Space (Annihilator):

- Differential semblance operator $A = \partial_{z_s}$ (see Symes (1994));
- t-moment operator after source signature deconvolution $A = tf^{-1}(t)$ (eg. deconvolution-based by Luo and Sava (2011), AWI by Warner (2014)).

MSWI

Matched Source Waveform Inversion (MSWI) is stated as follows,

$$J_{\mathrm{MS}}[v] = \frac{1}{2} \sum_{\mathbf{x}_r, \mathbf{x}_s} \int |A\bar{f}|^2 dt \tag{5}$$

$$s.t.\ \bar{S}[v, \bar{f}](\mathbf{x}_r, t; \mathbf{x}_s) = d(\mathbf{x}_r, t; \mathbf{x}_s). \tag{6}$$

Key feature: Even given wrong velocity, data fitting is perfect, hence no cycle skipping problem!

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Factorization of Tangent Operator

Under single arrival approximation, we have

$$\bar{S}[v, \bar{f}](\mathbf{x}_r, t; \mathbf{x}_s) \approx a(\mathbf{x}_r, \mathbf{x}_s)\bar{f}(\mathbf{x}_r, \mathbf{x}_s, t - \tau(\mathbf{x}_r, \mathbf{x}_s))$$

Then taking the first order variation of $\bar{S}[v, \bar{f}]$ formally gives us the desired factorization of operator

$$(D\bar{S}[v]\delta v)\bar{f}(\mathbf{x}_r, t; \mathbf{x}_s) \approx a(\mathbf{x}_r, \mathbf{x}_s)\frac{\partial}{\partial t}\bar{f}\left(\mathbf{x}_r, \mathbf{x}_s, t - \tau(\mathbf{x}_r, \mathbf{x}_s)\right)(-D\tau[v]\delta v)$$
$$\triangleq \bar{S}[v, Q[v, \delta v]\bar{f}].$$

where $D\tau[v]$ is the tangent operator of traveltime function, and $Q[v, \delta v]\bar{f}$ is bilinear operator with respect to δv and f with

$$Q[v, \delta v] = -(D\tau[v]\delta v)\frac{\partial}{\partial t}.$$

Backprojection of Traveltime Differences

Taking the first order perturbation of J_{MS}, we have

$$DJ_{\mathrm{MS}}[v]\delta v \approx \sum_{\mathbf{x}_r,\mathbf{x}_s} \int A^T A\bar{f}(D\tau[v]\delta v \bar{f}_t)dt$$

Hence the gradient is given by,

$$g \approx \sum_{\mathbf{x}_r,\mathbf{x}_s} D\tau[v]^T \left(\int (A^T A\bar{f})\bar{f}_t dt \right)$$

Here $D\tau[v]^T$ is the adjoint operator of $D\tau[v]$, which backprojects its arguments along rays.

Adjoint Source in the Gradient

Assume that data is noise-free and well approximated by geometric optics,

$$d(\mathbf{x}_r, t; \mathbf{x}_s) \approx a^*(\mathbf{x}_r, \mathbf{x}_s) f^*(t - \tau[v^*](\mathbf{x}_r, \mathbf{x}_s)). \tag{7}$$

Denote by $\Delta\tau(\mathbf{x}_r, \mathbf{x}_s) = \tau[v^*](\mathbf{x}_r, \mathbf{x}_s) - \tau[v](\mathbf{x}_r, \mathbf{x}_s)$, then

- For differential semblance operator $A = \partial_{z_s}$,

$$\int (A^T A\bar{f})\bar{f}_t dt \approx -\left(\frac{a^*}{a}\right)^2 \int \left(\frac{\partial f^*}{\partial t}\right)^2 dt \left(\frac{\partial}{\partial z_s}\right)^T \left(\frac{\partial}{\partial z_s}\Delta\tau[v](z_r, z_s)\right).$$

- For t-moment operator, we have

$$\int (A^T A\bar{f})\bar{f}_t dt \approx -(a^*/a)^2 \Delta\tau[v](z_r, z_s).$$

Traveltime Tomography

Travel tomography attempts to minimize the difference between the computed traveltime and picked traveltime from data,

$$J_{\mathrm{TT}} = \frac{1}{2} \sum_{\mathbf{x}_r, \mathbf{x}_s} \| \tau[v](\mathbf{x}_r, \mathbf{x}_s) - \tau[v^*](\mathbf{x}_r, \mathbf{x}_s) \|^2,$$

The gradient of J_{TT} is given by,

$$\nabla J_{\mathrm{TT}} = - \sum_{\mathbf{x}_r, \mathbf{x}_s} D\tau[v]^T (\Delta \tau[v](z_r, z_s)).$$

Local Convexity of Hessian

The Hessian of J_{MS} at consistent data $A\bar{f} = 0$ is given by

$$D^2 J[v^*](\delta v, \delta v) \approx \sum_{\mathbf{x}_r, \mathbf{x}_s} \int |[A, Q](v^*, \delta v)\bar{f}|^2 dt$$

where $[A, Q] = AQ - QA$.

- for $A = \partial_{z_s}$,

$$D^2 J[v^*](\delta v, \delta v) \approx \int \left(\frac{\partial f^*}{\partial t}\right)^2 dt \Big(\sum_{\mathbf{x}_r, \mathbf{x}_s} \left| \frac{\partial}{\partial z_s} D\tau[v^*]\delta v \right|^2 \Big)$$

- for $A = tf^{-1}(t)$,

$$D^2 J[v^*](\delta v, \delta v) \approx \sum_{\mathbf{x}_r, \mathbf{x}_s} |D\tau[v^*]\delta v|^2$$

Hessian of J_{MS} is proportional to the Hessian of a traveltime objective function, and is as convex as tomographic objective.

Relation with DSO formulation

DSO formulation seeks the balance between data fitting and annihilator term, i.e.

$$J_\alpha[v,\bar{f}] = \frac{1}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s}\int |\bar{S}[v,\bar{f}] - d|^2 dt + \frac{\alpha}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s}\int |A\bar{f}|^2 dt.$$

- As $\alpha \to 0$, the gradient and Hessian of $\frac{1}{\alpha}J_\alpha$ is the same as J_{MS}.
- As $\alpha \to +\infty$, $A\bar{f} = 0$, then it's equivalent to minimize J_{FWI}

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Example 1: Velocity Scan of J_{MS}

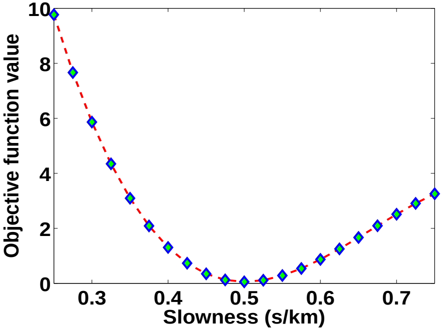


Figure: $J_{\mathrm{MS}}[v]$ for homogeneous velocity v, 0.25 s/km $\leq v^{-1} \leq$ 0.75 s/km. Correct velocity is 2 km/s.

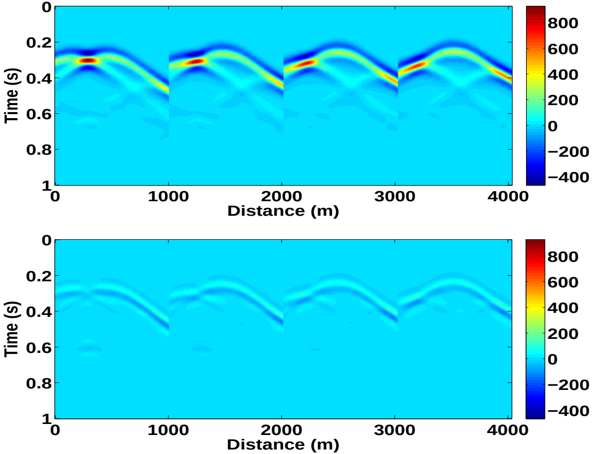
Example 2: Gaussian Lens

In this example, the target velocity consists of two Gaussian velocity anomalies embedded in a $v = 2$km/s background:

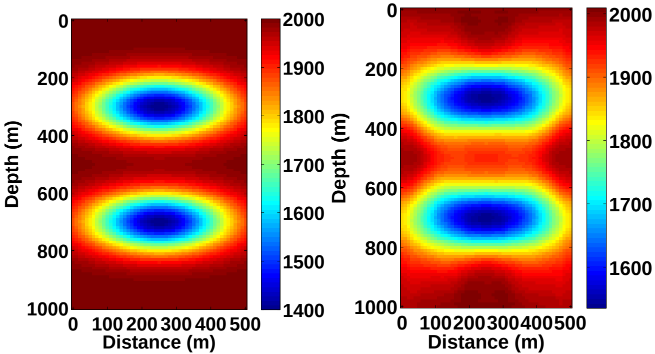
$$v(x,z) = 2 - 0.6e^{-\frac{(x-0.25)^2+(z-0.3)^2}{(0.2)^2}} - 0.6e^{-\frac{(x-0.25)^2+(z-0.7)^2}{(0.1)^2}},$$

where $x \in [0, 0.5]\mathrm{km}, z \in [0, 1]\mathrm{km}$. The initial model is given by the constant velocity $v_0 = 2$ km/s. Data is consisted of 50 shots and 99 receivers for each shot, which are uniformly distributed.

Example 2: Gaussian Lens



Example 2: Gaussian Lens



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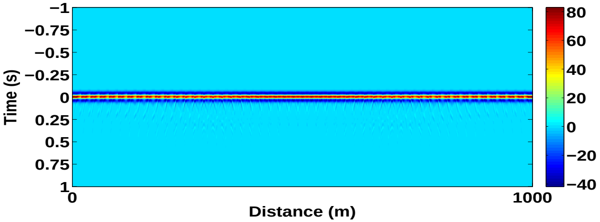


Figure: Extended source functions

Example 3: Big Gaussian

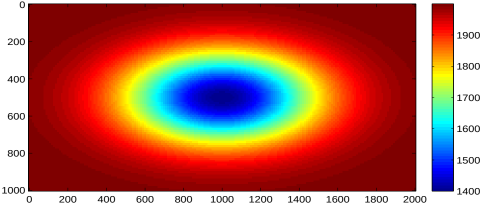


Figure: Low velocity Gaussian anomaly model with radius $250m \times 150m$ embedded in the constant background velocity $v_0 = 2000m/s$

Example 3: Big Gaussian

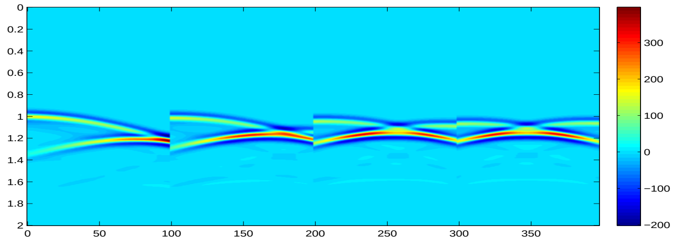
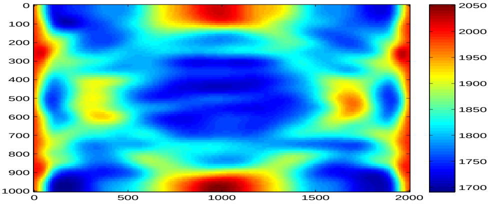


Figure: Receivers are uniformly distributed long $x_r = 1990$m from $z_r = 10$m to $x_r = 990$m for each shots, shots interval is 20m. The zero-phase ricker wavelet with main frequency $f = 10$Hz for generating the data.

Example 3: Big Gaussian



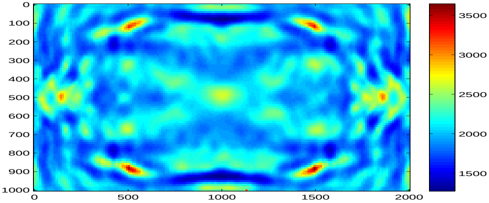


Figure: Inverted velocity after 50 iterations by MSWI (top) and FWI (bottom)

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Conclusion and Discussion

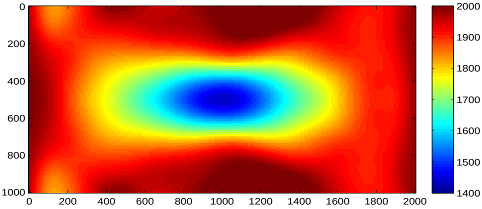
Conclusion

- Establish the relation between waveform inversion and traveltime tomography;
- Convexity of objective function is independent of frequency content.
- MSWI still get stuck in local minima due to the strong multipaths.

Discussion

- Does perfect data fitting (no cycle skipping) mean avoiding local minima;
- How do we choose the extended model space and the related annihilator.

To be cont'd



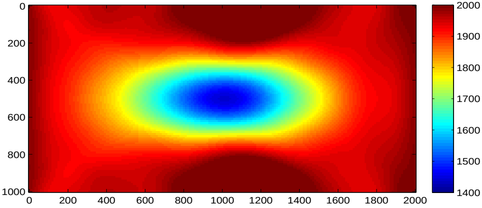


Figure: Inverted velocity after 100 iterations and 200 iterations

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