Research Article

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Fully degenerate Bell polynomials associated with degenerate Poisson random variables

https://doi.org/10.1515/math-2021-0022
received July 12, 2020; accepted February 4, 2021

Abstract: Many mathematicians have studied degenerate versions of quite a few special polynomials and numbers since Carlitz's work (Utilitas Math. **15** (1979), 51–88). Recently, Kim et al. studied the degenerate gamma random variables, discrete degenerate random variables and two-variable degenerate Bell polynomials associated with Poisson degenerate central moments, etc. This paper is divided into two parts. In the first part, we introduce a new type of degenerate Bell polynomials associated with degenerate Poisson random variables with parameter $\alpha > 0$, called the fully degenerate Bell polynomials. We derive some combinatorial identities for the fully degenerate Bell polynomials related to the nth moment of the degenerate Poisson random variable, special numbers and polynomials. In the second part, we consider the fully degenerate Bell polynomials associated with degenerate Poisson random variables with two parameters $\alpha > 0$ and $\beta > 0$, called the two-variable fully degenerate Bell polynomials. We show their connection with the degenerate Poisson central moments, special numbers and polynomials.

Keywords: Bell polynomials and numbers, degenerate Bell polynomials and numbers, Poisson random variable, degenerate Poisson random variable, the Poisson degenerate central moments

MSC 2020: 11B73, 11B83, 05A19

1 Introduction

Carlitz [1] initiated a study of degenerate versions of some special polynomials and numbers, called the degenerate Bernoulli and Euler polynomials and numbers. In recent years, many mathematicians have studied various degenerate versions of many special polynomials and numbers in some arithmetic and combinatorial aspects and probability theory (see [2–16]). Recently, Kim et al. studied degenerate gamma random variables, discrete degenerate random variables and two-variable degenerate Bell polynomials associated with Poisson degenerate central moments, etc. (see [11–14]).

A Poisson random variable indicates how many events occurred within a given period of time. A random variable X is a real valued function defined on a sample space. If X takes any values in a countable set, then X is called a discrete random variable.

A random variable X taking on one of the values $0, 1, 2, \dots$ is said to be the Poisson random variable with parameter α (>0) if the probability mass function of X is given by

$$p(i) = P\{X = i\} = e^{-\alpha}\frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \dots, \quad \text{(see [11, 12]).}$$

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Note that

$$\sum_{i=0}^{\infty} p(i) = e^{-\alpha} \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} = e^{-\alpha} e^{\alpha} = 1.$$

Kim et al. [11] considered the degenerate Poisson random variable $X(:X_\lambda)$ with parameter $\alpha(>0)$ if the probability mass function of X is given by

$$p_\lambda(i) = P\{X = i\} = e_\lambda^{-1}(\alpha) \frac{\alpha^i (1)_{i,\lambda}}{i!}, \quad i = 0, 1, 2, \ldots, \quad (\text{see } [11-13]).$$

We note that

$$\sum_{i=0}^{\infty} p_\lambda(i) = e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{i!} = e_\lambda^{-1}(\alpha) e_\lambda(\alpha) = 1.$$

Let us take an interesting example in which we consider the degenerate Poisson random variable with parameter $\alpha(>0)$. Let us assume that the probability of success in an experiment is p. We wondered if we can say the probability of success in the 20th trial is still p after failing 19 times in 20 trial experiments. Because there is a psychological burden to be successful. It seems plausible that the probability is less than p (see [2]).

Thus, we study a new type of degenerate Bell polynomials associated with the degenerate Poisson random variable with parameters in this paper. In Section 2, we introduce a new type of degenerate Bell polynomials and numbers associated with the degenerate Poisson random variable with parameter $\alpha > 0$, called the fully degenerate Bell polynomials and numbers (or single variable fully degenerate Bell polynomials). We show their connections with nth moment of the degenerate Poisson random variable with parameter $\alpha > 0$, and give several identities related to these polynomials including the degenerate Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, degenerate derangement numbers, degenerate Frobenius-Euler polynomials and numbers, etc. In Section 3, we will introduce the fully degenerate Bell polynomials associated with degenerate Poisson random variables with two parameters $\alpha > 0$ and $\beta > 0$, called the two-variable fully degenerate Bell polynomials. We show that they are equal to the two-variable fully degenerate Bell polynomials and the Poisson degenerate central moments. Also, we derive some explicit expressions for the two-variable degenerate Bell polynomials. Here we note that the two-variable fully degenerate Bell polynomials are generalization of the fully degenerate Bell polynomials associated with degenerate Poisson random variables with one parameter $\alpha > 0$.

When a pandemic such as Corona virus spreads throughout society, it changes the psychology of people on both an individual and group level, and in a broader sense the psychology of the community as a whole. In this respect, it is expected that relations with the fully degenerate Bell polynomials and moment of the degenerate Poisson random variable will be applied to predict how many people will be infected within a given period when a number of variables interact in a given environment.

Now, we give some definitions and properties needed in this paper.

For any nonzero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}} \quad (\text{see } [2-14]). \tag{1}$$

By Taylor expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad (\text{see } [2-14]), \tag{2}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda)\cdots(x - (n - 1)\lambda)$, $(n \geq 1)$.

Note that

$$\lim_{\lambda \to 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}.$$

It is well known that

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0) \quad (\text{see [3–5]}). \tag{3}$$

As an inversion formula of (3), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0) \quad (\text{see [3–5]}), \tag{4}$$

and

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \quad (\text{see [5]}), \tag{5}$$

and

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \quad (\text{see [3, 5]}). \tag{6}$$

For $u \in \mathbb{C}$ with $u \neq 1$, the classical Frobenius-Euler polynomials $h_n(x|u)$ are defined by means of the following generating function

$$\frac{1-u}{e^t - u}e^{xt} = \sum_{n=0}^{\infty} h_n(x|u)\frac{t^n}{n!} \quad (\text{see [7, 8]}).$$

In the special case when $x = 0$, $h_n(u) = h_n(0|u)$ are called nth Frobenius-Euler numbers. When $u = -1$, $h_n(x : -1) = E_n(x)$, are called the Euler polynomials (see [1]).

Kim et al. introduced the degenerate Frobenius-Euler polynomials defined by

$$\frac{1-u}{e_\lambda(t) - u}e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!} \quad (\text{see [7]}). \tag{7}$$

When $x = 0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called the degenerate Frobenius-Euler numbers.

As is well known, the Bell polynomials (also called Tochard polynomials or exponential polynomials) are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x)\frac{t^n}{n!} \quad (\text{see [17–20]}).$$

Kim et al. studied the degenerate Bell polynomials of as which are given by

$$e_\lambda^{-1}(\alpha)e_\lambda(\alpha e^t) = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(\alpha)\frac{t^n}{n!} \quad (\text{see [11]}).$$

When $x = 1$, $\mathrm{Bel}_{n,\lambda}(1) = \mathrm{Bel}_{n,\lambda}$ are called the degenerate Bell numbers. We note that $\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x) = \mathrm{Bel}_n(x)$.

2 Fully degenerate Bell polynomials associated with degenerate Poisson random variable with parameter $\alpha > 0$

In this section, we introduce a new type of degenerate Bell polynomials and numbers associated with degenerate Poisson random variable with parameter $\alpha > 0$, called the fully degenerate Bell polynomials and numbers. We give several combinatorial identities related to these polynomials and numbers.

From this section, for $\lambda \in \mathbb{R}$, let $X\,(:X_\lambda)$ be the degenerate Poisson random variable with parameter $\alpha > 0$ if the probability mass function of X is given by

$$P_\lambda(i) = P\{X = i\} = e_\lambda^{-1}(\alpha)\frac{\alpha^i(1)_{i,\lambda}}{i!} \quad \text{(see [11]).} \tag{8}$$

For $n \in \mathbb{N}$, we note that the expectation and the nth moments of X with parameter $\alpha > 0$ are

$$E[X] = e_\lambda^{-1}(\alpha)\sum_{i=0}^{\infty}\frac{\alpha^i(1)_{i,\lambda}}{i!}i \tag{9}$$

and

$$E[X^n] = \sum_{i=0}^{\infty} i^n p_\lambda(i) = e_\lambda^{-1}(\alpha)\sum_{i=0}^{\infty}\frac{\alpha^i(1)_{i,\lambda}}{i!}i^n, \tag{10}$$

respectively.

From (2), we also observe

$$E[e_\lambda^X(t)] = \sum_{i=0}^{\infty} e_\lambda^i(t)p_\lambda(i) = e_\lambda^{-1}(\alpha)\sum_{i=0}^{\infty} e_\lambda^i(t)\frac{\alpha^i}{i!}(1)_{i,\lambda} = e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t)). \tag{11}$$

In view of (11), naturally, we can define a new type of degenerate Bell polynomials, called the fully degenerate Bell polynomials as follows:

$$e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t)) = \sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^*(\alpha)\frac{t^n}{n!}. \tag{12}$$

When $\alpha = 1$, $\mathrm{Bel}_{n,\lambda}^*(1) := \mathrm{Bel}_{n,\lambda}^*$ is called the fully degenerate Bell numbers.

We note that $\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}^*(\alpha) = \mathrm{Bel}_n(\alpha)$.

Theorem 1. *Let X be a degenerate Poisson random variable with parameter $\alpha\,(>0)$. For $n \in \mathbb{N}$, we have*

$$\mathrm{Bel}_{n,\lambda}^*(\alpha) = E[(X)_{n,\lambda}].$$

Proof. From (9), we observe that

$$E[(X)_{n,\lambda}] = E[X(X-\lambda)\cdots(X-(n-1)\lambda)] = e_\lambda^{-1}(\alpha)\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}\alpha^k}{k!}(k)_{n,\lambda}. \tag{13}$$

On the other hand, from (2), (11) and (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^*(\alpha)\frac{t^n}{n!} &= E[e_\lambda^X(t)] = \sum_{i=0}^{\infty}e_\lambda^i(t)p_\lambda(i)\\ &= e_\lambda^{-1}(\alpha)\sum_{k=0}^{\infty}e_\lambda^k(t)\frac{\alpha^k}{k!}(1)_{k,\lambda}\\ &= e_\lambda^{-1}(\alpha)\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}(k)_{n,\lambda}\frac{t^n}{n!}\right)(1)_{k,\lambda}\frac{\alpha^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(e_\lambda^{-1}(\alpha)\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}\alpha^k}{k!}(k)_{n,\lambda}\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}E[(X)_{n,\lambda}]\frac{t^n}{n!}.\end{aligned} \tag{14}$$

Therefore, by comparing the coefficients on both sides of (14), we have the desired result. □

From Theorem 1 and (13), we obtain the following Dovinski-like formula for the fully degenerate Bell numbers as follows:

Corollary 2. *For $n \in \mathbb{N}$, we have*

$$\mathrm{Bel}^*_{n,\lambda} = e_\lambda^{-1}(1) \sum_{k=0}^{\infty} (1)_{k,\lambda} \frac{(k)_{n,\lambda}}{k!}.$$

In addition, when $\lim_{\lambda \to 0}$, *we get*

$$\mathrm{Bel}^*_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Theorem 3. *For $n \in \mathbb{N}$, we have*

$$\mathrm{Bel}^*_{n,\lambda}(\alpha) = \sum_{k=0}^{n} \left(\frac{\alpha}{1+\lambda\alpha}\right)^k (1)_{k,\lambda} S_{2,\lambda}(n,k).$$

In particular, for $\alpha = 1$,

$$\mathrm{Bel}^*_{n,\lambda} = \sum_{k=0}^{n} \left(\frac{1}{1+\lambda}\right)^k (1)_{k,\lambda} S_{2,\lambda}(n,k).$$

Proof. From (1) and (5), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!} &= e_\lambda^{-1}(\alpha) e_\lambda(\alpha e_\lambda(t)) = \left(\frac{1+\lambda\alpha e_\lambda(t)}{1+\lambda\alpha}\right)^{\frac{1}{\lambda}} \\
&= \left(1 + \frac{\lambda\alpha}{1+\lambda\alpha}(e_\lambda(t)-1)\right)^{\frac{1}{\lambda}} \\
&= \sum_{k=0}^{\infty} (1)_{k,\lambda}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k \frac{(e_\lambda(t)-1)^k}{k!} \\
&= \sum_{k=0}^{\infty} (1)_{k,\lambda}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\frac{\alpha}{1+\lambda\alpha}\right)^k (1)_{k,\lambda} S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Therefore, by comparing the coefficients on both sides of (15), we get the desired result. □

Theorem 4. *For $n \in \mathbb{N}$, we have*

$$\mathrm{Bel}^*_{n,\lambda}(\alpha) = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l}\sum_{j=0}^{l-k} \binom{l}{k} (-1)^j \lambda^{l-k-j} (1)_{k,\lambda} S_1(l-k,j)(k)_{n,\lambda}\right)\frac{\alpha^l}{l!}.$$

Proof. We observe

$$\begin{aligned}
e_\lambda^m(x) = e^{\frac{m}{\lambda}\log(1+\lambda x)} &= \sum_{k=0}^{\infty} \left(\frac{m}{\lambda}\right)^k \frac{(\log(1+\lambda x))^k}{k!} \\
&= \sum_{k=0}^{\infty} m^k \lambda^{-k} \sum_{n=k}^{\infty} S_1(n,k)\frac{\lambda^n x^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} m^k \lambda^{n-k} S_1(n,k)\right)\frac{x^n}{n!}.
\end{aligned} \tag{16}$$

By using Theorem 1 and (14), we have

$$\begin{aligned}\mathrm{Bel}^*_{n,\lambda}(\alpha) &= E[(X)_{n,\lambda}]\\ &= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}(-1)^j(\lambda)^{m-j}S_1(m,j)\right)\frac{\alpha^m}{m!}\times\sum_{k=0}^{\infty}(1)_{k,\lambda}(k)_{n,\lambda}\frac{\alpha^k}{k!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}\binom{l}{k}\sum_{j=0}^{l-k}(-1)^j\lambda^{l-k-j}S_1(l-k,j)(1)_{k,\lambda}(k)_{n,\lambda}\right)\frac{\alpha^l}{l!}\end{aligned} \tag{17}$$

Therefore, from (17), we get the desired result. □

Theorem 5. *For $n \in \mathbb{N}$, we have*

$$\mathrm{Bel}^*_{n,\lambda}(\alpha) = \sum_{l=0}^{\infty}\sum_{n_1=0}^{l}\sum_{k_2=0}^{l-n_1}\sum_{k_1=0}^{n_1}\binom{l}{n_1}\frac{(-1)^{k_2}n_1^k\lambda^{l-k_1-k_2+n-k}\alpha^l}{l!}S_1(l-n_1,k_2)S_1(n_1,k_1)S_1(n,k).$$

Proof. By using (16), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!} &= e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))\\ &= \sum_{n_2=0}^{\infty}\left(\sum_{k_2=0}^{n_2}(-1)^{k_2}(\lambda)^{n_2-k_2}S_1(n_2,k_2)\right)\frac{\alpha^{n_2}}{n_2!}\sum_{n_1=0}^{\infty}\left(\sum_{k_1=0}^{n_1}(\lambda)^{n_1-k_1}S_1(n_1,k_1)\right)\frac{\alpha^{n_1}}{n_1!}e_\lambda^{n_1}(t)\\ &= \sum_{l=0}^{\infty}\left(\sum_{n_1=0}^{l}\sum_{k_2=0}^{l-n_1}\sum_{k_1=0}^{n_1}\binom{l}{n_1}(-1)^{k_2}(\lambda)^{l-k_1-k_2}S_1(l-n_1,k_2)S_1(n_1,k_1)\right)\frac{\alpha^l}{l!}\\ &\quad\times\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}n_1^k(\lambda)^{n-k}S_1(n,k)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{n_1=0}^{l}\sum_{k_2=0}^{l-n_1}\sum_{k_1=0}^{n_1}\binom{l}{n_1}\frac{(-1)^{k_2}n_1^k\lambda^{l-k_1-k_2+n-k}\alpha^l}{l!}\right.\\ &\quad\left.\times S_1(l-n_1,k_2)S_1(n_1,k_1)S_1(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{18}$$

Thus, by comparing the coefficients on both sides of (18), we get the desired result. □

A derangement is a permutation with no fixed points. The number of derangements of an n-element set is called the nth derangement number and denoted by d_n. This number satisfies the following recurrences:

$$d_n = n\cdot d_{n-1} + (-1)^n, \quad n \geq 1. \tag{19}$$

By (19), we get

$$d_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}, \quad n \geq 0 \quad \text{(see [19])}. \tag{20}$$

From (20), we can derive the following generating function of the number of derangements of an n-element set

$$\frac{1}{1-t}e^{-t} = \left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^k\right)\left(\sum_{m=0}^{\infty}t^m\right) = \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}d_n\frac{t^n}{n!}. \tag{21}$$

Recently, Kim et al. considered the derangement polynomials by the generating function

$$\frac{1}{1-xt}e^{-t} = \sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!} \quad \text{(see [19])}. \tag{22}$$

When $x = 1$, $d_n(1) = d_n$, $n \geq 0$.

From (22), we naturally define the degenerate derangement polynomials by

$$\frac{1}{1-xt}e_\lambda(-t)=\sum_{n=0}^{\infty}d_{n,\lambda}(x)\frac{t^n}{n!}. \tag{23}$$

When $x=1$, $d_{n,\lambda}(1):=d_{n,\lambda}$ is called the degenerate derangement numbers.

We note that $\lim_{\lambda\to 0}d_{n,\lambda}(1)=d_n$, $n\geq 0$.

Theorem 6. *For $n\in\mathbb{N}\cup 0$, we get*

$$\frac{1}{1+\alpha}\sum_{l=0}^{n}\binom{n}{l}h^*_{l,\lambda}(-\alpha^{-1})\mathrm{Bel}^*_{n-l,\lambda}(\alpha)=e_\lambda^{-1}(\alpha)\sum_{k=0}^{n}\sum_{l=0}^{\infty}\frac{(-\alpha)^l}{l!}d_{l,\lambda}l^k\lambda^{n-k}S_1(n,k),$$

where $h_{n,\lambda}(u)$ are the degenerate Frobenius-Euler numbers.

Proof. By using (16) and (23),

$$\begin{aligned}\frac{1}{1+\alpha e_\lambda(t)}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!}&=\frac{1}{1+\alpha e_\lambda(t)}e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))\\&=e_\lambda^{-1}(\alpha)\sum_{l=0}^{\infty}d_{l,\lambda}\frac{(-\alpha e_\lambda(t))^l}{l!}\\&=e_\lambda^{-1}(\alpha)\sum_{l=0}^{\infty}d_{l,\lambda}\frac{(-\alpha)^l}{l!}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}l^k\lambda^{n-k}S_1(n,k)\right)\frac{t^n}{n!}\\&=e_\lambda^{-1}(\alpha)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{\infty}\frac{(-\alpha)^l}{l!}d_{l,\lambda}l^k\lambda^{n-k}S_1(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{24}$$

On the other hand, from (7), we get

$$\begin{aligned}\frac{1}{1+\alpha e_\lambda(t)}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!}&=\frac{\alpha^{-1}}{e_\lambda(t)+\alpha^{-1}}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!}\\&=\frac{\alpha^{-1}}{1+\alpha^{-1}}\frac{1-(-\alpha^{-1})}{e_\lambda(t)-(-\alpha^{-1})}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!}\\&=\frac{1}{1+\alpha}\sum_{l=0}^{\infty}h^*_{l,\lambda}(-\alpha^{-1})\frac{t^l}{l!}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha)\frac{t^n}{n!}\\&=\frac{1}{1+\alpha}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}h^*_{l,\lambda}(-\alpha^{-1})\mathrm{Bel}^*_{n-l,\lambda}(\alpha)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Thus, from (24) and (25), we get

$$\frac{1}{1+\alpha}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}h^*_{l,\lambda}(-\alpha^{-1})\mathrm{Bel}^*_{n-l,\lambda}(\alpha)\frac{t^n}{n!}=e_\lambda^{-1}(\alpha)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{\infty}\frac{(-\alpha)^l}{l!}d_{l,\lambda}l^k\lambda^{n-k}S_1(n,k)\right)\frac{t^n}{n!}. \tag{26}$$

Therefore, by comparing the coefficients on both sides of (26), we arrive at what we want. □

Theorem 7. *For $n\in\mathbb{N}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{Bel}^*_{n,\lambda}(\alpha)=\frac{1}{1+\lambda\alpha}\left(\sum_{l=0}^{n}\binom{n}{l}h^*_{l,\lambda}(1|-\lambda^{-1}\alpha^{-1})\mathrm{Bel}^*_{n-l,\lambda}(\alpha)-\mathrm{Bel}^*_{n,\lambda}(\alpha)\right),$$

where $h_{n,\lambda}(x|u)$ are called the degenerate Frobenius-Euler polynomials.

Proof. First, we note that

$$\frac{d}{d\alpha}e_\lambda(\alpha)=e_\lambda^{1-\lambda}(\alpha)\quad\text{and}\quad\frac{d}{d\alpha}e_\lambda^{-1}(\alpha)=-\frac{1}{1+\lambda\alpha}e_\lambda^{-1}(\alpha).\tag{27}$$

By using (1), (7) and (27), we get

$$\begin{aligned}\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\frac{t^n}{n!}\right)&=\frac{d}{d\alpha}(e_\lambda(\alpha e_\lambda(t))e_\lambda^{-1}(\alpha))\\&=e_\lambda^{-1}(\alpha)e_\lambda(t)e_\lambda^{1-\lambda}(\alpha e_\lambda(t))-\frac{1}{1+\lambda\alpha}e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))\\&=e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))\left(e_\lambda(t)e_\lambda^{-\lambda}(\alpha e_\lambda(t))-\frac{1}{1+\lambda\alpha}\right)\\&=\left(\frac{e_\lambda(t)}{1+\lambda\alpha e_\lambda(t)}-\frac{1}{1+\lambda\alpha}\right)\sum_{m=0}^{\infty}\mathrm{Bel}_{m,\lambda}^{*}(\alpha)\frac{t^m}{m!}\\&=\left(\frac{(\lambda\alpha)^{-1}}{1+(\lambda\alpha)^{-1}}\frac{1-(-(\lambda\alpha)^{-1})}{e_\lambda(t)-(-(\lambda\alpha)^{-1})}e_\lambda(t)-\frac{1}{1+\lambda\alpha}\right)\sum_{m=0}^{\infty}\mathrm{Bel}_{m,\lambda}^{*}(\alpha)\frac{t^m}{m!}\\&=\left(\frac{1}{1+\lambda\alpha}\sum_{l=0}^{\infty}h_{l,\lambda}^{*}(1|-\lambda^{-1}\alpha^{-1})\frac{t^l}{l!}-\frac{1}{1+\lambda\alpha}\right)\sum_{m=0}^{\infty}\mathrm{Bel}_{m,\lambda}^{*}(\alpha)\frac{t^m}{m!}\\&=\frac{1}{1+\lambda\alpha}\sum_{n=0}^{\infty}\binom{n}{l}h_{l,\lambda}^{*}(1|-\lambda^{-1}\alpha^{-1})\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha)\frac{t^n}{n!}-\frac{1}{1+\lambda\alpha}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\frac{t^n}{n!}\\&=\frac{1}{1+\lambda\alpha}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}^{*}(1|-\lambda^{-1}\alpha^{-1})\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha)-\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\right)\frac{t^n}{n!}.\end{aligned}\tag{28}$$

Therefore, by comparing the coefficients on both sides of (28), we get what we want. □

Theorem 8. *For $n\in\mathbb{N}$, we have*

$$\mathrm{Bel}_{n+1,\lambda}^{*}(\alpha)=e_\lambda^{-1}(\alpha)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\alpha^m\frac{1}{(m-1)!}(m-\lambda)_{n,\lambda}\right).$$

Proof. By using (2), we get

$$\begin{aligned}\frac{d}{dt}\left(\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\frac{t^n}{n!}\right)&=\frac{d}{dt}(e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t)))\\&=e_\lambda^{-1}(\alpha)\frac{d}{dt}(1+\lambda\alpha e_\lambda(t))^{\frac{1}{\lambda}}\\&=e_\lambda^{-1}(\alpha)\frac{d}{dt}\left(\sum_{m=0}^{\infty}\binom{\frac{1}{\lambda}}{m}(\lambda\alpha)^m e_\lambda^m(t)\right)\\&=e_\lambda^{-1}(\alpha)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\alpha^m\frac{1}{(m-1)!}e_\lambda^{m-\lambda}(t)\right)\\&=e_\lambda^{-1}(\alpha)\sum_{m=0}^{\infty}(1)_{m,\lambda}\alpha^m\frac{1}{(m-1)!}\sum_{n=0}^{\infty}(m-\lambda)_{n,\lambda}\frac{t^n}{n!}\\&=e_\lambda^{-1}(\alpha)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\alpha^m\frac{1}{(m-1)!}(m-\lambda)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{29}$$

On the other hand, we have

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\frac{t^n}{n!}\right)=\sum_{n=1}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha)\frac{t^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n+1,\lambda}^{*}(\alpha)\frac{t^n}{n!}. \tag{30}$$

Therefore, by comparing the coefficients of (29) and (30), we get the desired result. □

3 Two-variable fully degenerate Bell polynomials

In this section, as one of the generalizations of the fully degenerate Bell polynomials in Section 2, we will introduce the two-variable fully degenerate Bell polynomials associated with degenerate Poisson random variables with two parameters $\alpha > 0$ and $\beta > 0$, and show their connection with the degenerate Poisson central moments. We also derive some explicit expressions for the two-variable degenerate Bell polynomials.

For a Poisson random variable X with parameter $\alpha > 0$, Kim et al. [13] considered the Poisson degenerate central moments given by $E[(X-\alpha)_{n,\lambda}]$, $(n \geq 0)$, where $(X-\alpha)_{0,\lambda}=1$, $(X-\alpha)_{n,\lambda}=(X-\alpha)(X-\alpha-\lambda)\cdots(X-\alpha-(n-1)\lambda)$, $(n \geq 1)$.

Note that

$$\lim_{\lambda\to 0}E[(X-\alpha)_{n,\lambda}]=E[(X-\alpha)^n].$$

In this section, we give a definition of two-variable fully degenerate Bell polynomials as follows:

$$\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)\frac{t^n}{n!}=e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))e_\lambda^{\beta-\alpha}(t). \tag{31}$$

When $\beta=\alpha$, for $n \geq 0$

$$\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\alpha)=\mathrm{Bel}_{n,\lambda}^{*}(\alpha),\quad (n\geq 0).$$

Theorem 9. *Let X be a degenerate Poisson random variable with two parameters $\alpha > 0$ and $\beta > 0$. For $n \geq 0$, we have*

$$\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)=E[(X-\alpha+\beta)_{n,\lambda}].$$

Proof. By using (13), we have

$$\begin{aligned}E[e_\lambda^{X-\alpha+\beta}(t)]&=\sum_{k=0}^{\infty}e_\lambda^{k-\alpha+\beta}(t)p_\lambda(k)\\&=e_\lambda^{\beta-\alpha}(t)\sum_{k=0}^{\infty}e_\lambda^{k}(t)e_\lambda^{-1}(\alpha)\frac{\alpha^k(1)_{k,\lambda}}{k!}\\&=e_\lambda^{\beta-\alpha}(t)e_\lambda^{-1}(\alpha)\sum_{k=0}^{\infty}e_\lambda^{k}(t)\frac{\alpha^k(1)_{k,\lambda}}{k!}\\&=e_\lambda^{-\alpha+\beta}(t)e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)\frac{t^n}{n!}.\end{aligned} \tag{32}$$

On the other hand, from (10), we observe

$$E[e_\lambda^{X-\alpha+\beta}(t)]=E\left[\sum_{n=0}^{\infty}(X-\alpha+\beta)_{n,\lambda}\frac{t^n}{n!}\right]=\sum_{n=0}^{\infty}E[(X-\alpha+\beta)_{n,\lambda}]\frac{t^n}{n!}. \tag{33}$$

Therefore, by comparing the coefficients of (34) and (33), we obtain what we want. □

Theorem 10. *For $n \geq 0$, we have*

$$\mathrm{Bel}^*_{n,\lambda}(\alpha,\beta) = \sum_{l=0}^{n}\binom{n}{l}\mathrm{Bel}^*_{l,\lambda}(\alpha)(\beta-\alpha)_{n-l,\lambda}.$$

In particular,

$$\mathrm{Bel}^*_{n,\lambda}(\alpha,\alpha) = \mathrm{Bel}^*_{n,\lambda}(\alpha) = \sum_{l=0}^{n}\binom{n}{l}\mathrm{Bel}^*_{l,\lambda}(\alpha) \quad (n \geq 0).$$

Proof. From (2) and (11), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}^*_{n,\lambda}(\alpha,\beta) &= e_\lambda^{-1}(\alpha)e_\lambda(\alpha e_\lambda(t))e_\lambda^{\beta-\alpha}(t)\\ &= \sum_{l=0}^{\infty}\mathrm{Bel}^*_{l,\lambda}(\alpha)\frac{t^l}{l!}\sum_{m=0}^{\infty}(\beta-\alpha)_{m,\lambda}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathrm{Bel}^*_{l,\lambda}(\alpha)(\beta-\alpha)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

Therefore, by comparing the coefficients on both sides of (34), we obtain the desired result. □

The following equation is needed to prove the next theorem.

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(y)_{n-k,\lambda}\ (n \geq 0). \tag{35}$$

Theorem 11. *For $n \geq 0$, we have*

$$\mathrm{Bel}^*_{n,\lambda}(\alpha,\beta) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k(\beta-\alpha)_{n-l,\lambda}(1)_{k,\lambda}S_{2,\lambda}(l,k).$$

Proof. From Theorems 3 and 10, we have

$$\begin{aligned}\mathrm{Bel}^*_{n,\lambda}(\alpha,\beta) &= \sum_{l=0}^{n}\binom{n}{l}(\beta-\alpha)_{n-l,\lambda}\mathrm{Bel}^*_{l,\lambda}(\alpha)\\ &= \sum_{l=0}^{n}\binom{n}{l}(\beta-\alpha)_{n-l,\lambda}\sum_{k=0}^{l}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k(1)_{k,\lambda}S_{2,\lambda}(l,k)\\ &= \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(\beta-\alpha)_{n-l,\lambda}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k(1)_{k,\lambda}S_{2,\lambda}(l,k).\end{aligned} \tag{36}$$

Thus, we get the desired result. □

Corollary 12. *Let X be a degenerate Poisson random variable with parameter $\alpha(>0)$.*

For $n \geq 0$, we get the Poisson degenerate central moments of X as follows:

$$E[(X-\alpha)_{n,\lambda}] = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\left(\frac{\alpha}{1+\lambda\alpha}\right)^k(-\alpha)_{n-l,\lambda}(1)_{k,\lambda}S_{2,\lambda}(l,k).$$

In particular, $\alpha = 1$, we have

$$E[(X-1)_{n,\lambda}] = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\left(\frac{1}{1+\lambda}\right)^k(-1)_{n-l,\lambda}(1)_{k,\lambda}S_{2,\lambda}(l,k).$$

Theorem 13. *For $n \geq 0$, we have*

$$\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(\beta-\alpha)^{k}\lambda^{l-k}S_{1}(l,k)\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha).$$

Proof. By using (12), (16) and (31), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)\frac{t^{n}}{n!}&=e_{\lambda}^{-1}(\alpha)e_{\lambda}(\alpha e_{\lambda}(t))e_{\lambda}^{\beta-\alpha}(t)\\&=\sum_{j=0}^{\infty}\mathrm{Bel}_{j,\lambda}^{*}(\alpha)\frac{t^{j}}{j!}\sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(\beta-\alpha)^{k}\lambda^{l-k}S_{1}(l,k)\right)\frac{t^{l}}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(\beta-\alpha)^{k}\lambda^{l-k}S_{1}(l,k)\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{37}$$

Therefore, by comparing the coefficients on both sides of (37), we have what we want. □

Theorem 14. *For $n \geq 0$, we have*

$$\frac{1}{1+\alpha}\sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}^{*}(-\alpha^{-1})\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha,\beta)=e_{\lambda}^{-1}(\alpha)\sum_{l=0}^{\infty}\frac{(-\alpha)^{l}}{l!}(l+\beta-\alpha)^{k}\lambda^{n-k}d_{l,\lambda}S_{1}(n,k),$$

where $h_{n,\lambda}(u)$ are called the degenerate Frobenius-Euler numbers.

Proof. By using (16) and (23),

$$\begin{aligned}\frac{1}{1+\alpha e_{\lambda}(t)}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{*}(\alpha,\beta)\frac{t^{n}}{n!}&=e_{\lambda}^{-1}(\alpha)\frac{1}{1+\alpha e_{\lambda}(t)}e_{\lambda}(\alpha e_{\lambda}(t))e_{\lambda}^{\beta-\alpha}(t)\\&=e_{\lambda}^{-1}(\alpha)\left(\sum_{l=0}^{\infty}d_{l,\lambda}\frac{(-\alpha e_{\lambda}(t))^{l}}{l!}\right)e_{\lambda}^{\beta-\alpha}(t)\\&=e_{\lambda}^{-1}(\alpha)\sum_{l=0}^{\infty}d_{l,\lambda}\frac{(-\alpha)^{l}}{l!}e_{\lambda}^{l+\beta-\alpha}(t)\\&=e_{\lambda}^{-1}(\alpha)\sum_{l=0}^{\infty}d_{l,\lambda}\frac{(-\alpha)^{l}}{l!}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(l+\beta-\alpha)^{k}\lambda^{n-k}S_{1}(n,k)\right)\frac{t^{n}}{n!}\\&=e_{\lambda}^{-1}(\alpha)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{\infty}\frac{(-\alpha)^{l}}{l!}(l+\beta-\alpha)^{k}\lambda^{n-k}d_{l,\lambda}S_{1}(n,k)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{38}$$

On the other hand, by the same way of (25), we get

$$\frac{1}{1+\alpha e_{\lambda}(t)}\sum_{m=0}^{\infty}\mathrm{Bel}_{m,\lambda}^{*}(\alpha,\beta)\frac{t^{m}}{m!}=\frac{1}{1+\alpha}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}^{*}(-\alpha^{-1})\mathrm{Bel}_{n-l,\lambda}^{*}(\alpha,\beta)\frac{t^{n}}{n!}.\tag{39}$$

Thus, by comparing the coefficients of (38) and (39), we get the desired result. □

4 Conclusion

In this paper, we introduced the fully degenerate Bell polynomials associated with degenerate Poisson random variables with parameter $\alpha > 0$ and the two-variable fully degenerate Bell polynomials associated with degenerate Poisson random variables with two parameters $\alpha > 0$ and $\beta > 0$. We showed their connections with nth moment of the degenerate Poisson random variables and the Poisson degenerate central moments, respectively. We also expressed those polynomials and numbers in terms of the degenerate Stirling numbers of the second kind; the degenerate Stirling numbers of the first kind; the degenerate derangement numbers and the Stirling numbers of the first kind; and degenerate Frobenius-Euler polynomials.

It is important that the study of the degenerate version is widely applied not only to numerical theory and combinatorial theory, but also to symmetric identity, differential equations and probability theory. The Bell numbers have also been extensively studied in many different context in such branches of Mathematics [16–22]. With this in mind, as a future project, I would like to continue to study degenerate versions of certain special polynomials and numbers.

Acknowledgements: The author would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form. And the author would like to thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding information: This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, the Ministry of Education (NRF-2018R1D1A1B07049584).

Conflict of interest: Author states no conflict of interest.

Ethics approval and consent to participate: The author reveals that there is no ethical problem in the production of this paper.

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