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Fully inert subgroups of divisible Abelian groups

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Abstract. A subgroup H of an Abelian group G is said to be fully inert if the quotient $(H + \phi(H))/H$ is finite for every endomorphism ϕ of G. Clearly, this is a common generalization of the notions of fully invariant, finite and finite-index subgroups. We investigate the fully inert subgroups of divisible Abelian groups, and in particular, those Abelian groups that are fully inert in their divisible hull, called inert groups. We prove that the inert torsion-free groups coincide with the completely decomposable homogeneous groups of finite rank and we give a complete description of the inert groups in the general case. This yields a characterization of the fully inert subgroups of divisible Abelian groups.

1 Introduction

In this paper we introduce and investigate the notions of ϕ-inert and fully inert subgroups of an Abelian group G, where ϕ denotes an endomorphism of G. These notions have their origin in the non-commutative setting, as we will briefly survey now.

Let G be an arbitrary non-commutative group. A subgroup H of G is *inert* if $H \cap H^g$ has finite index in H for every g in G (as usual, H^g denotes the conjugate of H under g); an inert subgroup is close to being normal. Normal subgroups, finite subgroups, subgroups of finite index, and permutable subgroups are all examples of inert subgroups. The group G is called *totally inert* if every subgroup of G is inert. The class of totally inert groups includes Dedekind groups, FC-groups, and Tarski monsters, but it contains no infinite locally-finite simple group, so it is a highly complex class (see [3]). The notions of inert subgroups and totally inert groups have been introduced in [2] and [3] (Belyaev [2] gives credit to Kegel for coining the term "inert subgroup"); totally inert groups have been studied also by Robinson in [14] under the name of inertial groups.

Recently, Dardano and Rinauro [4] changed the above setting simultaneously in two directions: first, general automorphisms (not necessarily internal, as in [2]

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and [3]) were involved. Secondly, they moved the focus onto automorphisms, by considering *inertial automorphisms* ϕ of a group G, defined by the property that $[H : H \cap \phi(H)]$ and $[\phi(H) : H \cap \phi(H)]$ are finite (i.e., H and $\phi(H)$ are *commensurable*) for every subgroup H of G. A characterization of the inertial automorphisms of an Abelian group is provided in [4]; when the Abelian group is torsion, they coincide with the so-called *almost power automorphisms*, studied by Franciosi, de Giovanni and Newell [9].

We move now definitively to the Abelian setting, so from now on the groups are always assumed to be Abelian. Inspired by the notions described above, with the relevant modification of considering endomorphisms of groups and not only automorphisms, we introduced in [6] the following notion.

Definition 1.1. Let G be an Abelian group and $\phi : G \to G$ an endomorphism. A subgroup H of G is called *ϕ-inert* if $H \cap \phi(H)$ has finite index in $\phi(H)$, equivalently, if the factor group $(H + \phi(H))/H$ is finite.

Our motivation for investigating ϕ-inert subgroups comes from the study of the dynamical properties of a given endomorphism $\phi : G \to G$ of an Abelian group G, developed in a series of papers concerning algebraic entropy (see [1, 15], the more recent [5, 7], and the references there). The algebraic entropy of ϕ, roughly speaking, is an invariant measuring how chaotically ϕ acts on the family of finite subgroups of G. It turns out that, taking the family of ϕ-inert subgroups instead of the generally smaller family of finite subgroups, and with some slight changes in the definition of the entropy function, we obtain a better behaved dynamical invariant called *intrinsic algebraic entropy* (see [6]).

The aspects of innovation in the definition of ϕ-inert subgroup are the following. First, as noted above, endomorphisms are considered in place of automorphisms; this automatically imposes a second difference, namely, one does not ask finiteness of the index $[H : H \cap \phi(H)]$: that would rule out some natural endomorphisms (e.g., the zero map, or the endomorphisms with finite image in case H is infinite). The third and most important one is that both the subgroup and the endomorphism are isolated into a 'local condition' that is not imposed on all subgroups or all endomorphisms.

We denote by $\mathcal{I}_\phi(G)$ the family of all ϕ-inert subgroups of G. Obviously $\mathcal{I}_\phi(G)$ contains all the ϕ-invariant subgroups of G, as well as the finite subgroups and the subgroups of finite index. Passing to a 'global condition', we have the following definition.

Definition 1.2. Let G be an Abelian group. A subgroup H of G is called *fully inert* if it is ϕ-inert for every endomorphism ϕ of G.

We denote by $\mathcal{I}(G)$ the family of all fully inert subgroups of G, that is,

$$\mathcal{I}(G) = \bigcap_{\phi \in \mathrm{End}(G)} \mathcal{I}_\phi(G).$$

The notion of fully inert subgroup can be viewed as a slight generalization of that of fully invariant subgroup. Clearly, besides fully invariant subgroups, also finite subgroups and subgroups of finite index are fully inert.

If the group G has few endomorphisms, the family of its fully inert subgroups can be very large; for instance, if $\mathrm{End}(G) = \mathbb{Z}$ (and this can happen for torsion-free groups G of arbitrary cardinality), then all the subgroups of G are fully invariant, hence fully inert too. Therefore the most interesting cases are when the endomorphism ring of G is huge, this case occurs, for instance, when G is divisible. So our main concern in this paper is to investigate fully inert subgroups of divisible groups and, in particular, the groups which are fully inert in their divisible hulls (this property is independent on the choice of the specific divisible hull of G); for these groups we coin an *ad hoc* name.

Definition 1.3. An Abelian group G is called *inert* if it is a fully inert subgroup of its divisible hull.

This paper is organized as follows. In Section 2 we collect preliminary results on ϕ-inert subgroups, some of which are extracted from [6], and on fully inert subgroups. In Section 3 fully inert subgroups of divisible groups are investigated, as a preparation for the main results of the paper contained in Sections 4, 5 and 6, where we characterize inert groups as follows.

In Section 4 we will characterize fully inert subgroups of finite direct sums $D = D_1 \oplus \cdots \oplus D_n$ of divisible groups by showing that they are commensurable with subgroups of D of the form $A = A_1 \oplus \cdots \oplus A_n$, where each A_i is a fully inert subgroup of D_i satisfying a particular condition with respect to D_j, for $j \neq i$ (Lemma 4.1 and Proposition 4.2). We describe in Corollary 4.3 when a direct sum of finitely many groups is inert in terms of its direct summands and their interrelations. An easy consequence of this useful criterion is the fact that finite direct powers of inert groups are inert. These results and a theorem of Procházka [13] are used to deduce the major theorems of this section:

(i) the inert torsion-free groups are the divisible groups and the completely decomposable homogeneous groups of finite rank (Theorem 4.9),

(ii) every inert mixed group splits (Theorem 4.10).

In Section 5 we characterize torsion and inert mixed groups. We show that the inert torsion groups G are exactly those of the form $G = F \oplus \bigoplus_{p \in P} T_p$, where F

is a finite group, P is a set of primes, and T_p is a homogeneous direct sum of co-cyclic p-groups for each prime $p \in P$, that is, of the form $\bigoplus_{\alpha_p} \mathbb{Z}(p^{m_p})$, with $\alpha_p > 0$ a cardinal and $m_p \in \mathbb{N}_+ \cup \{\infty\}$ (see Theorem 5.3). For a torsion group G of this form we define the *induced type* $[m_p^*]_{p\in\mathbb{P}}$ as follows: $m_p^* = m_p$ for $p \in P$, $m_p^* = \infty$ for $p \notin P$.

In the mixed case, inert groups are splitting (as mentioned above) with both the torsion and the torsion-free summands inert groups; furthermore, (apart the divisible case) the type of the torsion-free completely decomposable homogeneous summand is less than or equal to the induced type $[m_p^*]_{p\in\mathbb{P}}$ of the torsion summand, as described above.

The paper ends with Section 6, which completes the characterization of fully inert subgroups of divisible groups started in Section 3.

Notation and terminology

We denote by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}(m)$ and $\mathbb{Z}(p^\infty)$ the set of natural numbers, the set of positive natural numbers, the set of prime numbers, the group of integers, the group of rationals, the cyclic group of size m, and the co-cyclic divisible p-group, respectively. The word ‘group’ will always mean ‘Abelian group’.

For a group G, we denote by $t(G)$ the torsion subgroup of G, by G_p the p-primary component of $t(G)$, by $P(G) = \{p \in \mathbb{P} \mid G_p \neq 0\}$ the set of the relevant primes of G, and by $\mathrm{D}(G)$ the divisible hull of G. By $\mathrm{rk}(G)$ we denote the torsion-free rank of G, and by $r_p(G)$ its p-rank, that is, the dimension of the p-socle $G[p]$ viewed as vector space over $\mathbb{Z}(p)$. Let us recall that a mixed group G is *splitting* if its torsion part $t(G)$ is a direct summand of G; it is understood that by ‘mixed group’ we mean a group which is neither torsion, nor torsion-free. For unexplained notation and terminology we refer to [10].

2 ϕ-inert and fully inert subgroups

We recall that, given an endomorphism $\phi : G \to G$ and a subgroup H of G, $T_n(\phi, H)$ denotes the subgroup $\sum_{0\leqslant k<n} \phi^k(H)$ for all $n \in \mathbb{N}_+$, and

$$T(\phi, H) = \sum_{k=0}^{\infty} \phi^k(H)$$

is the minimal ϕ-invariant subgroup of G containing H. The subgroups $T_n(\phi, H)$ and $T(\phi, H)$ are called the n-th partial ϕ-trajectory and the ϕ-trajectory of H, respectively. These notions, in case H is a finite subgroup of G, are the basic ingredients for defining the algebraic entropy of ϕ (see [7]). As usual, if H is a subgroup of a torsion-free group G, the pure closure of H will be denoted by H^*; recall that $H^*/H = t(G/H)$.

We start with some results already proved in [6], or easily deducible from them. We provide their essential details, for the sake of completeness.

Lemma 2.1. *Let G be a group, $\phi : G \to G$ an endomorphism and H a ϕ-inert subgroup of G. Then the following assertions hold:*

(a) *$T_n(\phi, H)/H$ is finite for all $n \in \mathbb{N}_+$,*

(b) *$T(\phi, H)/H$ is torsion,*

(c) *if G is torsion-free, then $T(\phi, H)$ is contained in H^*, which is ϕ-invariant in G.*

Proof. See [6, Lemma 2.1]. Item (a) is easily proved by induction on n, the case $n = 2$ following by the definition of ϕ-inert subgroup; part (b) follows from the equality $T(\phi, H) = \bigcup_{n\in\mathbb{N}_+} T_n(\phi, H)$ and from the fact that each quotient group $T_n(\phi, H)/H$ is torsion. The first claim in (c) is an immediate consequence of (b), as $H^*/H = t(G/H)$. Finally, let $x \in H^*$; then one has $nx \in H$ for some $n \neq 0$. As $T(\phi, H)$ is ϕ-invariant in G, $n\phi(x) \in T(\phi, H)$, thus $m\phi(x) \in H$ for some $m \neq 0$ by (b); this shows that $\phi(x) \in H^*$, which is therefore ϕ-invariant in G. □

The following lemma shows that both $\mathcal{I}(G)$ and $\mathcal{I}_\phi(G)$ (for any endomorphism $\phi : G \to G$) are sublattices of the lattice of all the subgroups of G. Notice that these lattices are bounded, as $\{0\}$ and G are always ϕ-inert for all $\phi \in \mathrm{End}(G)$.

Lemma 2.2. *Let G be a group, $\phi : G \to G$ an endomorphism.*

(a) *If H and H' are ϕ-inert subgroups of G, then $H \cap H'$ and $H + H'$ are both ϕ-inert.*

(b) *If H is ϕ-inert, then $\phi^n(H)$ is ϕ-inert and H is ϕ^n-inert for every $n \in \mathbb{N}$.*

(c) *If H and H' are fully inert subgroups of G, then $H \cap H'$ and $H + H'$ are both fully inert.*

Proof. (a) See [6, Lemma 2.6]. The fact that $H \cap H'$ is ϕ-inert depends on the existence of a natural embedding of $[(H \cap H') + \phi(H \cap H')]/(H \cap H')$ into $[(H + \phi(H))/H] \oplus [(H' + \phi(H'))/H']$. The fact that $H + H'$ is ϕ-inert depends on the fact that $[H + H' + \phi(H + H')]/(H + H')$ is a quotient of the direct sum $[(H + \phi(H))/H] \oplus [(H' + \phi(H'))/H']$.

(b) The first claim follows by the fact that $(\phi^n(H) + \phi^{n+1}(H))/\phi^n(H)$ is the image under ϕ^n of $(H + \phi(H))/H$; the latter claim follows from Lemma 2.1 (a).

(c) This is an immediate consequence of (a). □

As we will see in the next Example 2.7, $\mathcal{I}(G)$ is not a complete lattice. In order to prove this fact, we need the following result.

Lemma 2.3. *Let G be a torsion-free group of finite rank. Then every finitely generated subgroup H of G of maximal rank is fully inert.*

Proof. See [6, Lemma 2.4]; just observe that $(H+\phi(H))/H$ is torsion (as G/H is such) and finitely generated, being isomorphic to $\phi(H)/(H\cap\phi(H))$. □

In order to go on, we need to fix some more notation, which follows the standard notation, and recall some basic facts on the types (we refer to [10] for more information on the notions of characteristic and type and their properties). A *characteristic* is a sequence $(m_p)_{p\in\mathbb{P}}$ of natural numbers or symbols ∞ indexed by the set $\mathbb{P}$ of the prime numbers; two characteristics $(m_p)_{p\in\mathbb{P}}$ and $(m'_p)_{p\in\mathbb{P}}$ are equivalent if $m_p=\infty$ if and only if $m'_p=\infty$ and $m'_p\neq m_p$ for at most finitely many primes p where $m_p<\infty$ and $m'_p<\infty$. The equivalence class of a characteristic $(m_p)_{p\in\mathbb{P}}$, denoted by $[m_p]_{p\in\mathbb{P}}$ (or simply by $[m_p]$), is called the *type* of $(m_p)_{p\in\mathbb{P}}$. Given two characteristics $(k_p)_{p\in\mathbb{P}}$ and $(m_p)_{p\in\mathbb{P}}$, we set $[k_p]\leqslant[m_p]$ if $k_p>m_p$ for at most finitely many primes p where $k_p<\infty$.

A rational group A is a non-zero subgroup of $\mathbb{Q}$; without loss of generality, one can assume that $\mathbb{Z}\subseteq A$. The type of A, denoted by $\tau(A)$, is the type of the characteristic $(k_p)_{p\in\mathbb{P}}$ defined as follows: for each p, k_p is the supremum of the natural numbers k such that $1\in p^kA$. Two rational groups A and B are isomorphic if and only if $\tau(A)=\tau(B)$.

Recall that two subgroups H_1 and H_2 of a group G are called *commensurable* if $H_1\cap H_2$ has finite index in both H_1 and H_2. Then it is easy to see that $\tau(A)=\tau(B)$ amounts to saying that A and B are commensurable, that is, $A\cap B$ has finite index in both A and B.

Since every non-zero endomorphism of $\mathbb{Q}$ is an automorphism, the above observations yield the following example that will be essentially exploited in the proof of Theorem 4.9.

Example 2.4. Every subgroup of $\mathbb{Q}$ is fully inert.

The hypothesis in Lemma 2.3 that the subgroup is finitely generated cannot be removed, as the next example shows.

Example 2.5. Let $G=\mathbb{Q}\oplus\mathbb{Q}$ be the direct sum of two copies of the field $\mathbb{Q}$ of rational numbers, and consider $H=A\oplus B$, where A and B are non-isomorphic non-zero rational groups. We claim that H is not ϕ-inert, if ϕ denotes the automorphism of $\mathbb{Q} \oplus \mathbb{Q}$ defined by $\phi(x, y) = (y, x)$ for $x, y \in \mathbb{Q}$. Indeed,

$$H + \phi(H) = (A + B) \oplus (A + B),$$

therefore $(H + \phi(H))/H \cong (A + B)/A \oplus (A + B)/B$, which fails to be finite.

Even the hypothesis in Lemma 2.3 that the finitely generated subgroup H of G has maximal rank cannot be removed, as the next lemma shows.

Lemma 2.6. *A non-zero fully inert subgroup G of $\mathbb{Q}^n$ $(n \geqslant 1)$ has rank n.*

Proof. Assume, by way of contradiction, that $\mathrm{rk}(G) < n$; then $\mathbb{Q}^n = D_1 \oplus D_2$, with $G \leqslant D_1$ and $D_2 \neq 0$. There is a non-zero map $\phi : G \to D_2$ that can be extended to an endomorphism $\psi : \mathbb{Q}^n \to \mathbb{Q}^n$. Then

$$(G + \psi(G))/G = (G + \phi(G))/G \cong \phi(G)$$

is infinite, a contradiction. □

The following is the announced example showing that $\mathcal{I}(G)$ is not a complete lattice, so also $\mathcal{I}_\phi(G)$ is not complete: we will see that the intersection and the infinite sum of countably many fully inert subgroups need not be fully inert, in items (1) and (2) respectively.

Example 2.7. Let us consider again $G = \mathbb{Q} \oplus \mathbb{Q}$, as in Example 2.5.

(1) Let $H_n = n\mathbb{Z} \oplus \mathbb{Z}$ for every $n \in \mathbb{N}_+$. Since H_n is a finitely generated maximal rank subgroup of G, it is fully inert by Lemma 2.3, while

$$\bigcap_{n \in \mathbb{N}_+} H_n = \{0\} \oplus \mathbb{Z}$$

is not fully inert, by Lemma 2.6.

(2) Let $K_n = p^{-n}\mathbb{Z} \oplus q^{-n}\mathbb{Z}$ for every $n \in \mathbb{N}$, where p and q denote two different prime numbers. Since K_n is a finitely generated maximal rank subgroup of G, again by Lemma 2.3 it is fully inert, while

$$\sum_{n \in \mathbb{N}} K_n = \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/q]$$

is not fully inert, by Example 2.5.

The next result shows that a finite index subgroup H_1 of a subgroup H_2 of a group G is ϕ-inert or fully inert in G if and only if H_2 has the same property.

Lemma 2.8. *Let G be a group, $\phi : G \to G$ an endomorphism and H_1, H_2 subgroups of G such that $H_1 \subseteq H_2$ and H_2/H_1 is finite. Then:*

(a) *H_1 is ϕ-inert if and only if H_2 is ϕ-inert.*

(b) *H_1 is fully inert if and only if H_2 is fully inert.*

Proof. (a) Consider the canonical surjective homomorphism

$$\pi : \frac{H_2 + \phi(H_2)}{H_1} \to \frac{H_2 + \phi(H_2)}{H_2}.$$

Since $\ker \pi = H_2/H_1$ is finite, the quotient group $(H_2 + \phi(H_2))/H_1$ is finite if and only if $(H_2 + \phi(H_2))/H_2$ is finite.

Assume that H_2 is ϕ-inert. Then the quotient group $(H_2 + \phi(H_2))/H_2$ is finite and so $(H_2 + \phi(H_2))/H_1$ is finite. Since $H_1 + \phi(H_1) \subseteq H_2 + \phi(H_2)$, we conclude that $(H_1 + \phi(H_1))/H_1$ is finite, hence H_1 is ϕ-inert.

Conversely, assume that H_1 is ϕ-inert. As H_2/H_1 is finite, also $\phi(H_2)/\phi(H_1)$ is finite, and so $(H_2 + \phi(H_2))/(H_1 + \phi(H_1))$ is finite as well. Since

$$(H_2 + \phi(H_2))/(H_1 + \phi(H_1)) \cong ((H_2 + \phi(H_2))/H_1)/((H_1 + \phi(H_1))/H_1)$$

and $(H_1 + \phi(H_1))/H_1$ is finite by assumption, we have that also the quotient group $(H_2 + \phi(H_2))/H_1$ is finite. By the earlier observation, $(H_2 + \phi(H_2))/H_2$ is finite, that is, H_2 is ϕ-inert.

Statement (b) follows from (a). □

As an easy consequence of Lemma 2.8 we derive the following result.

Corollary 2.9. *Let G be a group, $\phi : G \to G$ an endomorphism and H_1, H_2 two commensurable subgroups of G. Then:*

(a) *H_1 is ϕ-inert if and only if $H_1 \cap H_2$ is ϕ-inert, if and only if H_2 is ϕ-inert.*

(b) *H_1 is fully inert if and only if $H_1 \cap H_2$ is fully inert, if and only if H_2 is fully inert.*

Proof. Apply Lemma 2.8 twice, using that $H_1/(H_1 \cap H_2)$ and $H_2/(H_1 \cap H_2)$ are finite by hypothesis. □

The next immediate consequence of the above corollary will be used to provide a complete characterization of inert p-groups (see Theorem 5.2).

Corollary 2.10. *If a subgroup H of a group G is commensurable with some fully invariant subgroup of G, then H is fully inert.*

The sufficient condition from this corollary is quite far from being necessary. Indeed, according to Example 2.4, every subgroup of $\mathbb{Q}$ is fully inert, while the only non-zero fully invariant subgroup of $\mathbb{Q}$ is $\mathbb{Q}$ itself, and so obviously no proper subgroup of $\mathbb{Q}$ is commensurable with $\mathbb{Q}$.

3 Fully inert subgroups of divisible groups

The following folklore fact in Abelian group theory will be freely used in the sequel.

Fact 3.1. The non-zero fully invariant subgroups of a non-trivial divisible group D are:

(i) D itself and the subgroups $D[p^n]$ $(n \geqslant 1)$, if D is a p-group,

(ii) the subgroups $T = \bigoplus_p T_p$, where T_p is fully invariant in the p-component D_p of D, if $D = \bigoplus_p D_p$ is torsion,

(iii) the fully invariant subgroups of $t(D)$ and D itself, in the general case.

Our aim in this section is to characterize the fully inert subgroups of the divisible groups. A basic result by Kulikov [11] says that a divisible group D containing a subgroup G, has a minimal divisible subgroup D_1 containing G, and any two minimal divisible subgroups D_1 and D_1' containing G are isomorphic over G (see also [10, Theorem 24.4]); furthermore, it is easy to see that $D_1/G \cong D_1'/G$. These minimal divisible subgroups are divisible hulls of G (recall that a *divisible hull* of G is any divisible group in which G embeds as an essential subgroup), and are direct summands in D; we will denote by $\mathrm{D}(G)$ a fixed divisible hull of G.

Proposition 3.2. *Let D be a divisible group, let G be a subgroup of D and let $D = D_1 \oplus D_2$, where D_1 is a minimal divisible subgroup of D containing G. Then G is fully inert in D if and only if it is fully inert in D_1 and* $\mathrm{Im}(\alpha)$ *is finite for every homomorphism $\alpha : G \to D_2$.*

Proof. Assume first that G is fully inert in D and let $\phi : D_1 \to D_1$ be an endomorphism of D_1. Let ψ be any extension of ϕ to D; then

$$(G + \phi(G))/G = (G + \psi(G))/G$$

is finite, so G is fully inert in D_1. Furthermore, given a fixed homomorphism $\alpha : G \to D_2$, it extends to a homomorphism $\beta : D \to D$; since $(G + \beta(G))/G$ is finite and $(G + \beta(G))/G = (G \oplus \alpha(G))/G \cong \alpha(G)$, it follows that $\alpha(G)$ must be finite.

Conversely, assume the subgroup G fully inert in D_1 and $\mathrm{Im}(\alpha)$ finite for every homomorphism $\alpha : G \to D_2$. Let $\phi : D \to D$ be an endomorphism of D, and let $\pi_i : D \to D_i$ be the canonical projections ($i = 1, 2$). Then

$$G + \phi(G) \subseteq (G + \pi_1(\phi(G))) \oplus \pi_2(\phi(G));$$

therefore

$$\frac{G + \phi(G)}{G} \subseteq \frac{G + \pi_1(\phi(G))}{G} \oplus \pi_2(\phi(G)).$$

The first summand of the right term is finite, as G is fully inert in D_1, and the second summand is finite by assumption, setting $\alpha = \pi_2 \cdot \phi$. Hence G is fully inert in D. □

It is immediate to check that, given a group G and two different divisible hulls of it, G is fully inert in the first one if and only if it is fully inert in the latter. In view of Proposition 3.2, the problem of finding the fully inert subgroups of a fixed divisible group is split into the following two problems:

(P1) characterize the groups G which are fully inert in their divisible hulls, that is, the inert groups,

(P2) characterize the pairs of groups (G, D) with G inert, D divisible, and such that $\mathrm{Im}(\alpha)$ is finite for every homomorphism $\alpha : G \to D$.

Problem (P1) will be completely settled in Section 5, and problem (P2) in Section 6. In the sequel of this section we consider a more general setting for (P2), without asking G to be inert (see Definition 3.3).

For every pair (G, D) of groups let

$$\pi(G, D) = \{p \in \mathbb{P} : r_p(G) > 0 \text{ and } r_p(D) > 0\}.$$

If D is non-trivial and divisible, one can easily check that $\mathrm{Hom}(G, D) = 0$ if and only if G is torsion and $\pi(G, D) = \emptyset$. To tackle (P2) (and its more general version) we need a more subtle notion.

Definition 3.3. Let G, D be non-trivial groups and let D be divisible.

(a) We call the pair (G, D) *almost orthogonal* if $\mathrm{Im}(\alpha)$ is finite for every homomorphism $\alpha \in \mathrm{Hom}(G, D)$.

(b) For a subgroup L of D we call the pair (G, D) *relatively almost orthogonal with respect to* L if $\mathrm{Im}(\alpha)/(\mathrm{Im}(\alpha) \cap L)$ is finite for every $\alpha \in \mathrm{Hom}(G, D)$.

Remark 3.4. The weaker version proposed in item (b) was tailored in order to fit inert groups, as a group G is inert if and only if $(G, \mathrm{D}(G))$ is relatively almost orthogonal with respect to G (see also Corollary 4.3 that describes when a finite direct sum is inert).

To completely determine *all* almost orthogonal pairs, it is relevant to know when a given group G admits divisible quotients and which divisible groups can be obtained as quotients of G. The groups that do not admit a divisible quotient were described in [8]. For the sake of completeness, and due to the fact that [8] is not easily accessible, we provide a proof of this theorem, formulated in a counter-positive form that is more appropriate for our purposes.

Theorem 3.5 ([8]). *For a group G the following conditions are equivalent:*

(a) *There exist a prime p and a surjective homomorphism $G \to \mathbb{Z}(p^\infty)$.*

(b) *Some non-trivial quotient of G is divisible.*

(c) *Either* $\mathrm{rk}(G)$ *is infinite, or* $\mathrm{rk}(G) = n$ *is finite and for every free subgroup F of rank n of G some p-primary component of G/F is unbounded.*

(d) *Either* $\mathrm{rk}(G)$ *is infinite, or* $\mathrm{rk}(G) = n$ *is finite and for some free subgroup F of rank n of G some p-primary component of G/F is unbounded.*

If $\mathrm{rk}(G)$ *is infinite, then G has a quotient isomorphic to $\mathbb{Q}^{(\aleph_0)}$, so every countable divisible group can be obtained as a quotient of G. If in* (c) *and* (d) *the second option occurs, then G has a quotient isomorphic to $\mathbb{Z}(p^\infty)$.*

Proof. The implications (a) ⇒ (b) and (c) ⇒ (d) are trivial.

To prove (d) ⇒ (a) assume first that $\mathrm{rk}(G)$ is infinite. Then G contains a free subgroup F of rank $\aleph_0$. Hence, one can define a surjective homomorphism $F \to \mathbb{Q}$ that can be extended to a surjective homomorphism $\phi : G \to \mathbb{Q}$ by the divisibility of $\mathbb{Q}$. Composing with the obvious surjective homomorphism $\mathbb{Q} \to \mathbb{Z}(p^\infty)$, we are done. Now assume that $\mathrm{rk}(G) = n$ is finite and for some free subgroup F of rank n of G the p-primary component H_p of G/F is unbounded for some prime p. It suffices to find a surjective homomorphism $H_p \to \mathbb{Z}(p^\infty)$; it will produce then a surjective homomorphism $G \to \mathbb{Z}(p^\infty)$ via the composition $G \to G/F \to H_p$. Our assumption that H_p is unbounded guarantees that, for any basic subgroup B of H_p, H_p/B is non-zero divisible (see [10, Section 33]), hence the conclusion follows.

(b) ⇒ (c) Assume, by way of contradiction, that (c) fails. Then $\mathrm{rk}(G) = n$ is finite and there exists a free subgroup F of rank n of G such that all p-primary components of G/F are bounded. Since every divisible group has as a quotient $\mathbb{Z}(p^\infty)$ for some p, there exists a surjective homomorphism $f : G \to \mathbb{Z}(p^\infty)$. Since $f(F)$ is finitely generated, hence a finite subgroup of $\mathbb{Z}(p^\infty)$, we can assume without loss of generality that $f(F) = 0$ (by taking a further quotient modulo the finite subgroup $f(F)$). This assumption entails $F \subseteq \ker f$, so that we get a surjective homomorphism $q : G/F \to \mathbb{Z}(p^\infty)$ factorizing f. This is impossible, since all primary components of the torsion group G/F are bounded. □

One can easily see that a group is divisible precisely when it admits no maximal proper subgroups. Using this fact, the following further condition, equivalent to those of Theorem 3.5, was given in [8]: there exists a subgroup N of G such that G has no maximal proper subgroups containing N.

From Theorem 3.5 we obtain a complete description of the almost orthogonal pairs.

Theorem 3.6. *Let (G, D) be a pair of groups, with D non-trivial divisible.*

(a) *If the pair (G, D) is almost orthogonal, then:*

  (a_1) $\mathrm{rk}(G)$ *and* $\pi(G, D)$ *are finite,*

  (a_2) *for all $p \in \pi(G, D)$, $t_p(G)$ is bounded and one has either $r_p(G) < \infty$ or $r_p(D) < \infty$.*

(b) *If G is torsion, conditions* (a_1) *and* (a_2) *are also sufficient for (G, D) to be almost orthogonal.*

(c) *If the pair (G, D) is almost orthogonal and G is not torsion, then:*

  (c_1) *D is torsion,*

  (c_2) *for every free subgroup F of G of maximal rank, $(G/F)_p$ is bounded for all primes p such that $r_p(D) > 0$ and $\pi(G/F, D)$ is finite.*

(d) *If G is not torsion, conditions* (a_1), (a_2), (c_1) *and* (c_2) *are also sufficient for (G, D) to be almost orthogonal.*

Proof. (a) The finiteness of $\mathrm{rk}(G)$ follows immediately from the proof of the implication (d) $\Rightarrow$ (a) in Theorem 3.5. If $\pi(G, D)$ is infinite, one can easily produce a homomorphism $\alpha \in \mathrm{Hom}(G, D)$ with infinite image; this implies finiteness of $\pi(G, D)$. If $t_p(G)$ is unbounded for some $p \in \pi(G, D)$, then by Theorem 3.5 there exists a surjective homomorphism $\beta : G \to \mathbb{Z}(p^\infty)$ which produces a homomorphism $\alpha \in \mathrm{Hom}(G, D)$ with infinite image. Finally, if both $r_p(G)$ and $r_p(D)$ are infinite, then there exists a homomorphism $\gamma : G[p] \to D[p]$ with infinite image, that can be extended to a homomorphism $\alpha \in \mathrm{Hom}(G, D)$ with infinite image.

(b) Take any homomorphism $\alpha \in \mathrm{Hom}(G, D)$. Split α into its finitely many non-trivial restrictions α_p on the p-primary components with $p \in \pi(G, D)$. For a fixed $p \in \pi(G, D)$, $\mathrm{Im}(\alpha_p)$ is a bounded subgroup of D_p. Moreover, our hypothesis ensures that either the domain of α_p is finite, or $r_p(D)$ is finite; thus in both cases $\mathrm{Im}(\alpha_p)$ is finite.

(c) The assumption that the pair (G, D) is almost orthogonal ensures that $\mathrm{rk}(G) = n$ is finite, by (a), and the assumption that G is not torsion ensures that $n > 0$. Then (c_1) is obvious. To prove (c_2), fix a free subgroup F of G of rank n.

Assume that $r_p(D) > 0$ for some prime p, while the p-primary component H_p of G/F is unbounded. Then by Theorem 3.5 there exists a surjective homomorphism $G \to \mathbb{Z}(p^\infty)$, which gives a homomorphism $\alpha \in \mathrm{Hom}(G, D)$ with infinite image, a contradiction. Now assume that $\pi(G/F, D)$ is infinite. For every $p \in \pi(G/F, D)$ there exists a non-trivial homomorphism $\alpha_p \in \mathrm{Hom}(H_p, D_p)$. This gives rise to a homomorphism $\alpha' : G/F \to \bigoplus_p D_p \leqslant D$ with infinite image. Composing with the canonical projection $G \to G/F$ one gets a homomorphism $\alpha \in \mathrm{Hom}(G, D)$ with infinite image, a contradiction. This proves that $\pi(G/F, D)$ is finite.

(d) Let conditions (a_1), (a_2), (c_1) and (c_2) hold true. By hypothesis, one has $\mathrm{rk}(G) = n \geqslant 1$. Then conditions ($c_1$) and ($c_2$) guarantee that for every free subgroup F of rank n of G the pair $(G/F, D)$ is orthogonal. For any homomorphism $\alpha \in \mathrm{Hom}(G, D)$ one can find a free subgroup F of rank n of $\ker\alpha$, since D is torsion. Therefore, α factorizes through $\alpha_1 : G/F \to D$. Since $\mathrm{Im}(\alpha_1)$ is finite, $\mathrm{Im}(\alpha)$ is finite as well. □

The next results provide some necessary conditions for a subgroup G of a divisible group D to be fully inert.

Lemma 3.7. *Let G be a fully inert subgroup of a divisible group D. Then G contains all divisible subgroups D_0 of D admitting a surjective homomorphism $G \to D_0$.*

Proof. Let D_0 be a divisible subgroup of D and let $\psi : G \to D_0$ be a surjective homomorphism. By the divisibility of D, one can extend ψ to an endomorphism $\phi : D \to D$ that obviously satisfies $D_0 \subseteq \phi(G)$. Since G is fully inert, $(G + \phi(G))/G$ is finite. On the other hand,

$$(G + \phi(G))/G \supseteq (G + D_0)/G \cong D_0/(D_0 \cap G).$$

Since $D_0/(D_0 \cap G)$ must be divisible, this is possible only if $D_0 \subseteq G$. □

Lemma 3.7 makes it relevant again to know when a given group G admits non-trivial divisible quotients and which divisible groups can be obtained as quotients of G. This was described in Theorem 3.5 above and allows us to prove some of the most relevant necessary conditions satisfied by fully inert subgroups of divisible groups.

Theorem 3.8. *Let G be a proper fully inert subgroup of a divisible group D. Then:*

(a) *G has finite (torsion-free) rank,*

(b) *if G has as a quotient an unbounded p-group for some prime p, then G contains the p-primary component of D.*

Proof. (a) Let us assume, by way of contradiction, that $\mathrm{rk}(G)$ is infinite. According to Theorem 3.5, for every countable divisible subgroup D_0 of D there exists a surjective homomorphism $G \to D_0$. By Lemma 3.7 this yields $D_0 \subseteq G$. Since D is generated by its countable divisible subgroups, this proves that $G = D$, a contradiction.

(b) Suppose that the quotient G/H is an unbounded p-group for some prime p. By Theorem 3.5, there exists a surjective homomorphism $G \to \mathbb{Z}(p^\infty)$. Hence, applying again Lemma 3.7, we can conclude that G contains all subgroups of D isomorphic to $\mathbb{Z}(p^\infty)$. Since the p-primary component of D is generated by these subgroups, we are done. □

4 Properties of the inert groups related to direct sums

The aim of this section is to investigate fully inert subgroups of finite direct sums of divisible groups $D = D_1 \oplus \cdots \oplus D_n$; the subgroups of D of the form

$$G = G_1 \oplus \cdots \oplus G_n,$$

with G_i a subgroup of D_i for all i, are called *box-like* subgroups. From this investigation we will derive the characterization of inert torsion-free groups (Theorem 4.9) and the fact that inert mixed groups are splitting (Theorem 4.10).

Recall that a torsion-free group H of rank n ($n \in \mathbb{N}_+$) is *completely decomposable* if $H = A_1 \oplus \cdots \oplus A_n$, where A_i has rank 1 for every $i = 1, \dots, n$. The group H is *almost completely decomposable* if it has a completely decomposable subgroup of finite index. A torsion-free group is *homogeneous* if all its rank-one pure subgroups have the same type. Thus, a completely decomposable group $A_1 \oplus \cdots \oplus A_n$ is *homogeneous* if $A_i \cong A_j$ for all $i, j \in \{1, \dots, n\}$.

The next lemma shows that the study of fully inert subgroups of finite direct sums of divisible groups can be reduced to box-like subgroups.

Lemma 4.1. *Let $D_1, \dots, D_n$ be divisible groups and let H be a subgroup of $D = D_1 \oplus \cdots \oplus D_n$. Let $A_i = H \cap D_i$ for $i = 1, \dots, n$ and $A = A_1 \oplus \cdots \oplus A_n$. Then the following conditions are equivalent:*

(a) *H is fully inert.*

(b) *$[H : A] < \infty$ and A is fully inert.*

Proof. Clearly, we may assume that $n = 2$; the general case follows by induction.

Denote by π_i ($i = 1, 2$) the canonical projection of D onto D_i. Assume that H is fully inert. Then the quotients

$$\pi_i(H)/(H \cap \pi_i(H)) \cong (H + \pi_i(H))/H$$

are finite for $i = 1, 2$, as each π_i can be considered also as an endomorphism of D. Then their quotients $\pi_i(H)/A_i$ are finite for $i = 1, 2$. Since H/A is contained in $(\pi_1(H) \oplus \pi_2(H))/A \cong \pi_1(H)/A_1 \oplus \pi_2(H)/A_2$, we conclude that H/A is finite. By Lemma 2.8, A is fully inert.

If $[H : A] < \infty$ and A is fully inert, again Lemma 2.8 implies that H is fully inert. □

Now we characterize the fully inert box-like subgroups of finite direct sums of divisible groups.

Proposition 4.2. *Let $D_1, \dots, D_n$ be divisible groups and let G_i be a subgroup of D_i for $i = 1, \dots, n$. Then $G = G_1 \oplus \dots \oplus G_n$ is fully inert in $D = D_1 \oplus \dots \oplus D_n$ if and only if*

(a) *the subgroup G_i is fully inert in D_i for $i = 1, \dots, n$,*

(b) *for every pair of indices $i \neq j$, (G_i, D_j) is relatively almost orthogonal with respect to G_j.*

Proof. We assume again that $n = 2$; the general case follows by induction. Assume first that G is fully inert in D.

(a) To prove that the subgroup G_1 is fully inert in D_1 consider an endomorphism $\phi' : D_1 \to D_1$ and extend it to an endomorphism $\phi : D \to D$ by letting ϕ vanish on D_2. Then $G + \phi(G) = (G_1 + \phi'(G_1)) \oplus G_2$, hence

$$(G_1 + \phi'(G_1)/G_1 \cong ((G_1 + \phi'(G_1)) \oplus G_2)/(G_1 \oplus G_2) = (G + \phi(G))/G$$

is finite, as G is fully inert in D. Analogously one checks that G_2 is fully inert in D_2.

(b) To prove that the pair (G_2, D_1) is relatively almost orthogonal with respect to G_1, assume that for some homomorphism $\alpha \in \mathrm{Hom}(G_2, D_1)$ the quotient $\mathrm{Im}(\alpha)/(\mathrm{Im}(\alpha) \cap G_1) \cong (G_1 + \alpha(G_2))/G_1$ is infinite. Extend α to a homomorphism $\alpha_1 : D_2 \to D_1$ and extend α_1 to an endomorphism $\phi : D \to D$ letting ϕ vanish on D_1. Then

$$\phi(G) = \alpha(G_2),$$

so $(G + \phi(G))/G = ((G_1 + \alpha(G_2)) \oplus G_2)/(G_1 \oplus G_2) \cong (G_1 + \alpha(G_2))/G_1$ is infinite. This contradicts the hypothesis that G is fully inert in D. Analogously one checks that (G_1, D_2) is relatively almost orthogonal with respect to G_2.

Now assume that (a) and (b) are fulfilled. Let $\pi_1 : D \to D_1$ and $\pi_2 : D \to D_2$ be the canonical projections, and let $\phi : D \to D$ be an endomorphism. Then, with a little abuse of notation, we can write ϕ as a sum $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$, where $\phi_1 : D_1 \to D_1$, $\phi_2 : D_2 \to D_2$, $\phi_3 : D_2 \to D_1$ and $\phi_4 : D_1 \to D_2$. From the equality

$$\frac{G+\phi(G)}{G} = \frac{G_1+\phi_1(G_1)+\phi_3(G_2)}{G_1} \oplus \frac{G_2+\phi_2(G_2)+\phi_4(G_1)}{G_2}$$

we get the conclusion. Indeed, $(G_1+\phi_1(G_1))/G_1$ and $(G_2+\phi_2(G_2))/G_2$ are finite because of (a). The finiteness of $(G_1+\phi_3(G_2))/G_1$ and $(G_2+\phi_4(G_1))/G_2$ is ensured by (b). □

As a first corollary we obtain a complete description of when a direct sum of groups is inert.

Corollary 4.3. *A finite direct sum of groups $G = G_1 \oplus \cdots \oplus G_n$ is inert if and only if*

(a) *all subgroups G_i are inert,*

(b) *for every pair of indices $i \neq j$, $(G_i, \mathrm{D}(G_j))$ is relatively almost orthogonal with respect to G_j.*

Proof. Apply Proposition 4.2. □

Remark 4.4. It is worth observing that a subgroup G of a divisible group D is fully inert exactly if it is relatively almost orthogonal with respect to itself. So Proposition 4.2 and Corollary 4.3 could be stated just using conditions (b) for all indices i and j, possibly equal. We prefer the presentation given, as it allows a clear proof.

From Proposition 4.2 we obtain the following immediate consequence.

Corollary 4.5. *For a subgroup G of a divisible group D the following conditions are equivalent:*

(a) *G is fully inert in D.*

(b) *G^n is fully inert in D^n for all $n \in \mathbb{N}_+$.*

(c) *There exists an $n \in \mathbb{N}_+$ such that G^n is fully inert in D^n.*

We shall see in the sequel that direct sums of two inert groups need not be inert. Nevertheless, taking $D = \mathrm{D}(G)$ in the above corollary, we get a positive result as far as finite direct powers are concerned.

Corollary 4.6. *For every group G the following conditions are equivalent:*

(a) *G is inert.*

(b) *G^n is inert for all $n \in \mathbb{N}_+$.*

(c) *There exists an $n \in \mathbb{N}_+$ such that G^n is inert.*

Example 4.7. According to Example 2.4, every rational group is inert. Hence Corollary 4.6 implies that every finite rank homogeneous completely decomposable group is inert. We shall see in Theorem 4.9 that these groups are the only inert torsion-free groups.

We need first the following result providing a complete description of the fully inert subgroups A of finite direct sums D^n of a divisible group D. According to Lemma 4.1, it is not restrictive to assume that A is a box-like subgroup.

Proposition 4.8. *Let D be a divisible group, let $A_1, \dots, A_n$ be subgroups of D (with $n \geqslant 2$), and let $A = A_1 \oplus \dots \oplus A_n \subseteq D^n$. Then the following conditions are equivalent:*

(a) *A is a fully inert subgroup of D^n.*

(b) *All subgroups A_i are fully inert in D and pairwise commensurable.*

(c) *There exists a fully inert subgroup A_0 of D such that A is a finite extension of (hence, commensurable with) A_0^n.*

Proof. (a) $\Rightarrow$ (b) Assume that the subgroup A is fully inert. Then the subgroups A_i $(i = 1, \dots, n)$ are fully inert in D by Proposition 4.2. Fix two different indices i and j. Then (A_i, D) must be relatively almost orthogonal with respect to A_j by Proposition 4.2. Taking as a test homomorphism the inclusion $A_i \hookrightarrow D$, we deduce that $A_i/(A_i \cap A_j)$ is finite. Similarly one proves that $A_j/(A_i \cap A_j)$ is finite, so A_i and A_j are commensurable.

(b) $\Rightarrow$ (c) Assume that A_i $(i = 1, \dots, n)$ are pairwise commensurable and fully inert in D. Then $A_0 = A_1 \cap \dots \cap A_n$ is fully inert in D by Lemma 2.8, so A_0^n is fully inert in D^n from Corollary 4.5. As A_0^n has finite index in A, we deduce that A is commensurable with A_0^n.

(c) $\Rightarrow$ (a) It follows from Corollary 4.5 that A_0^n is a fully inert subgroup of D^n. Now Lemma 2.8 applies. □

We can now describe the inert torsion-free groups H. It turns out that they are not commensurable with the only fully invariant subgroup of their divisible hull $D(H)$, which is, by Fact 3.1, $D(H)$ itself. This is a remarkable difference with respect to the torsion case (see next Theorem 5.3).

Theorem 4.9. *A torsion-free group H is inert if and only if it is either divisible, or completely decomposable homogenous of finite rank.*

Proof. Divisible groups are trivially inert, so the sufficiency follows from Example 4.7. To check the necessity, assume that H is inert. It suffices to consider the case when H is not divisible. In view of Theorem 3.8, H has finite rank n.

By Lemma 2.6, we can assume without loss of generality that H is fully inert in $\mathbb{Q}^n$. Then, by Lemma 4.1 and Proposition 4.8, H is a finite extension of a group of the form A^n, where A is a rational group. This means that H is almost completely decomposable. To conclude, it suffices to recall the well-known fact that a finite extension of a completely decomposable homogeneous group is again completely decomposable homogeneous (see [12], [10, Theorem 86.6]). □

As an application of Lemma 4.1 and Corollary 4.3, we show that any inert mixed group G splits.

Theorem 4.10. *If G is an inert group, then $G = t(G) \oplus H$ and*

(a) *the subgroups $t(G)$ and H are both inert groups,*

(b) *moreover, $(H, D(t(G)))$ is relatively almost orthogonal with respect to $t(G)$.*

Conversely, if $G = t(G) \oplus H$ is a splitting group satisfying (a) *and* (b)*, then G is inert.*

Proof. Let $D = t(D) \oplus D_1$ be the splitting decomposition of the divisible hull D of G. In order to prove that G is splitting, it is enough to note that $t(G) \oplus (G \cap D_1)$ has finite index in G, according to Lemma 4.1. Now the conclusion follows from a result by Procházka [13], stating that finite extensions of splitting groups are still splitting (see also [10, vol. II, Exercise 2, p. 188]), and (b) follows from Corollary 4.3.

The sufficiency follows directly from Corollary 4.3, noting that

$$\mathrm{Hom}(t(G), D_1) = 0,$$

so that $(t(G), D_1)$ is almost orthogonal (so also relatively orthogonal with respect to any subgroup of D_1). □

5 Characterization of the inert groups

In this section we continue the step-by-step description of the inert groups that started with the torsion-free case in Theorem 4.9 above. The torsion case will be tackled in Section 5.1 and the mixed case will be concluded in Section 5.2, making use of Theorem 4.10.

5.1 Inert torsion groups

By Corollary 2.10, every group G which is commensurable with some fully invariant subgroup of $\mathrm{D}(G)$ is inert. The main goal of this subsection is to show that the groups obtained in this way cover all possible inert torsion groups.

Lemma 5.1 characterizes, in terms of Ulm–Kaplansky invariants, the p-groups that are commensurable with a fully invariant subgroup of their divisible hull. For the notion of Ulm–Kaplansky invariants and their connection with direct sum of cyclic groups we refer to [10].

Lemma 5.1. *Let G be a reduced p-group with divisible hull $\mathrm{D}(G) = D$. The following facts are equivalent:*

(a) *There exists an $n \in \mathbb{N}_+$ such that G and $D[p^n]$ are commensurable.*

(b) *There exists an $n \in \mathbb{N}_+$ such that $D[p^n]/G[p^n]$ and $G/G[p^n]$ are finite.*

(c) *G is bounded with at most one infinite Ulm–Kaplansky invariant.*

(d) *G is commensurable with a fully invariant proper subgroup of its divisible hull.*

Proof. (a) $\Leftrightarrow$ (b) This is an immediate consequence of the definition of commensurable subgroups, taking into account that $D[p^n] \cap G = G[p^n]$.

(b) $\Rightarrow$ (c) The hypothesis that the quotient $G/G[p^n]$ is finite ensures that G is bounded, so that there exists a positive integer k such that $G = \bigoplus_{1 \leqslant i \leqslant k} B_i$ with $B_i \cong \mathbb{Z}(p^i)^{(\alpha_i)}$, where the α_i are the Ulm–Kaplansky invariants of G. Then $D[p^n]/G[p^n]$ finite ensures that α_i is finite for each $0 < i < n$, while $G/G[p^n]$ finite ensures that α_i is finite for each $i > n$, hence only α_n is possibly infinite.

(c) $\Rightarrow$ (d) Let $G = \bigoplus_{1 \leqslant i \leqslant k} B_i$ with $B_i \cong \mathbb{Z}(p^i)^{(\alpha_i)}$, as above. If all the α_i are finite, i.e., if G is finite, then G has finite index in $D[p^k]$. If α_n is infinite for some $1 \leqslant n \leqslant k$ and all the remaining invariants are finite, then both $G/G[p^n]$ and $D[p^n]/G[p^n]$ are finite, so we are done.

(d) $\Rightarrow$ (a) Follows from the fact that $D[p^n]$ ($n \in \mathbb{N}_+$) are the only proper fully invariant subgroups of D (see Fact 3.1). □

We can now give the characterization of inert p-groups.

Theorem 5.2. *A p-group G is inert if and only if it is either divisible or bounded with at most one infinite Ulm–Kaplansky invariant.*

Proof. If G is divisible, then it is obviously inert; furthermore, if G is bounded with at most one infinite Ulm–Kaplansky invariant, it is commensurable with a fully invariant subgroup of $D(G)$, according to Lemma 5.1. Hence G is inert, by Corollary 2.10.

Conversely, suppose G is inert and not divisible. Then G is bounded by Theorem 3.8(b). Thus, we can write $G = \bigoplus_{1 \leqslant i \leqslant k} B_i$, where $B_i \cong \mathbb{Z}(p^i)^{(\alpha_i)}$, $k \in \mathbb{N}_+$ and the α_i are suitable cardinals for all i. To conclude the proof, we have to show that at most one α_i is infinite. Suppose, looking for a contradiction, that α_i and α_j

are infinite for some $i < j$. Let $B'_i \leqslant B_i$ and $B'_j \leqslant B_j$ be countable direct summands, and let $\phi' : D(B'_j) \to D(B'_i)$ be an isomorphism. Extend ϕ' to an endomorphism $\phi : D(G) \to D(G)$ and notice that the infinite group

$$\mathbb{Z}(p^{j-i})^{(\mathbb{N})} \cong D(B'_i)[p^j]/B'_i = \phi(B'_j)/B'_i$$
$$= \phi(B'_j)/\phi(B'_j) \cap G \cong (G + \phi(B'_j))/G$$

embeds into $(G + \phi(G))/G$, contradicting the fact that G is inert. □

An inert p-group G can be written, in a unique way up to isomorphism, in the form

$$G = F \oplus \mathbb{Z}(p^i)^{(\alpha)},$$

where $1 \leqslant i \leqslant \infty$, α is a positive cardinal, F is a finite group such that:

(a) if $i = \infty$, then F is zero,

(b) if $i < \infty$, then F has no summands isomorphic to $\mathbb{Z}(p^i)$,

(c) if both i and α are finite, then $p^{i-1}F = 0$ (i.e., α equals the Ulm–Kaplansky invariant of maximal index).

We say that the inert p-group G is then written in its *canonical form*.

Passing to the global case, we can derive easily the characterization of inert torsion groups.

Theorem 5.3. *For a torsion group G the following conditions are equivalent:*

(a) *G is inert.*

(b) *Each p-primary component G_p of G is an inert p-group, and all but finitely many of them are either divisible or have a single non-zero Ulm–Kaplansky invariant.*

(c) *G is of the form*

$$G \cong F \oplus \bigoplus_{p\in P(G)} \mathbb{Z}(p^{m_p})^{(\alpha_p)},$$

where F is a finite group, $m_p \in \mathbb{N}_+ \cup \{\infty\}$ and the α_p are positive cardinals for each $p \in P(G)$, $F_p = 0$ whenever $m_p = \infty$, and $F_p \oplus \mathbb{Z}(p^{m_p})^{(\alpha_p)}$ is the canonical form of G_p.

(d) *G is commensurable with a fully invariant subgroup of its divisible hull.*

Proof. (a) ⇒ (b) The proof that G_p is an inert p-group for every prime p is straightforward. So each G_p is either divisible, or bounded with at most one infinite Ulm–Kaplansky invariant, by Theorem 5.2. The p-components G_p which are bounded with more than one non-zero Ulm–Kaplansky invariant are not fully invariant in their divisible hull $D(G_p)$; for these p, there exists an endomorphism ϕ_p of $D(G_p)$ such that the quotient $(G_p + \phi_p(G_p))/G_p$ is non-trivial. If this occurs for an infinite set of primes p, it gives rise to the endomorphism $\phi = \bigoplus_p \phi_p$ of $D(G) = \bigoplus_p D(G_p)$, such that $(G + \phi(G))/G$ is infinite, a contradiction.

The implications (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) are obvious, and (d) $\Rightarrow$ (a) follows by Corollary 2.10. □

We have seen in Theorem 4.9 that item (d) fails to be true in the case of inert torsion-free groups; we will see in Remark 5.6 that item (d) fails also for inert mixed groups.

When the inert torsion group G is written as in Theorem 5.3 (c), we say that G is in its *canonical form*; note that all the m_p and α_p are uniquely determined by G, and F is unique up to isomorphism. For the purposes of Section 5.2, we introduce the following convention:

Definition 5.4. For an inert torsion group $G = F \oplus \bigoplus_{p \in P(G)} \mathbb{Z}(p^{m_p})^{(\alpha_p)}$ written in its canonical form, we define the *induced type* $[m_p^*]_{p \in \mathbb{P}}$ as follows:

$$m_p^* = \begin{cases} m_p & \text{for } p \in P(G), \\ \infty & \text{for } p \notin P(G). \end{cases}$$

5.2 Inert mixed group

Our aim in this subsection is to obtain a complete description of the inert groups in the mixed case.

Theorem 5.5. *A mixed group G is inert if and only if either G is divisible, or the following conditions are satisfied:*

(a) *$G = T \oplus H$ is splitting, where T is a torsion group and H is a torsion-free group,*

(b) *$T = F \oplus \bigoplus_{p \in P} \mathbb{Z}(p^{m_p})^{(\alpha_p)}$, where F is a finite group, P is a set of primes, $m_p \in \mathbb{N}_+ \cup \{\infty\}$ and the α_p are positive cardinals for each p, and $F_p = 0$ whenever $m_p = \infty$,*

(c) *$H \cong A^n$, where A is a rational group of type $\tau(A) = [k_p]$,*

(d) *$[k_p] \leqslant [m_p^*]$.*

Proof. Let us assume that G is not divisible and let $D = \mathrm{D}(G)$ be the divisible hull of G. First, assume G inert, so that G has finite torsion-free rank by Theorem 3.8. By Proposition 4.10, (a) holds and both T and H are inert groups. Therefore (b) and (c) follow by Theorem 5.3 and Theorem 4.9, respectively. Clearly, we can identify $t(D)$ with $\mathrm{D}(T)$, and write $D = t(D) \oplus D_1$, where $D_1 = \mathrm{D}(H)$. Our next aim is to show that (d) is equivalent to asking $(H, t(D))$ to be relatively almost orthogonal with respect to T. According to Theorem 4.10 (ii), this will imply the validity of (d), in case G is inert.

Indeed, (d) fails when either $k_p = \infty$ and $m_p < \infty$ for some prime, or $k_p > m_p$ for infinitely many primes p. In both cases there exists a homomorphism

$$\alpha : \bigoplus_p \mathbb{Z}(p^{k_p}) \to t(D)$$

such that $\mathrm{Im}(\alpha)/(\mathrm{Im}(\alpha) \cap T)$ is infinite.

Since there exist epimorphisms

$$H \to A \to \bigoplus_p \mathbb{Z}(p^{k_p}),$$

calling $\psi : H \to t(D)$ the composition of these epimorphisms with α, we get that $\mathrm{Im}(\psi)/(\mathrm{Im}(\psi) \cap T)$ is infinite, witnessing that $(H, t(D))$ fails to be relatively almost orthogonal with respect to T whenever (d) fails.

Let us assume now that (a), (b), (c), (d) hold true. By Theorem 4.9 and Theorem 5.3, (b) and (c) imply that T and H are inert groups. Let us consider a homomorphism $\alpha : H \to t(D)$. Since $t(D)$ is torsion, $\ker \alpha \neq 0$. We can assume without loss of generality that $\mathbb{Z}$ is contained in $\ker \alpha$, so

$$\alpha(A) \cong A/\ker \alpha = \bigoplus_p \mathbb{Z}(p^{l_p}),$$

with $[l_p] \leqslant [k_p]$. Then (d) yields $[l_p] \leqslant [m^*_p]$. This means that $\alpha(A)/(\alpha(A) \cap T)$ is finite. Hence, $(H, t(D))$ is relatively almost orthogonal with respect to T. Therefore G is inert, according to Proposition 4.10. □

Remark 5.6. If G is a inert mixed non-divisible group, then G cannot be commensurable with any fully invariant subgroup of $\mathrm{D}(G)$, as they are all torsion, by Fact 3.1. This shows that no counterpart of Theorem 5.3 (d) remains valid in the mixed case.

The groups satisfying only the hypotheses (b), (c) and (d) of Theorem 5.5 do not split, in general, as the next example shows.

Example 5.7. Every Abelian group G with $t(G) \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p)$ and $G/t(G) = A$ of rank one and type $\tau(A) = [1, 1, 1, \dots]$ satisfies the hypotheses (b), (c) and (d) of Theorem 5.5. We will show that there are $2^{\aleph_0}$ non-isomorphic groups G with these properties; only the splitting group among them is inert. From the exact sequence

$$0 \to \mathbb{Z} \to A \to \bigoplus_p \mathbb{Z}(p) \to 0$$

we derive the long exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}\Big(\mathbb{Z}, \bigoplus_p \mathbb{Z}(p)\Big) \to \mathrm{Ext}^1_{\mathbb{Z}}\Big(\bigoplus_p \mathbb{Z}(p), \bigoplus_p \mathbb{Z}(p)\Big)$$
$$\to \mathrm{Ext}^1_{\mathbb{Z}}\Big(A, \bigoplus_p \mathbb{Z}(p)\Big) \to \mathrm{Ext}^1_{\mathbb{Z}}\Big(\mathbb{Z}, \bigoplus_p \mathbb{Z}(p)\Big) = 0.$$

The first group in the above sequence is isomorphic to $\bigoplus_p \mathbb{Z}(p)$, so it is countable; the second group has cardinality the continuum, since it is isomorphic to $\prod_p \mathbb{Z}(p)$, so the third group, consisting of the isomorphism classes of the groups G satisfying the properties described above, also has cardinality the continuum.

6 Characterization of fully inert subgroups of divisible groups

The complete characterization of inert groups now makes possible the solution of problem (P2) posed in Section 3, namely, characterize the pairs of groups (G, D) with G inert and D divisible, such that $\mathrm{Im}(\alpha)$ is finite for every homomorphism $\alpha : G \to D$ (that is, almost orthogonal).

Theorem 6.1. *Let G be a non-divisible inert group and D a divisible group. Then the pair (G, D) is almost orthogonal if and only if the following assertions hold:*

(a) *If $G \cong F \oplus \bigoplus_{p\in P} \mathbb{Z}(p^{m_p})^{(\alpha_p)}$ is torsion and in its canonical form, then:*
   - (a$_1$) *if $m_p = \infty$, then $r_p(D) = 0$,*
   - (a$_2$) *if α_p is infinite, then $r_p(D)$ is finite,*
   - (a$_3$) *$r_p(D) = 0$ for almost all $p \in P$ (i.e., $\pi(G, D)$ is finite).*

(b) *If $G \cong A^n$ is torsion-free, where A is a rational group of type $\tau(A) = [k_p]$, then:*
   - (b$_1$) *D is torsion,*
   - (b$_2$) *$r_p(D) = 0$ for almost all p such that $k_p > 0$,*
   - (b$_3$) *$r_p(D) = 0$ for all p such that $k_p = \infty$.*

(c) *If G is mixed, then D satisfies all conditions stated in* (a) *and* (b) *above.*

Proof. The proof is a straightforward application of Theorem 5.5, which gives the structure of the inert group G, and Theorem 3.6, giving the equivalent conditions for (G, D) to be an almost orthogonal pair in this case. □

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