Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY
MNRAS **461,** 3239–3258 (2016)
Advance Access publication 2016 June 27
doi:10.1093/mnras/stw1505

Fully non-linear cosmological perturbations of multicomponent fluid and field systems

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Accepted 2016 June 20. Received 2016 May 8; in original form 2015 September 9

ABSTRACT

We present fully non-linear and exact cosmological perturbation equations in the presence of multiple components of fluids and minimally coupled scalar fields. We ignore the tensor-type perturbation. The equations are presented without taking the temporal gauge condition in the Friedmann background with general curvature and the cosmological constant. We include the anisotropic stress. Even in the absence of anisotropic stress of individual component, the multiple component nature introduces the anisotropic stress in the collective fluid quantities. We prove the Newtonian limit of multiple fluids in the zero-shear gauge and the uniform-expansion gauge conditions, present the Newtonian hydrodynamic equations in the presence of general relativistic pressure in the zero-shear gauge, and present the fully non-linear equations and the third-order perturbation equations of the non-relativistic pressure fluids in the CDM-comoving gauge.

Key words: gravitation – hydrodynamics – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

Cosmological perturbation theory is an important tool in mediating cosmological models with observations. The spatially homogeneous and isotropic world model based on the cosmological constant modified Einstein's gravity with small perturbations (Friedmann 1922; Lifshitz 1946) enjoyed unexpected and unprecedented success in explaining most of the cosmologically relevant observations. Minor or major (depending on one's view) shortcomings of this ΛCDM (cosmological constant and cold dark matter) model are, besides the small-scale problem of the CDM model, that substantial amounts of dark energy and dark matter are needed to maintain the model confronted by observations. The cosmological constant is a minimal form of the dark energy (often modelled using a scalar field), but this should be considered as a pure theoretical modification of Einstein's gravity introduced by Einstein himself (Einstein 1917). Besides the dark atoms (invisible to light detectors), substantial amount of the dark matter is also currently in a state of unidentified character.

With the help of such minor/major unknowns the concordance cosmological model (the ΛCDM Friedmann world model with small perturbations) is agreed among researchers to be quite successful in delineating the structure and evolution of the large-scale observable universe. As the scale becomes smaller and the time goes over into later stage, the non-linear processes cannot be ignored, but these processes are often treated in the Newtonian context using either non-linear perturbations or numerical simulations (Vishniac 1983; Blumenthal et al. 1984; Bernardeau et al. 2002; L'Huillier, Park & Kim 2014). Non-linear perturbation theory in Einstein's gravity also has been developed in parallel with increasing observational precisions (Bartolo et al. 2004; Noh & Hwang 2004; Malik & Wands 2009).

Fully non-linear formulation of cosmological perturbation theory in the context of Einstein's gravity was recently introduced (Hwang & Noh 2013c; Noh 2014). The formulation ignores spatially transverse-tracefree (tensor-type) perturbation, but have not taken the slicing (temporal gauge) condition. Besides simple derivation of the perturbation equations to any non-linear order, the formulation was used to show the Newtonian limit, the first-order post-Newtonian approximation and the Newtonian hydrodynamic equations with general relativistic (gravitating) pressure (Hwang & Noh 2013a,b; Noh & Hwang 2013). The formulation considered a fluid (without anisotropic stress) or a scalar field system.

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Here we extend the fully non-linear perturbation formulation to include the multiple component fluids now including the anisotropic stress of individual fluid and multiple component minimally coupled scalar fields.

We set $c \equiv 1 \equiv \hbar$ for the scalar fields.

2 FULLY NON-LINEAR PERTURBATIONS OF A FLUID

Our metric convention in the perturbation theory is (Bardeen 1988; Hwang & Noh 2013c)

$$ds^2 = -a^2(1+2\alpha)(dx^0)^2 - 2a\chi_i dx^0 dx^i + a^2(1+2\varphi)\gamma_{ij}dx^i dx^j, \tag{1}$$

where the spatial index of χ_i is raised and lowered by γ_{ij} as the metric; γ_{ij} is the comoving part of the three-space metric of the Robertson–Walker spacetime metric. Here we *assume* a to be a function of time only, and α, φ and χ_i are functions of space and time with arbitrary amplitudes. The spatial part of the metric is simple because we already have taken the spatial gauge condition *without* losing any generality, and have *ignored* the transverse-tracefree tensor-type perturbation; for our spatial gauge condition see Bardeen (1988), section VI of Noh & Hwang (2004), and section 2 of Hwang & Noh (2013c).

The energy-momentum tensor of a fluid is (Ehlers 1961; Ellis 1971, 1973)

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\widetilde{h}_{ab} + \widetilde{q}_a\widetilde{u}_b + \widetilde{q}_b\widetilde{u}_a + \widetilde{\pi}_{ab}, \tag{2}$$

where $\widetilde{u}_a$ is the normalized four-vector with $\widetilde{u}^a\widetilde{u}_a \equiv -1$ and $\widetilde{h}_{ab} \equiv \widetilde{g}_{ab} + \widetilde{u}_a\widetilde{u}_b$ is the projection tensor. Tildes indicate covariant quantities; $\widetilde{\mu}$, $\widetilde{p}$, $\widetilde{q}_a$ and $\widetilde{\pi}_{ab}$ are the covariant energy density, pressure, flux vector and anisotropic stress tensor, respectively, with $\widetilde{q}_a\widetilde{u}^a \equiv 0$ and $\widetilde{\pi}_{ab}\widetilde{u}^a \equiv 0 \equiv \widetilde{\pi}^c_c$. The fluid quantities in equation (2) can be read as

$$\widetilde{\mu} = \widetilde{T}_{ab}\widetilde{u}^a\widetilde{u}^b, \quad \widetilde{p} = \frac{1}{3}\widetilde{T}_{ab}\widetilde{h}^{ab}, \quad \widetilde{q}_a = -\widetilde{T}_{cd}\widetilde{u}^c\widetilde{h}^d_a, \quad \widetilde{\pi}_{ab} = \widetilde{T}_{cd}\widetilde{h}^c_a\widetilde{h}^d_b - \widetilde{p}\widetilde{h}_{ab}. \tag{3}$$

We will take the energy-frame condition $\widetilde{q}_a \equiv 0$ without losing any generality; in this case $\widetilde{u}_a$ becomes the fluid four-vector.

In the perturbation theory we may introduce (Hwang & Noh 2013c)

$$\widetilde{\mu} \equiv \mu + \delta\mu, \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{\pi}_{ij} \equiv a^2\Pi_{ij}, \quad \widetilde{u}_i \equiv a\widehat{\gamma}\frac{\widehat{v}_i}{c}, \quad \widehat{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{\widehat{v}^k\widehat{v}_k}{c^2(1+2\varphi)}}}, \tag{4}$$

where μ and p are the background energy density and pressure, respectively; spatial indices of $\widehat{v}_i$ and Π_{ij} are raised and lowered by γ_{ij} as the metric; the perturbed fluid quantities $\delta\mu$, δp, $\widehat{v}_i$ and Π_{ij} are functions of space and time with arbitrary amplitudes; $\widehat{\gamma}$ is the Lorentz factor, a cosmological version of it. In the following we will keep $\widetilde{\mu}$ and $\widetilde{p}$ without decomposition. We can decompose χ_i, v_i and Π_{ij} to scalar-, vector- and tensor-type perturbations as (Bardeen 1980, 1988; Noh & Hwang 2004; Hwang & Noh 2013c)

$$\chi_i \equiv \chi_{,i} + \chi^{(v)}_i, \quad \widehat{v}_i \equiv -\widehat{v}_{,i} + \widehat{v}^{(v)}_i, \quad \Pi_{ij} \equiv \frac{1}{a^2}\left(\Pi_{,i|j} - \frac{1}{3}\gamma_{ij}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(i|j)} + \Pi^{(t)}_{ij}, \tag{5}$$

with the vector-type perturbations satisfying $\chi^{(v)i}{}_{|i} = \widehat{v}^{(v)i}{}_{|i} = \Pi^{(v)i}{}_{|i} \equiv 0$ and the tensor-type perturbation $\Pi^{(t)i}{}_i \equiv 0 \equiv \Pi^{(t)j}{}_{i|j}$; a vertical bar indicates a covariant derivative based on the metric γ_{ij}; we have $t_{(ab)} \equiv \frac{1}{2}(t_{ab} + t_{ba})$. To the non-linear order our scalar-, vector- and tensor-type perturbations are coupled in the equation level.

In Hwang & Noh (2013c) we presented non-linear perturbation equations for vanishing anisotropic stress in a flat background. Here, we consider background world model with general curvature (Noh 2014), and include the anisotropic stress. The exact and fully non-linear perturbation equations, without taking the temporal gauge (slicing or hypersurface) condition are the following.

Definition of κ:

$$\kappa \equiv 3\frac{\dot{a}}{a}\left(1 - \frac{1}{\mathcal{N}}\right) - \frac{1}{\mathcal{N}(1+2\varphi)}\left[3\dot{\varphi} + \frac{c}{a^2}\left(\chi^k{}_{|k} + \frac{\chi^k\varphi_{,k}}{1+2\varphi}\right)\right]. \tag{6}$$

ADM energy constraint:

$$-\frac{3}{2}\left(\frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3c^2}\widetilde{\mu} + \frac{\overline{K}c^2}{a^2(1+2\varphi)} - \frac{\Lambda c^2}{3}\right) + \frac{\dot{a}}{a}\kappa + \frac{c^2\Delta\varphi}{a^2(1+2\varphi)^2}$$
$$= \frac{1}{6}\kappa^2 - \frac{4\pi G}{c^2}\left(\widetilde{\mu} + \widetilde{p}\right)\left(\widehat{\gamma}^2 - 1\right) + \frac{3}{2}\frac{c^2\varphi^{|i}\varphi_{,i}}{a^2(1+2\varphi)^3} - \frac{c^2}{4}\overline{K}^i_j\overline{K}^j_i - \frac{1}{(1+2\varphi)^2}\frac{4\pi G}{c^4}\Pi_{ij}\widehat{v}^i\widehat{v}^j. \tag{7}$$

ADM momentum constraint:

$$\frac{2}{3}\kappa_{,i} + \frac{c}{a^2\mathcal{N}(1+2\varphi)}\left[\frac{1}{2}\left(\Delta\chi_i + \chi^k{}_{|ik}\right) - \frac{1}{3}\chi^k{}_{|ki}\right] + \frac{8\pi G}{c^4}\left(\widetilde{\mu} + \widetilde{p}\right)a\widehat{\gamma}^2\widehat{v}_i = \frac{c}{a^2\mathcal{N}(1+2\varphi)}\left\{\left(\frac{\mathcal{N}_{,j}}{\mathcal{N}} - \frac{\varphi_{,j}}{1+2\varphi}\right)\left[\frac{1}{2}\left(\chi^j{}_{|i} + \chi_i{}^{|j}\right)\right.\right.$$
$$\left.\left.-\frac{1}{3}\delta^j_i\chi^k{}_{|k}\right] - \frac{\varphi^{,j}}{(1+2\varphi)^2}\left(\chi_i\varphi_{,j} + \frac{1}{3}\chi_j\varphi_{,i}\right) + \frac{\mathcal{N}}{1+2\varphi}\nabla_j\left[\frac{1}{\mathcal{N}}\left(\chi^j\varphi_{,i} + \chi_i\varphi^{|j} - \frac{2}{3}\delta^j_i\chi^k\varphi_{,k}\right)\right]\right\} - \frac{a}{1+2\varphi}\frac{8\pi G}{c^4}\Pi_{ij}\widehat{v}^j. \tag{8}$$

Trace of ADM propagation:

$$-3\left[\frac{1}{\mathcal{N}}\left(\frac{\dot{a}}{a}\right)^{\cdot}+\frac{\dot{a}^2}{a^2}+\frac{4\pi G}{3c^2}\left(\widetilde{\mu}+3\widetilde{p}\right)-\frac{\Lambda c^2}{3}\right]+\frac{1}{\mathcal{N}}\dot{\kappa}+2\frac{\dot{a}}{a}\kappa+\frac{c^2\Delta\mathcal{N}}{a^2\mathcal{N}(1+2\varphi)}=\frac{1}{3}\kappa^2+\frac{8\pi G}{c^2}\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)$$
$$-\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left(\chi^i\kappa_{,i}+c\frac{\varphi^{|i}\mathcal{N}_{,i}}{1+2\varphi}\right)+c^2\overline{K}^i_j\overline{K}^j_i+\frac{1}{1+2\varphi}\frac{4\pi G}{c^2}\left(\Pi^i_i+\frac{1}{1+2\varphi}\Pi_{ij}\frac{\widehat{v}^i\widehat{v}^j}{c^2}\right). \quad (9)$$

Tracefree ADM propagation:

$$\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t}+3\frac{\dot{a}}{a}-\kappa+\frac{c\chi^k}{a^2\mathcal{N}(1+2\varphi)}\nabla_k\right)\left\{\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left[\frac{1}{2}\left(\chi^i{}_{|j}+\chi_j{}^{|i}\right)-\frac{1}{3}\delta^i_j\chi^k{}_{|k}-\frac{1}{1+2\varphi}\left(\chi^i\varphi_{,j}+\chi_j\varphi^{|i}-\frac{2}{3}\delta^i_j\chi^k\varphi_{,k}\right)\right]\right\}$$
$$-\frac{c^2}{a^2(1+2\varphi)}\left[\frac{1}{1+2\varphi}\left(\nabla^i\nabla_j-\frac{1}{3}\delta^i_j\Delta\right)\varphi+\frac{1}{\mathcal{N}}\left(\nabla^i\nabla_j-\frac{1}{3}\delta^i_j\Delta\right)\mathcal{N}\right]=\frac{8\pi G}{c^2}\left(\widetilde{\mu}+\widetilde{p}\right)\left[\frac{\widehat{\gamma}^2\widehat{v}^i\widehat{v}_j}{c^2(1+2\varphi)}-\frac{1}{3}\delta^i_j\left(\widehat{\gamma}^2-1\right)\right]$$
$$+\frac{c^2}{a^4\mathcal{N}^2(1+2\varphi)^2}\left[\frac{1}{2}\left(\chi^{i|k}\chi_{j|k}-\chi_{k|j}\chi^{k|i}\right)+\frac{1}{1+2\varphi}\left(\chi^{k|i}\chi_k\varphi_{,j}-\chi^{i|k}\chi_j\varphi_{,k}+\chi_{k|j}\chi^k\varphi^{|i}-\chi_{j|k}\chi^i\varphi^{|k}\right)\right.$$
$$\left.+\frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_j\varphi^{|k}\varphi_{,k}-\chi^k\chi_k\varphi^{|i}\varphi_{,j}\right)\right]-\frac{c^2}{a^2(1+2\varphi)^2}\left[\frac{3}{1+2\varphi}\left(\varphi^{|i}\varphi_{,j}-\frac{1}{3}\delta^i_j\varphi^{|k}\varphi_{,k}\right)+\frac{1}{\mathcal{N}}\left(\varphi^{|i}\mathcal{N}_{,j}+\varphi_{,j}\mathcal{N}^{|i}-\frac{2}{3}\delta^i_j\varphi^{|k}\mathcal{N}_{,k}\right)\right]$$
$$+\frac{1}{1+2\varphi}\frac{8\pi G}{c^2}\left(\Pi^i_j-\frac{1}{3}\delta^i_j\Pi^k_k\right). \quad (10)$$

We have

$$\mathcal{N}\equiv\sqrt{1+2\alpha+\frac{\chi^k\chi_k}{a^2(1+2\varphi)}},\quad \overline{K}^i_j\overline{K}^j_i=\frac{1}{a^4\mathcal{N}^2(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i|j}\left(\chi_{i|j}+\chi_{j|i}\right)-\frac{1}{3}\chi^i{}_{|i}\chi^j{}_{|j}\right.$$
$$\left.-\frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{|j}\left(\chi_{i|j}+\chi_{j|i}\right)-\frac{1}{3}\chi^i{}_{|i}\chi^j\varphi_{,j}\right]+\frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{|j}\varphi_{,j}+\frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}. \quad (11)$$

Equations (6)–(10) follow from the Einstein's equation in the ADM (Arnowitt–Deser–Misner) formulation (Arnowitt, Deser & Misner 1962), presented in equations A4, A6–A9 of Hwang & Noh (2013c). These equations without the anisotropic stress are derived in Hwang & Noh (2013c) and Noh (2014). In order to include the anisotropic stress, it is convenient to have the ADM fluid quantities in the following

$$E=\widetilde{\mu}+\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)+\frac{1}{(1+2\varphi)^2}\frac{\widehat{v}^i\widehat{v}^j}{c^2}\Pi_{ij},\quad S=3\widetilde{p}+\left(\widehat{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)+\frac{1}{1+2\varphi}\Pi^i_i,$$
$$J_i=a\left(\widetilde{\mu}+\widetilde{p}\right)\widehat{\gamma}^2\frac{\widehat{v}_i}{c}+\frac{a}{1+2\varphi}\frac{\widehat{v}^j}{c}\Pi_{ij},\quad \overline{S}^i_j=\left(\widetilde{\mu}+\widetilde{p}\right)\left[\frac{\widehat{\gamma}^2}{1+2\varphi}\frac{\widehat{v}^i\widehat{v}_j}{c^2}-\frac{1}{3}\delta^i_j\left(\widehat{\gamma}^2-1\right)\right]+\frac{1}{1+2\varphi}\left(\Pi^i_j-\frac{1}{3}\delta^i_j\Pi^k_k\right), \quad (12)$$

with

$$\Pi^i_i=\frac{1}{1+2\varphi}\frac{\widehat{v}^i\widehat{v}^j}{c^2}\Pi_{ij}. \quad (13)$$

Equations (6)–(13) are valid not only for a single component fluid but also for multicomponent system as we regard the fluid quantities as the collective ones.

The ADM and the covariant conservation equations follow from equations A10, A11, C11 and C12 of Hwang & Noh (2013c). Now including the anisotropic stress we have the followings.

ADM energy conservation:

$$\frac{1}{\mathcal{N}}\left[\widetilde{\mu}+\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)\right]^{\cdot}+\frac{c}{a^2\mathcal{N}}\frac{\chi^i}{1+2\varphi}\left[\widetilde{\mu}+\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)\right]_{|i}$$
$$+\left(\widetilde{\mu}+\widetilde{p}\right)(3H-\kappa)\frac{1}{3}\left(4\widehat{\gamma}^2-1\right)+\left(\frac{\widetilde{\mu}+\widetilde{p}}{a(1+2\varphi)}\widehat{\gamma}^2\widehat{v}^i\right)_{|i}+\left(\frac{3\varphi_{,i}}{1+2\varphi}+2\frac{\mathcal{N}_{,i}}{\mathcal{N}}\right)\frac{\widetilde{\mu}+\widetilde{p}}{a(1+2\varphi)}\widehat{\gamma}^2\widehat{v}^i$$
$$+\frac{\widehat{\gamma}^2(\widetilde{\mu}+\widetilde{p})}{ca^2\mathcal{N}(1+2\varphi)^2}\left[\chi^{i|j}\widehat{v}_i\widehat{v}_j-\frac{1}{3}\chi^j{}_{|j}\widehat{v}^i\widehat{v}_i-\frac{2}{1+2\varphi}\left(\widehat{v}^i\widehat{v}^j\chi_i\varphi_{,j}-\frac{1}{3}\widehat{v}^i\widehat{v}_i\chi^j\varphi_{,j}\right)\right]=-\Pi^{(\mathrm{ADM})}. \quad (14)$$

ADM momentum conservation:

$$\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t}+3H-\kappa\right)\left[a(\widetilde{\mu}+\widetilde{p})\widehat{\gamma}^2\widehat{v}_i\right]+\frac{c}{a^2\mathcal{N}}\frac{\chi^j}{1+2\varphi}\left[a(\widetilde{\mu}+\widetilde{p})\widehat{\gamma}^2\widehat{v}_i\right]_{|j}+c^2\widetilde{p}_{,i}+c^2\left(\widetilde{\mu}+\widetilde{p}\right)\frac{\mathcal{N}_{,i}}{\mathcal{N}}+\left(\frac{\widetilde{\mu}+\widetilde{p}}{1+2\varphi}\widehat{\gamma}^2\widehat{v}^j\widehat{v}_i\right)_{|j}$$
$$+\frac{c}{a\mathcal{N}}\left(\frac{\chi^j}{1+2\varphi}\right)_{|i}\left(\widetilde{\mu}+\widetilde{p}\right)\widehat{\gamma}^2\widehat{v}_j+\frac{\widetilde{\mu}+\widetilde{p}}{1+2\varphi}\widehat{\gamma}^2\widehat{v}^j\left[\frac{1}{1+2\varphi}(3\widehat{v}_i\varphi_{,j}-\widehat{v}_j\varphi_{,i})+\frac{1}{\mathcal{N}}(\widehat{v}_i\mathcal{N}_{,j}+\widehat{v}_j\mathcal{N}_{,i})\right]=-c\Pi_i^{(\mathrm{ADM})}. \tag{15}$$

Covariant energy conservation:

$$\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i+\frac{c}{a}\chi^i\right)\nabla_i\right]\widetilde{\mu}+(\widetilde{\mu}+\widetilde{p})\left\{(3H-\kappa)\mathcal{N}+\frac{(\mathcal{N}\widehat{v}^i)_{|i}}{a(1+2\varphi)}+\frac{\mathcal{N}\widehat{v}^i\varphi_{,i}}{a(1+2\varphi)^2}\right.$$
$$\left.+\frac{1}{\widehat{\gamma}}\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i+\frac{c}{a}\chi^i\right)\nabla_i\right]\widehat{\gamma}\right\}=-\Pi^{(\mathrm{ADM})}+\frac{\widehat{v}^i}{ca(1+2\varphi)}\Pi_i^{(\mathrm{ADM})}. \tag{16}$$

Covariant momentum conservation:

$$\frac{\partial}{\partial t}\left(a\widehat{\gamma}\widehat{v}_i\right)+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\left(a\widehat{\gamma}\widehat{v}_i\right)+c^2\widehat{\gamma}\mathcal{N}_{,i}+\frac{1-\widehat{\gamma}^2}{\widehat{\gamma}}\frac{c^2\mathcal{N}\varphi_{,i}}{1+2\varphi}+\frac{c}{a}\widehat{\gamma}\widehat{v}^k\nabla_i\left(\frac{\chi_k}{1+2\varphi}\right)+\frac{1}{\widetilde{\mu}+\widetilde{p}}$$
$$\times\left\{c^2\frac{\mathcal{N}}{\widehat{\gamma}}\widetilde{p}_{,i}+a\widehat{\gamma}\widehat{v}_i\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\right]\widetilde{p}\right\}=\frac{c\mathcal{N}}{(\widetilde{\mu}+\widetilde{p})\widehat{\gamma}}\left[\frac{a\widehat{\gamma}^2\widehat{v}_i}{c}\Pi^{(\mathrm{ADM})}-\left(\delta_i^j+\frac{\widehat{\gamma}^2\widehat{v}_i\widehat{v}^j}{c^2(1+2\varphi)}\right)\Pi_j^{(\mathrm{ADM})}\right]. \tag{17}$$

We have

$$\Pi^{(\mathrm{ADM})}\equiv\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t}+\frac{c\chi^k}{a^2\mathcal{N}(1+2\varphi)}\nabla_k+4\frac{\dot{a}}{a}-\frac{4}{3}\kappa\right)\left(\frac{\widehat{v}^i\widehat{v}^j}{c^2(1+2\varphi)^2}\Pi_{ij}\right)$$
$$+\frac{c}{a^2\mathcal{N}(1+2\varphi)^2}\left(\chi^i{}_{|j}-\frac{2\varphi^{|i}\chi_j}{1+2\varphi}\right)\left(\Pi_i^j-\frac{1}{3}\delta_i^j\Pi_k^k\right)+\left(\nabla_i+2\frac{\mathcal{N}_{,i}}{\mathcal{N}}+\frac{3\varphi_{,i}}{1+2\varphi}\right)\left(\frac{\widehat{v}^j}{a(1+2\varphi)^2}\Pi_j^i\right), \tag{18}$$

$$\Pi_i^{(\mathrm{ADM})}\equiv\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t}+\frac{c\chi^k}{a^2\mathcal{N}(1+2\varphi)}\nabla_k+3\frac{\dot{a}}{a}-\kappa\right)\left(\frac{a\widehat{v}^j}{c(1+2\varphi)}\Pi_{ij}\right)+\frac{1}{a\mathcal{N}(1+2\varphi)}\left(\frac{\chi^j}{1+2\varphi}\right)_{|i}\widehat{v}^k\Pi_{jk}$$
$$+\frac{c\mathcal{N}_{,i}}{\mathcal{N}(1+2\varphi)^2}\frac{\widehat{v}^j\widehat{v}^k}{c^2}\Pi_{jk}+\left(\nabla_j+3\frac{\varphi_{,j}}{1+2\varphi}+\frac{\mathcal{N}_{,j}}{\mathcal{N}}\right)\left(\frac{c}{1+2\varphi}\Pi_i^j\right)-\frac{c\varphi_{,i}}{(1+2\varphi)^2}\Pi_j^j. \tag{19}$$

In order to derive the covariant conservation equations we used the relations among conservation equations as the following

$$\widetilde{T}^b_{a;b}\widetilde{n}^a=\frac{1}{a\mathcal{N}}\widetilde{T}^b_{0;b}+\frac{\chi^i}{a^2\mathcal{N}(1+2\varphi)}\widetilde{T}^b_{i;b},\quad \widetilde{T}^b_{a;b}\widetilde{h}^{(n)a}{}_i=\widetilde{T}^b_{i;b},$$
$$\widetilde{T}^b_{a;b}\widetilde{u}^a=\widehat{\gamma}\widetilde{T}^b_{a;b}\widetilde{n}^a+\frac{\widehat{\gamma}\widehat{v}^i}{ca(1+2\varphi)}\widetilde{T}^b_{i;b},\quad \widetilde{T}^b_{a;b}\widetilde{h}^a_i=\frac{a\widehat{\gamma}\widehat{v}_i}{c}\widetilde{T}^b_{a;b}\widetilde{u}^a+\widetilde{T}^b_{i;b}, \tag{20}$$

where $\widetilde{n}_a$ is the normal four-vector and $\widetilde{h}^{(n)}_{ab}\equiv\widetilde{g}_{ab}+\widetilde{n}_a\widetilde{n}_b$ its projection tensor. See equation C17 of Hwang & Noh (2013c).

In the multiple component system the above collective conservation equations will be replaced by the conservation equations of individual component, see next section. The minimally coupled scalar field does not have the anisotropic stress. Even in the absence of anisotropic stress of individual component, as the multiple nature of fluids and fields generates the anisotropic stress of the collective fluid it is important to keep the anisotropic stress terms in Einstein's equations.

As the dimensions we consider

$$[\widetilde{g}_{ab}]=[\widetilde{u}_a]=[a]=[\gamma_{ij}]=[\alpha]=[\varphi]=[\chi^i]=[\chi_i^{(v)}]=[\widehat{v}_i/c]=[\widehat{v}_i^{(v)}/c]=[\widehat{\gamma}]=1,$$
$$[x^a]=[cdt]\equiv[ad\eta]=L,\quad [G\widetilde{\varrho}]=T^{-2},\quad [\Lambda]=[\overline{K}]=L^{-2},\quad [\chi]=[\widehat{v}/c]=L,\quad [\kappa]=T^{-1},$$
$$[\widetilde{T}_{ab}]=[\widetilde{\mu}]=[\widetilde{\varrho}c^2]=[\widetilde{p}]=[\widetilde{q}_a]=[\widetilde{\pi}_{ab}]=[\Pi_{ij}]=[\Pi]/L^2=[\Pi_i^{(v)}]/L, \tag{21}$$

where $\overline{K}$ is the normalized background three-space curvature with $R^{(3)}\equiv6\overline{K}$ (Noh 2014).

In the above set of equations we have not taken the temporal gauge (hypersurface or slicing) condition yet. As the temporal gauge condition we can impose any one of the following conditions (Bardeen 1988; Hwang 1991; Hwang & Noh 2013c)

comoving gauge : $v\equiv0$,

zero−shear gauge : $\chi\equiv0$,

uniform−curvature gauge : $\varphi\equiv0$,

$$\text{uniform}-\text{expansion gauge}: \qquad \kappa \equiv 0,$$

$$\text{uniform}-\text{density gauge}: \qquad \delta \equiv 0, \tag{22}$$

or combinations of these to all perturbation orders. We can also impose different gauge conditions at different perturbation orders; see equation 272 in Noh & Hwang (2004) for an example of gauge-invariant combination with different gauge-invariant combination taken to different perturbation order; section VI.C.2 of that paper shows the method of constructing such a mixed gauge-invariant combination. With the imposition of any of these slicing condition the remaining perturbation variables are free from the remnant (both spatial and temporal) gauge mode, and have unique gauge-invariant combinations (Bardeen 1988; Hwang 1991; Noh & Hwang 2004; Hwang & Noh 2013c). In the presence of multiple component of fluids and fields we will have additional choice of temporal gauge condition based on the individual component, see equations (42) and (119).

3 MULTICOMPONENT FLUIDS

Now, we consider the case of multiple components of fluids. We will show that even in the absence of anisotropic stress of individual component the multiple fluid nature leads to non-vanishing anisotropic stress of the collective fluid; this may be a well known result in the Newtonian fluid context. Equations (6)–(10) remain valid with the fluid quantities considered as the collective ones. In fact all relations derived for a single component remain valid in the multicomponent situation with the fluid quantities replaced by the collective ones to be presented in this section; the conservation equations (14)–(17) are also valid in this sense, but the conservation equations for individual component are more convenient. In the multiple fluids we additionally need to (i) express the collective fluid quantities in terms of the individual ones, and (ii) provide the (energy and momentum) conservation equations for the individual component.

3.1 Fluid quantities

In the presence of N fluids we have

$$\widetilde{T}_{ab} = \sum_J \widetilde{T}_{Jab}, \tag{23}$$

with the fluid quantities of individual component introduced as

$$\widetilde{T}_{Iab} \equiv \widetilde{\mu}_I \widetilde{u}_{Ia} \widetilde{u}_{Ib} + \widetilde{p}_I \widetilde{h}_{Iab} + \widetilde{q}_{Ia} \widetilde{u}_{Ib} + \widetilde{q}_{Ib} \widetilde{u}_{Ia} + \widetilde{\pi}_{Iab}, \tag{24}$$

where $\widetilde{u}_{Ia}$ is the normalized four vectors with $\widetilde{u}^c_I \widetilde{u}_{Ic} \equiv -1$, and $\widetilde{h}_{Iab} \equiv \widetilde{g}_{ab} + \widetilde{u}_{Ia} \widetilde{u}_{Ib}$ is the projection tensor of each component. Indices I, J, $\ldots = 1, 2 \ldots N$ indicate the fluid component. We have $\widetilde{q}_{Ia} \widetilde{u}^a_I \equiv 0$ and $\widetilde{\pi}_{Iab} \widetilde{u}^a_I \equiv 0 \equiv \widetilde{\pi}_{Ic}{}^c$. The fluid quantities of individual component can be read as

$$\widetilde{\mu}_I = \widetilde{T}_{Iab} \widetilde{u}^a_I \widetilde{u}^b_I, \quad \widetilde{p}_I = \frac{1}{3} \widetilde{T}_{Iab} \widetilde{h}^{ab}_I, \quad \widetilde{q}_{Ia} = -\widetilde{T}_{Icd} \widetilde{u}^c_I \widetilde{h}_{Ia}{}^d, \quad \widetilde{\pi}_{Iab} = \widetilde{T}_{Icd} \widetilde{h}_{Ia}{}^c \widetilde{h}_{Ib}{}^d - \widetilde{p}_I \widetilde{h}_{Iab}. \tag{25}$$

Without losing generality we can take the energy-frame condition $\widetilde{q}_{Ia} \equiv 0$ for each component.

From equations (3) and (25) we have

$$\widetilde{\mu} = \sum_J \left\{ \widetilde{\mu}_J + \left(\widetilde{\mu}_J + \widetilde{p}_J \right) \left[\left(\widetilde{u}^c_J \widetilde{u}_c \right)^2 - 1 \right] + \widetilde{\pi}_{Jab} \widetilde{u}^a \widetilde{u}^b \right\},$$

$$\widetilde{p} = \sum_J \left\{ \widetilde{p}_J + \frac{1}{3} \left(\widetilde{\mu}_J + \widetilde{p}_J \right) \left[\left(\widetilde{u}^c_J \widetilde{u}_c \right)^2 - 1 \right] + \frac{1}{3} \widetilde{\pi}_{Jab} \widetilde{u}^a \widetilde{u}^b \right\},$$

$$\widetilde{u}_a = -\frac{1}{\widetilde{\mu} + \sum_K \widetilde{p}_K} \sum_J \left[\left(\widetilde{\mu}_J + \widetilde{p}_J \right) \widetilde{u}_{Ja} \widetilde{u}_{Jb} + \widetilde{\pi}_{Jab} \right] \widetilde{u}^b,$$

$$\widetilde{\pi}_{ab} = \left(\widetilde{h}^c_a \widetilde{h}^d_b - \frac{1}{3} \widetilde{h}^{cd} \widetilde{h}_{ab} \right) \sum_J \left[\left(\widetilde{\mu}_J + \widetilde{p}_J \right) \widetilde{u}_{Jc} \widetilde{u}_{Jd} + \widetilde{\pi}_{Jcd} \right], \tag{26}$$

thus

$$\widetilde{\mu} - 3\widetilde{p} = \sum_J \left(\widetilde{\mu}_J - 3\widetilde{p}_J \right). \tag{27}$$

We introduce the fluid four-vector of individual component as

$$\widetilde{u}_{Ii} \equiv \frac{a \widehat{\gamma}_I \widehat{v}_{Ii}}{c}, \tag{28}$$

thus we have

$$\widetilde{u}_{I0} = -\widehat{\gamma}_I \left(a\mathcal{N} + \frac{\widehat{v}_{Ik}\chi^k}{c(1+2\varphi)} \right), \quad \widetilde{u}^i_I = \frac{\widehat{\gamma}_I}{a(1+2\varphi)} \left(\frac{\widehat{v}^i_I}{c} + \frac{\chi^i}{a\mathcal{N}} \right), \quad \widetilde{u}^0_I = \frac{1}{a\mathcal{N}} \widehat{\gamma}_I,$$

$$\widehat{\gamma}_I \equiv \frac{1}{\sqrt{1 - \frac{\widehat{v}^k_I \widehat{v}_{Ik}}{c^2(1+2\varphi)}}}, \tag{29}$$

where spatial index of $\widehat{v}_{Ii}$ is raised and lowered by γ_{ij} as the metric. For anisotropic stress we similarly introduce

$$\widetilde{\pi}_{Iij} \equiv a^2 \Pi_{Iij}. \tag{30}$$

For the collective component, we can simply remove the subindex indicating the component.

The fluid quantities in equation (26) give

$$\widetilde{\mu} = \sum_J \left\{ \widetilde{\mu}_J + (\widetilde{\mu}_J + \widetilde{p}_J) \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] + \frac{\widehat{\gamma}^2}{(1+2\varphi)^2} \frac{1}{c^2} \left(\widehat{v}^i - \widehat{v}^i_J \right) \left(\widehat{v}^j - \widehat{v}^j_J \right) \Pi_{Jij} \right\},$$

$$\widetilde{p} = \sum_J \left\{ \widetilde{p}_J + \frac{1}{3} (\widetilde{\mu}_J + \widetilde{p}_J) \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] + \frac{1}{3} \frac{\widehat{\gamma}^2}{(1+2\varphi)^2} \frac{1}{c^2} \left(\widehat{v}^i - \widehat{v}^i_J \right) \left(\widehat{v}^j - \widehat{v}^j_J \right) \Pi_{Jij} \right\},$$

$$\widehat{v}_i = \frac{1}{\widehat{\gamma}^2} \frac{\sum_J \left[(\widetilde{\mu}_J + \widetilde{p}_J) \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}_j \widehat{v}^j_J}{c^2} \right) \widehat{v}_{Ji} - \frac{1}{1+2\varphi} \left(\widehat{v}^j - \widehat{v}^j_J \right) \Pi_{Jij} \right]}{\sum_K \left[(\widetilde{\mu}_K + \widetilde{p}_K) \widehat{\gamma}^2_K \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}_k \widehat{v}^k_K}{c^2} \right)^2 + \frac{1}{(1+2\varphi)^2} \frac{1}{c^2} \left(\widehat{v}^k - \widehat{v}^k_K \right) \left(\widehat{v}^\ell - \widehat{v}^\ell_K \right) \Pi_{Kk\ell} \right]},$$

$$\Pi_{ij} = \sum_J (\widetilde{\mu}_J + \widetilde{p}_J) \left\{ \widehat{\gamma}^2_J \left[\widehat{\gamma}^2 \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right) \frac{\widehat{v}_i}{c} - \frac{\widehat{v}_{Ji}}{c} \right] \left[\widehat{\gamma}^2 \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^\ell \widehat{v}_{J\ell}}{c^2} \right) \frac{\widehat{v}_j}{c} - \frac{\widehat{v}_{Jj}}{c} \right] \right.$$
$$\left. - \frac{1}{3} \left[(1+2\varphi) \gamma_{ij} + \widehat{\gamma}^2 \frac{\widehat{v}_i \widehat{v}_j}{c^2} \right] \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] \right\}$$
$$- \frac{\widehat{\gamma}^2}{3(1+2\varphi)} \sum_J \left(\gamma_{ij} - \frac{2\widehat{\gamma}^2}{1+2\varphi} \frac{\widehat{v}_i \widehat{v}_j}{c^2} \right) \frac{1}{c^2} \left(\widehat{v}^k - \widehat{v}^k_J \right) \left(\widehat{v}^\ell - \widehat{v}^\ell_J \right) \Pi_{Jk\ell}. \tag{31}$$

These are the relations between the collective and individual fluid quantities. Using these relations equations (6)–(10) remain valid even in the multiple component fluid case.

For vanishing anisotropic stress of individual component, we have

$$\widetilde{\mu} = \sum_J \left\{ \widetilde{\mu}_J + (\widetilde{\mu}_J + \widetilde{p}_J) \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] \right\},$$

$$\widetilde{p} = \sum_J \left\{ \widetilde{p}_J + \frac{1}{3} (\widetilde{\mu}_J + \widetilde{p}_J) \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] \right\},$$

$$\widehat{v}_i = \frac{1}{\widehat{\gamma}^2} \frac{\sum_J (\widetilde{\mu}_J + \widetilde{p}_J) \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}_j \widehat{v}^j_J}{c^2} \right) \widehat{v}_{Ji}}{\sum_K (\widetilde{\mu}_K + \widetilde{p}_K) \widehat{\gamma}^2_K \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}_k \widehat{v}^k_K}{c^2} \right)^2},$$

$$\Pi_{ij} = \sum_J (\widetilde{\mu}_J + \widetilde{p}_J) \left\{ \widehat{\gamma}^2_J \left[\widehat{\gamma}^2 \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right) \frac{\widehat{v}_i}{c} - \frac{\widehat{v}_{Ji}}{c} \right] \left[\widehat{\gamma}^2 \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^\ell \widehat{v}_{J\ell}}{c^2} \right) \frac{\widehat{v}_j}{c} - \frac{\widehat{v}_{Jj}}{c} \right] \right.$$
$$\left. - \frac{1}{3} \left[(1+2\varphi) \gamma_{ij} + \widehat{\gamma}^2 \frac{\widehat{v}_i \widehat{v}_j}{c^2} \right] \left[\widehat{\gamma}^2 \widehat{\gamma}^2_J \left(1 - \frac{1}{1+2\varphi} \frac{\widehat{v}^k \widehat{v}_{Jk}}{c^2} \right)^2 - 1 \right] \right\}. \tag{32}$$

In the following, unless mentioned otherwise, for simplicity we ignore the anisotropic stress of individual component $\widetilde{\pi}_{Icd}$.

As the right-hand side of $\widehat{v}_i$ in equation (32) still has $\widehat{v}_k$, in order to express $\widehat{v}_i$ in terms of individual fluid component only, we need to solve this relation perturbatively by iteration. The other collective fluid quantities also contain $\widehat{v}_i$ which should be replaced by thus obtained individual ones.

For example, to the second order, we have

$$\widehat{v}_i = \frac{\sum_J \left(\widetilde{\mu}_J + \widetilde{p}_J\right) \widehat{v}_{Ji}}{\sum_K \left(\widetilde{\mu}_K + \widetilde{p}_K\right)}, \tag{33}$$

and to the third order, we have

$$\widetilde{v}_i = \frac{\sum_J \left(\widetilde{\mu}_J + \widetilde{p}_J\right) \left[1 + \frac{1}{c^2}\left(\widehat{v}_k \widehat{v}^k - \widehat{v}_k \widehat{v}^k_J + \widehat{v}_{Jk}\widehat{v}^k_J\right)\right] \widehat{v}_{Ji}}{\sum_K \left(\widetilde{\mu}_K + \widetilde{p}_K\right)\left[1 + \frac{1}{c^2}\widehat{v}_{K\ell}\left(\widehat{v}^\ell_K - 2\widehat{v}^\ell\right)\right]}, \tag{34}$$

where $\widehat{v}_j$s in the right-hand side can be replaced by the one in equation (33). This can be continued to higher order perturbations. The rest of fluid quantities to the third order are

$$\widetilde{\mu} = \sum_J \left[\widetilde{\mu}_J + \frac{\widetilde{\mu}_J + \widetilde{p}_J}{1+2\varphi}\frac{1}{c^2}\left(\widehat{v}^k - \widehat{v}^k_J\right)\left(\widehat{v}_k - \widehat{v}_{Jk}\right)\right], \quad \widetilde{p} = \sum_J \left[\widetilde{p}_J + \frac{1}{3}\frac{\widetilde{\mu}_J + \widetilde{p}_J}{1+2\varphi}\frac{1}{c^2}\left(\widehat{v}^k - \widehat{v}^k_J\right)\left(\widehat{v}_k - \widehat{v}_{Jk}\right)\right],$$

$$\Pi_{ij} = \sum_J \left(\widetilde{\mu}_J + \widetilde{p}_J\right)\frac{1}{c^2}\left[\left(\widehat{v}_i - \widehat{v}_{Ji}\right)\left(\widehat{v}_j - \widehat{v}_{Jj}\right) - \frac{1}{3}\gamma_{ij}\left(\widehat{v}^k - \widehat{v}^k_J\right)\left(\widehat{v}_k - \widehat{v}_{Jk}\right)\right], \tag{35}$$

where $\widehat{v}_j$ in the right-hand sides can be replaced by the one in equation (33).

3.2 Conservation equations of individual component

Now, we provide the energy and momentum conservation equations followed by the individual fluid. The energy and momentum conservation equations follow from $\widetilde{T}^b_{a;b} = 0$, thus

$$\widetilde{T}^{\ b}_{Ia;b} \equiv \widetilde{I}_{Ia}, \quad \sum_J \widetilde{I}_{Ja} = 0, \tag{36}$$

where $\widetilde{I}_{Ia}$ indicates the interaction terms among fluids.

The ADM and the covariant (energy and momentum) conservation equations for individual component are presented in equations E10–E13 of Hwang & Noh (2013c). Using the ADM and the covariant quantities presented in the appendices B and C of Hwang & Noh (2013c) we can derive these equations in the fully non-linear forms as the following.

ADM energy conservation:

$$\frac{1}{\mathcal{N}}\left[\widetilde{\mu}_I + \left(\widetilde{\mu}_I + \widetilde{p}_I\right)\left(\widehat{\gamma}^2_I - 1\right)\right]^{\cdot} + \frac{c}{a^2\mathcal{N}}\frac{\chi^i}{1+2\varphi}\left[\widetilde{\mu}_I + \left(\widetilde{\mu}_I + \widetilde{p}_I\right)\left(\widehat{\gamma}^2_I - 1\right)\right]_{|i}$$
$$+ \left(\widetilde{\mu}_I + \widetilde{p}_I\right)(3H - \kappa)\frac{1}{3}\left(4\widehat{\gamma}^2_I - 1\right) + \left(\frac{\widetilde{\mu}_I + \widetilde{p}_I}{a(1+2\varphi)}\widehat{\gamma}^2_I \widehat{v}^i_I\right)_{|i} + \left(\frac{3\varphi_{,i}}{1+2\varphi} + 2\frac{\mathcal{N}_{,i}}{\mathcal{N}}\right)\frac{\widetilde{\mu}_I + \widetilde{p}_I}{a(1+2\varphi)}\widehat{\gamma}^2_I \widehat{v}^i_I$$
$$+ \frac{\widehat{\gamma}^2_I(\widetilde{\mu}_I + \widetilde{p}_I)}{ca^2\mathcal{N}(1+2\varphi)^2}\left[\chi^{i|j}\widehat{v}_{Ii}\widehat{v}_{Ij} - \frac{1}{3}\chi^j_{\ |j}\widehat{v}^i_I\widehat{v}_{Ii} - \frac{2}{1+2\varphi}\left(\widehat{v}^i_I\widehat{v}^j_I\chi_i\varphi_{,j} - \frac{1}{3}\widehat{v}^i_I\widehat{v}_{Ii}\chi^j\varphi_{,j}\right)\right] = -\frac{c}{a\mathcal{N}}\left(\widetilde{I}_{I0} + \frac{\chi^i}{a(1+2\varphi)}I_{Ii}\right). \tag{37}$$

ADM momentum conservation:

$$\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t} + 3H - \kappa\right)\left[a(\widetilde{\mu}_I + \widetilde{p}_I)\widehat{\gamma}^2_I\widehat{v}_{Ii}\right] + \frac{c}{a^2\mathcal{N}}\frac{\chi^j}{1+2\varphi}\left[a(\widetilde{\mu}_I + \widetilde{p}_I)\widehat{\gamma}^2_I\widehat{v}_{Ii}\right]_{|j} + c^2\widetilde{p}_{I,i} + c^2\left(\widetilde{\mu}_I + \widetilde{p}_I\right)\frac{\mathcal{N}_{,i}}{\mathcal{N}}$$
$$+ \left(\frac{\widetilde{\mu}_I + \widetilde{p}_I}{1+2\varphi}\widehat{\gamma}^2_I\widehat{v}^j_I\widehat{v}_{Ii}\right)_{|j} + \frac{c}{a\mathcal{N}}\left(\frac{\chi^j}{1+2\varphi}\right)_{|i}\left(\widetilde{\mu}_I + \widetilde{p}_I\right)\widehat{\gamma}^2_I\widehat{v}_{Ij}$$
$$+ \frac{\widetilde{\mu}_I + \widetilde{p}_I}{1+2\varphi}\widehat{\gamma}^2_I\widehat{v}^j_I\left[\frac{1}{1+2\varphi}(3\widehat{v}_{Ii}\varphi_{,j} - \widehat{v}_{Ij}\varphi_{,i}) + \frac{1}{\mathcal{N}}(\widehat{v}_{Ii}\mathcal{N}_{,j} + \widehat{v}_{Ij}\mathcal{N}_{,i})\right] = c^2 I_{Ii}. \tag{38}$$

Covariant energy conservation:

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i_I + \frac{c}{a}\chi^i\right)\nabla_i\right]\widetilde{\mu}_I + \left(\widetilde{\mu}_I + \widetilde{p}_I\right)\left\{(3H - \kappa)\mathcal{N} + \frac{(\mathcal{N}\widehat{v}^i_I)_{|i}}{a(1+2\varphi)} + \frac{\mathcal{N}\widehat{v}^i_I\varphi_{,i}}{a(1+2\varphi)^2}\right.$$
$$\left. + \frac{1}{\widehat{\gamma}_I}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i_I + \frac{c}{a}\chi^i\right)\nabla_i\right]\widehat{\gamma}_I\right\} = -\frac{c}{a}\widetilde{I}_{I0} - \frac{\mathcal{N}}{a(1+2\varphi)}\left(\widehat{v}^i_I + \frac{c}{a\mathcal{N}}\chi^i\right)I_{Ii}. \tag{39}$$

Covariant momentum conservation:

$$\frac{\partial}{\partial t}\left(a\widehat{\gamma}_I\widehat{v}_{Ii}\right)+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}_I^k+\frac{c}{a}\chi^k\right)\nabla_k\left(a\widehat{\gamma}_I\widehat{v}_{Ii}\right)+c^2\widehat{\gamma}_I\mathcal{N}_{,i}+\frac{1-\widehat{\gamma}_I^2}{\widehat{\gamma}_I}\frac{c^2\mathcal{N}\varphi_{,i}}{1+2\varphi}+\frac{c}{a}\widehat{\gamma}_I\widehat{v}_I^k\nabla_i\left(\frac{\chi_k}{1+2\varphi}\right)$$
$$+\frac{1}{\widetilde{\mu}_I+\widetilde{p}_I}\left\{c^2\frac{\mathcal{N}}{\widehat{\gamma}_I}\widetilde{p}_{I,i}+a\widehat{\gamma}_I\widehat{v}_{Ii}\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}_I^k+\frac{c}{a}\chi^k\right)\nabla_k\right]\widetilde{p}_I\right\}$$
$$=\frac{c^2}{\widetilde{\mu}_I+\widetilde{p}_I}\left[\frac{\mathcal{N}}{\widehat{\gamma}_I}I_{Ii}+\widehat{\gamma}_I\frac{\widehat{v}_{Ii}}{c}\widetilde{I}_{I0}+\frac{\mathcal{N}\widehat{\gamma}_I}{1+2\varphi}\frac{\widehat{v}_{Ii}}{c}\left(\frac{\widehat{v}_I^j}{c}+\frac{\chi^j}{a\mathcal{N}}\right)I_{Ij}\right]. \quad (40)$$

We have introduced $I_{Ii}\equiv\widetilde{I}_{Ii}$ where the spatial index of I_{Ii} is raised and lowered by γ_{ij}; I_{Ii} is the perturbed order quantity whereas $\widetilde{I}_0$ includes the background order quantity as $\widetilde{I}_{I0}=I_{I0}+\delta I_{I0}$. Either set of conservation equations together with equations (6)–(10) complete the equations we need in the multiple fluid system; equation (32) provides the collective fluid quantities in terms of the individual ones.

In the presence of anisotropic stress of individual component, we can similarly add the $\Pi^{(\rm ADM)}$ and $\Pi_i^{((\rm ADM)}$ terms as in equations (14)–(17) to equations (37)–(40) after replacing the fluid quantities in equations (18) and (19) by individual ones. Equation (31) provides the collective fluid quantities in terms of the individual ones.

The vector variables v_{Ii} and I_{Ii} can be decomposed into the scalar- and vector-type perturbations as

$$v_{Ii}=-v_{I,i}+v_{Ii}^{(v)},\quad I_{Ii}=I_{I,i}+I_{Ii}^{(v)}, \quad (41)$$

with $v_I^{(v)i}{}_{|i}\equiv 0\equiv I_I^{(v)i}{}_{|i}$.

As in the single component case, to the linear order the scalar-type perturbation δ_I, δp_I, v_I, δI_{I0} and δI_I depend on the temporal gauge transformation whereas the vector-type perturbations are gauge invariant (Bardeen 1988). In addition to the fundamental gauge conditions in equation (22), in the multicomponent case, for a chosen I-component, we have the following gauge conditions available

$$I-\text{component}-\text{comoving gauge}:\qquad v_I\equiv 0,$$
$$\text{uniform}-I-\text{component}-\text{density gauge}:\qquad \delta_I\equiv 0, \quad (42)$$

to the fully non-linear order.

4 NEWTONIAN LIMIT

In Hwang & Noh (2013a) we have shown the Newtonian theory as the infinite-speed-of-light limit. The Newtonian limit is available in both the zero-shear gauge and the uniform-expansion gauge. In this section we will present the Newtonian limit in multiple component fluids in the same gauge conditions. As in Hwang & Noh (2013a) we will *assume* a flat background, and *ignore* the anisotropic stress of the individual fluid. Even in this Newtonian limit we will show that anisotropic stress of collective fluid is generated from the multiple component nature of the fluids system.

In both gauge conditions, the infinite-speed-of-light limit ($c\to\infty$) implies

$$\alpha\ll 1,\quad \varphi\ll 1,\quad \frac{\widehat{v}_I^i\widehat{v}_{Ii}}{c^2}\ll 1,\quad \frac{a^2H^2}{k^2c^2}\ll 1,\quad \frac{\widetilde{p}_I}{\widetilde{\varrho}_Ic^2}\ll 1, \quad (43)$$

where k is the wavenumber with $\Delta\to -k^2$, $H\equiv\dot{a}/a$, and $\widetilde{\mu}_I\equiv\widetilde{\varrho}_Ic^2$. The first two conditions are the weak-gravity conditions, and the remaining ones are the slow-motion, the small-scale (smaller than the horizon), and negligible pressure conditions, respectively. As we consider the infinite-speed-of-light limit, we will ignore the above quantities but we will keep $c^2\widetilde{p}_{I,i}$.

Equation (8) gives

$$\frac{2}{3}\kappa_{,i}+c\frac{\Delta}{a^2\mathcal{N}}\left(\frac{2}{3}\chi_{,i}+\frac{1}{2}\chi_i^{(v)}\right)-\frac{c\mathcal{N}_{,j}}{a^2\mathcal{N}^2}\left[\chi^{,j}{}_{,i}-\frac{1}{3}\delta_i^j\Delta\chi+\frac{1}{2}\left(\chi^{(v)j}{}_{,i}+\chi_i^{(v),j}\right)\right]=-\frac{8\pi G}{c^2}a\widetilde{\varrho}\left(-\widehat{v}_{,i}+\widehat{v}_i^{(v)}\right), \quad (44)$$

with

$$\mathcal{N}=\sqrt{1+\frac{1}{a^2}(\chi^{,k}+\chi^{(v)k})(\chi_{,k}+\chi_k^{(v)})}, \quad (45)$$

where we decompose into scalar- and vector-type perturbations using equation (5). Here we have properly considered the anisotropic stress of the collective component which arises from the multicomponent nature of the fluids, see equation (56). This equation can be analysed perturbatively in the two gauge conditions.

In the zero-shear gauge ($\chi\equiv 0$), in a perturbative manner we can show

$$\frac{1}{a}\chi_i^{(v)}\sim\mathcal{O}\left(\frac{a^2H^2}{k^2c^2}\frac{\widehat{v}_i}{c}\right), \quad (46)$$

to all perturbation order, thus $\chi_i^{(v)}/a$ is negligible in our limit. Thus, to fully non-linear order equation (44) can be written as

$$\frac{2}{3}\kappa_{,i}+\frac{c\Delta}{2a^2}\chi_i^{(v)}=-\frac{8\pi G}{c^2}a\widetilde{\varrho}\widehat{v}_i, \quad (47)$$

and, we have

$$\kappa = -\frac{12\pi G a}{c^2}\Delta^{-1}\nabla_i\left(\widetilde{\varrho}\widehat{v}^i\right), \quad \chi_i^{(v)} = -\frac{16\pi G}{c^3}a^3\Delta^{-1}\left[\widetilde{\varrho}\widehat{v}_i - \Delta^{-1}\nabla_i\nabla_j\left(\widetilde{\varrho}\widehat{v}^j\right)\right]. \tag{48}$$

In the uniform-expansion gauge ($\kappa \equiv 0$), in a perturbative manner we can show

$$\frac{1}{a}\chi_{,i} \sim \frac{1}{a}\chi_i^{(v)} \sim \mathcal{O}\left(\frac{a^2H^2}{k^2c^2}\frac{\widehat{v}_i}{c}\right), \tag{49}$$

to all perturbation orders, thus χ_i/a is negligible in our limit. Thus, to fully non-linear order equation (44) can be written as

$$\chi_{,i} + \frac{3}{4}\chi_i^{(v)} = -\frac{12\pi G}{c^3}a^3\Delta^{-1}\left(\widetilde{\varrho}\widehat{v}_i\right), \tag{50}$$

and, we have

$$\chi = -\frac{12\pi G}{c^3}a^3\Delta^{-2}\nabla_i\left(\widetilde{\varrho}\widehat{v}^i\right), \quad \chi_i^{(v)} = -\frac{16\pi G}{c^3}a^3\Delta^{-1}\left[\widetilde{\varrho}\widehat{v}_i - \Delta^{-1}\nabla_i\nabla_j\left(\widetilde{\varrho}\widehat{v}^j\right)\right]. \tag{51}$$

Thus we have shown that for $c \to \infty$ we have

$$\chi_i^{(v)} = 0, \tag{52}$$

in both gauge conditions. This does not imply that we ignore the vector-type perturbation. We will keep $\widehat{v}_i^{(v)}$ in $\widehat{v}_i$. As χ_i/a is negligible we have $\mathcal{N} = 1 + \alpha$ in our limit, and we will keep $c^2\mathcal{N}_{,i} = c^2\alpha_{,i}$ in the momentum conservation equation.

In the zero-shear gauge ($\chi \equiv 0$) we have $\chi_i = 0$. Using κ in equation (48), equations (6), (7), (9) and (10), respectively, give

$$\dot{\varphi} - \frac{\dot{a}}{a}\alpha = \frac{4\pi G a}{c^2}\Delta^{-1}\nabla_i\left(\widetilde{\varrho}\widehat{v}^i\right), \quad 4\pi G\delta\varrho + c^2\frac{\Delta}{a^2}\varphi = 0, \quad 4\pi G\delta\varrho - c^2\frac{\Delta}{a^2}\alpha = 0, \quad \varphi = -\alpha. \tag{53}$$

The energy conservation and the momentum conservation equations in equations (37)–(40) give

$$\dot{\widetilde{\varrho}}_I + 3\frac{\dot{a}}{a}\widetilde{\varrho}_I + \frac{1}{a}\nabla_i\left(\widetilde{\varrho}_I\widehat{v}_I^i\right) = 0, \tag{54}$$

$$\dot{\widehat{v}}_{Ii} + \frac{\dot{a}}{a}\widehat{v}_{Ii} + \frac{1}{a}\widehat{v}_I^k\nabla_k\widehat{v}_{Ii} = -\frac{c^2}{a}\alpha_{,i} - \frac{\widetilde{p}_{I,i}}{a\widetilde{\varrho}_I}. \tag{55}$$

In the uniform-expansion gauge ($\kappa \equiv 0$) we have $\chi_i = \chi_{,i}$. Using χ in equation (51), we can show that the same equations used in the zero-shear gauge above lead to exactly the same equations in equations (53)–(55). Thus the following analysis applies for both gauges.

Fluid quantities in equation (53) are collective ones. The relations between individual fluid and collective one are presented in equation (32). In the $c \to \infty$ limit, we have

$$\widetilde{\varrho} = \sum_K \widetilde{\varrho}_K, \quad \widetilde{p} = \sum_K \widetilde{p}_K, \quad \widetilde{\varrho}\widehat{v}_i = \sum_K \widetilde{\varrho}_K\widehat{v}_{Ki}, \quad \Pi_{ij} = -\widetilde{\varrho}\widehat{v}_i\widehat{v}_j + \sum_K \widetilde{\varrho}_K\widehat{v}_{Ki}\widehat{v}_{Kj}. \tag{56}$$

Notice that we have non-vanishing anisotropic stress of the collective fluid generated from the multiple nature of fluid system.

Now, we *identify* the Newtonian densities, pressures and velocities of individual component, and the gravitational potential as

$$\widetilde{\varrho}, \quad \widetilde{\varrho}_I, \quad \widetilde{p}, \quad \widetilde{p}_I, \quad \boldsymbol{v} = v^i \equiv \widehat{v}^i, \quad \boldsymbol{v}_I = v_I^i \equiv \widehat{v}_I^i, \quad \frac{1}{c^2}U \equiv -\alpha = \varphi. \tag{57}$$

Using these variables, equations (53)–(55) give

$$\dot{\widetilde{\varrho}}_I + 3\frac{\dot{a}}{a}\widetilde{\varrho}_I + \frac{1}{a}\nabla\cdot\left(\widetilde{\varrho}_I\boldsymbol{v}_I\right) = 0, \tag{58}$$

$$\dot{\boldsymbol{v}}_I + \frac{\dot{a}}{a}\boldsymbol{v}_I + \frac{1}{a}\boldsymbol{v}_I\cdot\nabla\boldsymbol{v}_I = \frac{1}{a}\nabla U - \frac{1}{a\widetilde{\varrho}_I}\nabla\widetilde{p}_I. \tag{59}$$

$$4\pi G\delta\varrho = -\frac{\Delta}{a^2}U, \tag{60}$$

which are the same as the energy conservation, momentum conservation, and the Poisson's equations, respectively, in Newtonian context.

For the collective fluid we can show that

$$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} + \frac{1}{a}\nabla\cdot\left(\widetilde{\varrho}\boldsymbol{v}\right) = 0, \tag{61}$$

$$\dot{\boldsymbol{v}} + \frac{\dot{a}}{a}\boldsymbol{v} + \frac{1}{a}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \frac{1}{a}\nabla U - \frac{1}{a\widetilde{\varrho}}\left(\nabla\widetilde{p} + \nabla^j\Pi_{ij}\right), \tag{62}$$

with

$$\widetilde{\varrho} = \sum_K \widetilde{\varrho}_K, \quad \widetilde{p} = \sum_K \widetilde{p}_K, \quad \widetilde{\varrho}\boldsymbol{v} = \sum_K \widetilde{\varrho}_K\boldsymbol{v}_K, \quad \Pi_{ij} = -\widetilde{\varrho}v_iv_j + \sum_K \widetilde{\varrho}_K v_{Ki}v_{Kj}. \tag{63}$$

We still have one more relation to check. Using the identification in equation (57) the first equation in equation (53) gives

$$\dot{U} + \frac{\dot{a}}{a}U = 4\pi Ga\Delta^{-1}\nabla\cdot\left(\widetilde{\varrho}\boldsymbol{v}\right), \tag{64}$$

which follows from equations (60) and (61). In the presence of relativistic pressure, see the paragraph including equation (82). Thus, in our infinite-speed-of-light limit we have checked that the complete set of Einstein's equations is consistent.

This completes the proof of the Newtonian limit of Einstein's gravity in the cosmological context. By setting $a = 1$ with $\varrho = 0$, thus $\delta\varrho = \widetilde{\varrho}$, we properly recover the well known Newtonian limit in the Minkowski background.

5 NEWTONIAN LIMIT WITH RELATIVISTIC PRESSURE

In Hwang & Noh (2013b) we have presented Newtonian equations with general relativistic pressure, thus $\widetilde{p} \sim \widetilde{\varrho}c^2$, for a single component fluid in the zero-shear gauge. It is not necessary that such a limit should exist at all, but we have shown that such a limit does exist with correct special relativistic hydrodynamics limit for vanishing gravity. Here, we extend the case to the multicomponent fluid system.

We take the zero-shear gauge, thus set $\chi \equiv 0$. We impose the infinite-speed-of-light conditions in equation (43) except for the last one concerning the pressure. Thus, we consider a situation with $\widetilde{p}_I \sim \widetilde{\varrho}_Ic^2$. The analysis proceeds similarly as in the Newtonian limit studied in Section 4.

Equation (8) gives

$$\frac{2}{3}\kappa_{,i} + c\frac{\Delta}{a^2\mathcal{N}}\frac{1}{2}\chi_i^{(v)} - \frac{c\mathcal{N}_{,j}}{a^2\mathcal{N}^2}\frac{1}{2}\left(\chi^{(v)j}{}_{,i} + \chi_i^{(v),j}\right) = -\frac{8\pi G}{c^2}a\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}_i, \tag{65}$$

with

$$\mathcal{N} = \sqrt{1 + \frac{1}{a^2}\chi^{(v)k}\chi_k^{(v)}}. \tag{66}$$

In a perturbative manner we can show

$$\frac{1}{a}\chi_i^{(v)} \sim \mathcal{O}\left(\frac{a^2H^2}{k^2c^2}\frac{\widehat{v}_i}{c}\right), \tag{67}$$

to all perturbation orders, thus $\chi_i^{(v)}/a$ is negligible in our limit. Thus, to fully non-linear order equation (65) becomes

$$\frac{2}{3}\kappa_{,i} + \frac{c\Delta}{2a^2}\chi_i^{(v)} = -\frac{8\pi G}{c^2}a\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}_i, \tag{68}$$

and, we have

$$\kappa = -\frac{12\pi Ga}{c^2}\Delta^{-1}\nabla_i\left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}^i\right], \quad \chi_i^{(v)} = -\frac{16\pi G}{c^3}a^3\Delta^{-1}\left\{\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}_i - \Delta^{-1}\nabla_i\nabla_j\left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}^j\right]\right\}. \tag{69}$$

Thus, we have

$$\chi_i^{(v)} = 0, \tag{70}$$

but we keep $\widehat{v}_i^{(v)}$ in $\widehat{v}_i$. As χ_i/a is negligible we have $\mathcal{N} = 1 + \alpha$ with $\alpha \ll 1$; thus we have $\mathcal{N} = 1$, but we will keep $c^2\mathcal{N}_{,i} = c^2\alpha_{,i}$ in the momentum conservation equation.

Using κ in equation (69), equations (6), (7), (9) and (10), respectively, give

$$\dot{\varphi} - \frac{\dot{a}}{a}\alpha = \frac{4\pi Ga}{c^2}\Delta^{-1}\nabla_i\left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\widehat{v}^i\right], \tag{71}$$

$$4\pi G\delta\varrho + c^2\frac{\Delta}{a^2}\varphi = -\frac{\dot{a}}{a}\kappa, \tag{72}$$

$$4\pi G\left(\delta\varrho + 3\frac{\delta p}{c^2}\right) - c^2\frac{\Delta}{a^2}\alpha = \dot{\kappa} + 2\frac{\dot{a}}{a}\kappa, \tag{73}$$

$$\varphi = -\alpha. \tag{74}$$

Although κ is higher order in c^{-2} it is important to keep this term because $\dot{\kappa}$ is no longer higher order in the presence of relativistic pressure, see equation (80). Fluid quantities in equations (71)–(73) are collective ones. From equation (32) we have

$$\widetilde{\varrho} = \sum_K \widetilde{\varrho}_K, \quad \widetilde{p} = \sum_K \widetilde{p}_K, \quad \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right) \widehat{v}_i = \sum_K \left(\widetilde{\varrho}_K + \frac{\widetilde{p}_K}{c^2}\right) \widehat{v}_{Ki}, \quad \Pi_{ij} = -\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right) \widehat{v}_i \widehat{v}_j + \sum_K \left(\widetilde{\varrho}_K + \frac{\widetilde{p}_K}{c^2}\right) \widehat{v}_{Ki} \widehat{v}_{Kj}. \tag{75}$$

The energy conservation and the momentum conservation equations in equations (37)–(40) give

$$\dot{\widetilde{\varrho}}_I + 3\frac{\dot{a}}{a}\left(\widetilde{\varrho}_I + \frac{\widetilde{p}_I}{c^2}\right) + \frac{1}{a}\nabla_i\left(\widetilde{\varrho}_I \widehat{v}_I^i\right) = \frac{1}{ac^2}\left(\widehat{v}_I^i \nabla_i \widetilde{p}_I - \widetilde{p}_I \nabla_i \widehat{v}_I^i\right), \tag{76}$$

$$\dot{\widehat{v}}_{Ii} + \frac{\dot{a}}{a}\widehat{v}_{Ii} + \frac{1}{a}\widehat{v}_I^k \nabla_k \widehat{v}_{Ii} = -\frac{c^2}{a}\alpha_{,i} - \frac{1}{\widetilde{\varrho}_I + \widetilde{p}_I/c^2}\left(\frac{1}{a}\nabla_i \widetilde{p}_I + \widehat{v}_{Ii}\frac{\dot{\widetilde{p}}_I}{c^2}\right). \tag{77}$$

We identify the Newtonian densities, pressures and velocities of individual component, and the gravitational potential, the same as those in equation (57) in the Newtonian limit. Using the Newtonian variables, equations (76) and (77) become

$$\dot{\widetilde{\varrho}}_I + 3\frac{\dot{a}}{a}\left(\widetilde{\varrho}_I + \frac{\widetilde{p}_I}{c^2}\right) + \frac{1}{a}\nabla \cdot \left[\left(\widetilde{\varrho}_I + \frac{\widetilde{p}_I}{c^2}\right)\mathbf{v}_I\right] = \frac{2}{ac^2}\mathbf{v}_I \cdot \nabla \widetilde{p}_I, \tag{78}$$

$$\dot{\mathbf{v}}_I + \frac{\dot{a}}{a}\mathbf{v}_I + \frac{1}{a}\mathbf{v}_I \cdot \nabla \mathbf{v}_I = \frac{1}{a}\nabla U - \frac{1}{\widetilde{\varrho}_I + \widetilde{p}_I/c^2}\left(\frac{1}{a}\nabla \widetilde{p}_I + \mathbf{v}_I\frac{\dot{\widetilde{p}}_I}{c^2}\right). \tag{79}$$

Now, we derive the Poisson's equation. From the κ relation in equation (69), using equations (78) and (79), we can show

$$\dot{\kappa} = \frac{12\pi G}{c^2}\delta p. \tag{80}$$

Thus, as we mentioned, in our limit it is important to keep $\dot{\kappa}$ term. Using this, the Poisson's equation follows from equations (72)–(74) as

$$4\pi G \delta\varrho = -\frac{\Delta}{a^2}U. \tag{81}$$

Equations (78), (79) and (81) are the mass and momentum conservation equations for the individual fluid component, and the Poisson's equation in the presence of the general relativistic pressure. Notice the absence of pressure term in the Poisson's equation.

We have checked all the equations in Einstein's gravity except for equation (71). Using equations (72)–(74), (78) and (80) we can show that the left-hand side of equation (71) gives

$$\frac{4\pi G}{c^2}a\Delta^{-1}\nabla \cdot \left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\mathbf{v}\right] - \frac{8\pi G}{c^2}a\Delta^{-1}\sum_J\left(\mathbf{v}_J \cdot \nabla\frac{\widetilde{p}_J}{c^2}\right), \tag{82}$$

where the second term is missing in the right-hand side of the same equation; the form of missing term varies depending on the way of derivation. In Hwang & Noh (2013b) we have attributed this failure as due to the higher order (in c^{-2}) nature of equation (71). Accepting this argument we have checked the consistency of all the Einstein's equations.

For the collective fluid we can show

$$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right) + \frac{1}{a}\nabla \cdot \left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\mathbf{v}_I\right] = \frac{2}{a}\sum_J \mathbf{v}_J \cdot \nabla\frac{\widetilde{p}_J}{c^2}, \tag{83}$$

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{a}\nabla U + \frac{1}{\widetilde{\varrho} + \widetilde{p}/c^2}\left(\frac{1}{a}\nabla \widetilde{p} + \frac{1}{a}\nabla_j \Pi^{ij}\right) = \frac{1}{\widetilde{\varrho} + \widetilde{p}/c^2}\left[-\mathbf{v}_I\frac{\dot{\widetilde{p}}}{c^2} + \frac{2}{a}\sum_J(\mathbf{v}_J - \mathbf{v})\,\mathbf{v}_J \cdot \nabla\frac{\widetilde{p}_J}{c^2}\right], \tag{84}$$

with

$$\widetilde{\varrho} = \sum_K \widetilde{\varrho}_K, \quad \widetilde{p} = \sum_K \widetilde{p}_K, \quad \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\mathbf{v} = \sum_K\left(\widetilde{\varrho}_K + \frac{\widetilde{p}_K}{c^2}\right)\mathbf{v}_K, \quad \Pi_{ij} = -\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)v_i v_j + \sum_K\left(\widetilde{\varrho}_K + \frac{\widetilde{p}_K}{c^2}\right)v_{Ki}v_{Kj}. \tag{85}$$

Terms in the right-hand sides of equations (83) and (84) are the non-trivial contributions from relativistic pressure. In the absence of gravity equations (78) and (79) or equations (83) and (84) have the proper special relativistic limits studied in equation 2.10.16 of Weinberg (1972) and equations 2.3 and 2.4 of Peacock (1999).

6 NON-RELATIVISTIC PRESSURE FLUIDS IN THE CDM-COMOVING GAUGE

We consider the multicomponent fluids with negligible pressure. By this, we ignore $\widetilde{p}_I/\widetilde{\mu}_I$ but keep $c^2\widetilde{p}_{I,i}$ term in the momentum conservation equation. We consider the scalar-type perturbation in a flat background and ignore interaction terms among fluids. As one of the components we consider the cold-dark-matter (CDM), thus set $\widetilde{p}_c \equiv 0$. We take the CDM-comoving gauge by setting $\widehat{v}_c \equiv 0$ as the temporal gauge condition. The momentum conservation equation in equation (40) for the CDM component gives

$$\mathcal{N} = 1. \tag{86}$$

Equation (39) for $I = c$, equation (9), and equations (39) and (40) for other I components give

$$\dot{\widetilde{\varrho}}_c + \widetilde{\varrho}_c(3H - \kappa) + \frac{c\chi^{,i}}{a^2(1+2\varphi)}\widetilde{\varrho}_{c,i} = 0, \tag{87}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\delta\varrho + \frac{c\chi^{,i}}{a^2(1+2\varphi)}\kappa_{,i} = \frac{1}{3}\kappa^2 + 8\pi G\widetilde{\varrho}\left(\widehat{\gamma}^2 - 1\right) + c^2\overline{K}^i_j\overline{K}^j_i + \frac{1}{1+2\varphi}\frac{4\pi G}{c^2}\left(\Pi^i_i - \frac{1}{1+2\varphi}\Pi_{ij}\frac{\widehat{v}^i\widehat{v}^j}{c^2}\right), \tag{88}$$

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\widehat{v}^i_I + \frac{c}{a}\chi^{,i}\right)\nabla_i\right]\widetilde{\varrho}_I + \widetilde{\varrho}_I\left\{3H - \kappa + \frac{\widehat{v}^i_{I,i}}{a(1+2\varphi)} + \frac{\widehat{v}^i_I\varphi_{,i}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}_I}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\widehat{v}^i_I + \frac{c}{a}\chi^{,i}\right)\nabla_i\right]\widehat{\gamma}_I\right\} = 0, \tag{89}$$

$$\frac{\partial}{\partial t}\left(a\widehat{\gamma}_I\widehat{v}_{Ii}\right) + \frac{1}{1+2\varphi}\left(\widehat{v}^k_I + \frac{c}{a}\chi^{,k}\right)\nabla_k\left(\widehat{\gamma}_I\widehat{v}_{Ii}\right) + \frac{1}{\widehat{\gamma}_I\widetilde{\varrho}_I}\widetilde{p}_{I,i} + \frac{1-\widehat{\gamma}_I^2}{\widehat{\gamma}_I}\frac{c^2\varphi_{,i}}{1+2\varphi} + \frac{c}{a}\widehat{\gamma}_I\widehat{v}^k_I\nabla_i\left(\frac{\chi_{,k}}{1+2\varphi}\right) = 0. \tag{90}$$

Instead of equation for $\dot{\widehat{v}}_c$, we have equation of $\dot{\kappa}$. Together with equations (6)–(8) and (10), and the collective fluid quantities given in equation (32) we have the complete set of equations. Except for negligible pressure these equations are still fully relativistic.

6.1 Newtonian limit in the general scale

We further assume the weak gravity ($\alpha \ll 1$ and $\varphi \ll 1$), and the slow-motion ($\widehat{v}^i_I\widehat{v}_{Ii}/c^2 \ll 1$). Compared with the infinite-speed-of-light limit, here we do *not* assume the small-scale $[a^2H^2/(c^2k^2) \ll 1]$ condition. Equation (8) gives

$$\frac{2}{3}\kappa_{,i} + \frac{2}{3}\frac{c\Delta}{a^2}\chi_{,i} = -\frac{8\pi G}{c^2}a\widetilde{\varrho}\widehat{v}_i, \tag{91}$$

where we have properly considered the anisotropic stress term using equation (32).

By identifying

$$\kappa = -c\frac{\Delta}{a^2}\chi \equiv -\frac{1}{a}\nabla\cdot\boldsymbol{v}_c \equiv -\frac{\Delta}{a}v_c, \tag{92}$$

thus $c\chi/a = v_c$, equation (91) is satisfied. Although we have introduced a new velocity component $\boldsymbol{v}_c(\equiv\nabla v_c)$, it does not necessarily imply that $\boldsymbol{v}_c$ is the velocity of the CDM component in the multiple component situation. Equations (93) and (94) below do support such an identification, but equations (95) and (96) look different from the Newtonian equations in (58) and (59) which are valid only in the sub-horizon-scale. This is because of the gauge condition we take; here we take $\widehat{v}_{ci}(\equiv -\widehat{v}_{c,i}) = 0$ as our CDM-comoving gauge condition where $v_c \neq \widehat{v}_c \equiv 0$. The variables as well as the equations depend on the gauge choice. By making a due reservation on identifying the nature of variables here we are emphasizing the relativistic nature of our equations where the meanings can be achieved only by analogy or correspondence with the Newtonian situation. With this identification with $\boldsymbol{v}_I = v^i_I \equiv \widehat{v}^i_I$, equations (87)–(90) become

$$\dot{\widetilde{\varrho}}_c + 3\frac{\dot{a}}{a}\widetilde{\varrho}_c + \frac{1}{a}\nabla\cdot\left(\widetilde{\varrho}_c\boldsymbol{v}_c\right) = 0, \tag{93}$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{v}}_c + \frac{\dot{a}}{a}\boldsymbol{v}_c\right) + 4\pi G\delta\varrho + \frac{1}{a^2}\nabla\cdot(\boldsymbol{v}_c\cdot\nabla\boldsymbol{v}_c) = 0, \tag{94}$$

$$\dot{\widetilde{\varrho}}_I + 3\frac{\dot{a}}{a}\widetilde{\varrho}_I + \frac{1}{a}\nabla\cdot\left[\widetilde{\varrho}_I(\boldsymbol{v}_I + \boldsymbol{v}_c)\right] = 0, \tag{95}$$

$$\dot{\boldsymbol{v}}_I + \frac{\dot{a}}{a}\boldsymbol{v}_I + \frac{1}{a}\boldsymbol{v}_I\cdot\nabla\boldsymbol{v}_I + \frac{1}{a}(\boldsymbol{v}_c\cdot\nabla\boldsymbol{v}_I + \boldsymbol{v}_I\cdot\nabla\boldsymbol{v}_c) + \frac{1}{a\widetilde{\varrho}_I}\nabla\widetilde{p}_I = 0, \tag{96}$$

where $\delta\varrho = \sum_J\delta\varrho_J$ from equation (32). Compared with the equations (58)–(60) in the Newtonian limit ($c \to \infty$ limit) in the zero-shear gauge and the uniform-expansion gauge, we have the additional presence of $\boldsymbol{v}_c$ contributions in equations (95) and (96), the potential is missing in equation (96), and the Poisson's equation is absorbed in equation (94). For the remaining Einstein equations, we can show that equation (6) simply corresponds to equation (92), and both equation (7) and equation (10) together with equation (94) give

$$c^2\frac{\Delta}{a^2}\varphi = -4\pi G\delta\varrho + \frac{\dot{a}}{a}\frac{1}{a}\nabla\cdot\boldsymbol{v}_c + \frac{1}{4a^2}\left[(\nabla\cdot\boldsymbol{v}_c)^2 - v_c^{,ij}v_{c,ij}\right], \tag{97}$$

which can be considered as a relation determining the curvature perturbation φ from the fluid quantities. Notice that we have not assumed the small-scale limit. Thus the above results are valid in general scales including the super-horizon scale, whereas the proper Newtonian limit in the other two gauge conditions in Section 4 is valid only far inside the horizon.

The single component case was studied in Hwang, Noh & Park (2014). In that case the above results remain valid by keeping only the c-component. In that case we further have not assumed the slow-motion limit ($|\boldsymbol{v}_c|^2/c^2 \ll 1$). In Hwang et al. (2014) we have considered a fluid with non-relativistic pressure. In that case, equation (94) is replaced by

$$\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{v}}+\frac{\dot{a}}{a}\boldsymbol{v}\right)+4\pi G\delta\varrho+\frac{1}{a^2}\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})+\frac{1}{a^2}\nabla\cdot\left(\frac{\nabla\widetilde{p}}{\widetilde{\varrho}}\right)=0. \quad (98)$$

6.2 Third-order perturbations

To the third-order perturbations, equations (87)–(90) become

$$\delta\dot{\varrho}_c+3\frac{\dot{a}}{a}\delta\varrho_c+\frac{c}{a^2}(1-2\varphi)\chi^{,i}\delta\varrho_{c,i}-(\varrho_c+\delta\varrho_c)\kappa=0, \quad (99)$$

$$\dot{\kappa}+2\frac{\dot{a}}{a}\kappa+\frac{c}{a^2}(1-2\varphi)\chi^{,i}\kappa_{,i}-4\pi G\delta\varrho=\frac{1}{3}\kappa^2+8\pi G\widetilde{\varrho}\left(\widehat{\gamma}^2-1\right)+c^2\overline{K}^i_j\overline{K}^j_i, \quad (100)$$

$$\begin{aligned}&\delta\dot{\varrho}_I+3\frac{\dot{a}}{a}\delta\varrho_I-\varrho_I(1+\delta_I)\kappa+\frac{1}{a}\varrho_I\left(1+\delta_I-2\varphi-2\delta_I\varphi+4\varphi^2\right)\widehat{v}^i_{I,i}\\&\quad=-\frac{1}{a}(1-2\varphi)\left(\widehat{v}^i_I+\frac{c}{a}\chi^{,i}\right)\delta\varrho_{I,i}+\frac{1}{c^2}(1-2\varphi)\widehat{v}^i_I\delta p_{I,i}+\frac{1}{c^2}\frac{\dot{a}}{a}\varrho_I(1+\delta_I-2\varphi)\widehat{v}^i_I\widehat{v}_{Ii}\\&\qquad+\frac{1}{ca^2}\varrho_I\widehat{v}^i_I\widehat{v}^k_I\chi_{,ik}+\frac{1}{c^2}\varrho_I\dot{\varphi}\widehat{v}^i_I\widehat{v}_{Ii}-\frac{1}{a}\varrho_I(1+\delta_I-4\varphi)\widehat{v}^i_I\varphi_{,i},\end{aligned} \quad (101)$$

$$\begin{aligned}&\dot{\widehat{v}}_{Ii}+\frac{\dot{a}}{a}\widehat{v}_{Ii}+\left(1-\delta_I+\delta_I^2\right)\frac{\delta p_{I,i}}{a\varrho_I}\\&\quad=-\frac{1}{a}\left[(1-2\varphi)\widehat{v}^k_I\left(\frac{1}{2}\widehat{v}_{Ik}+\frac{c}{a}\chi_{,k}\right)\right]_{,i}+\frac{1}{c^2}\widehat{v}^k_I\widehat{v}_{Ik}\left(\frac{\dot{a}}{a}\widehat{v}_{Ii}+\frac{3}{2}\frac{\delta p_{I,i}}{a\varrho_I}\right)+\frac{1}{c^2}\widehat{v}_{Ii}\widehat{v}^k_I\frac{\delta p_{I,k}}{a\varrho_I},\end{aligned} \quad (102)$$

where

$$\begin{aligned}\delta\varrho&=\sum_J\left\{\delta\varrho_J+\frac{1}{c^2}\varrho_J\left[(1+\delta_J-2\varphi)\widehat{v}^i_J\widehat{v}_{Ji}+\frac{2}{\sum_L\varrho_L}\left(\sum_K\delta\varrho_K\widehat{v}^i_K\right)\left(\frac{1}{\sum_L\varrho_L}\sum_M\varrho_M\widehat{v}_{Mi}-\widehat{v}_{Ji}\right)\right.\right.\\&\qquad\left.\left.+\frac{1}{\sum_L\varrho_L}\left(1+\delta_J-2\varphi-\frac{\sum_M\delta\varrho_M}{\sum_L\varrho_L}\right)\left(\sum_K\varrho_K\widehat{v}^i_K\right)\left(\frac{1}{\sum_L\varrho_L}\sum_M\varrho_M\widehat{v}_{Mi}-2\widehat{v}_{Ji}\right)\right]\right\},\\\widehat{\gamma}^2-1&=\frac{1}{c^2}\frac{1}{\left(\sum_L\varrho_L\right)^2}\left(\sum_J\varrho_J\widehat{v}^i_J\right)\left[\left(1-2\varphi-2\frac{\sum_M\delta\varrho_M}{\sum_L\varrho_L}\right)\left(\sum_K\varrho_K\widehat{v}_{Ki}\right)-2\sum_K\delta\varrho_K\widehat{v}_{Ki}\right],\\c^2\overline{K}^i_j\overline{K}^j_i&=\frac{c^2}{a^4}(1-4\varphi)\left[\chi^{,ij}\chi_{,ij}-\frac{1}{3}(\Delta\chi)^2\right].\end{aligned} \quad (103)$$

As we took the CDM comoving gauge ($\widehat{v}_c\equiv 0$), we have $\widehat{v}_{Ji}=\widehat{v}^{(v)}_{Ji}$ for $J=c$.

In order to close the above set of equations, we need relations determining χ and φ to the second order perturbation; in the single component case we needed φ only to the linear order (Hwang & Noh 2005). From equations (8) and (7), respectively, we have

$$c\frac{\Delta}{a^2}\chi_{,i}=-(1+2\varphi)\left(\kappa_{,i}+\frac{12\pi G}{c^2}a\widetilde{\varrho}\widehat{v}_i\right)-\frac{c}{a^2}\left(\varphi_{,i}\Delta\chi+\frac{3}{2}\chi_{,i}\Delta\varphi+2\varphi^{,j}\chi_{,ij}+\frac{1}{2}\chi^{,j}\varphi_{,ij}\right), \quad (104)$$

$$c^2\frac{\Delta}{a^2}\varphi=-(1+4\varphi)\left(4\pi G\delta\varrho+\frac{\dot{a}}{a}\kappa\right)+\frac{1}{6}\kappa^2-4\pi G\varrho\left(\widehat{\gamma}^2-1\right)+\frac{3}{2}\frac{c^2}{a^2}\varphi^{,i}\varphi_{,i}-\frac{c^2}{4}\overline{K}^i_j\overline{K}^j_i, \quad (105)$$

to the second-order perturbation; to the same order, from equation (33) we have $\widetilde{\varrho}\widehat{v}_i=\sum_J\widehat{\varrho}_J\widehat{v}_{Ji}$. From equation (6), we have

$$\dot{\varphi}=-\frac{1}{3}\left(\kappa+c\frac{\Delta}{a^2}\chi\right), \quad (106)$$

to the linear order.

7 MINIMALLY COUPLED SCALAR FIELDS

We consider an action given as (Hwang & Noh 2000)

$$S = \int d^4 x \sqrt{-\widetilde{g}} \left[\frac{1}{16\pi G} \widetilde{R} - \frac{1}{2} \widetilde{\phi}^{I;c} \widetilde{\phi}_{I,c} - \widetilde{V}(\widetilde{\phi}^J) \right], \tag{107}$$

where $\widetilde{\phi}_I$ are minimally coupled scalar fields with $I, J \ldots = 1, 2, \ldots N$. In this section, except for Section 7.3, we assume the summation convention for the repeated indices of $I, J, \ldots$, and set $c \equiv 1 \equiv \hbar$ in the presence of the scalar field. The energy–momentum tensor becomes

$$\widetilde{T}_{ab} = \widetilde{\phi}^I_{,a} \widetilde{\phi}_{I,b} - \left(\frac{1}{2} \widetilde{\phi}^{I;c} \widetilde{\phi}_{I,c} + \widetilde{V} \right) \widetilde{g}_{ab}. \tag{108}$$

The equation of motion for each field is

$$\widetilde{\phi}_I{}^{;c}{}_c = \widetilde{V}_{,I}, \tag{109}$$

where $\widetilde{V}_{,I} \equiv \partial \widetilde{V} / (\partial \widetilde{\phi}^I)$.

As in the multicomponent fluids, equations (6)–(10) remain valid as the fluid quantities considered as the collective ones. Thus we need the collective fluid quantities expressed in terms of individual field quantities. Instead of the (energy and momentum) conservation equations we have the equation of motion of each component.

7.1 Collective fluid quantities

From equations (108) and (3) we have

$$\widetilde{\mu} = \frac{1}{2} \dot{\widetilde{\phi}}^I \dot{\widetilde{\phi}}_I + \frac{1}{2} \widetilde{h}^{cd} \widetilde{\phi}^I_{,c} \widetilde{\phi}_{I,d} + \widetilde{V}, \quad \widetilde{p} = \frac{1}{2} \dot{\widetilde{\phi}}^I \dot{\widetilde{\phi}}_I - \frac{1}{6} \widetilde{h}^{cd} \widetilde{\phi}^I_{,c} \widetilde{\phi}_{I,d} - \widetilde{V},$$
$$\widetilde{q}_a = - \dot{\widetilde{\phi}}^I \widetilde{h}^b_a \widetilde{\phi}_{I,b}, \quad \widetilde{\pi}_{ab} = \left(\widetilde{h}^c_a \widetilde{h}^d_b - \frac{1}{3} \widetilde{h}_{ab} \widetilde{h}^{cd} \right) \widetilde{\phi}^I_{,c} \widetilde{\phi}_{I,d}, \tag{110}$$

where $\dot{\widetilde{\phi}}_I \equiv \widetilde{\phi}_{I,c} \widetilde{u}^c$.

As we take the energy-frame ($\widetilde{q}_a \equiv 0$) we have $\dot{\widetilde{\phi}}^I \widetilde{h}^b_a \widetilde{\phi}_{I,b} = 0$. Using equation (4) we have

$$\widehat{v}_i = - \frac{1}{a \widehat{\gamma}} \frac{\dot{\widetilde{\phi}}^I \widetilde{\phi}_{I,i}}{\dot{\widetilde{\phi}}^J \dot{\widetilde{\phi}}_J}. \tag{111}$$

We can show

$$\dot{\widetilde{\phi}}_I = \widetilde{\phi}_{I,c} \widetilde{u}^c = \widehat{\gamma} \left(\frac{D}{Dt} + \frac{\widehat{v}^i}{a(1+2\varphi)} \nabla_i \right) \widetilde{\phi}_I, \tag{112}$$

where we introduced

$$\frac{D}{Dt} \equiv \frac{1}{N} \left(\partial_0 - N^i \nabla_i \right) = \frac{1}{\mathcal{N}} \left(\frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1+2\varphi)} \nabla_i \right). \tag{113}$$

The collective fluid quantities in equations (110) and (111) become

$$\widehat{v}_i = - \frac{1}{a \widehat{\gamma}^2} \frac{\left[\left(\frac{D}{Dt} + \frac{\widehat{v}^j}{a(1+2\varphi)} \nabla_j \right) \widetilde{\phi}^I \right] \widetilde{\phi}_{I,i}}{\left[\left(\frac{D}{Dt} + \frac{\widehat{v}^k}{a(1+2\varphi)} \nabla_k \right) \widetilde{\phi}^J \right] \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^\ell}{a(1+2\varphi)} \nabla_\ell \right) \widetilde{\phi}_J \right]},$$
$$\widetilde{\mu} = \widehat{\gamma}^2 \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^i}{a(1+2\varphi)} \nabla_i \right) \widetilde{\phi}^I \right] \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^j}{a(1+2\varphi)} \nabla_j \right) \widetilde{\phi}_I \right] - \frac{1}{2} \frac{D\widetilde{\phi}^I}{Dt} \frac{D\widetilde{\phi}_I}{Dt} + \frac{1}{2} \frac{\widetilde{\phi}^{I|i} \widetilde{\phi}_{I,i}}{a^2(1+2\varphi)} + \widehat{V},$$
$$\widetilde{p} = \frac{1}{3} \widehat{\gamma}^2 \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^i}{a(1+2\varphi)} \nabla_i \right) \widetilde{\phi}^I \right] \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^j}{a(1+2\varphi)} \nabla_j \right) \widetilde{\phi}_I \right] + \frac{1}{6} \frac{D\widetilde{\phi}^I}{Dt} \frac{D\widetilde{\phi}_I}{Dt} - \frac{1}{6} \frac{\widetilde{\phi}^{I|i} \widetilde{\phi}_{I,i}}{a^2(1+2\varphi)} - \widehat{V},$$
$$\Pi_{ij} = \left\{ \widehat{\gamma}^2 \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^k}{a(1+2\varphi)} \nabla_k \right) \widetilde{\phi}^I \right] \widehat{v}_i + \frac{1}{a} \widetilde{\phi}^I_{,i} \right\} \left\{ \widehat{\gamma}^2 \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^\ell}{a(1+2\varphi)} \nabla_\ell \right) \widetilde{\phi}_I \right] \widehat{v}_j + \frac{1}{a} \widetilde{\phi}_{I,j} \right\}$$
$$- \frac{1}{3} \left[(1+2\varphi) \gamma_{ij} + \widehat{\gamma}^2 \widehat{v}_i \widehat{v}_j \right] \left\{ \widehat{\gamma}^2 \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^k}{a(1+2\varphi)} \nabla_k \right) \widetilde{\phi}^I \right] \left[\left(\frac{D}{Dt} + \frac{\widehat{v}^\ell}{a(1+2\varphi)} \nabla_\ell \right) \widetilde{\phi}_I \right] \right.$$
$$\left. - \frac{D\widetilde{\phi}^I}{Dt} \frac{D\widetilde{\phi}_I}{Dt} + \frac{\widetilde{\phi}^{I|k} \widetilde{\phi}_{I,k}}{a^2(1+2\varphi)} \right\}. \tag{114}$$

As the right-hand side of $\widehat{v}_i$ in equation (114) contains $\widehat{v}_k$, in order to express $\widehat{v}_i$ purely in terms of individual field quantities, we have to solve the relation perturbatively by iteration. The other collective fluid quantities also contain $\widehat{v}_i$ which should be replaced by thus obtained individual ones.

For example, to the linear and second order, we have

$$\widehat{v}_i = -\frac{1}{a}\frac{\dot{\phi}^I\widetilde{\phi}_{I,i}}{\dot{\phi}^J\dot{\phi}_J}, \quad \widehat{v}_i = -\frac{\mathcal{N}}{a}\frac{\dot{\widetilde{\phi}}{}^I\widetilde{\phi}_{I,i}}{\dot{\widetilde{\phi}}{}^J\dot{\widetilde{\phi}}_J}, \tag{115}$$

respectively, and to the third order, we have

$$\widehat{v}_i = -\frac{1}{a}\left(1-\widehat{v}^m\widehat{v}_m\right)\frac{\left[\left(\frac{D}{Dt}+\frac{1}{a}\widehat{v}^j\nabla_j\right)\widetilde{\phi}^I\right]\widetilde{\phi}_{I,i}}{\left[\left(\frac{D}{Dt}+\frac{1}{a}\widehat{v}^k\nabla_k\right)\widetilde{\phi}^J\right]\left[\left(\frac{D}{Dt}+\frac{1}{a}\widehat{v}^\ell\nabla_\ell\right)\widetilde{\phi}_J\right]}, \tag{116}$$

where we can replace $\widehat{v}^k$s in the right-hand side by $\widehat{v}^k$ to the linear order in equation (115). This can be continued to any higher order perturbation.

7.2 Equations of motion

The equation of motion in equation (109) gives

$$\begin{aligned}
-\widetilde{\phi}_I{}^{;c}{}_c &= \frac{D^2\widetilde{\phi}_I}{Dt^2} + (3H-\kappa)\frac{D\widetilde{\phi}_I}{Dt} - \frac{(\mathcal{N}\sqrt{1+2\varphi}\widetilde{\phi}_I{}^{|i})_{|i}}{a^2\mathcal{N}(1+2\varphi)^{3/2}} \\
&= \frac{1}{\mathcal{N}^2}\left\{\ddot{\widetilde{\phi}}_I + \left(3H\mathcal{N}-\mathcal{N}\kappa-\frac{\dot{\mathcal{N}}}{\mathcal{N}}-\frac{\chi^i\mathcal{N}_{,i}}{a^2\mathcal{N}(1+2\varphi)}\right)\dot{\widetilde{\phi}}_I + \frac{2\chi^i}{a^2(1+2\varphi)}\dot{\widetilde{\phi}}_{I,i}\right. \\
&\quad - \frac{1}{a^2(1+2\varphi)}\left(\mathcal{N}^2\gamma^{ij}-\frac{\chi^i\chi^j}{a^2(1+2\varphi)}\right)\widetilde{\phi}_{I,i|j} + \left[-\frac{\mathcal{N}^2}{a^2(1+2\varphi)}\left(\frac{\mathcal{N}^{|i}}{\mathcal{N}}+\frac{\varphi^{|i}}{1+2\varphi}\right)\right. \\
&\quad \left.\left. + \left(3H\mathcal{N}-\mathcal{N}\kappa-\frac{\dot{\mathcal{N}}}{\mathcal{N}}-\frac{\chi^k\mathcal{N}_{,k}}{a^2\mathcal{N}(1+2\varphi)}\right)\frac{\chi^i}{a^2(1+2\varphi)} + \left(\frac{\chi^i}{a^2(1+2\varphi)}\right)^{\cdot} + \frac{\chi^k}{a^4(1+2\varphi)}\left(\frac{\chi^i}{1+2\varphi}\right)_{|k}\right]\widetilde{\phi}_{I,i}\right\} \\
&= -\widetilde{V}_{,I}.
\end{aligned} \tag{117}$$

Equations (6)–(10) and (117) together with the collective fluid quantities in equation (114) provide a complete set of non-linear and exact perturbation equations in the presence of the multiple fields. In the perturbation theory we may introduce

$$\widetilde{\phi}_I \equiv \phi_I + \delta\phi_I, \tag{118}$$

where ϕ_Is are the background scalar fields, and $\delta\phi_I$s are functions of space and time with arbitrary amplitudes.

In addition to the fundamental gauge conditions in equation (22), in the multicomponent field case, for a chosen I-component we have the following gauge condition available

$$\text{uniform}-I-\text{component}-\text{field gauge}: \qquad \delta\phi_I = 0, \tag{119}$$

to the fully non-linear order. Under this gauge condition for a chosen I-component, equation (117) gives

$$\ddot{\phi}_I + \left(3H\mathcal{N}-\mathcal{N}\kappa-\frac{\dot{\mathcal{N}}}{\mathcal{N}}-\frac{\chi^i\mathcal{N}_{,i}}{a^2\mathcal{N}(1+2\varphi)}\right)\dot{\phi}_I = -\mathcal{N}^2V_{,I}. \tag{120}$$

7.3 Fluid formulation

For the scalar fields, instead of the equations of motion, we can also use the energy and momentum conservation equations of the individual component in the fluid formulation. In order for this, we need to introduce fluid quantities of the individual component of the field. In the following we do not assume the summation convention for $I, J, \ldots$. We may introduce

$$\widetilde{T}_{Iab} = \widetilde{\phi}^I{}_{,a}\widetilde{\phi}_{I,b} - \left(\frac{1}{2}\widetilde{\phi}^{I;c}\widetilde{\phi}_{I,c}+\widetilde{V}_I\right)\widetilde{g}_{ab}, \tag{121}$$

with

$$\widetilde{T}_{ab} = \sum_J \widetilde{T}_{Jab}, \quad \widetilde{V} \equiv \sum_J \widetilde{V}_J. \tag{122}$$

Decomposition of the potential into the individual ones $\widetilde{V}_I$ has ceratin degree of arbitrariness, but does not imply that we ignore the interaction among the fields. From equations (121) and (25) we have

$$\widetilde{\mu}_I = \frac{1}{2}\left(\widetilde{\phi}_{I,b}\widetilde{u}^{Ib}\right)^2 + \widetilde{V}_I, \quad \widetilde{p}_I = \frac{1}{2}\left(\widetilde{\phi}_{I,b}\widetilde{u}^{Ib}\right)^2 - \widetilde{V}_I,$$

$$\widetilde{u}_{Ia} = -\frac{\widetilde{\phi}_{I,a}}{\widetilde{\phi}_{I,b}\widetilde{u}^{Ib}}, \quad \widetilde{\pi}_{Iab} = 0, \tag{123}$$

where we have used the energy-frame condition $\widetilde{q}_{Ia} \equiv 0$. Using the fluid quantities of individual field in equation (123) we can show that the collective fluid quantities in equation (110) can be identified with the ones in the fluid formulation in equation (26). Therefore, instead of the equations of motion in equation (117) we can use the energy and momentum conservation equations in equations (37)–(40) using the individual fluid quantities identified in equation (123).

The interaction terms can be read from

$$\widetilde{T}_{Ia;b}^{\ \ b} = \frac{\partial \widetilde{V}}{\partial \widetilde{\phi}^I}\widetilde{\phi}^I_{,a} - \sum_K \frac{\partial \widetilde{V}_I}{\partial \widetilde{\phi}^K}\widetilde{\phi}^K_{,a} \equiv \widetilde{I}_{Ia}, \tag{124}$$

where we have used the equations of motion in equation (109). For $\widetilde{V}_I = \widetilde{V}_I(\widetilde{\phi}^I)$ for all Is, we have $\widetilde{I}_{Ia} = 0$.

From equation (123) we have fluid quantities of individual component

$$\widetilde{\mu}_I = \frac{1}{2}\widehat{\gamma}_I^2\left[\left(\frac{D}{Dt} + \frac{\widehat{v}_I^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I\right]^2 + \widetilde{V}_I, \quad \widetilde{p}_I = \frac{1}{2}\widehat{\gamma}_I^2\left[\left(\frac{D}{Dt} + \frac{\widehat{v}_I^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I\right]^2 - \widetilde{V}_I,$$

$$\widehat{v}_{Ii} = -\frac{\widetilde{\phi}_{I,i}}{a\widehat{\gamma}_I^2\left(\frac{D}{Dt} + \frac{\widehat{v}_I^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I}, \quad \Pi_{Iij} = 0. \tag{125}$$

Using these relations we can derive equation (114) from equation (32). In order to show this we need the following two relations

$$\left(\frac{D}{Dt} + \frac{\widehat{v}^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I = \widehat{\gamma}_I^2\left(1 - \frac{\widehat{v}_\ell\widehat{v}_I^\ell}{1+2\varphi}\right)\left(\frac{D}{Dt} + \frac{\widehat{v}_I^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I,$$

$$\widehat{\gamma}_I^2\left[\left(\frac{D}{Dt} + \frac{\widehat{v}_I^k}{a(1+2\varphi)}\nabla_k\right)\widetilde{\phi}_I\right]^2 = \left(\frac{D\widetilde{\phi}_I}{Dt}\right)^2 - \frac{\widetilde{\phi}_I^{\ |i}\widetilde{\phi}_{I,i}}{a^2(1+2\varphi)}, \tag{126}$$

which follow from identities

$$\widetilde{\phi}_{I,a}\widetilde{u}^a = -\widetilde{\phi}_{I,b}\widetilde{u}_I^b\widetilde{u}_{Ia}\widetilde{u}^a, \quad \left(\widetilde{\phi}_{I,a}\widetilde{u}_I^a\right)^2 = -\widetilde{\phi}_{I,a}\widetilde{\phi}_I^{\ ;a}, \tag{127}$$

respectively.

8 LINEAR ORDER

Here we present the equations to linear perturbation order in the Friedmann background world model. This may clarify our notations used and allow easy comparison with other notations used in the literature. To the background order, equations (7), (9), (37) and (39) give

$$H^2 = \frac{8\pi G}{3c^2}\mu - \frac{\overline{K}c^2}{a^2} + \frac{\Lambda c^2}{3},$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\mu + 3p) + \frac{\Lambda c^2}{3},$$

$$\dot{\mu}_I + 3H(\mu_I + p_I) = -\frac{c}{a}I_{I0}. \tag{128}$$

To the linear order in perturbation, equations (6)–(10) and (37)–(40) give (Bardeen 1988; Hwang 1991; Hwang & Noh 2013c)

$$\kappa = 3H\alpha - 3\dot{\varphi} - c\frac{\Delta}{a^2}\chi, \tag{129}$$

$$\frac{4\pi G}{c^2}\delta\mu + H\kappa + c^2\frac{\Delta + 3\overline{K}}{a^2}\varphi = 0, \tag{130}$$

$$\kappa + c\frac{\Delta + 3\overline{K}}{a^2}\chi - \frac{12\pi G}{c^4}a\left(\mu + p\right)\widehat{v} = 0, \tag{131}$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\alpha = \frac{4\pi G}{c^2}\left(\delta\mu + 3\delta p\right), \tag{132}$$

$$\varphi + \alpha - \frac{1}{c}\left(\dot{\chi} + H\chi\right) = -\frac{8\pi G}{c^4}\Pi, \tag{133}$$

$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right) + \left(\mu + p\right)\left(3H\alpha - \kappa - \frac{\Delta}{a}\widehat{v}\right) = 0, \tag{134}$$

$$\frac{1}{a^4}\left[a^4\left(\mu + p\right)\widehat{v}\right]^{\cdot} = \frac{c^2}{a}\left[\delta p + \left(\mu + p\right)\alpha + \frac{2}{3}\frac{\Delta + 3\overline{K}}{a^2}\Pi\right], \tag{135}$$

$$\delta\dot{\mu}_I + 3H\left(\delta\mu_I + \delta p_I\right) + \left(\mu_I + p_I\right)\left(3H\alpha - \kappa - \frac{\Delta}{a}\widehat{v}_I\right) = -\frac{c}{a}\delta I_{I0}, \tag{136}$$

$$\frac{1}{a^4}\left[a^4\left(\mu_I + p_I\right)\widehat{v}_I\right]^{\cdot} = \frac{c^2}{a}\left[\delta p_I + \left(\mu_I + p_I\right)\alpha + \frac{2}{3}\frac{\Delta + 3\overline{K}}{a^2}\Pi_I - I_I\right], \tag{137}$$

for the scalar-type perturbation. Equations (8), (10) and (38) give

$$c\frac{\Delta + 2\overline{K}}{a^2}\chi_i^{(v)} = -\frac{16\pi G}{c^4}a\left(\mu + p\right)\widehat{v}_i^{(v)}, \tag{138}$$

$$\frac{c}{a}\left(\dot{\chi}_i^{(v)} + H\chi_i^{(v)}\right) = \frac{8\pi G}{c^2}\Pi_i^{(v)}, \tag{139}$$

$$\frac{1}{a^4}\left[a^4\left(\mu + p\right)\widehat{v}_i^{(v)}\right]^{\cdot} = -c^2\frac{\Delta + 2\overline{K}}{2a^2}\Pi_i^{(v)}, \tag{140}$$

$$\frac{1}{a^4}\left[a^4\left(\mu_I + p_I\right)\widehat{v}_{Ii}^{(v)}\right]^{\cdot} = -c^2\frac{\Delta + 2\overline{K}}{2a^2}\Pi_{Ii}^{(v)} + \frac{c^2}{a}I_{Ii}^{(v)}, \tag{141}$$

for the vector-type perturbation. For fluid quantities equation (32) gives

$$\mu = \sum_J \mu_J, \quad p = \sum_J p_J, \tag{142}$$

to the background order, and

$$\delta\mu = \sum_J \delta\mu_J, \quad \delta p = \sum_J \delta p_J,$$

$$\widehat{v} = \frac{\sum_J(\mu_J + p_J)\widehat{v}_J}{\sum_K(\mu_K + p_K)}, \quad \widehat{v}_i^{(v)} = \frac{\sum_J(\mu_J + p_J)\widehat{v}_{Ji}^{(v)}}{\sum_K(\mu_K + p_K)},$$

$$\Pi = \sum_J \Pi_J, \quad \Pi_i^{(v)} = \sum_J \Pi_{Ji}^{(v)}, \tag{143}$$

to the perturbed order.

8.1 Scalar fields

For the scalar fields, equations (117) and (114) give

$$\ddot{\phi}_I + 3H\dot{\phi}_I + V_{,I} = 0, \tag{144}$$

$$\mu = \frac{1}{2}\sum_J \dot{\phi}_J^2 + V, \quad p = \frac{1}{2}\sum_J \dot{\phi}_J^2 - V, \tag{145}$$

to the background order, and

$$\delta\ddot{\phi}_I + 3H\delta\dot{\phi}_I - \frac{\Delta}{a^2}\delta\phi_I + \sum_J V_{,IJ}\delta\phi_J = 2\ddot{\phi}_I\alpha + \dot{\phi}_I\left(\dot{\alpha} + 3H\alpha + \kappa\right), \tag{146}$$

$$\delta\mu = \sum_J \left(\dot{\phi}^J\delta\dot{\phi}_J - \dot{\phi}_J^2\alpha + V_{,J}\delta\phi^J\right), \quad \delta p = \sum_J \left(\dot{\phi}^J\delta\dot{\phi}_J - \dot{\phi}_J^2\alpha - V_{,J}\delta\phi^J\right),$$

$$\widehat{v} = \frac{1}{a}\frac{\sum_J \dot{\phi}^J\delta\phi_J}{\sum_K \dot{\phi}_K^2}, \quad \widehat{v}_i^{(v)} = 0, \quad \Pi_{ij} = 0, \tag{147}$$

to the linear perturbation order. Using the individual components, equation (125) give

$$\mu_I = \frac{1}{2}\dot{\phi}_I^2 + V_I, \quad p_I = \frac{1}{2}\dot{\phi}_I^2 - V_I, \quad \delta\mu_I = \dot{\phi}_I\delta\dot{\phi}_I - \dot{\phi}_I^2\alpha + \sum_J V_{I,J}\delta\phi^J,$$

$$\delta p_I = \dot{\phi}_I\delta\dot{\phi}_I - \dot{\phi}_I^2\alpha - \sum_J V_{I,J}\delta\phi^J, \quad \widehat{v}_I = \frac{\delta\phi_I}{a\dot{\phi}_I}, \quad \widehat{v}_i^{(v)} = 0, \quad \Pi_{ij} = 0. \tag{148}$$

The interaction term follows from equation (124) as

$$I_{I0} = a\left(\frac{\partial V}{\partial\phi^I}\dot{\phi}^I - \sum_K \frac{\partial V_I}{\partial\phi^K}\dot{\phi}^K\right), \tag{149}$$

$$\delta I_{I0} = a\left[\frac{\partial V}{\partial\phi^I}\delta\dot{\phi}^I + \sum_J \frac{\partial^2 V}{\partial\phi^J\partial\phi^I}\dot{\phi}^I\delta\phi^J - \sum_K\left(\frac{\partial V_I}{\partial\phi^K}\delta\dot{\phi}^K + \sum_J \frac{\partial^2 V_I}{\partial\phi^J\partial\phi^K}\dot{\phi}^K\delta\phi^J\right)\right],$$

$$I_I = \frac{\partial V}{\partial\phi^I}\delta\phi^I - \sum_K \frac{\partial V_I}{\partial\phi^K}\delta\phi^K, \quad I_{Ii}^{(v)} = 0. \tag{150}$$

As we have shown below equation (124), for $\widetilde{V}_I = \widetilde{V}_I(\widetilde{\phi}^I)$ for all Is, we have $\widetilde{I}_{Ia} = 0$.

For fluids system, see Kodama & Sasaki (1986, 1987). For minimally coupled scalar fields system, see Sasaki & Stewart (1996) and Hwang & Noh (2000). For fluids-field system, see Hwang & Noh (2002).

8.2 Using relative variables

Kodama & Sasaki (1984) introduced the following relative variables

$$S_{IJ} \equiv \frac{\delta\mu_I}{\mu_I + p_I} - \frac{\delta\mu_J}{\mu_J + p_J}, \quad \widehat{v}_{IJ} \equiv \widehat{v}_I - \widehat{v}_J, \quad e_{IJ} \equiv \frac{e_I}{\mu_I + p_I} - \frac{e_J}{\mu_J + p_J}, \quad \Pi_{IJ} \equiv \frac{\Pi_I}{\mu_I + p_I} - \frac{\Pi_J}{\mu_J + p_J},$$

$$\delta I_{IJ0} \equiv \frac{\delta I_{I0}}{\mu_I + p_I} - \frac{\delta I_{J0}}{\mu_J + p_J}, \quad I_{IJi} \equiv \frac{I_{Ii}}{\mu_I + p_I} - \frac{I_{Ji}}{\mu_J + p_J}, \tag{151}$$

where

$$e_I \equiv \delta p_I - c_I^2\delta\mu_I, \quad c_I^2 \equiv \frac{\dot{p}_I}{\dot{\mu}_I}. \tag{152}$$

From equations (136) and (137), respectively, we have (Kodama & Sasaki 1984)

$$\dot{S}_{IJ} - \frac{\Delta}{a}\widehat{v}_{IJ} + 3He_{IJ} = -\frac{c}{a}\delta I_{IJ0} + \frac{c}{2a}\left(\frac{1+c_I^2}{\mu_I + p_I}I_{I0} + \frac{1+c_J^2}{\mu_J + p_J}I_{J0}\right)S_{IJ}$$

$$+\frac{c}{a}\left(\frac{1+c_I^2}{\mu_I + p_I}I_{I0} - \frac{1+c_J^2}{\mu_J + p_J}I_{J0}\right)\left[\frac{1}{2}\sum_K \frac{\mu_K + p_K}{\mu + p}\left(S_{IK} + S_{JK}\right) + \frac{\delta\mu}{\mu + p}\right], \tag{153}$$

$$\dot{\widehat{v}}_{IJ} + H\left[1 - \frac{3}{2}\left(c_I^2 + c_J^2\right)\right]\widehat{v}_{IJ} - 3H\left(c_I^2 - c_J^2\right)\left[\frac{1}{2}\sum_K \frac{\mu_K + p_K}{\mu + p}\left(\widehat{v}_{IK} + \widehat{v}_{JK}\right) + \widehat{v}\right] - \frac{c^2}{2a}\left(c_I^2 + c_J^2\right) S_{IJ}$$

$$- \frac{c^2}{a}\left(c_I^2 - c_J^2\right)\left[\frac{1}{2}\sum_K \frac{\mu_K + p_K}{\mu + p}\left(S_{IK} + S_{JK}\right) + \frac{\delta\mu}{\mu + p}\right] - \frac{c}{a}e_{IJ} - \frac{2}{3}\frac{c^2}{a}\frac{\Delta + 3K}{a^2}\Pi_{IJ} = -\frac{c}{a}I_{IJ}$$

$$+ \frac{1}{2a}\left(\frac{1 + c_I^2}{\mu_I + p_I}I_{I0} + \frac{1 + c_J^2}{\mu_J + p_J}I_{J0}\right)\widehat{v}_{IJ} + \frac{c}{a}\left(\frac{1 + c_I^2}{\mu_I + p_I}I_{I0} - \frac{1 + c_J^2}{\mu_J + p_J}I_{J0}\right)\left[\frac{1}{2}\sum_K \frac{\mu_K + p_K}{\mu + p}\left(\widehat{v}_{IK} + \widehat{v}_{JK}\right) + \widehat{v}\right]. \quad (154)$$

The entropic perturbation, e, can be written as (Kodama & Sasaki 1984)

$$e \equiv \delta p - c_s^2 \delta\mu \equiv e_{\rm int} + e_{\rm rel}, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}} = \frac{1}{\mu + p}\sum_K c_K^2\left(\mu_K + p_K + \frac{c}{3Ha}I_{K0}\right), \quad e_{\rm int} \equiv \sum_K e_K,$$

$$e_{\rm rel} \equiv \sum_K \left(c_K^2 - c_s^2\right)\delta\mu_K = \frac{1}{2}\sum_{K,L}\frac{(\mu_K + p_K)(\mu_L + p_L)}{\mu + p}\left(c_K^2 - c_L^2\right)S_{KL} - \frac{c}{3Ha}\sum_K c_K^2 I_{K0}\frac{\delta\mu}{\mu + p}. \quad (155)$$

Instead of equations (136) and (137), equations (153) and (154) together with equations (129)–(135) provide the complete set of equations. Various analytic solutions in the system of radiation and dust (baryon plus dark matter) are studied in Kodama & Sasaki (1986, 1987).

Similar expressions using the relative variables in the case of scalar fields can be found in Hwang & Noh (2000, 2002).

9 DISCUSSION

We have been pursuing fully non-linear and exact formulation of the relativistic cosmological perturbation theory in the context of Friedmann cosmological background. In our previous works we have presented the basic perturbation equations in the presence of a fluid or a minimally coupled scalar field (Hwang & Noh 2013c; Noh 2014) and applied the equations to various situations: Newtonian limit, Newtonian limit with general relativistic pressure, and post-Newtonian limit (Hwang & Noh 2013a; Noh & Hwang 2013).

In this work we extend our previous works to include the multiple component fluids and fields. The followings are new results in this work.

(1) Fully non-linear and exact perturbation equations with multiple components of fluids (including the anisotropic stress of individual fluids) and minimally coupled scalar fields. We consider the general background curvature and the cosmological constant. The basic sets of the equations are equations (6)–(10) for Einstein's equations, with equations (37)–(40) for fluids, and equation (117) for fields. Relations among the collective and individual fluid quantities are presented in equation (31) for fluids, and equation (114) for fields. These are all exact equations readily applicable to fully non-linear perturbations by the simple Taylor expansion. The fluids formulation in Section 3 also applies to the fields system in Section 7 as well as the mixed system with the multiple fluids and fields. The available temporal gauge (slicing or hypersurface) conditions are presented in equations (22), (42) and (119). These slicing conditions can be imposed to any order perturbation with the remaining variables equivalently gauge-invariant to fully non-linear orders.

(2) Newtonian limit in the zero-shear gauge and the uniform-expansion gauge: see equations (58)–(64).

(3) Newtonian equations modified by the general relativistic pressures in the zero-shear gauge: see equations (78), (79), (81) and (83)–(85).

(4) Exact and third-order perturbation equations in the CDM-comoving gauge. We considered multiple ideal fluids in the presence of one CDM (zero-pressure fluid) component: see equations (87)–(90). The equations in the CDM-comoving gauge is convenient to study the non-linear power spectra (Jeong et al. 2011). To the third order, see equations (99)–(106).

In Noh & Hwang (2013) we have shown that the cosmological first-order post-Newtonian (1PN) equations (modulo anisotropic stress) in Hwang, Noh & Puetzfeld (2008) can be easily derived from our fully non-linear cosmological formulation. Our multicomponent formulation may allow simple derivation of the 1PN equations in the presence of multiple fluids as well. This subject is also left for a future study.

Perturbations up to third order are needed to get the leading non-linear order power spectra. In the literature this is well developed in the single component, zero-pressure, non-rotational case only. The result shows that pure Einstein effects are negligible in such a case in all scales (Jeong et al. 2011; Hwang, Jeong & Noh 2016). Although the higher order non-linear perturbation equations even in more general case can be derived by brute force as well, our fully non-linear perturbation formulation provides a much easier route. It is still not clear whether we may have room for pure *general relativistic non-linear* effects in the current concordance cosmology paradigm. One potential general relativistic effect expected on the *observed* galaxy power spectrum is the projection effect through the deflection of light coming from galaxies (Yoo, Fitzpatrick & Zaldarriaga 2009; Yoo 2010; Jeong, Schmidt & Hirata 2012; Yoo 2014; Jeong & Schmidt 2015). In this context the null-geodesic and Boltzmann equation approaches in the similarly fully non-linear and exact formulation may be feasible but still not investigated.

ACKNOWLEDGEMENTS

We wish to thank the anonymous referee for clarifying comments and suggestions. JH was supported by Basic Science Research Program through the NRF of Korea funded by the Ministry of Science, ICT and Future Planning (No. 2013R1A2A2A01068519). HN was supported by NRF of Korea funded by the Korean Government (No. 2015R1A2A2A01002791). CGP was supported by Basic Science Research Program through the National Research Foundation (NRF) of Korea funded by the Ministry of Science, ICT and Future Planning (No. 2013R1A1A1011107).

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