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FULLY NONLINEAR, UNIFORMLY ELLIPTIC EQUATIONS UNDER NATURAL STRUCTURE CONDITIONS

BY

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ABSTRACT. We derive first and second derivative estimates for classical solutions of fully nonlinear, uniformly elliptic equations which are subject to natural structure conditions analogous to those proposed and treated by Ladyzhenskaya and Ural'tseva for quasilinear equations. As an application we extend recent work of Evans and Lions on the Bellman equation for families of linear operators to families of quasilinear operators.

1. Introduction. We are concerned in this paper with second order, nonlinear partial differential equations of the general form

$$F[u] = F(x, u, Du, D^2u) = 0 \tag{1.1}$$

in open subsets Ω of Euclidean n space $\mathbf{R}^n$. Here F is a real function on $\Gamma = \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the $n(n+1)/2$-dimensional space of real symmetric $n \times n$ matrices. The function u is assumed twice differentiable in Ω in an appropriate sense with $Du = [D_i u]$, $D^2u = [D_{ij}u]$ denoting, respectively, the gradient and Hessian of u.

The operator F and the equation (1) are called *elliptic* in Ω if

$$F(x, z, p, r) < F(x, z, p, r + \eta) \tag{1.2}$$

for all $(x, z, p, r) \in \Gamma$ and $\eta \geq 0$, $\neq 0$, $\in \mathbb{S}^n$. If $\mathfrak{U}$ is a subset of $\Omega \times \mathbf{R} \times \mathbf{R}^n$, F and (1) are called *uniformly elliptic* on $\mathfrak{U}$ if there exists a constant μ and positive functions λ, Λ on Γ such that

$$\lambda \operatorname{tr} \eta \leq F(x, z, p, r + \eta) - F(x, z, p, r) \leq \Lambda \operatorname{tr} \eta, \qquad \Lambda/\lambda \leq \mu, \tag{1.3}$$

for all $(x, z, p, r) \in \mathfrak{U} \times \mathbb{S}^n$ and $\eta \geq 0$, $\in \mathbb{S}^n$. When F is differentiable with respect to r, ellipticity of F is equivalent to positivity of the matrix

$$F_r = [F_{ij}] = [\partial F/\partial r_{ij}]$$

on Γ, while the uniform ellipticity condition (1.3) is equivalent to the usual condition

$$\lambda |\xi|^2 \leq F_{ij}\xi_i\xi_j \leq \Lambda |\xi|, \qquad \Lambda/\lambda \leq \mu, \tag{1.4}$$

for all $\xi \in \mathbf{R}^n$, $(x, z, p) \in \mathfrak{U}$, $r \in \mathbb{S}^n$. In (1.4) and throughout this paper we adopt the standard summation convention that repeated indices indicate summation from 1 to n. We also observe that the uniform ellipticity of F on $\mathfrak{U}$ implies that F is Lipschitz continuous with respect to r on $\mathfrak{U} \times \mathbb{S}^n$. To see this we write, for any

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matrix $\eta \in \mathbb{S}^n$, $\eta = \eta^+ - \eta^-$ where η^+ and η^- are nonnegative. Then using (1.3) we have

$$(1.5) \qquad |F(x, z, p, r + \eta) - F(x, z, p, r)| \leq \Lambda \operatorname{tr}(\eta^+ + \eta^-) \leq \sqrt{n}\Lambda|\eta| \leq \mu\sqrt{n}\lambda|\eta|,$$

provided $\eta^{\pm}$ are chosen appropriately.

Quasilinear equations, that is, those for which the function F is linear in the variable r, have been extensively studied in the literature, the basic theory being described for example in the monographs [**5**, **8**]. Fully nonlinear uniformly elliptic equations for which the function F is convex or concave with respect to all the variables z, p and r have been treated recently by several authors, notably Evans and Friedman [**3**], P. L. Lions [**11**] and Evans [**1**, **2**] in conjunction with the Bellman equation of stochastic control theory. The main purpose of the present work is the derivation of first and second derivative estimates for solutions of equation (1) where the convexity of concavity of F with respect to z and p is replaced by *natural structure conditions* analogous to the natural conditions proposed by Ladyzhenskaya and Ural′tseva [**8**] for quasilinear uniformly elliptic equations. These conditions are formulated explicitly in the ensuing sections, but roughly stated they require that the ratio F/λ grows at most quadratically in $|p|$ for large $|p|$ and behaves under differentiation similarly to a polynomial in r and p with coefficients depending on x and z. The interior second derivative estimates here should also be compared with those of Ivanov [**6**] who assumes smallness conditions instead of concavity.

As an application of our estimates we treat the Dirichlet problem for the Bellman equation for a family of *quasilinear* equations generalizing the previous work of Lions [**11**] and Evans [**2**] for families of linear equations. However, unlike the linear case the genuine quasilinear case has no apparent relevance to stochastic control theory. We also present a simplified version of the second derivative Hölder estimates of Evans [**1**, **2**] along the lines proposed by the author [**14**]. The second derivative bounds are established in §6 in conjunction with their Hölder estimates by a technique based on interpolation. Similar bounds could alternatively be derived independently utilizing key features from [**3**, **4**, **6** and **11**], although one would nevertheless still require a modulus of continuity estimate for first derivatives such as provided in §5.

2. Preliminaries. As usual we denote by $C^k(\Omega)$, $(C^k(\overline{\Omega}))$ the linear space of functions k times differentiable in Ω whose kth order derivatives are continuous (uniformly continuous) in Ω. For $\alpha \in (0, 1]$, the Hölder spaces $C^{k,\alpha}(\Omega)$, $(C^{k,\alpha}(\overline{\Omega}))$ consist of those functions in $C^{k,\alpha}(\Omega)$ whose kth order derivatives are locally uniformly Hölder continuous in Ω, $(\overline{\Omega})$ with exponent α. We introduce the following seminorms on $C^k(\Omega)$:

$$(2.1) \qquad [u]_{j;\Omega} = \sup_{\Omega} |D^j u|, \quad [u]_{j,\beta;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^\beta},$$

$$[u]^*_{j;\Omega} = \sup_{\Omega' \Subset \Omega} d^j_{\Omega'}[u]_{j;\Omega'}, \quad [u]^*_{j,\beta;\Omega} = \sup_{\Omega' \Subset \Omega} d^{j+\beta}_{\Omega'}[u]_{j,\beta;\Omega'},$$

for $j = 0, 1, \ldots, k$, $\beta \in (0, 1]$ where $d_{\Omega'} = \operatorname{dist}(\Omega', \Omega)$. Setting

$$(2.2)\qquad |u|_{k,\alpha;\Omega} = \sum_{j=0}^{k} [u]_{j;\Omega} + [u]_{k,\alpha;\Omega}, \quad |u|^*_{k,\alpha;\Omega} = \sum_{j=0}^{k} [u]^*_{j;\Omega} + [u]^*_{k,\alpha;\Omega}$$

and defining

$$C^{k,\alpha}_*(\Omega) = \left\{u \in C^{k,\alpha}(\Omega) \,\middle|\, |u|^*_{k,\alpha;\Omega} < \infty\right\},$$

we obtain that the spaces $C^{k,\alpha}(\bar{\Omega})$, $C^{k,\alpha}_*(\Omega)$ are Banach spaces under their respective norms (2.2). A useful property of the weighted seminorms in (2.1) is the following interpolation inequality which is proved, for example, in **[5]**.

LEMMA 2.1. *Suppose $j + \beta < k + \alpha$ and $u \in C^{k,\alpha}(\Omega)$. Then for any $\varepsilon > 0$, there exists a constant C depending only on ε, k, j, α, β such that*

$$(2.3)\qquad [u]^*_{j,\beta;\Omega} \leqslant \varepsilon[u]^*_{k,\alpha;\Omega} + C|u|_{0;\Omega}.$$

For the various Hölder estimates of this paper we shall require the weak Harnack inequality from **[13]** which is based on the estimates of Krylov and Safonov **[6]** for linear equations. In our formulation $\mathcal{A} = [a^{ij}]$ will denote a positive $\mathbb{S}^n$ valued function on Ω satisfying

$$(2.4)\qquad \lambda|\xi|^2 \leqslant a^{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2$$

for all $\xi \in \mathbf{R}^n$ where λ and Λ are positive constants. We also denote by $B_R(y)$ the open ball in $\mathbf{R}^n$ with centre y and radius R and abbreviate $B_R(y) = B_R$ when the centre is clearly understood.

LEMMA 2.2. *Let $u \in C^2(\Omega)$ satisfy the differential inequality*

$$(2.5)\qquad Lu = a^{ij}D_{ij}u \leqslant \lambda\left(\mu_0|Du|^2 + g\right)$$

in Ω where $\mu_0 \in \mathbf{R}$ and $g \in L^n(\Omega)$. Then if $u \geqslant 0$ on a ball $B_{2R} \subset \Omega$, there exist positive constants κ, C depending only on n, Λ/λ and $\mu_0 \sup u$ such that

$$(2.6)\qquad \Phi_{\kappa,R}(u) = \left(\frac{1}{|B_R|}\int_{B_R} u^\kappa\right)^{1/\kappa} \leqslant C\left\{\inf_{B_R} u + R\|g\|_{L^n(B_{2R})}\right\}.$$

As a consequence of Lemma 2.2 we have the following Hölder estimate (also proved in **[13]**).

LEMMA 2.3. *Let $u \in C^2(\Omega)$ satisfy*

$$(2.7)\qquad |Lu| \leqslant \lambda\left(\mu_0|Du|^2 + g\right)$$

in Ω where $\mu_0 \in \mathbf{R}$ and $g \in L^n(\Omega)$. Then for any ball $B_R \subset \Omega$ and $\sigma \in (0,1)$ we have

$$(2.8)\qquad \operatorname*{osc}_{B_{\sigma R}} u \leqslant C\sigma^\alpha\left\{\operatorname*{osc}_{B_R} u + R\|g\|_{L^n(B_R)}\right\}$$

where C and α are positive constants depending only on n, Λ/λ and $\mu_0 M_0$ where $M_0 = |u|_{0;\Omega}$.

For second derivative Hölder estimates in §6 we shall also require the following result from matrix theory due to Motzkin and Wasow **[12]**.

LEMMA 2.4. *Let $\mathcal{A} = [a^{ij}]$ be a symmetric $n \times n$ matrix satisfying* (2.4). *Then there exist a natural number N, unit vectors $\gamma_k \in \mathbf{R}^n$, $k = 1, \ldots, N$, and positive constants λ^*, Λ^*, all depending only on λ and Λ, such that*

$$\mathcal{A} = \sum_{k=1}^{n} \beta_k \gamma_k \otimes \gamma_k, \qquad a^{ij} = \sum_{k=1}^{N} \beta_k \gamma_{ki} \gamma_{kj}, \tag{2.9}$$

where $\beta_k \in \mathbf{R}$ satisfy $\lambda^* \leqslant \beta_k \leqslant \Lambda^*$, $k = 1, \ldots, N$. Furthermore the γ_k can be chosen to include the coordinate vectors e_i, $i = 1, \ldots, n$, and the vectors $(e_i \pm e_j)/\sqrt{2}$, $1 \leqslant i < j \leqslant n$.

For further material from the theory of quasilinear elliptic equations, the reader will be referred directly to [**5**].

3. Hölder estimates for solutions. Letting $\mathcal{U}_k = \Omega \times (-K, K) \times \mathbf{R}^n$ for $K \in \mathbf{R}$, we adopt the following structural conditions in this section:

F1: F is uniformly elliptic on $\mathcal{U}_K$ for all $K \in \mathbf{R}$, with (1.3) holding for $\mu = \mu(K)$;

F2: $|F(x, z, p, 0)|/\lambda(x, z, p, 0) \leqslant \mu_0(1 + |p|^2)$ for all $x, z, p \in \mathcal{U}_K$, $K \in \mathbf{R}$ where $\mu_0 = \mu_0(K) \in \mathbf{R}$.

A Hölder estimate for solutions of (1.1) now follows readily from Lemma 2.3.

THEOREM 3.1. *Let $u \in C^2(\Omega)$ satisfy $F[u] = 0$ in Ω with* F1 *and* F2 *holding. Then*

$$[u]^*_{0,\alpha;\Omega} \leqslant C \tag{3.1}$$

where $\alpha > 0$ depends on n, $\mu(M_0)$, $\mu_0(M_0)$, $M_0 = |u|_{0;\Omega}$ and C depends, in addition, on diam Ω.

PROOF. We first assume that F is differentiable with respect to r so that inequality (1.4) holds. Using the mean value theorem we can then write (1.1) in the form

$$F_{ij}(x, u, Du, s) D_{ij} u + F(x, u, Du, 0) = 0 \tag{3.2}$$

where $s = s(x) \in \mathbb{S}^n$. By Lemma 2.3 we thus have for any ball $B_R \subset \Omega$ and $\sigma \in (0, 1)$.

$$\operatorname*{osc}_{B_{\sigma R}} u \leqslant C\sigma^{\alpha}, \tag{3.3}$$

where C and α depend on the quantities specified in the theorem statement. The estimate (3.1) follows directly from (3.3).

In the general case we mollify F with respect to r by defining for $h > 0$,

$$F_h(x, z, p, r) = \int_{|\eta| \leqslant 1} F(x, z, p, r + \eta h) \rho(\eta) \, d\eta$$

where $\rho \geqslant 0, \in C_0^{\infty}(\mathbf{R}^{n^2})$ satisfies $\int \rho = 1$. Using (1.5) we have

$$\begin{aligned} |F_h(x, z, p, r) - F(x, z, p, r)| &\leqslant \int_{|\eta| \leqslant 1} |F(x, z, p, r + \eta h) - F(x, z, p, r)| \rho(\eta) \, d\eta \\ &\leqslant \sqrt{n} \, \mu \lambda h. \end{aligned}$$

Accordingly the operator F_h satisfies F1 and F2 with μ_0 replaced by $\mu_0 + \sqrt{\eta}\,\mu h$. Applying (3.1) now to F_h and letting $h \to 0$, we thus obtain (3.1) for general F.

4. Gradient estimates. Let us now assume that the function F is differentiable in Γ and augment conditions F1 and F2 by

F3: $|p||F_p|,\ |F_z|,\ |F_x| \leqslant \lambda\mu_1(1 + |p|^2 + |r|)$ for all $x, z, p \in \mathcal{U}_K$, $r \in \mathbb{S}^n$, $K \in \mathbf{R}$ where $\mu_1 = \mu_1(K) \in \mathbf{R}$.

For quasilinear operators, F_1, F_2 and F_3 reduce to the natural growth conditions of Ladyzhenskaya and Ural'tseva. Also condition F2 is a consequence of F1 and F3. The following estimate extends the corresponding result for the quasilinear case **[10, 13]**.

THEOREM 4.1. *Let $u \in C^3(\Omega)$ satisfy $F[u] = 0$ in Ω with* F1, F2 *and* F3 *holding. Then*

$$[u]^*_{1;\Omega} \leqslant C \tag{4.1}$$

where C depends on n, μ, μ_0, μ_1, M_0 and diam Ω.

PROOF. Our proof corresponds closely to that devised by Ladyzhenskaya and Ural'tseva for the quasilinear case which utilizes an earlier technique of Bernstein. We first differentiate (1) with respect to x_k to obtain

$$F_{ij}D_{ijk}u + F_{p_i}D_{ik}u + F_zD_ku + F_{x_k} = 0. \tag{4.2}$$

Multiplying (4.2) by D_ku and summing over k we then get

$$-F_{ij}D_{ik}uD_{jk}u + \tfrac{1}{2}F_{ij}D_{ij}v + \tfrac{1}{2}F_{p_i}D_iv + \delta Fv = 0 \tag{4.3}$$

where

$$v = |Du|^2, \qquad \delta F = F_z + (D_ku/v)F_{x_k}.$$

To proceed further we consider a change of dependent variable. Let $\Omega' \subset \Omega$, $M = \sup_{\Omega'} u$, $m = \inf_{\Omega'} u$ and let $\phi \in C^2[m, M]$ satisfy $\phi, \phi' > 0$, $\phi'' < 0$. Then if $\bar{v} \in C^2(\Omega')$ is defined by $v = \phi(u)\bar{v}$, we have, by differentiation,

$$D_iv = \phi D_i\bar{v} + \phi'\bar{v}D_iu,$$
$$F_{ij}D_{ij}v = \phi F_{ij}D_{ij}\bar{v} + 2\phi' F_{ij}D_iuD_j\bar{v} + \phi''\bar{v}F_{ij}D_iuD_ju + \phi'\bar{v}F_{ij}D_{ij}u,$$

so that by substitution into (4.3),

$$\begin{aligned} -2F_{ij}D_{ik}uD_{jk}u &+ \phi F_{ij}D_{ij}\bar{v} + 2\phi' F_{ij}D_iuD_j\bar{v} + \phi'\bar{v}\mathcal{E} \\ &+ \phi'\bar{v}F_{ij}D_{ij}u + \phi F_{p_i}D_i\bar{v} + \phi'\bar{v}F_{p_i}D_iu + 2\delta Fv = 0, \end{aligned} \tag{4.4}$$

where

$$\mathcal{E} = F_{ij}D_iuD_ju \geqslant \lambda v \tag{4.5}$$

corresponds to the Bernstein $\mathcal{E}$ function for quasilinear equations. Gradient bounds follow from (4.4) by judicious choice of the auxiliary function ϕ. For local bounds we take $\Omega' = \Omega \cap B_R(y)$, $y \in \Omega$, and introduce a cut-off function η by defining

$$\eta(x) = \left(1 - |x - y|^2/R^2\right)^2. \tag{4.6}$$

Then setting $w = \eta v$, $\bar{w} = \eta \bar{v}$, we have, by differentiation,

$$D_i \bar{w} = \eta D_i \bar{v} + \bar{v} D_i \eta,$$

$$F_{ij} D_{ij} \bar{w} = \eta F_{ij} D_{ij} \bar{v} + (2/\eta) F_{ij} D_i \eta D_j \bar{w} - \big((2/\eta) F_{ij} D_i \eta D_j \eta - F_{ij} D_{ij} \eta\big) \bar{v},$$

so that, writing $\chi = \phi'/\phi$ and substituting into (4.4), we obtain in Ω',

$$\begin{aligned} &-2\eta F_{ij} D_{ik} u D_{jk} u + \phi\{F_{ij} D_{ij} \bar{w} + B_i D_i w\} + (\chi' + \chi^2)\mathfrak{S} w \\ &\quad + \big(\chi F_{ij} D_{ij} u + \chi F_{p_i} D_i u + 2\delta F\big) w \\ &\quad + \big((2/\eta) F_{ij} D_i \eta D_j \eta - F_{ij} D_{ij} \eta - 2\chi F_{ij} D_i u D_j \eta - F_{p_i} D_i \eta\big) v = 0, \end{aligned} \tag{4.7}$$

where

$$B_1 = -(2/\eta) F_{ij} D_j \eta + 2\chi F_{ij} D_j u + F_{p_i}.$$

Using the structure conditions F1 and F3, the terms in (4.7) may be estimated as follows:

$$F_{ij} D_{ik} u D_{jk} u \geqslant \lambda |D^2 u|^2, \qquad F_{ij} D_{ij} u \leqslant n\Lambda |D^2 u|, \tag{4.8}$$

$$F_{p_i} D_i u, \delta F \leqslant \lambda \mu_1 \big(1 + v + |D^2 u|\big), \quad \frac{2}{\eta} F_{ij} D_i \eta D_j \eta - F_{ij} D_{ij} \eta \leqslant \frac{C(n)\Lambda}{R^2},$$

$$-F_{ij} D_i u D_j \eta \leqslant \frac{8\Lambda\sqrt{v}}{R}, \qquad -F_{p_i} D_i \eta \leqslant \frac{\lambda \mu_1 |D\eta|}{\sqrt{v}} \big(1 + v + |D^2 u|\big).$$

Setting $\Omega_R = \{x \in \Omega' \mid w(x) \geqslant 1/R^2\}$ and using Cauchy's inequality,

$$ab \leqslant \varepsilon a^2 + b^2/\varepsilon$$

for appropriate a, b, $\varepsilon > 0$, we may deduce from (4.7) the differential inequality

$$-\phi\big(F_{ij} D_{ij} \bar{w} + B_i D_i \bar{w}\big) \leqslant \chi' \mathfrak{S} w + A\lambda v w\big(\chi^2 + 1\big)$$

in Ω_R, where A is a constant depending only on n, μ and μ_1. Consequently, if

$$\operatorname*{osc}_{\Omega'} u = M - m < \pi/2A, \tag{4.9}$$

we may choose

$$\chi(z) = -\tan A(z - m) \tag{4.10}$$

so that

$$F_{ij} D_{ij} \bar{w} + B_i D_i \bar{w} \geqslant 0 \tag{4.11}$$

in Ω_R. It then follows by the classical maximum principle [**5**] that

$$|Du(y)| \leqslant 1/R \tag{4.12}$$

provided $R < \operatorname{dist}(y, \partial\Omega)$ and (4.9) holds. Combining (4.12) with the Hölder estimate (3.1) in Theorem 3.1, we subsequently infer (4.1).

REMARKS. (i) It is clear from the proof of Theorem 4.1 that the structure condition F3 can be replaced by the more general condition

F3: $\qquad |p||F_p|, \qquad \delta F \leqslant \lambda \mu_1 \big(1 + |p|^2 + |r|\big).$

Furthermore, by adding an appropriate multiple of (1.1) to (4.7) we may replace F_p, δF in F3* by $F_p + \mathfrak{s}F$, $\delta F + tF$ for any functions $\mathfrak{s}\colon \Gamma \to \mathbf{R}^n$, $t\colon \Gamma \to \mathbf{R}$. Further interior and global gradient bounds for elliptic fully nonlinear equations may also be derived as above by adaptation of the Bernstein method for the quasilinear case as described, for example, in [**5**, Chapter 14].

(ii) A refinement of (4.1) will be useful for global regularity considerations in §7. By replacing u by $u/(M_0 + d_{\Omega'})$, we can, by inspection of the above proof, obtain the estimate

$$[u]_{1;\Omega'} \leq C(M_0/d_{\Omega'} + 1) \tag{4.13}$$

where C depends on the same quantities as in (4.1).

5. Hölder estimates for derivatives. Both the Hölder estimates for first derivatives in this section and second derivatives in the next section will be necessary for our derivation of second derivative bounds. In formulating appropriate structural hypotheses we may take account of already established gradient bounds. Consequently we set

$$\tilde{\mathfrak{U}}_K = \{x \in \mathfrak{U} \mid |z| + |p| \leq K\}$$

and assume the structural conditions:

$\tilde{\mathrm{F}}1$: F is uniformly elliptic on $\tilde{\mathfrak{U}}_K$ for all $K \in \mathbf{R}$ with (1.4) holding for $\mu = \mu(K)$;

$\tilde{\mathrm{F}}3$: $|F_p|, |F_z|, |F_x| \leq \lambda\mu_1(1 + |r|)$ for all $x, z, p \in \tilde{\mathfrak{U}}_K$, $r \in \mathbb{S}^n$, $K \in \mathbf{R}$, where $\mu_1 = \mu_1(K)$.

The following theorem then extends the corresponding basic result of Ladyzhenskaya and Ural'tseva for quasilinear elliptic equations.

THEOREM 5.1. *Let $u \in C^3(\Omega)$ satisfy $F[u] = 0$ in Ω with $\tilde{\mathrm{F}}_1$, $\tilde{\mathrm{F}}_3$ holding. Then*

$$[Du]^*_{\alpha;\Omega} \leq C \tag{5.1}$$

where $\alpha > 0$ depends on n, $\mu(M_1)$, $\mu_1(M_1)$, $M_1 = |u|_{1;\Omega}$ and C depends, in addition, on diam Ω.

PROOF. We basically follow the proof of Ladyzhenskaya and Ural'tseva for the quasilinear case with the weak Harnack inequality, Lemma 2.2 being used in place of divergence structure results. Similar ideas will be carried over to the second derivative estimation in the next section. Let ε be a positive constant and set

$$w^{\pm} = w_l^{\pm} = \pm D_l u + \varepsilon v \tag{5.2}$$

where $1 \leq l \leq n$ and $v = |Du|^2$. Combining (4.2) and (4.3) we see that the functions $w^{\pm}$ satisfy the equations

$$-2\varepsilon F_{ij}D_{ik}uD_{jk}u + F_{ij}D_{ij}w^{\pm} + F_{p_i}D_i w^{\pm} + 2\delta F v \pm \left(F_z D_l u + F_{x_l}\right) = 0, \tag{5.3}$$

so that, using the ellipticity of F,

$$F_{ij}D_{ik}uD_{jk}u \geq \lambda |D^2 u|^2,$$

together with $\tilde{F}3$, we obtain the inequalities

$$-F_{ij}D_{ij}w^{\pm} \leqslant \lambda C\left(|Dw^{\pm}|^2+1\right) \tag{5.4}$$

where $C = C(\mu_1, M_1, \varepsilon)$. Now suppose $B_{2R} \subset \Omega$ and set $W^{\pm} = \sup_{B_{2R}} w^{\pm}$. Applying Lemma 2.2, we obtain

$$\begin{aligned}\Phi_{\kappa,R}(W^{\pm}-w^{\pm}) &= \left[R^{-n}\int_{B_R}(W^{\pm}-w^{\pm})^{\kappa}\right]^{1/\kappa} \\ &\leqslant C\left(W^{\pm}-\sup_{B_R} w^{\pm}+R^2\right)\end{aligned} \tag{5.5}$$

where $\kappa, C > 0$ depend on n, μ, μ_1, M_1 and ε. With the constant ε chosen sufficiently small, for example, $\varepsilon \leqslant (10nM_1)^{-1}$, the remainder of the proof follows that for the quasilinear case as described in [**5**, Chapter 12].

6. Second derivative estimates. This section embodies the main contribution of this work, which is an interior bound for the second derivatives of (1). Our method involves a careful estimation of Hölder norms in terms of second derivative bounds, followed by a subtle interpolation argument which permits us to handle cubic terms in the second derivatives. Our approach to the Hölder estimation is a simplified version of Evans [**2**] along the lines proposed in [**14**] for the special case when F is a function of r only.

As in the preceding section structural hypotheses are formulated under the assumption that gradient bounds are already known. We will thus assume in addition to $\tilde{F}1$ and $\tilde{F}3$ that the function F is twice differentiable in Γ with second derivatives satisfying the following condition:

F4:
$$\begin{aligned}&|F_{rx}|, |F_{rp}|, |F_{rz}| \leqslant \lambda\mu_2,\\ &|F_{pp}|, |F_{pz}|, |F_{px}|, |F_{zz}|, |F_{zx}|, |F_{xx}| \leqslant \lambda\mu_2(1+|r|)\\ &\text{for all } x, z, p \in \tilde{\mathfrak{U}}_K, r \in \mathbb{S}^n, K \in \mathbf{R}, \text{ where } \mu_2 = \mu_2(K).\end{aligned}$$

We further assume

F5: F is a *concave* function of r, that is,
$$F_{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial r_{ij}\partial r_{kl}}\eta_{ij}\eta_{kl} \leqslant 0 \quad \text{for all } x, z, p, r \in \Gamma.$$

Clearly any quasilinear elliptic equation with C^2 coefficients satisfies $\tilde{F}1$, $\tilde{F}3$, F4, F5. It will also be apparent from the proof below that certain of the bounds in F4 can be replaced by the concavity of F with respect to additional variables (see Remark (i) below). Indeed we shall show that the case considered by Evans [**2**] where F is concave and uniformly Lipschitz in r, p and z is considerably simpler.

The basic estimate is the following

THEOREM 6.1. *Let $u \in C^4(\Omega)$ satisfy $F[u] = 0$ in Ω with* $\tilde{F}1$, $\tilde{F}3$, F4 *and* F5 *holding. Then*

$$[Du]^*_{1,\alpha} \leqslant C \tag{6.1}$$

where $\alpha > 0$ depends on n and μ, and C depends in addition on $\mu_1, \mu_2, M_1 = |\mu|_{1;\Omega}$ and diam Ω.

PROOF. We commence by fixing a unit vector $\gamma \in \mathbf{R}^n$ and differentiating (1) twice in the direction γ. We thus obtain

$$F_{ij}D_{ij\gamma}u + F_{p_i}D_{i\gamma}u + F_zD_\gamma u + \gamma_iF_{x_i} = 0, \tag{6.2}$$

$$\begin{aligned} F_{ij}D_{ij\gamma\gamma}u &+ F_{ij,kl}D_{ij\gamma}uD_{kl\gamma}u + 2F_{ij,p_k}D_{ij\gamma}uD_{k\gamma}u \\ &+2F_{ij,z}D_{ij\gamma}uD_\gamma u + 2\gamma_kF_{ij,x_k}D_{ij\gamma}u + F_{p_i}D_{i\gamma\gamma}u \\ &+F_{p_ip_j}D_{i\gamma}uD_{j\gamma}u + 2F_{p_iz}D_{i\gamma}uD_\gamma u + 2\gamma_jF_{p_i,x_j}D_{i\gamma}u \\ &+F_zD_{\gamma\gamma}u + F_{zz}(D_\gamma u)^2 + 2\gamma_iF_{zx_i}D_\gamma u + \gamma_i\gamma_jF_{x_ix_j} = 0. \end{aligned} \tag{6.3}$$

Next we let Ω' be a subdomain of Ω and set

$$M_2 = \sup_{\Omega'} |D^2u|, \qquad h_\gamma = \frac{1}{2}\left(1 + \frac{D_{\gamma\gamma}u}{1 + M_2}\right)$$

so that $0 < h_\gamma < 1$. Using the structure conditions $\tilde{\text{F}}3$, F4, F5 we obtain from (6.3) the inequality

$$-F_{ij}D_{ij}h_\gamma \leqslant C\lambda\{|D^3u| + (1 + M_2)^2\}, \tag{6.4}$$

where C depends on n, μ_1, μ_2 and M_1. Let us now choose directions $\gamma_1,\dots,\gamma_N$ in accordance with Lemma 2.4 applied to the matrix $a^{ij} = \lambda^{-1}F_{ij}$, multiply (6.4) for $\gamma = \gamma_k$ by $h_k = h_{\gamma_k}$ and sum over k. We thus obtain

$$\sum_{k=1}^N F_{ij}D_ih_kD_jh_k - \frac{1}{2}F_{ij}D_{ij}v \leqslant C\lambda\{|D^3u| + (1 + M_2)^2\}, \tag{6.5}$$

where $v = \sum_{k=1}^N(h_k)^2$. Consequently for $\varepsilon \in (0,1)$ and $w = w_k = h_k + \varepsilon v$, $k = 1,\dots,N$, we have, by combining (6.4) and (6.5),

$$\varepsilon\sum_{k=1}^N F_{ij}D_ih_kD_jh_k - \frac{1}{2}F_{ij}D_{ij}w \leqslant C\lambda\{|D^3u| + (1 + M_2)^2\}. \tag{6.6}$$

Using the ellipticity of F and the choice of γ_k in Lemma 2.4, we estimate

$$\sum_{k=1}^N F_{ij}D_ih_kD_jh_k \geqslant \lambda\sum_{k=1}^N |Dh_k|^2 \geqslant \lambda\left(\frac{|D^3u|}{1 + M_2}\right)^2,$$

so that from (6.6) we obtain an inequality

$$-F_{ij}D_{ij}w \leqslant \lambda\bar{\mu}, \tag{6.7}$$

where

$$\bar{\mu} = (C/\varepsilon^2)(1 + M_2^2), \qquad C = C(n, \mu_1, \mu_2, M_1).$$

We are now in a position to apply the weak Harnack inequality Lemma 2.2. Let B_R, B_{2R} be concentric balls in Ω' and set, for $s = 1, 2$, $k = 1,\dots,N$,

$$W_k^{(s)} = \sup_{B_{sR}} w, \quad M_k^{(s)} = \sup_{B_{sR}} h_k, \quad m_k^{(s)} = \inf_{B_{sR}} h_k,$$

$$\omega(sR) = \sum_{k=1}^N \operatorname*{osc}_{B_{sR}} h_k = \sum_{k=1}^N \left(M_k^{(s)} - m_k^{(s)}\right).$$

Applying Lemma 2.2 to the function $W_k^{(2)} - w_k$ we obtain

$$\Phi_{\kappa,R}\left(W_k^{(2)} - w_k\right) \le C\left\{W_k^{(2)} - W_k^{(1)} + \bar{\mu}R^2\right\}, \tag{6.8}$$

where κ, C are positive constants depending only on n and μ. Using the inequalities

$$W_k^{(2)} - w_k \ge M_k^{(2)} - h_k - 2\varepsilon\omega(2R), \quad W_k^{(2)} - W_k^{(1)} \le M_k^{(2)} - M_k^{(1)} + 2\varepsilon\omega(2R),$$

we can deduce from (6.8) a similar inequality for the functions h_k, namely

$$\Phi_{\kappa,R}\left(M_k^{(2)} - h_k\right) \le C\left\{M_k^{(2)} - M_k^{(1)} + \varepsilon\omega(2R) + \bar{\mu}R^2\right\}. \tag{6.9}$$

Let us now fix some index l and sum the inequalities (6.9) over $k \ne l$. We thus obtain

$$\begin{aligned} \Phi_{\kappa,R}\Big(\sum_{k\ne l}\left(M_k^{(2)} - h_k\right)\Big) &\le N^{1/\kappa}\sum_{k\ne l}\Phi\left(M_k^{(2)} - h_k\right) \\ &\le C\left\{(1+\varepsilon)\omega(2R) - \omega(R) + \bar{\mu}R^2\right\}, \end{aligned} \tag{6.10}$$

where $C = C(n, \mu)$ as before. To compensate for not having the corresponding inequality to (6.7) for the functions $-h_k$, we involve (1) itself, which, by virtue of Lemma 2.4, expresses a functional relationship between the functions h_k. In fact, by the concavity of F with respect to r, F5, we have for any x, $y \in B_R$,

$$\begin{aligned} &F_{ij}\left(y, u(y), Du(y), D^2u(y)\right)\left(D_{ij}u(y) - D_{ij}u(x)\right) \\ &\quad\le F\left(y, u(y), Du(y), D^2u(x)\right) - F\left(y, u(y), Du(y), D^2u(y)\right) \\ &\quad= F\left(y, u(y), Du(y), D^2u(x)\right) - F\left(x, u(x), Du(x), D^2u(x)\right) \\ &\quad\le \lambda\mu_1(1 + M_2)\{|x - y| + |u(x) - u(y)| + |Du(x) - Du(y)|\} \\ &\quad\le 4\lambda\mu_1 R(1 + M_2)(1 + M_1 + M_2) \end{aligned} \tag{6.11}$$

by $\tilde{\text{F}}3$. Furthermore, by Lemma 2.4,

$$\begin{aligned} &F_{ij}\left(y, u(y), Du(y), D^2u(y)\right)\left(D_{ij}u(y) - D_{ij}u(x)\right) \\ &\quad= \lambda\sum_{k=1}\beta_k(y)\left(D_{\gamma_k\gamma_k}u(y) - D_{\gamma_k\gamma_k}u(x)\right) \\ &\quad= 2\lambda(1 + M_2)\sum_{k=1}^{N}\beta_k\left(h_k(y) - h_k(x)\right), \end{aligned} \tag{6.12}$$

where $0 < \lambda^* \le \beta_k \le \Lambda^*$, $k = 1, \ldots, N$, and λ^*, Λ^* depend only on n and μ. The combination of (6.1) and (6.12) now yields

$$\sum_{k=1}^{N}\beta_k\left(h_k(y) - h_k(x)\right) \le 4\mu_1 R(1 + M_1 + M_2),$$

so that for fixed l,

$$\begin{aligned} h_l(y) - m_l^{(2)} &\le \frac{1}{\lambda^*}\Big\{4\mu_1 R(1 + M_1 + M_2) + \Lambda^*\sum_{k\ne l}\left(M_k^{(2)} - h_l(y)\right)\Big\} \\ &\le C\Big\{(1 + M_2)R + \sum_{k\ne l}\left(M^{(2)} - h_l(y)\right)\Big\} \end{aligned}$$

where $C = C(n, \mu, \mu_1, M_1)$. Consequently, using (6.10) we obtain, for $l = 1, \ldots, N$,

$$\Phi_{\kappa,R}\big(h_l - m_l^{(2)}\big) \leqslant C\{(1+\varepsilon)\omega(2R) - \omega(R) + \tilde{\mu}R + \bar{\mu}R^2\} \tag{6.13}$$

where $C = C(n, \mu)$ and $\tilde{\mu} = C(1 + M_2)$, $C = C(\mu_1, M_1)$. By adding (6.10) and (6.13) for $l = k$ and summing over k, we then obtain

$$\omega(2R) \leqslant C\{(1+\varepsilon)\omega(2R) - \omega(R) + \tilde{\mu}R + \bar{\mu}R^2\},$$

whence

$$\omega(R) \leqslant \delta\omega(2R) + C\big(\varepsilon\omega(2R) + \tilde{\mu}R + \bar{\mu}R^2\big)$$

for $\delta = 1 - 1/C$. Finally, by choosing ε sufficiently small we get the oscillation estimate

$$\omega(R) \leqslant \bar{\delta}\omega(2R) + C\big(\tilde{\mu}R + \bar{\mu}R^2\big), \tag{6.14}$$

where $0 < \bar{\delta} < 1$, $C, \bar{\delta}$ depend only on n and μ, and $\tilde{\mu}$, $\bar{\mu}$ are as indicated above. A Hölder estimate for ω, and, hence, for the second derivatives of the solution u now follows from (6.14) by a standard argument (see [**5**, Chapter 8]). In any ball $B_R \subset \Omega'$ and $0 < \sigma < 1$, we obtain

$$\sum_{k=1}^{N} \operatorname*{osc}_{B_{\sigma R}} h_k \leqslant C\sigma^\alpha\big(1 + \tilde{\mu}R + \bar{\mu}R^2\big),$$

where C and α are positive constants depending only on n and μ. Consequently,

$$\operatorname*{osc}_{B_{\sigma R}} D^2u \leqslant C\sigma^\alpha(1 + M_2)\big(1 + \tilde{\mu}R + \bar{\mu}R^2.\big) \tag{6.15}$$

From (6.15) we can infer an interior Hölder estimate

$$[D^2u]^*_{\alpha;\Omega} \leqslant C, \tag{6.16}$$

where $\alpha = \alpha(n, u)$ and $C = C(n, \mu, \mu_1, \mu_2, |u|_{2;\Omega}, \operatorname{diam}\Omega)$. Indeed for this estimation there was no need to take account of the dependence on M_2 in the above argument. However for the establishment of second derivative bounds this dependence is crucial, as we shall now demonstrate.

Let us take Ω' to be a ball $B = B_\delta(y) \subset \Omega$ and suppose that

$$(1 + M_2)\delta \leqslant 1. \tag{6.17}$$

The quantities $\tilde{\mu}R$, $\bar{\mu}R^2$ are then bounded independently of M_2 in any ball $B_R \subset B$, and we obtain from (6.16) that for any $0 < \sigma < 1$, $0 < R < \delta$,

$$\operatorname*{osc}_{B_{\sigma R}} D^2u \leqslant C\sigma^\alpha\Big(1 + \sup_{B_R}|D^2u|\Big) \tag{6.18}$$

where $C = C(n, \mu, \mu_1, \mu_2, M_1)$. Consequently,

$$[Du]^*_{1,\alpha;\beta} \leqslant C\big(\delta + [Du]^*_{1;B}\big),$$

and, hence, by the interpolation inequality, Lemma 2.1, we obtain

$$[Du]^*_{1;B} \leqslant C(\delta + |Du|_{0;B})$$

so that, in particular,

$$|D^2u(y)| \leqslant C\big(1 + \delta^{-1}|Du|_{0;B}\big). \tag{6.19}$$

By replacing u by $u - x \cdot Du(y)$ we can assume without loss of generality that $Du(y) = 0$. Consequently, by Theorem 5.1,

$$|Du|_{0;B} = \sup_B |Du(x) - Du(y)| \leq C\left(\frac{\delta}{d}\right)^{\bar{\alpha}},$$

where C and $\bar{\alpha}$ are positive constants depending only on n, μ, μ_1 and M_1, and $d = \min\{1, d_y\}$, where $d_y = \operatorname{dist}(y, \partial\Omega)$. Substituting into (6.19) we thus have

$$|D^2u(y)| \leq Cd^{-\bar{\alpha}}(1 + \delta^{\bar{\alpha}-1}). \tag{6.20}$$

The proof is now completed through an appropriate choice of the ball B. To do this we set

$$M_2^* = [Du]_{1;\Omega} = \sup_{x \in \Omega} \left(d_x |D^2u(x)|\right)$$

and choose y such that

$$d_y |D^2u(y)| \geq M_2^*/2.$$

Then if $M_2^* \geq 2$, we choose

$$\delta = 1/\left(1 + 2|D^2u(y)|\right)$$

so that

$$M_2 = \sup_{B_\delta(y)} |D^2u| \leq 2|D^2u(y)|,$$

and, hence, (6.17) is fulfilled. By inserting our choice of δ in (6.20) we obtain the desired bound for $D^2u(y)$, namely

$$|D^2u(y)| \leq Cd^{-1}. \tag{6.21}$$

Combining (6.21) with the Hölder estimate (6.18), we finally obtain the assertion of Theorem 6.1.

REMARKS. (i) If we designate points in $\Gamma = \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbb{S}^n$ by $X = (X_1, \ldots, X_\nu) = (x, z, p, r)$, where $\nu = n + 1 + n + \frac{1}{2}n(n+1) = \frac{1}{2}(n^2 + 5n + 2)$, then F4 and F5 in the hypotheses of Theorem 6.1 may be replaced by the more general condition

$$\text{F4*:} \quad \frac{\partial^2 F}{\partial X_i \, \partial X_j} Y_i Y_j \leq \mu_2 \lambda\left\{(1 + |r|)^3 Z_1^2 + (1 + |r|) Z_2^2 + (1 + |r|)^{3/2} Z_1 Z_3 + (1 + |r|)^{1/2} Z_2 Z_3\right\}$$

for all $Y \in \mathbf{R}^\nu$ where

$$Z_1 = |(Y_1, \ldots, Y_{n+1})|, \quad Z_2 = |Y_{n+2}, \ldots, Y_{2n+1}|, \quad Z_3 = |Y_{2n+2}, \ldots, Y_\nu|.$$

To pass from (6.3) to (6.4) we apply F4* with $Y = (\gamma, D_\gamma u, DD_\gamma u, D^2 D_\gamma u)$. In particular, if F is concave with respect to z, p, r, the second derivatives involving these variables only may be omitted from F4. If F is concave with respect to all variables, F4 becomes superfluous and (6.1) is independent of the second derivatives of F. Note that F4* implies F5. An alternative approach to second derivative bounds which encompasses nonconcave F will be presented in a further paper.

(ii) In the course of the proof of Theorem 6.1 we established a Hölder estimate for the second derivatives of solutions in terms of bounds on their second derivatives, namely the estimate (6.16). We may formulate this result, essentially due to Evans [**2**], as follows.

THEOREM 6.2. *Let $u \in C^4(\Omega)$ be a solution of equation* (1.1) *in Ω where F is concave in r, and suppose that for constant μ, μ_1, and μ_2,*

$$\Lambda \leqslant \mu\lambda, \qquad |F_p|, |F_z|, |F_x| \leqslant \mu_1\lambda,$$
$$|F_{pr}|, |F_{zr}|, |F_{xr}|, |F_{pp}|, |F_{pz}|, |F_{px}|, |F_{zz}|, |F_{zx}|, |F_{xx}| \leqslant \mu_2\lambda \tag{6.22}$$

for all $x \in \Omega$, $z = u(x)$, $p = Du(x)$, $r = D^2u(x)$. Then we have the estimate $[D^2u]^*_\alpha \leqslant C$, where $\alpha > 0$ depends only on n and μ, and C depends, in addition, on μ_1, μ_2, $|u|_{2;\Omega}$ and $\operatorname{diam}\Omega$.

This result may also be generalized in accordance with the preceding remark. It is also worth pointing out that in many of the preceding theorems of this paper the sets $\mathfrak{U}_\kappa$ and $\tilde{\mathfrak{U}}_\kappa$ can be replaced by the set

$$\Gamma_u = \{(x, u(x), Du(x), D^2u(x)) \mid x \in \Omega\}$$

where u is the solution under consideration (although for Theorem 3.1 we should assume (1.4) rather than (1.3)).

7. Global estimates. By means of standard barrier techniques the interior estimates of the preceding sections may be extended to the boundary of the domain Ω. We first consider a boundary gradient estimate which, for later purposes, we formulate as follows.

LEMMA 7.1. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $g \in C^2(\bar{\Omega})$ satisfy $F[u] = 0$ in Ω, $u = g$ on $\partial\Omega$ and suppose that* F1 *and* F2 *hold and Ω satisfies a uniform exterior sphere condition. Then*

$$\sup_{\substack{x\in\Omega,\\ y\in\partial\Omega}} |u(x) - g(y)| \leqslant C|x - y| \tag{7.1}$$

where C depends on n, μ, μ_0, $M_0 = |u|_{0;\Omega}$, $|g|_{2;\Omega}$ and Ω.

PROOF. The situation is reduced to the quasilinear case as in the proof of Theorem 3.1. In this case (7.1) follows from well-known barrier arguments (see [**8**, Chapter 6, Lemma 2.1] or [**5**, Theorem 13.1]).

The combination of Lemma 7.1 and Theorem 4.1 yields the following global gradient bound.

THEOREM 7.2. *Let $u \in C^3(\Omega) \cap C^0(\bar{\Omega})$, $g \in C(\bar{\Omega})$ satisfy $F[u] = 0$ in Ω, $u = g$ on $\partial\Omega$ and suppose that* F1, F2, F3 (*or* F3*) *hold and Ω satisfies a uniform exterior sphere condition. Then $u \in C^{0,1}(\bar{\Omega})$ and*

$$|u|_{1;\Omega} = \sup_\Omega |Du| \leqslant C \tag{7.2}$$

where C depends on n, μ, μ_0, μ_1, M_0, $|g|_{2;\Omega}$ and Ω.

PROOF. We fix a point $x_0 \in \Omega$ and set $d = \frac{1}{2}d_{x_0}$, $B_1 = B_d(x_0)$, $B_2 = B_{2d}(x_0)$. Then using Theorem 4.1, in particular (4.13), we obtain, for any $x \in B_1$,

$$|u(x) - u(x_0)| \leqslant (C/d)|x - x_0| \operatorname*{osc}_{B_2} u$$

where $C = C(n, \mu, \mu_0, \mu_1, M_0)$ as in (4.1). But by Lemma 7.1,

$$\operatorname*{osc}_{B_2} u \leqslant Cd,$$

where $C = C(n, \mu, \mu_0, M_0, |g|_{2;\Omega}, \Omega)$, whence (7.2) follows.

By combining Lemma 7.1 and Theorem 3.1 we would obtain a global Hölder estimate under the hypotheses of Lemma 7.1. However this result is not as general as that obtained through direct extension of Lemma 2.3 to the boundary, which would only require that $g \in C^{\beta}(\overline{\Omega})$ for some $\beta > 0$ and $\partial\Omega$ satisfy a uniform exterior cone condition, or, more generally, that there exist positive constants κ_1, κ_2 such that $\operatorname{meas}(\Omega - B_R(y))/\operatorname{meas} B_R(y) \geqslant \kappa_1$ for all $0 < R \leqslant \kappa_2$ [**13**].

We turn now to estimates for the second derivatives at the boundary. Setting $\mathbf{R}^n_+ = \{x = (x', x_n) \in \mathbf{R}^n \mid x_n > 0\}$, we first prove an estimate for flat boundary portions.

LEMMA 7.3. *Let $\Omega = B_R(0) \cap \mathbf{R}^n_+$ be a half ball and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $F[u] = 0$ in Ω, $u = 0$ on $\partial\Omega$ in $\partial\mathbf{R}^n_+$ with conditions* $\tilde{\text{F}}1$, $\tilde{\text{F}}3$ *holding. Then for $0 \leqslant x_n \leqslant R$, $k = 1, \ldots, n-1$, we have*

$$|D_k u(0, x_n)| \leqslant C x_n \tag{7.3}$$

where C depends on n, μ, μ_1, M_1 and R.

PROOF. We begin, similarly to the proof of Theorem 5.1, by setting $v' = \sum_{k=1}^{n-1}(D_k u)^2$, $w'_l = w' = D_l u + v'$, $l = 1, \ldots, n-1$, thereby obtaining, instead of (5.3),

$$-2\sum_{k=1}^{n-1} F_{ij} D_{ik} u D_{jk} u + F_{ij} D_{ij} w' + 2F_z w' + 2\left(F_x + \sum_{k=1}^{n-1} D_k u F_{x_k}\right) = 0. \tag{7.4}$$

Using (1.1) in the form (3.2) we can write

$$D_{nn} u = \frac{1}{F_{nn}(x, u, Du, s)}\left\{\sum_{i+j<2n} F_{ij}(x, u, Du, s) D_{ij} u + F(x, u, Du, 0)\right\},$$

so that by $\tilde{\text{F}}1$, $\tilde{\text{F}}3$ we may estimate

$$|D_{nn} u| \leqslant n\mu\left(\sum_{i+j<2n}\left(D_{ij}u\right)^2\right)^{1/2} + \mu_1(\operatorname{diam}\Omega + M_1). \tag{7.5}$$

(Note that without loss of generality we can assume $F(x_0, 0, 0, 0) = 0$ for some $x_0 \in \Omega$.) Using (7.5), F3 and the ellipticity of F,

$$\sum_{k=1}^{n-1} F_{ij} D_{ik} u D_{jk} u \geqslant \lambda \sum_{i+j<2n}\left(D_{ij}u\right)^2,$$

we obtain from (7.4) the corresponding inequality to (5.3), namely

$$-F_{ij}D_{ij}w' \leqslant \lambda C\left(|Dw'|^2 + 1\right) \tag{7.6}$$

where $C = C(n, \mu, \mu_1, M_1)$. By the barrier argument of Theorem 13.1 of [5] (with an exterior sphere fixed at 0), we thus have $w'(0, x_n) \leqslant Cx_n$ for $0 \leqslant x_n \leqslant R$, where $C = C(n, \mu, \mu_1, M_1, R)$ and, hence, after replacing u with $-u$, we infer (7.3).

If, in Lemma 7.3, the function u was twice differentiable at 0, (7.3) would imply $|D_{nk}u(0)| \leqslant C$ for $k = 1, \ldots, n-1$. Since $D_{ij}u(0) = 0$ for $i, j = 1, \ldots, n-1$, we then here by (7.5) an estimate for all the second derivatives at 0, namely

$$|D^2u(0)| \leqslant C \tag{7.7}$$

where $C = C(n, \mu, \mu_1, M_1, R)$. This result may then be extended to C^3 domains by means of a diffeomorphism which preserves the form of the structural conditions $\tilde{F}1$, $\tilde{F}3$. Accordingly we obtain the following boundary second derivative estimate.

THEOREM 7.4. *Let $u \in C^3(\Omega) \cap C^2(\bar{\Omega})$, $g \in C^3(\bar{\Omega})$ satisfy $F[u] = 0$ with $\tilde{F}1$, $\tilde{F}3$ holding and $\partial\Omega \in C^3$. Then*

$$\sup_{\partial\Omega} |D^2u| \leqslant C \tag{7.8}$$

where C depends on $n, \mu, \mu_1, M_1, \Omega$ and $|g|_{3;\Omega}$.

By combining Lemma 7.3 and Theorem 6.1, we also obtain a global second derivative bound.

THEOREM 7.5. *Let $u \in C^4(\Omega) \cap C^1(\bar{\Omega})$, $g \in C^3(\bar{\Omega})$ satisfy $F[u] = 0$ in Ω, $u = g$ on $\partial\Omega$, with $\tilde{F}1$, $\tilde{F}3$, F4 (or F4*), F5 holding and $\partial\Omega \in C^3$. Then $u \in C^{1,1}(\bar{\Omega})$ and*

$$[u]_{2;\Omega} = \sup_{\Omega} |D^2u| \leqslant C \tag{7.9}$$

where C depends on $n, \mu, \mu_1, M_1, |g|_{3;\Omega}$ and Ω.

PROOF. Suppose first that Ω is the half ball $B_R(0) \cap \mathbf{R}^n_+$ with $u = 0$ on $\partial\Omega \cap \partial\mathbf{R}^n_+$, as in Lemma 7.3, and let $y = (0, y_n)$ where $y_n < R/4$. By Theorem 6.1 there exists a positive constant $\kappa < 1$, depending only on $n, \mu, \mu_1, \mu_2, M_1$ and Ω, such that $(1 + \sup_{B_\delta(y)} |D^2u|)\delta \leqslant 1$ for $\delta = \kappa y_n$ and, hence, condition (6.17) in the proof is fulfilled in the ball $B = B_\delta(y)$. Using (7.5) to eliminate the derivative $D_{nn}u$ from (6.18), we then obtain, in place of (6.19),

$$|D^2u(y)| \leqslant C\left(1 + \delta^{-1}\sum_{k=1}^{n-1} |D_ku|_{0;B}\right) \leqslant C\left(1 + \delta^{-1}y_n\right) \leqslant C \tag{7.10}$$

by Lemma 7.3 and our choice of δ. Finally, by means of diffeomorphisms which locally flatten $\partial\Omega$ and preserve the form of $\tilde{F}1$, $\tilde{F}3$, F4 and F5 and by replacement of u by $u - g$, we get (7.9) for general $\Omega \subset C^3$.

8. Applications. By combining Theorems 4.1, 6.1, 7.2 and 7.5, we obtain the following interior and global estimates for solutions of fully nonlinear, uniformly elliptic equations satisfying the natural structure conditions.

THEOREM 8.1. *Let $u \in C^4(\Omega)$ satisfy $F[u]=0$ in Ω with* F1–F5 *holding. Then for any subdomain $\Omega' \Subset \Omega$, we have*

$$|u|_{2,\alpha;\Omega'} \leqslant C, \tag{8.1}$$

where $\alpha > 0$ depends only on n, μ, and C depends, in addition, on μ_0, μ_1, μ_2, $|u|_{0;\Omega}$ and $\operatorname{dist}(\Omega', \partial\Omega)$. *Furthermore, if $u \in C^0(\overline{\Omega})$ and $u = g$ on $\partial\Omega$, where $g \in C^2(\overline{\Omega})$ and $\partial\Omega$ satisfies a uniform exterior sphere condition, then $u \in C^{0,1}$, $Du \in C^{1,\alpha}_*(\Omega)$, and*

$$|Du|^*_{1,\alpha;\Omega} \leqslant C, \tag{8.2}$$

where C depends on n, μ, μ_1, μ_2, $|u|_{0;\Omega}$, $|g|_{2;\Omega}$ and Ω. If $u \in C^1(\overline{\Omega})$, $g \in C^3(\overline{\Omega})$ and $\partial\Omega \in C^3$, then $u \in C^{1,1}(\overline{\Omega})$, $D^2u \in C^\alpha_*(\Omega)$ and

$$|D^2u|^*_{0,\alpha;\Omega} \leqslant C, \tag{8.3}$$

where C depends on n, μ, μ_0, μ_1, μ_2, $|u|_{0;\Omega}$, $|g|_{3;\Omega}$ and Ω. Finally, conditions F3, F4, F5 *may be replaced by the more general* F3*, F4*.

Using the method of continuity, we can establish existence theorems for the Dirichlet problem for (1.1) from the estimates of Theorem 8.1. As in Evans [**2**], the equation can be modified near the boundary to offset the lack of global Hölder estimates for the second derivatives. In particular, for $m = 1, 2, \ldots,$ we let $\eta_m \in C^2_0(\Omega)$ satisfy $0 \leqslant \eta_m \leqslant 1$ in Ω, $\eta_m(x) = 1$ for $\operatorname{dist}(x, \partial\Omega) \geqslant 1/m$, $|D\eta| \leqslant cm$, $|D^2\eta| \leqslant cm^2$, for some constant c depending only on n. Assuming that we can take $\lambda \equiv 1$ in (1.3) and (1.4), we set

$$F^{(m)}[u] = (1-\eta_m)\Delta u + \eta_m F(x, u, Du, D^2u). \tag{8.4}$$

Now let Ω be a $C^{2,\alpha}$ domain, $g \in C^{2,\alpha}(\overline{\Omega})$, and $F \in C^{2,\alpha}(\Gamma)$ for some $\alpha > 0$. In order to apply the method of continuity to solve the Dirichlet problem

$$F^{(m)}[u] = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega, \tag{8.5}$$

it suffices that:

(i) When $F^{(m)}$ is considered as a map from $E = \{u \in C^{2,\alpha}(\overline{\Omega}) \mid u = \text{ on } \partial\Omega\}$ into $C^\alpha(\overline{\Omega})$, its Fréchet derivative $F^{(m)}_u$, given by

$$F^{(m)}_u[v] = F^{(m)}_{ij}(x, u, Du, D^2u)D_{ij}v + F_{p_i}(x, u, Du, D^2u)D_iv$$
$$+ F_z(x, u, Du, D^2u)v,$$

for $v \in E$, has bounded inverse for each $u \in C^{2,\alpha}(\overline{\Omega})$; and

(ii) The set of solutions of the problems

$$F^{(m)}[u] = tF[\psi], \quad u = g \text{ on } \partial\Omega, \qquad 0 \leqslant t \leqslant 1, \tag{8.6}$$

is apriori bounded in $C^{2,\alpha}(\overline{\Omega})$ for some $\psi \in C^{2,\alpha}(\overline{\Omega})$, with $\psi = g$ on $\partial\Omega$.

By virtue of the Schauder theory for linear equations (see [**5**, Chapter 8]), (i) is satisfied if

$$F_z \leqslant 0 \tag{8.7}$$

for all $x, z, p, r \in \Gamma$, in which case the solution of the Dirichlet problem (8.5) is unique (if it exists). By combining (8.1) with the Schauder estimates in neighbourhoods of $\partial\Omega$, we see that (ii) is satisfied provided the solutions of (8.6) are uniformly bounded in Ω. Furthermore, if the solutions of (8.6) are uniformly bounded with respect to both t and m, we can then conclude from Theorem 8.7 and Lemma 7.1 the solvability of the Dirichlet problem, $F[u] = 0$, $u = g$ in $\partial\Omega$ in the space $C^{0,1}(\bar{\Omega}) \cap C^{4,\alpha}(\Omega)$. By further approximation this result can be extended to more general boundary data. Two possible conditions which would each imply the uniform boundedness of solutions of (8.6) are the restriction of (8.7),

$$F_z \leq -c_0 \tag{8.8}$$

for all $x, z, p, r \in \Gamma$ and some positive constant c_0, and the restriction of F2,

$$|F(x, z, p, 0)| \leq c_1\lambda(1 + |p|) \tag{8.9}$$

for all $x, z, p \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and some constant c_1 (see [**5**, Chapter 9]). In these cases we formulate resultant existence theorems as follows.

THEOREM 8.2. *Let Ω satisfy a uniform exterior sphere condition, $g \in C^2(\bar{\Omega})$, $F \in C^{2,\alpha}(\Gamma)$ for some $\alpha > 0$ and suppose that the operator F satisfies (for $\lambda \equiv 1$) F1, F2, F3 (or F3*), F4 (or F4*), F5, (8.6) and either (8.7) or (8.8). Then there exists a unique solution $u \in C^{0,1}(\bar{\Omega}) \cap C^{4,\alpha}(\Omega)$ of the Dirichlet problem $F[u] = 0$ in Ω, $u = g$ on $\partial\Omega$.*

When we only assume $g \in C^0(\partial\Omega)$ in Theorem 8.2, we would obtain a unique solution $u \in C^0(\bar{\Omega}) \cap C^{4,\alpha}(\Omega)$. Note also that the restriction $\lambda \equiv 1$ is not that severe as we really only require that (1.1) have an equivalent form which satisfies the structure conditions with $\lambda \equiv 1$.

To complete this paper we consider the application of the preceding results to the Bellman equation corresponding to a family of quasilinear operators. Let Q_k, $k = 1,\ldots,N$, be quasilinear operators of the form

$$Q_k[u] = a_k^{ij}(x, u, Du)D_{ij}u + b_k(x, u, Du) \tag{8.10}$$

with coefficients $a_k^{ij}, b_k \in C^2(\Omega \times \mathbf{R} \times \mathbf{R}^n)$, $k = 1,\ldots,N$. The operator F is defined by

$$F[u] = \inf Q_k[u], \tag{8.11}$$

and the equation $F[u] = 0$ is called the Bellman equation associated with the family $\{Q_k\}$. Writing

$$Q_k(x, g, p, r) = a_k^{ij}(x, z, p)r_{ij} + b_k(x, z, p),$$

we may then write F in the form (1.1) where the function F is given by

$$F(x, z, p, r) = \inf Q_k(x, z, p, r). \tag{8.12}$$

In order to apply our previous results we need to approximate F by smoother functions. Accordingly let G be a concave function in $C^2(\mathbf{R}^N)$ and consider, in place of (8.12), the function F given by

$$F(x, z, p, r) = G(Q_1,\ldots,Q_N). \tag{8.13}$$

By differentiation we obtain

$$F_{r_{ij}} = G_k a_k^{ij}, \qquad F_p = G_k\left(a_{kp}^{ij}r_{ij} + b_{kp}\right),$$
$$F_z = G_k\left(a_{kz}^{ij}r_{ij} + b_{kz}\right), \qquad F_x = G_k\left(a_{kx}^{ij}r_{ij} + b_{kx}\right).$$

Hence, if we assume there exist positive constants θ, Θ such that

$$(8.14) \qquad \theta \leqslant \sum G_k \leqslant \Theta,$$

and also $G(0) = 0$, we obtain that F satisfies any of the structure conditions F1, F2, F3, $\tilde{\text{F}}$1, $\tilde{\text{F}}$3, $\tilde{\text{F}}$3* (with μ, μ_0, μ_1 replaced by $\mu\Theta/\theta$, $\mu_0\Theta/\theta$, $\mu_1\Theta/\theta$, respectively) whenever all the operators Q_k satisfy the same conditions uniformly in k. For the second derivatives of F we have, in the notation of Remark (i) in §6,

$$\frac{\partial^2 F}{\partial X_i \partial X_j} = G_{kl}\frac{\partial Q_k}{\partial X_i}\frac{\partial Q_l}{\partial X_j} + G_k\frac{\partial^2 Q_k}{\partial X_i \partial X_j},$$

so that since G is concave, F4* is satisfied, with μ_2 replaced by $c\mu_2\Theta/\theta$, where now

$$(8.15) \qquad \mu_2 = \max_{\substack{1 \leqslant k \leqslant N \\ 1 \leqslant i,j \leqslant n}} \sup_{\mathfrak{U}_K} \frac{1}{\lambda_k}\left(|Da_k^{ij}| + |Db_k| + |D^2 a_k^{ij}| + |D^2 b_k|\right),$$

and c depends on n. In order to approximate the Bellman equation, we take for G the mollification of (8.12) given by

$$G(x) = G_h(x) = h^{-n}\int_{\mathbf{R}^N} \inf y_k \rho\left(\frac{x-y}{h}\right) dy,$$

where $\rho \geqslant 0$, $\in C_0^\infty(\mathbf{R}^N)$ satisfies $\int\rho = 1$ and $h > 0$. It is readily shown that G is concave and, furthermore, satisfies (8.14) with $\theta = \Theta = 1$. Since $G_h \to G$ uniformly as $h \to 0$, we obtain the existence of solutions to the Dirichlet problem for the Bellman equation as limits of solutions for the equations $G_h[u] = 0$ as $h \to 0$. In fact, as a consequence of Theorems 8.1, 8.2 and the preceding remarks, we have the following existence theorem.

THEOREM 8.3. *Let Ω satisfy a uniform exterior sphere condition, $g \in C^2(\bar{\Omega})$, $Q_k \in C^2(\Gamma)$ for some $\alpha > 0$, $k = 1,\dots,N$, and suppose that the operators Q_k satisfy the natural conditions* F1, F2, F3 (*or* F3*) *with $\lambda_k \equiv 1$, uniformly in $k = 1,\dots,N$. Suppose, in addition, that $a_k^{ij}(x,z,p) = a_k^{ij}(x,p)$ and either*

$$(8.16) \qquad b_{kz} \leqslant -c_0$$

or

$$(8.17) \qquad b_{kz} \leqslant 0, \qquad |b_k| \leqslant c_1(1 + |p|)$$

for all $x, z, p \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, where c_0 and c_1 are positive constants. Then there exists a unique solution $u \in C^{0,1}(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$, for some $\beta > 0$, of the Dirichlet problem

$$(8.18) \qquad F[u] = \inf Q_k[u] = 0, \qquad u = g \quad \text{on } \Omega.$$

If only $g \in C^0(\partial\Omega)$, we obtain a unique solution $u \in C^0(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$.

REMARKS. (i) When the operators Q_k are all linear, Theorem 8.3 reduces to the result of Evans [**2**] on the classical Dirichlet problem for the Bellman equation. (For an alternative approach, in the absence of second derivative Hölder estimates, see [**4**] and [**11**].) In this case the estimation of derivatives simplifies, as the quantities $\bar{\mu}$ and $\tilde{\mu}$ in the proof of Theorem 6.1 will be independent of M_2, and first and second derivative bounds will subsequently follow immediately from (6.15) by the interpolation Lemma 2.1.

(ii) By means of standard uniqueness and regularity arguments, Theorem 8.1 can be extended to hold for solutions $u \in C^2(\Omega)$ rather than $C^4(\Omega)$. Theorem 8.1 can also be similarly extended to embrace the Bellman equation for a family of quasilinear operators, $Q_k \in C^2(\Gamma)$, satisfying the natural conditions F1, F2, F3 (or F3*).

(iii) We also note that by the use of Lemma 2.2 in its full generality, the estimates of this paper may be set in Sobolev spaces $W^{k,n}(\Omega)$ instead of the classical spaces $C^k(\Omega)$ and the structure conditions extended accordingly.

(iv) Finally, we mention that the results and methods of this work can be extended to fully nonlinear, uniformly parabolic equations of the form

$$\partial u/\partial t = F(x, t, u, Du, D^2u) \tag{8.19}$$

in domains $D \subset \mathbf{R}^{n+1}(x, t)$, where F is now a function on $\hat{\Gamma} = D \times \mathbf{R} \times \mathbf{R}^n \times \mathbb{S}^n$. In the parabolic analogue of the structure conditions F4, we need to subject the derivative F_t to the same conditions as F_{xx}.

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