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Fully representable and *-semisimple topological partial *-algebras

by

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Abstract. We continue our study of topological partial *-algebras, focusing our attention on *-semisimple partial *-algebras, that is, those that possess a multiplication core and sufficiently many *-representations. We discuss the respective roles of invariant positive sesquilinear (ips) forms and representable continuous linear functionals, and focus on the case where the two notions are completely interchangeable (fully representable partial *-algebras) with the aim of characterizing a *-semisimple partial *-algebra. Finally we describe various notions of bounded elements in such a partial *-algebra, in particular, those defined in terms of a positive cone (order bounded elements). The outcome is that, for an appropriate order relation, one recovers the $\mathcal{M}$-bounded elements introduced in previous works.

1. Introduction. Studies on partial *-algebras have provided so far a considerable amount of information about their representation theory and their structure. Many results have been obtained for concrete partial *-algebras, i.e., partial *-algebras of closable operators (the so-called partial O*-algebras), but a substantial body of knowledge has been gathered also for abstract partial *-algebras. A full analysis has been developed by Inoue and two of us some time ago and it can be found in the monograph [1], where earlier articles are quoted.

In a recent paper [4], we have started the analysis of certain types of bounded elements in a partial *-algebra $\mathfrak{A}$ and their influence on the representation theory of $\mathfrak{A}$. It was shown, in particular, that the crucial condition is that $\mathfrak{A}$ possesses sufficiently many invariant positive sesquilinear forms (ips-forms). The latter, in turn, generate *-representations, that is, *-homomorphisms into a partial O*-algebra, via the well-known GNS construction. As in the particular case of a partial O*-algebra, a spectral theory can then be developed, provided the partial *-algebra has sufficiently many

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bounded elements. In this connection, we have introduced in [4] the notion of $\mathcal{M}$*-bounded elements*, associated to a sufficiently large family $\mathcal{M}$ of ips-forms.

We continue this study in the present work, focusing on topological partial *-algebras that possess what we call a *multiplication core*, that is, a dense subset of universal right multipliers with all the regularity properties necessary for a decent representation theory. In particular, we will require that our partial *-algebra has sufficiently many *-representations, a property usually characterized, for topological *-algebras, in terms of the so-called **-radical*. When the latter is reduced to $\{0\}$, the partial *-algebra is called **-semisimple*, and this is the main subject of the paper.

According to what we just said, *-semisimplicity is defined in terms of a family $\mathcal{M}$ of ips-forms. Since it may be difficult to identify such a family in practice, we examine in what sense ips-forms may be replaced by a special class of continuous linear functionals, called *representable*. This leads to identify a class of topological partial *-algebras for which representable linear functionals and ips-forms can be freely replaced by one another, since every representable linear functional comes (as for *-algebras with unit) from an ips-form. These partial *-algebras are called *fully representable* (extending the analogous concept discussed in [7] for locally convex quasi *-algebras), and the interplay of this notion with *-semisimplicity is investigated.

This being done, we may come back to bounded elements of a *-semisimple partial *-algebra, more precisely to elements bounded with respect to some positive cone, thus defined in purely algebraic terms. Early work in that direction has been done by Vidav [13] and Schmüdgen [9], then generalized in our paper [4]. Here we consider several types of order on a partial *-algebra and analyze the corresponding notion of order bounded elements. The outcome is that, under appropriate conditions, the correct notion reduces to that of $\mathcal{M}$-bounded ones introduced in [4]. Therefore, when the partial *-algebra has sufficiently many such elements, the whole spectral theory developed in [3] and [4] can be recovered.

The paper is organized as follows. Section 2 is devoted to some preliminaries about partial *-algebras, taken mostly from [1] and [3, 4]. In addition, we introduce the notion of multiplication core and draw some consequences. We introduce in Section 3 the notion of *-semisimple partial *-algebra and discuss some of its properties. In Section 4, we compare the roles of ips-forms and representable linear functionals, with particular reference to fully representable partial *-algebras, and discuss the relationship of the latter notion with that of *-semisimple partial *-algebra. Finally, Section 5 is devoted to various notions of bounded elements, from $\mathcal{M}$-bounded to order bounded ones.

2. Preliminaries. The following preliminary definitions will be needed. For more details we refer to [1, 8].

A *partial* **-algebra* $\mathfrak{A}$ is a complex vector space with a conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$, having the property that $(x, y) \in \Gamma$ if, and only if, $(y^*, x^*) \in \Gamma$ and $(x \cdot y)^* = y^* \cdot x^*$. From now on we will write simply xy instead of $x \cdot y$ whenever $(x, y) \in \Gamma$. For every $y \in \mathfrak{A}$, the set of left (resp. right) multipliers of y is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$ (resp. $R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$). We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$.

In general, a partial *-algebra is not associative, but in several situations a weaker form of associativity holds. More precisely, we say that $\mathfrak{A}$ is *semi-associative* if $y \in R(x)$ implies $yz \in R(x)$ for every $z \in R\mathfrak{A}$ and

$$(xy)z = x(yz).$$

The partial *-algebra $\mathfrak{A}$ *has a unit* if there exists an element $e \in \mathfrak{A}$ such that $e = e^*$, $e \in R\mathfrak{A} \cap L\mathfrak{A}$ and $xe = ex = x$ for every $x \in \mathfrak{A}$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$, $D(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}$ and the (*weak*) partial multiplication $X_1 \,\Box\, X_2 := X_1^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, whenever $X_2 \mathcal{D} \subset \mathcal{D}(X_1^{\dagger *})$ and $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$.

It is easy to check that $X_1 \in L^{\mathrm{w}}(X_2)$ if and only if there exists $Z \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that

$$\langle X_2 \xi \,|\, X_1^\dagger \eta \rangle = \langle Z \xi \,|\, \eta \rangle, \qquad \forall \xi, \eta \in \mathcal{D}. \tag{1}$$

In this case $Z = X_1 \,\Box\, X_2$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is neither associative nor semi-associative. If I denotes the identity operator of $\mathcal{H}$, then $I_{\mathcal{D}} := I \upharpoonright \mathcal{D}$ is the unit of the partial *-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

If $\mathfrak{N} \subseteq \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, we denote by $R^{\mathrm{w}}\mathfrak{N}$ the set of right multipliers of all elements of $\mathfrak{N}$. We recall that

$$\begin{aligned} R\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) &= R^{\mathrm{w}}\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \\ &= \{A \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : A \text{ bounded and } A : \mathcal{D} \to \mathcal{D}^*\}, \end{aligned} \tag{2}$$

where

$$\mathcal{D}^* = \bigcap_{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})} D(X^{\dagger *}).$$

We denote by $\mathcal{L}_b^\dagger(\mathcal{D}, \mathcal{H})$ the bounded part of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, i.e., $\mathcal{L}_b^\dagger(\mathcal{D}, \mathcal{H}) = \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : X \text{ is a bounded operator}\} = \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : \overline{X} \in \mathcal{B}(\mathcal{H})\}$.

A †-invariant subspace $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is called a *(weak) partial O*-algebra* if $X \,\Box\, Y \in \mathfrak{M}$ for every $X, Y \in \mathfrak{M}$ such that $X \in L^{\mathrm{w}}(Y)$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is the maximal partial O*-algebra on $\mathcal{D}$.

If $X, Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, their *strong* product $X \circ Y$ is defined by $(X \circ Y)\xi = \overline{X} Y \xi$, for every $\xi \in \mathcal{D}$, whenever $Y : \mathcal{D} \to D(\overline{X})$ and $X^\dagger : \mathcal{D} \to D(\overline{Y^\dagger})$.

The set $\mathcal{L}^\dagger(\mathcal{D}) := \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : X, X^\dagger : \mathcal{D} \to \mathcal{D}\}$ is a *-algebra; more precisely, it is the maximal O*-algebra on $\mathcal{D}$ (for the theory of O*-algebras and their representations we refer to [8]).

We will need the following topologies on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$:

- The *strong topology* t_s on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, defined by the seminorms
$$p_\xi(X) = \|X\xi\|, \qquad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi \in \mathcal{D}.$$
- The *strong* topology* t_{s^*} on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, defined by the seminorms
$$p^*_\xi(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\}, \qquad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi \in \mathcal{D}.$$

A **-representation* of a partial *-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$ is a linear map $\pi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}(\pi),\mathcal{H})$ such that: (i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathfrak{A}$; (ii) $x \in L(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x) \,\Box\, \pi(y) = \pi(xy)$. The subspace $\mathcal{D}(\pi)$ is called the *domain* of π. The *-representation π is said to be *bounded* if $\overline{\pi(x)} \in \mathcal{B}(\mathcal{H})$ for every $x \in \mathfrak{A}$.

Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$, where $D(\varphi)$ is a subspace of $\mathfrak{A}$. Then

$$\varphi(x,y) = \overline{\varphi(y,x)}, \qquad \forall x,y \in D(\varphi), \tag{3}$$

$$|\varphi(x,y)|^2 \le \varphi(x,x)\varphi(y,y), \quad \forall x,y \in D(\varphi). \tag{4}$$

We put
$$N_\varphi = \{x \in D(\varphi) : \varphi(x,x) = 0\}.$$
We have
$$N_\varphi = \{x \in D(\varphi) : \varphi(x,y) = 0,\ \forall y \in D(\varphi)\},$$
and so N_φ is a subspace of $D(\varphi)$ and the quotient space $D(\varphi)/N_\varphi := \{\lambda_\varphi(x) \equiv x + N_\varphi : x \in D(\varphi)\}$ is a pre-Hilbert space with respect to the inner product
$$\langle \lambda_\varphi(x) \,|\, \lambda_\varphi(y)\rangle = \varphi(x,y), \qquad x,y \in D(\varphi).$$
We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by completion of $D(\varphi)/N_\varphi$.

A positive sesquilinear form φ on $\mathfrak{A} \times \mathfrak{A}$ is said to be *invariant*, and called an *ips-form*, if there exists a subspace $B(\varphi)$ of $\mathfrak{A}$ (called a *core* for φ) with the properties

(ips_1) $B(\varphi) \subset R\mathfrak{A}$;
(ips_2) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
(ips_3) $\varphi(xa,b) = \varphi(a,x^*b)$ for all $x \in \mathfrak{A}$ and $a,b \in B(\varphi)$;
(ips_4) $\varphi(x^*a,yb) = \varphi(a,(xy)b)$ for all $x \in L(y)$ and $a,b \in B(\varphi)$.

In other words, an ips-form is an *everywhere defined* biweight, in the sense of [1].

To every ips-form φ on $\mathfrak{A}$, with core $B(\varphi)$, there corresponds a triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$, where $\mathcal{H}_\varphi$ is a Hilbert space, λ_φ is a linear map from $B(\varphi)$ into $\mathcal{H}_\varphi$ and π_φ is a *-representation of $\mathfrak{A}$ in the Hilbert space $\mathcal{H}_\varphi$. We refer to [1] for more details on this celebrated GNS construction.

Let $\mathfrak{A}$ be a partial *-algebra and π a *-representation of $\mathfrak{A}$ in $\mathcal{D}(\pi)$. For $\xi \in \mathcal{D}(\pi)$ we put

$$\varphi_\pi^\xi(x, y) := \langle \pi(x)\xi \,|\, \pi(y)\xi \rangle, \qquad x, y \in \mathfrak{A}. \tag{5}$$

Then φ_π^ξ is a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$.

Let $\mathfrak{B} \subseteq R\mathfrak{A}$ and assume that $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$. Then it is easily seen that φ_π^ξ satisfies conditions (ips_3) and (ips_4) above. However, φ_π^ξ is not necessarily an ips-form since $\pi(\mathfrak{B})\xi$ may fail to be dense in $\mathcal{H}$. For this reason, the following notion of *regular* *-representation was introduced in [11].

Definition 2.1. A *-representation π of $\mathfrak{A}$ with domain $\mathcal{D}(\pi)$ is called $\mathfrak{B}$-*regular* if φ_π^ξ is an ips-form with core $\mathfrak{B}$ for every $\xi \in \mathcal{D}(\pi)$.

Remark 2.2. The notion of regular *-representation was given in [2] for a larger class of positive sesquilinear forms (biweights) referring to the *natural* core

$$B(\varphi_\pi^\xi) = \{a \in R\mathfrak{A} : \pi(a) \in \mathcal{D}^{**}(\pi)\}$$

(we refer to [1] for precise definitions). If $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$, then $\mathfrak{B}$-regularity implies that φ_π^ξ is an also ips-form with core $B(\varphi_\pi^\xi)$. We will come back to this point in Proposition 2.8.

Let $\mathfrak{A}$ be a partial *-algebra. We assume that $\mathfrak{A}$ is a locally convex Hausdorff vector space under the topology τ defined by a (directed) set $\{p_\alpha\}_{\alpha \in \mathcal{I}}$ of seminorms. Assume that (1)

- (cl) for every $x \in \mathfrak{A}$, the linear map $\mathsf{L}_x : R(x) \mapsto \mathfrak{A}$ with $\mathsf{L}_x(y) = xy$, $y \in R(x)$, is closed with respect to τ, in the sense that, if $\{y_\alpha\} \subset R(x)$ is a net such that $y_\alpha \to y$ and $xy_\alpha \to z \in \mathfrak{A}$, then $y \in R(x)$ and $z = xy$.

For short, we will say that, in this case, $\mathfrak{A}$ is a *topological partial* **-algebra*. If the involution $x \mapsto x^*$ is continuous, we say that $\mathfrak{A}$ is a **-topological* partial *-algebra.

Starting from the family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{I}}$, we can define a second topology τ^* on $\mathfrak{A}$ by introducing the set of seminorms $\{p_\alpha^*(x)\}$, where

$$p_\alpha^*(x) = \max\{p_\alpha(x), p_\alpha(x^*)\}, \qquad x \in \mathfrak{A}.$$

(1) Condition (cl) was called $(\mathsf{t1})$ in [3].

The involution $x \mapsto x^*$ is automatically τ^*-continuous. By (cl) it follows that, for every $x \in \mathfrak{A}$, both maps $\mathsf{L}_x, \mathsf{R}_x$ are τ^*-closed. Hence, $\mathfrak{A}[\tau^*]$ is a *-topological partial *-algebra.

In this paper we consider the following particular classes of topological partial *-algebras.

Definition 2.3. Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with locally convex topology τ.

- A subspace $\mathfrak{B}$ of $R\mathfrak{A}$ is called a *multiplication core* if

  (d_1) $e \in \mathfrak{B}$ if $\mathfrak{A}$ has a unit e;
  (d_2) $\mathfrak{B} \cdot \mathfrak{B} \subseteq \mathfrak{B}$;
  (d_3) $\mathfrak{B}$ is τ^*-dense in $\mathfrak{A}$;
  (d_4) for every $b \in \mathfrak{B}$, the map $x \mapsto xb$, $x \in \mathfrak{A}$, is τ-continuous;
  (d_5) one has $b^*(xc) = (b^*x)c$ for all $x \in \mathfrak{A}$ and $b, c \in \mathfrak{B}$.

- $\mathfrak{A}[\tau]$ is called $\mathfrak{A}_0$-*regular* if it possesses (2) a multiplication core $\mathfrak{A}_0$ which is a *-algebra and, for every $b \in \mathfrak{A}_0$, the map $x \mapsto bx$, $x \in \mathfrak{A}$, is τ-continuous ([4, Def. 4.1]).
- If $\mathfrak{A}$ is $\mathfrak{A}_0$-regular and if, in addition, the involution $x \mapsto x^*$ is τ-continuous for all $x \in \mathfrak{A}$, then the couple $(\mathfrak{A}, \mathfrak{A}_0)$ is a locally convex *quasi* **-algebra.*

Remark 2.4. A simple limiting argument shows that, if $\mathfrak{B}$ is an algebra (i.e., it is also associative), then $\mathfrak{A}$ is a $\mathfrak{B}$-right module, i.e.,

$$(xa)b = x(ab), \qquad \forall x \in \mathfrak{A},\ a, b \in \mathfrak{B}.$$

If $\mathfrak{A}$ is $\mathfrak{A}_0$-regular then, in a similar way,

$$(xa)b = x(ab), \qquad (ax)b = a(xb), \qquad \forall x \in \mathfrak{A},\ a, b \in \mathfrak{A}_0.$$

Remark 2.5. We warn the reader that an $\mathfrak{A}_0$-regular topological partial *-algebra $\mathfrak{A}[\tau]$ is not necessarily a locally convex partial *-algebra in the sense of [1, Def. 2.1.8]. Neither need it be topologically regular in the sense of [4, Def. 2.1.8], which is a more restrictive notion.

Remark 2.6. Let $\mathfrak{A}[\tau]$ be an $\mathfrak{A}_0$-regular topological partial *-algebra. Then, for every $b \in \mathfrak{A}_0$, the maps $x \mapsto xb$ and $x \mapsto bx$, $x \in \mathfrak{A}$, are also τ^*-continuous. However, the density of $\mathfrak{A}_0$ in $\mathfrak{A}[\tau^*]$ may fail. Thus $\mathfrak{A}[\tau^*]$ need not be an $\mathfrak{A}_0$-regular *-topological partial *-algebra.

Examples 2.7. The three notions given in Definition 2.3 are really different.

1. Take $\mathfrak{A} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then $R\mathfrak{A}$ is given in equation (2), so that we have an example where $R\mathfrak{A} \cdot R\mathfrak{A} \not\subset R\mathfrak{A}$.

(2) In [4] it was only supposed that $\mathfrak{A}_0$ is τ-dense in $\mathfrak{A}$.

2. Take again $\mathfrak{A} = \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathsf{t}_{s^*}]$. Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is $\mathfrak{A}_0$-regular for $\mathfrak{A}_0 = \mathcal{L}_b^\dagger(\mathcal{D})$.

3. Assume $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathsf{t}_{s^*}]$ is self-adjoint, i.e., $\mathcal{D} = \mathcal{D}^*$ (for instance, when $\mathcal{D} = D^\infty(A)$ for a self-adjoint operator A). Then $R\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) = \{X \in \mathcal{L}_b^\dagger(\mathcal{D},\mathcal{H}) : X : \mathcal{D} \to \mathcal{D}\}$ is an algebra, but it is not *-invariant. Hence it is a multiplication core, since it is t_{s^*}-dense in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, but $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is not $R\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$-regular.

The case of a locally convex quasi *-algebra $(\mathfrak{A},\mathfrak{A}_0)$ was studied in [7] and a number of interesting properties have been derived. Some of these extend to the general case of a partial *-algebra, as we shall see later.

PROPOSITION 2.8. *Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra and $\mathfrak{B}$ a multiplication core. Then every (τ,t_s)-continuous *-representation of $\mathfrak{A}$ is $\mathfrak{B}$-regular.*

Proof. First we may assume that $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$. Indeed, put

$$D(\pi_1) := \Big\{\xi_0 + \sum_{i=1}^n \pi(b_i)\xi_i : b_i \in \mathfrak{B},\ \xi_i \in \mathcal{D}(\pi);\ i = 0,1,\dots,n\Big\},$$

$$\pi_1(x)\Big(\xi_0 + \sum_{i=1}^n \pi(b_i)\xi_i\Big) := \pi(x)\xi_0 + \sum_{i=1}^n (\pi(x)\,\Box\,\pi(b_i))\xi_i.$$

Then, exactly as in [4], we can prove that π_1 is a *-representation of $\mathfrak{A}$ with $\pi_1(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi_1))$.

If π is (τ,t_s)-continuous, then π_1 is (τ,t_s)-continuous too (recall that the domains are different!). Indeed, if $x_\alpha \xrightarrow{\tau} x$, then $\pi(x_\alpha)\xi \to \pi(x)\xi$ for every $\xi \in \mathcal{D}(\pi)$. The continuity of the right multiplication then implies that $x_\alpha b \xrightarrow{\tau} xb$ for every $b \in \mathfrak{B}$. Thus, by the continuity of π, for every $b \in \mathfrak{B}$ we get $\pi(x_\alpha b)\xi \to \pi(xb)\xi$ for every $\xi \in \mathcal{D}(\pi)$ or, equivalently, $(\pi(x_\alpha)\Box\pi(b))\xi \to (\pi(x)\,\Box\,\pi(b))\xi$ for every $\xi \in \mathcal{D}(\pi)$. Hence

$$\begin{aligned}\pi_1(x_\alpha)\Big(\sum_{i=1}^n \pi(b_i)\xi_i\Big) &= \sum_{i=1}^n (\pi(x_\alpha)\,\Box\,\pi(b_i))\xi_i \\ &\to \sum_{i=1}^n (\pi(x)\,\Box\,\pi(b_i))\xi_i = \pi_1(x)\Big(\sum_{i=1}^n \pi(b_i)\xi_i\Big).\end{aligned}$$

Thus, every (τ,t_s)-continuous *-representation π extends to a (τ,t_s)-continuous *-representation π_1 with $\pi_1(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi_1))$. Finally we prove the $\mathfrak{B}$-regularity of π. If $x \in \mathfrak{A}$ then there exists a net $\{b_\alpha\} \subset \mathfrak{B}$ such that $b_\alpha \xrightarrow{\tau} x$. We then have

$$\|\lambda_{\varphi_\pi^\xi}(x) - \lambda_{\varphi_\pi^\xi}(b_\alpha)\|^2 = \varphi_\pi^\xi(x - b_\alpha, x - b_\alpha) \le p(x - b_\alpha)^2 \to 0,$$

where p is some τ-continuous seminorm. This implies that $\lambda_{\varphi_\pi^\xi}(\mathfrak{B})$ is dense in $\mathcal{H}_{\varphi_\pi^\xi}$. Hence φ_π^ξ is an ips-form with core $\mathfrak{B}$. ∎

Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and φ a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ for which conditions (ips_1), (ips_3) and (ips_4) are satisfied (with respect to $\mathfrak{B}$). Suppose that φ is τ-continuous, i.e., there exist p_α and $\gamma > 0$ such that

$$|\varphi(x,y)| \leq \gamma p_\alpha(x)p_\alpha(y), \quad \forall x,y \in \mathfrak{A}.$$

Then (ips_2) is also satisfied, and therefore $\mathfrak{B}$ is a core for φ, so that φ is an ips-form. We denote by $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ the set of *all* τ-continuous ips-forms with core $\mathfrak{B}$.

Using the continuity of multiplication and Remark 2.4, it is easily seen that if $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and $a \in \mathfrak{B}$, then $\varphi_a \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$, where

$$\varphi_a(x,y) := \varphi(xa, ya), \quad x,y \in \mathfrak{A}.$$

3. Topological partial *-algebras with sufficiently many *-representations. Throughout this paper we are mostly concerned with topological partial *-algebras possessing sufficiently many continuous *-representations. In the case of topological *-algebras this situation can be studied by introducing the so-called (topological) *-radical of the algebra. Thus we extend this notion to topological partial *-algebras.

Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra. We define the **-radical* of $\mathfrak{A}$ as

$$\mathcal{R}^*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0 \text{ for all } (\tau, \mathsf{t}_s)\text{-continuous } ^*\text{-representations } \pi\}.$$

We put $\mathcal{R}^*(\mathfrak{A}) = \mathfrak{A}$ if $\mathfrak{A}[\tau]$ has no (τ, t_s)-continuous *-representations.

REMARK 3.1. The *-radical was defined in [4, Sec. 5] as

$$\mathcal{R}^*_*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0 \text{ for all } (\tau, \mathsf{t}_{s^*})\text{-continuous } ^*\text{-representations } \pi\}.$$

However, the two definitions are equivalent. Indeed, since every (τ, t_{s^*})-continuous *-representation is (τ, t_s)-continuous, we have $\mathcal{R}^*_*(\mathfrak{A}) \subset \mathcal{R}^*(\mathfrak{A})$. In order to prove that $\mathcal{R}^*(\mathfrak{A}) \subset \mathcal{R}^*_*(\mathfrak{A})$, assume that $x \notin \mathcal{R}^*_*(\mathfrak{A})$, i.e., there is a (τ, t_{s^*})-continuous *-representation π such that $\pi(x) \neq 0$. But π is also (τ, t_s)-continuous, hence $x \notin \mathcal{R}^*(\mathfrak{A})$ as well.

The *-radical enjoys the following immediate properties:

(1) If $x \in \mathcal{R}^*(\mathfrak{A})$, then $x^* \in \mathcal{R}^*(\mathfrak{A})$.
(2) If $x \in \mathfrak{A}$, $y \in \mathcal{R}^*(\mathfrak{A})$ and $x \in L(y)$, then $xy \in \mathcal{R}^*(\mathfrak{A})$.

From now on, we denote by $\mathrm{Rep}_c(\mathfrak{A})$ the set of all (τ, t_s)-continuous *-representations of $\mathfrak{A}$. It $\mathfrak{A}$ has a multiplication core $\mathfrak{B}$, we may always suppose that $\pi(x) \in \mathcal{L}^\dagger(\mathcal{D}(\pi))$ for every $x \in \mathfrak{B}$, as results from the proof of Proposition 2.8.

Proposition 3.2. *Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with unit e. Let $\mathfrak{B}$ be a multiplication core. For an element $x \in \mathfrak{A}$ the following statements are equivalent:*

(i) $x \in \mathcal{R}^*(\mathfrak{A})$.

(ii) $\varphi(x, x) = 0$ *for every* $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$.

Proof. (i)⇒(ii): Let $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and π_φ be the corresponding GNS representation. Then, for every $x \in \mathfrak{A}$,

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) = \varphi_a(x, x) \le p(x)^2, \quad a \in \mathfrak{B},$$

for some continuous τ-seminorm p (depending on a). Hence π_φ is (τ, t_s)-continuous. If $x \in \mathcal{R}^*(\mathfrak{A})$, then $\pi_\varphi(x) = 0$. Thus $\varphi(xa, xa) = 0$ for every $a \in \mathfrak{B}$. From $e \in \mathfrak{B}$, we get the assertion.

(ii)⇒(i): Let $\pi \in \mathrm{Rep}_c(\mathfrak{A})$. We assume $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$. For $x, y \in \mathfrak{A}$ and $\xi \in \mathcal{D}$, put, as before,

$$\varphi_\pi^\xi(x, y) := \langle \pi(x)\xi \,|\, \pi(y)\xi \rangle, \quad x, y \in \mathfrak{A}.$$

Then

$$|\varphi_\pi^\xi(x, y)| = |\langle \pi(x)\xi \,|\, \pi(y)\xi\rangle| \le \|\pi(x)\xi\| \, \|\pi(y)\xi\| \le p(x)p(y)$$

for some τ-continuous seminorm p. Hence, φ_π^ξ is continuous.

Thus, $\varphi_\pi^\xi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and, by assumption, $\|\pi(x)\xi\|^2 = \varphi_\pi^\xi(x, x) = 0$. The arbitrariness of ξ implies that $\pi(x) = 0$. ∎

As in topological *-algebras, the *-radical contains all elements x whose *square* x^*x (if well-defined) vanishes.

Proposition 3.3. *Let $\mathfrak{A}$ be a topological partial *-algebra. Let $x \in \mathfrak{A}$ with $x^* \in L(x)$. If $x^*x = 0$, then $x \in \mathcal{R}^*(\mathfrak{A})$.*

Proof. If π is a (τ, t_s)-continuous *-representation of $\mathfrak{A}$, $\pi(x^*) \,\square\, \pi(x) = \pi(x)^\dagger \,\square\, \pi(x)$ is well-defined and equals 0. Hence, for every $\xi \in \mathcal{D}(\pi)$,

$$\begin{aligned}\|\pi(x)\xi\|^2 &= \langle \pi(x)\xi \,|\, \pi(x)\xi\rangle = \langle \pi(x)^\dagger \,\square\, \pi(x)\xi \,|\, \xi\rangle = \langle \pi(x^*) \,\square\, \pi(x)\xi \,|\, \xi\rangle \\ &= \langle \pi(x^*x)\xi \,|\, \xi\rangle = 0.\end{aligned}$$

Therefore $\pi(x) = 0$. ∎

Remark 3.4. A sort of converse to the previous result was stated in [4, Proposition 5.3]. Unfortunately, the proof given there contains a gap.

Definition 3.5. A topological partial *-algebra $\mathfrak{A}[\tau]$ is called *-*semisimple* if for every $x \in \mathfrak{A} \setminus \{0\}$ there exists a (τ, t_s)-continuous *-representation π of $\mathfrak{A}$ such that $\pi(x) \neq 0$ or, equivalently, if $\mathcal{R}^*(\mathfrak{A}) = \{0\}$.

By Propositition 3.2, $\mathfrak{A}[\tau]$ is *-semisimple if, and only if, for some multiplication core $\mathfrak{B}$, the family $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ of ips-forms is *sufficient* in the following sense [4].

Definition 3.6. A family $\mathcal{M}$ of continuous ips-forms on $\mathfrak{A} \times \mathfrak{A}$ is *sufficient* if $x \in \mathfrak{A}$ and $\varphi(x,x) = 0$ for every $\varphi \in \mathcal{M}$ imply $x = 0$.

The sufficiency of the family $\mathcal{M}$ can be described in several different ways.

Lemma 3.7. *Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. Then the following statements are equivalent:*

(i) *$\mathcal{M}$ is sufficient.*
(ii) *$\varphi(xa,b) = 0$, for all $\varphi \in \mathcal{M}$ and $a,b \in \mathfrak{B}$, implies $x = 0$.*
(iii) *$\varphi(xa,a) = 0$, for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$, implies $x = 0$.*
(iv) *$\varphi(xa,y) = 0$, for all $\varphi \in \mathcal{M}$, $y \in \mathfrak{A}$ and $a \in \mathfrak{B}$, implies $x = 0$.*
(v) *$\varphi(xa,xa) = 0$, for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$, implies $x = 0$.*

We omit the easy proof.

Of course, if the family $\mathcal{M}$ is sufficient, so is any larger family $\mathcal{M}' \supset \mathcal{M}$. In this case, the maximal sufficient family (having $\mathfrak{B}$ as core) is obviously the set $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ of *all* continuous ips-forms with core $\mathfrak{B}$. Hence if a sufficient family $\mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ exists, $\mathfrak{A}[\tau]$ is *-semisimple.

Example 3.8. As mentioned before, the space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a $\mathcal{L}^\dagger_b(\mathcal{D})$-regular partial *-algebra when endowed with the strong* topology t_{s^*}. The set of positive sesquilinear forms $\mathcal{M} := \{\varphi_\xi : \xi \in \mathcal{D}\}$, where $\varphi_\xi(X,Y) = \langle X\xi \,|\, Y\xi\rangle$, $X,Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, is a sufficient family of ips-forms with core $\mathcal{L}^\dagger_b(\mathcal{D})$. Indeed, if $\varphi_\xi(X,X) = 0$ for every $\xi \in \mathcal{D}$, then $\|X\xi\|^2 = 0$ and therefore $X = 0$.

Example 3.9. As shown in [6], the space $L^p(X)$, $X = [0,1]$, endowed with its usual norm topology, is $L^\infty(X)$-regular and it is *-semisimple if $p \geq 2$. Indeed, in this case the family of all continuous ips-forms is given by $\mathcal{M} = \{\varphi_w : w \in L^{p/(p-2)},\ w \geq 0\}$, where

$$\varphi_w(f,g) = \int_X f(t)\overline{g(t)}w(t)\,dt, \qquad f,g \in L^p(X),$$

and it is sufficient.

If $1 \leq p < 2$, the set of all continuous ips-forms reduces to $\{0\}$. Hence, in this case, $\mathcal{R}^*(L^p(X)) = L^p(X)$.

Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. If $\mathfrak{A}$ possesses a sufficient family $\mathcal{M}$ of ips-forms, an *extension* of the multiplication of $\mathfrak{A}$ can be introduced in a similar way to [4, Sec. 4].

We say that the *weak* product $x \,\square\, y$ is well-defined (with respect to $\mathcal{M}$) if there exists $z \in \mathfrak{A}$ such that

$$\varphi(ya, x^*b) = \varphi(za,b), \qquad \forall a,b \in \mathfrak{B},\ \forall \varphi \in \mathcal{M}.$$

In this case, we put $x \,\square\, y := z$ and the sufficiency of $\mathcal{M}$ guarantees that z is unique. The weak multiplication $\square$ clearly depends on $\mathcal{M}$: the larger $\mathcal{M}$, the stronger the weak multiplication, in the sense that if $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and $x \,\square\, y$ exists with respect to $\mathcal{M}'$, then $x \,\square\, y$ exists with respect to $\mathcal{M}$ too.

A handy criterion for the existence of a weak product is provided by the following

PROPOSITION 3.10. *Let $\mathfrak{B}$ be an algebra. Then the weak product $x \,\square\, y$ is defined (with respect to $\mathcal{M}$) if, and only if, there exists a net $\{b_\alpha\}$ in $\mathfrak{B}$ such that $b_\alpha \xrightarrow{\tau} y$ and $xb_\alpha \xrightarrow{\tau_{\mathrm{w}}^{\mathcal{M}}} z \in \mathfrak{A}$.*

Here $\tau_{\mathrm{w}}^{\mathcal{M}}$ is the weak topology determined by $\mathcal{M}$, with seminorms $x \mapsto |\varphi(xa, b)|$, $\varphi \in \mathcal{M}$, $a, b \in \mathfrak{B}$. It is easy to prove that $\mathfrak{A}$ is also a partial *-algebra with respect to the weak multiplication.

Since the following condition holds in typical examples, e.g. $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ (see [4, Prop. 3.2]), we will often suppose here that it is satisfied:

(wp) xy exists if, and only if, $x \,\square\, y$ exists. In this case $xy = x \,\square\, y$.

In this situation it is possible to define a stronger multiplication on $\mathfrak{A}$: we say that the *strong* product $x \bullet y$ is well-defined (and that $x \in L^s(y)$ or $y \in R^s(x)$) if $x \in L(y)$ and:

(sm$_1$) $\varphi((xy)a, z^*b) = \varphi(ya, (x^*z^*)b)$, $\forall z \in L(x)$, $\forall \varphi \in \mathcal{M}$, $\forall a, b \in \mathfrak{B}$;
(sm$_2$) $\varphi((y^*x^*)a, vb) = \varphi(x^*a, (yv)b)$, $\forall v \in R(y)$, $\forall \varphi \in \mathcal{M}$, $\forall a, b \in \mathfrak{B}$.

The same considerations on the dependence on $\mathcal{M}$ of weak multiplication apply, of course, to strong multiplication.

DEFINITION 3.11. Let $\mathfrak{A}$ be a partial *-algebra. A *-representation π of $\mathfrak{A}$ is called *quasi-symmetric* if, for every $x \in \mathfrak{A}$,

$$\bigcap_{z \in L(x)} D((\pi(x)^* \lceil \pi(z^*)\mathcal{D}(\pi))^*) = D(\overline{\pi(x)}),$$

$$\bigcap_{v \in R(x)} D((\pi(x^*)^* \lceil \pi(v)\mathcal{D}(\pi))^*) = D(\overline{\pi(x)^\dagger}).$$

Of course, the same definition can be given for any partial O*-algebra $\mathfrak{M}$ (by considering the identical *-representation).

REMARK 3.12. The conditions given in the previous definition are certainly satisfied if, for every $x \in \mathfrak{A}$, there exist $s \in L(x)$, $t \in R(x)$ such that $(\pi(x)^* \lceil \pi(s^*)\mathcal{D}(\pi))^* = \overline{\pi(x)}$ and $(\pi(x^*)^* \lceil \pi(t)\mathcal{D}(\pi))^* = \overline{\pi(x^*)}$. These stronger conditions are satisfied, in particular, by any symmetric O*-algebra $\mathfrak{M}$ (*symmetric* means that $(I + X^*\overline{X})^{-1}$ is in the bounded part of $\mathfrak{M}$, for every $X \in \mathfrak{M}$). The proof given in [4, Theorem 3.5] shows that in this case $\mathfrak{M}$ is quasi-symmetric, whence the name. In particular, $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is quasi-symmetric.

PROPOSITION 3.13. *Let $\mathfrak{A}$ be a *-semisimple topological partial *-algebra with multiplication core $\mathfrak{B}$ containing the unit e of $\mathfrak{A}$. If the strong product $x \bullet y$ of $x, y \in \mathfrak{A}$ is well-defined with respect to $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$, then for every quasi-symmetric $\pi \in \mathrm{Rep}_c(\mathfrak{A})$ with $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ and $\pi(e) = I_{\mathcal{D}(\pi)}$, the strong product $\pi(x) \circ \pi(y)$ is well-defined too and*

$$\pi(x \bullet y) = \pi(x) \circ \pi(y).$$

Proof. Let $\pi \in \mathrm{Rep}_c(\mathfrak{A})$ satisfy the required assumptions. Let $\xi \in \mathcal{D}(\pi)$ and define φ_π^ξ as in the proof of Proposition 3.2. Then $\varphi_\pi^\xi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and since $x \bullet y$ is well-defined (with respect to $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$), we have

$$\varphi_\pi^\xi((xy)a, z^*b) = \varphi_\pi^\xi(ya, (x^*z^*)b), \quad \forall z \in L(x),\ \forall a, b \in \mathfrak{B},$$
$$\varphi_\pi^\xi((y^*x^*)a, vb) = \varphi_\pi^\xi(x^*a, (yv)b), \quad \forall v \in R(y),\ \forall a, b \in \mathfrak{B}.$$

Equivalently,

$$\langle(\pi(x) \,\Box\, \pi(y))\pi(a)\xi \,|\, \pi(z^*)\pi(b)\xi\rangle = \langle\pi(y)\pi(a)\xi \,|\, (\pi(x^*) \,\Box\, \pi(z^*))\pi(b)\xi\rangle, \quad \forall z \in L(x),\ \forall a, b \in \mathfrak{B},$$
$$\langle(\pi(y^*) \,\Box\, \pi(x^*))\pi(a)\xi \,|\, \pi(v)\pi(b)\xi\rangle = \langle\pi(x^*)\pi(a)\xi \,|\, (\pi(y) \,\Box\, \pi(v))\pi(b)\xi\rangle, \quad \forall v \in R(y),\ \forall a, b \in \mathfrak{B}.$$

By taking $a = b = e$ and using the polarization identity, one gets, for every $\xi, \eta \in \mathcal{D}(\pi)$,

$$\langle(\pi(x) \,\Box\, \pi(y))\xi \,|\, \pi(z^*)\eta\rangle = \langle\pi(y)\xi \,|\, (\pi(x^*) \,\Box\, \pi(z^*))\eta\rangle, \quad \forall z \in L(x),$$
$$\langle(\pi(y^*) \,\Box\, \pi(x^*))\xi \,|\, \pi(v)\eta\rangle = \langle\pi(x^*)\xi \,|\, (\pi(y) \,\Box\, \pi(v))\eta\rangle, \quad \forall v \in R(y).$$

From these relations, it follows that

$$\pi(y) : \mathcal{D}(\pi) \to D((\pi(x)^* \lceil \pi(z^*)\mathcal{D}(\pi))^*, \quad \forall z \in L(x),$$
$$\pi(x^*) : \mathcal{D}(\pi) \to D((\pi(y^*)^* \lceil \pi(v)\mathcal{D}(\pi))^*, \quad \forall v \in R(y).$$

By the assumption, it follows that $\pi(y) : \mathcal{D}(\pi) \to D(\overline{\pi(x)})$ and $\pi(x^*) : \mathcal{D}(\pi) \to D(\overline{\pi(y^*)})$. Thus, $\pi(x) \circ \pi(y)$ is well-defined. ∎

4. Representable functionals versus ips-forms. So far we have used ips-forms in order to characterize *-semisimplicity of a topological partial *-algebra. The reason is that ips-forms allow a GNS-like construction. However, from a general point of view it is not easy to find conditions for the existence of sufficient families of ips-forms, whereas there exist well-known criteria for the existence of continuous *linear* functionals that separate points of $\mathfrak{A}$. But continuous linear functionals, which are positive in a certain sense, do not give rise in general to a GNS construction. This can be done if they are *representable* [5] in the sense specified below. It is then natural to consider, in more detail, conditions for the representability of continuous positive linear functionals.

DEFINITION 4.1. Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. A continuous linear functional ω on $\mathfrak{A}$ is $\mathfrak{B}$*-positive* if $\omega(a^*a) \geq 0$ for every $a \in \mathfrak{B}$.

The continuity of ω implies that $\omega(x) \geq 0$ for every x which belongs to the τ-closure $\mathfrak{A}^+(\mathfrak{B})$ of the set

$$\mathfrak{B}^{(2)} = \Big\{\sum_{k=1}^{n} x_k^* x_k : x_k \in \mathfrak{B},\, n \in \mathbb{N}\Big\}.$$

In the very same way as in [7, Theorem 3.2] one can prove

THEOREM 4.2. *Assume that* $\mathfrak{A}^+(\mathfrak{B}) \cap (-\mathfrak{A}^+(\mathfrak{B})) = \{0\}$. *Let* $a \in \mathfrak{A}^+(\mathfrak{B})$, $a \neq 0$. *Then there exists a continuous linear functional* ω *on* $\mathfrak{A}$ *with the properties:*

(i) $\omega(x) \geq 0$ *for all* $x \in \mathfrak{A}^+(\mathfrak{B})$;
(ii) $\omega(a) > 0$.

The set $\mathfrak{A}^+(\mathfrak{B})$ will play an important role in Theorem 4.12 and in the analysis of order bounded elements in Section 5.2.

DEFINITION 4.3. Let ω be a linear functional on $\mathfrak{A}$, and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. We say that ω is *representable* (with respect to $\mathfrak{B}$) if the following requirements are satisfied:

(r_1) $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{B}$;
(r_2) $\omega(b^*(x^*a)) = \overline{\omega(a^*(xb))}$ for all $a, b \in \mathfrak{B}$ and $x \in \mathfrak{A}$;
(r_3) for all $x \in \mathfrak{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}$ for all $a \in \mathfrak{B}$.

In this case, one can prove that there exists a triple $(\pi_\omega^{\mathfrak{B}}, \lambda_\omega^{\mathfrak{B}}, \mathcal{H}_\omega^{\mathfrak{B}})$ such that

(a) $\pi_\omega^{\mathfrak{B}}$ is a *-representation of $\mathfrak{A}$ in $\mathcal{H}_\omega$;
(b) $\lambda_\omega^{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathcal{H}_\omega^{\mathfrak{B}}$ with $\lambda_\omega^{\mathfrak{B}}(\mathfrak{B}) = \mathcal{D}(\pi_\omega^{\mathfrak{B}})$ and $\pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) = \lambda_\omega^{\mathfrak{B}}(xa)$ for all $x \in \mathfrak{A}$ and $a \in \mathfrak{B}$.
(c) $\omega(b^*(xa)) = \langle \pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) \,|\, \lambda_\omega^{\mathfrak{B}}(b)\rangle$ for all $x \in \mathfrak{A}$ and $a, b \in \mathfrak{B}$.

In particular, if $\mathfrak{A}$ has a unit e and $e \in \mathfrak{B}$, we have:

(a_1) $\pi_\omega^{\mathfrak{B}}$ is a cyclic *-representation of $\mathfrak{A}$ with cyclic vector ξ_ω;
(b_1) $\lambda_\omega^{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathcal{H}_\omega^{\mathfrak{B}}$ with $\lambda_\omega^{\mathfrak{B}}(\mathfrak{B}) = \mathcal{D}(\pi_\omega^{\mathfrak{B}})$, $\xi_\omega = \lambda_\omega^{\mathfrak{B}}(e)$ and $\pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) = \lambda_\omega^{\mathfrak{B}}(xa)$ for all $x \in \mathfrak{A}$ and $a \in \mathfrak{B}$.
(c_1) $\omega(a) = \langle \pi_\omega^{\mathfrak{B}}(x)\xi_\omega \,|\, \xi_\omega\rangle$ for all $x \in \mathfrak{A}$.

The GNS construction then depends on the subspace $\mathfrak{B}$. We denote by $\mathcal{R}(\mathfrak{A},\mathfrak{B})$ the set of linear functionals on $\mathfrak{A}$ which are representable with respect to the same $\mathfrak{B}$.

Remark 4.4. It is worth recalling (also to fix notation) that the Hilbert space $\mathcal{H}_\omega$ is defined by considering the following subspace of $\mathfrak{B}$:

$$N_\omega = \{x \in \mathfrak{B} : \omega(y^*x) = 0, \forall y \in \mathfrak{B}\}.$$

The quotient $\mathfrak{B}/N_\omega = \{\lambda^0_\omega(x) := x + N_\omega : x \in \mathfrak{B}\}$ is a pre-Hilbert space with inner product

$$\langle \lambda^0_\omega(x) \,|\, \lambda^0_\omega(y)\rangle = \omega(y^*x), \qquad x, y \in \mathfrak{B}.$$

Then $\mathcal{H}_\omega$ is the completion of $\lambda^0_\omega(\mathfrak{B})$. The representability of ω implies that $\lambda^0_\omega : \mathfrak{B} \to \mathcal{H}_\omega$ extends to a linear map $\lambda_\omega : \mathfrak{A} \to \mathcal{H}_\omega$.

Remark 4.5. We notice that if π is a *-representation of $\mathfrak{A}$ on the domain $\mathcal{D}(\pi)$, and $\mathfrak{B}$ is a subspace of $R\mathfrak{A}$ such that $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$, then, for every $\xi \in \mathcal{D}(\pi)$, the linear functional ω^ξ_π defined by $\omega^\xi_\pi(x) = \langle \pi(x)\xi \,|\, \xi\rangle$ is representable, whereas the corresponding sesquilinear form $\varphi^\xi_\pi(x, y) = \langle \pi(x)\xi \,|\, \pi(y)\xi\rangle$ is not necessarily an ips-form; the latter fact leads to the notion of regular representation discussed above.

Example 4.6. A continuous linear functional ω whose restriction to $\mathfrak{B}$ is positive need not be representable. As an example, consider $\mathfrak{A} = L^1(I)$, with I a bounded interval on the real line, and $\mathfrak{B} = L^\infty(I)$. The linear functional

$$\omega(f) = \int_I f(t)\, dt, \qquad f \in L^1(I),$$

is continuous, but it is not representable, since (r_3) fails if $f \in L^1(I) \setminus L^2(I)$.

Since multiplication cores play an important role for topological partial *-algebras, we restrict our attention to the case where $\mathfrak{B}$ is a multiplication core and we omit explicit reference to $\mathfrak{B}$ whenever it appears. We will denote by $\mathcal{R}_c(\mathfrak{A},\mathfrak{B})$ the set of τ-continuous linear functionals that are representable (with respect to $\mathfrak{B}$).

Since representable functionals and ips-forms both define GNS-like representations, it is natural to consider the interplay of these two notions, with particular reference to the topological case. For more details consult also [5, 7].

Proposition 4.7. *Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ which is an algebra. If φ is a τ-continuous ips-form on $\mathfrak{A}$ then, for every $b \in \mathfrak{B}$, the linear functional ω^b_φ defined by*

$$\omega^b_\varphi(x) = \varphi(xb, b), \qquad x \in \mathfrak{A},$$

is representable and the corresponding map $a \in \mathfrak{B} \mapsto \lambda^0_{\omega^b_\varphi}(a) \in \mathcal{H}_{\omega^b_\varphi}$ *is continuous.*

Conversely, assume that $\mathfrak{A}$ *has a unit* $e \in \mathfrak{B}$. *Then, if* ω *is a representable linear functional on* $\mathfrak{A}$ *and the map* $a \in \mathfrak{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$ *is continuous, the positive sesquilinear form* φ_ω *defined on* $\mathfrak{B} \times \mathfrak{B}$ *by*

$$\varphi_\omega(a,b) := \omega(b^*a)$$

is τ*-continuous on* $\mathfrak{B} \times \mathfrak{B}$ *and it extends to a continuous ips-form* $\widetilde{\varphi}_\omega$ *on* $\mathfrak{A}$.

Proof. We only prove the second part of the statement.

For every $a, b \in \mathfrak{B}$ we have

$$\begin{aligned}|\varphi_\omega(a,b)| &= |\omega(b^*a)| = |\langle \pi_\omega(a)\lambda^0_\omega(e) \,|\, \pi_\omega(b)\lambda^0_\omega(e)\rangle| \\ &\le \|\pi_\omega(a)\lambda^0_\omega(e)\| \, \|\pi_\omega(b)\lambda^0_\omega(e)\| = \|\lambda^0_\omega(a)\| \, \|\lambda^0_\omega(b)\| \le p(a)p(b)\end{aligned}$$

for some continuous seminorm p. Hence φ_ω extends uniquely to $\mathfrak{A} \times \mathfrak{A}$. Let $\widetilde{\varphi}_\omega$ denote this extension. It is easily seen that $\widetilde{\varphi}_\omega$ is a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ and

$$|\widetilde{\varphi}_\omega(x,y)| \le p(x)p(y), \qquad \forall x, y \in \mathfrak{A}.$$

Hence the map $x \mapsto \lambda_{\widetilde{\varphi}_\omega}(x) \in \mathcal{H}_{\widetilde{\varphi}_\omega}$ is also continuous, since

$$\|\lambda_{\widetilde{\varphi}_\omega}(x)\|^2 = \widetilde{\varphi}_\omega(x,x) \le p(x)^2, \qquad \forall x \in \mathfrak{A}.$$

Thus, if $x = \tau\text{-}\lim_\alpha b_\alpha$, $b_\alpha \in \mathfrak{B}$, we get

$$\|\lambda_{\widetilde{\varphi}_\omega}(x) - \lambda_{\widetilde{\varphi}_\omega}(b_\alpha)\|^2 = \widetilde{\varphi}_\omega(x - b_\alpha, x - b_\alpha) \le p(x - b_\alpha)^2 \to 0.$$

Conditions (ips_3) and (ips_4) are readily checked. Concerning (ips_4), for instance, let $x \in L(y)$ and $a, b \in \mathfrak{B}$. Then

$$\begin{aligned}\widetilde{\varphi}_\omega(a,(xy)b) &= \omega(((xy)b)^*a) = \omega(b^*(xy)^*a) = \langle \pi_\omega(xy)^\dagger \lambda^0_\omega(a) \,|\, \lambda^0_\omega(b)\rangle \\ &= \langle (\pi_\omega(y)^\dagger \,\Box\, \pi_\omega(x)^\dagger)\lambda^0_\omega(a) \,|\, \lambda^0_\omega(b)\rangle = \langle \pi_\omega(x)^\dagger \lambda^0_\omega(a) \,|\, \pi_\omega(y)\lambda^0_\omega(b)\rangle \\ &= \widetilde{\varphi}_\omega(x^*a, yb). \ \blacksquare\end{aligned}$$

Remark 4.8. If ω is a representable linear functional on $\mathfrak{A}$ and the map $a \in \mathfrak{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$ is continuous, then ω is continuous. The converse is false in general.

However, the continuity of ω implies the τ^*-closability of the map λ^0_ω : $a \in \mathfrak{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$, as the next proposition shows.

Proposition 4.9. *Let* ω *be continuous and* $\mathfrak{B}$*-positive. Then the map* $\lambda^0_\omega : a \in \mathfrak{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$ *is* τ^**-closable.*

Proof. Let $a_\delta \xrightarrow{\tau^*} 0$, $a_\delta \in \mathfrak{B}$, and suppose that the net $\{\lambda^0_\omega(a_\delta)\}$ is Cauchy in $\mathcal{H}_\omega$. Hence it converges to some $\xi \in \mathcal{H}_\omega$ and

$$\langle \lambda^0_\omega(b) \,|\, \lambda^0_\omega(a_\delta)\rangle \to \langle \lambda^0_\omega(b) \,|\, \xi\rangle, \qquad \forall b \in \mathfrak{B}.$$

Moreover,

$$\langle \lambda^0_\omega(b) \,|\, \lambda^0_\omega(a_\delta)\rangle = \omega(a_\delta^* b) \to 0, \qquad \forall b \in \mathfrak{B},$$

since $a_\delta^* \xrightarrow{\tau} 0$ and since the right multiplication by $b \in \mathfrak{B}$ and by ω are both τ-continuous. Thus $\langle \lambda^0_\omega(b) \,|\, \xi\rangle = 0$ for every $b \in \mathfrak{B}$. This implies that $\xi = 0$, and therefore $\lambda^0_\omega(a_\delta) \to 0$. ∎

Actually, it is easy to see that the closability of the map λ^0_ω is equivalent to the closability of φ_ω. Indeed, closability of $a \in \mathfrak{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$ means that if $a_\delta \xrightarrow{\tau^*} 0$ and $\{\lambda^0_\omega(a_\delta)\}$ is a Cauchy net, then $\lambda^0_\omega(a_\delta) \to 0$. But $\{\lambda^0_\omega(a_\delta)\}$ is a Cauchy net if and only if $\varphi_\omega(a_\delta - a_\gamma, a_\delta - a_\gamma) \to 0$. This leads to the conclusion $\|\lambda^0_\omega(a_\delta)\|^2 = \varphi_\omega(a_\delta, a_\delta) \to 0$.

Therefore, Proposition 4.9 generalizes [7, Prop. 2.7], which says that, for a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, the sesquilinear form φ_ω is closable if $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.

Thus, if ω is continuous and $\mathfrak{B}$-positive, the map λ^0_ω has a closure $\overline{\lambda^0_\omega}$ defined on

$$D(\overline{\lambda^0_\omega}) = \{x \in \mathfrak{A} : \exists \{a_\delta\} \subset \mathfrak{B},\ a_\delta \xrightarrow{\tau^*} x,\ \{\lambda^0_\omega(a_\delta)\} \text{ is a Cauchy net}\}.$$

From the discussion above, it follows that $D(\overline{\lambda^0_\omega})$ coincides with the domain $D(\overline{\varphi_\omega})$ of the closure of φ_ω.

For the case of a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, the following assumption was made in [7]:

(fr) $\bigcap_{\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})} D(\overline{\varphi_\omega}) = \mathfrak{A}$.

Quasi *-algebras satisfying (fr) are called *fully representable* (hence the acronym). Some concrete examples have been described in [7] and several interesting structure properties have been derived. We maintain the same definition and the same name in the case of topological partial *-algebras and, in complete analogy, we say that a topological partial *-algebra $\mathfrak{A}[\tau]$ with multiplication core $\mathfrak{B}$ is *fully representable* if

(fr) $D(\overline{\varphi_\omega}) = \mathfrak{A}$ for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$.

We have the following generalization of Proposition 3.6 of [7].

PROPOSITION 4.10. *Let $\mathfrak{A}$ be a semi-associative *-topological partial *-algebra with multiplication core $\mathfrak{B}$. Assume that $\mathfrak{A}$ is fully representable and let $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$. Then $\overline{\varphi_\omega}$ is an ips-form on $\mathfrak{A}$ with core $\mathfrak{B}$, with the property*

$$\overline{\varphi_\omega}(xa, b) = \omega(b^* x a), \qquad \forall x \in \mathfrak{A},\ a, b \in \mathfrak{B}.$$

Proof. The continuity of the involution implies that, for every $a \in \mathfrak{B}$, the map $x \mapsto a^* x$ is continuous on $\mathfrak{A}$. Hence the linear functional ω_a defined by $\omega(a^* x a)$ is continuous. We now prove that ω_a is representable; for this we need to check properties (r_1)–(r_3). We have

$$\omega_a(b^*b) = \omega(a^*(b^*b)a) = \omega((a^*b^*)(ba)) \geq 0, \quad \forall b \in \mathfrak{B},$$

i.e., (r_1) holds. Furthermore, for every $b, c \in \mathfrak{B}$, we have

$$\begin{aligned}\omega_a(c^*(xb)) &= \omega(a^*(c^*(xb))a) = \omega(a^*((c^*x)b))a) \\ &= \overline{\omega(a^*(b^*(x^*c))a)} = \overline{\omega_a(b^*(x^*c))}.\end{aligned}$$

As for (r_3), for every $x \in \mathfrak{A}$ and $b \in \mathfrak{B}$, we have

$$\begin{aligned}|\omega_a(x^*b)| &= |\omega(a^*(x^*b)a)| = |\omega((a^*x^*)(ba))| \\ &\leq \gamma_{x,a}\omega(a^*(b^*b)a)^{1/2} = \gamma_{x,a}\omega_a(b^*b)^{1/2}.\end{aligned}$$

Thus $\omega_a \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ for every $a \in \mathfrak{B}$. By Proposition 4.9, $\overline{\varphi_{\omega_a}}$ is well-defined and, by the assumption, $D(\overline{\varphi_{\omega_a}}) = \mathfrak{A}$. Hence, if $x \in \mathfrak{A}$, there exists a net $\{x_\alpha\} \subset \mathfrak{B}$ such that $x_\alpha \xrightarrow{\tau^*} x$ and $\varphi_{\omega_a}(x_\alpha - x_\beta, x_\alpha - x_\beta) \to 0$ or, equivalently, $\varphi_\omega((x_\alpha - x_\beta)a, (x_\alpha - x_\beta)a) \to 0$. Hence, by the definition of closure, for all $b \in \mathfrak{A}$,

$$\overline{\varphi_\omega}(xa, b) = \lim_\alpha \varphi_\omega(x_\alpha a, b) = \lim_\alpha \omega(b^*(x_\alpha a)) = \omega(b^*(xa)),$$

by the continuity of ω. This easily implies that $\overline{\varphi_\omega}(xa, b) = \overline{\varphi_\omega}(a, x^*b)$ for all $x \in \mathfrak{A}$ and $a, b \in \mathfrak{B}$, so that $\overline{\varphi_\omega}$ satisfies (ips_3).

Let now $x \in L(y)$ and $a, b \in \mathfrak{B}$. Let $\{x_\beta^*\}$ and $\{y_\alpha\}$ be nets in $\mathfrak{B}$, τ^*-converging, respectively, to x^* and y and such that $\varphi_\omega((x_\beta^* - x_{\beta'}^*)b, (x_\beta^* - x_{\beta'}^*)b) \to 0$ and $\varphi_\omega((y_\alpha - y'_\alpha)a, (y_\alpha - y'_\alpha)a) \to 0$. Then we get

$$\begin{aligned}\overline{\varphi_\omega}((xy)a, b) &= \omega(b^*(xy)a) = \langle \pi_\omega(xy)\lambda_\omega^0(a) \mid \lambda_\omega^0(b)\rangle \\ &= \langle \pi_\omega(x) \,\square\, \pi_\omega(y)\lambda_\omega^0(a) \mid \lambda_\omega^0(b)\rangle = \langle \pi_\omega(y)\lambda_\omega^0(a) \mid \pi_\omega(x^*)\lambda_\omega^0(b)\rangle \\ &= \lim_{\alpha,\beta} \langle \pi_\omega(y_\alpha)\lambda_\omega^0(a) \mid \pi_\omega(x_\beta^*)\lambda_\omega^0(b)\rangle \\ &= \lim_{\alpha,\beta} \varphi_\omega(y_\alpha a, x_\beta^* b) = \overline{\varphi_\omega}(ya, x^*b).\end{aligned}$$

Thus, (ips_4) holds. To complete the proof, we need to show that $\lambda_{\overline{\varphi_\omega}}(\mathfrak{B})$ is dense in the Hilbert space $\mathcal{H}_{\overline{\varphi_\omega}}$. This part of the reasoning is completely analogous to that given in [7, Proposition 3.6] and we omit it. ∎

If $\mathfrak{A}$ is semi-associative and fully representable, every continuous representable linear functional ω comes from a closed ips-form $\overline{\varphi_\omega}$, but $\overline{\varphi_\omega}$ need not be continuous, in general, unless more assumptions are made on the topology τ.

COROLLARY 4.11. *Let $\mathfrak{A}[\tau]$ be a fully representable semi-associative *-topological partial *-algebra with multiplication core $\mathfrak{B}$. Assume that $\mathfrak{A}[\tau]$ is a Fréchet space. Then, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$, $\overline{\varphi_\omega}$ is a continuous ips-form.*

Proof. The map $\overline{\lambda_\omega^0}$ is closed and everywhere defined. The closed graph theorem then implies that $\overline{\lambda_\omega^0}$ is continuous. The statement follows from Proposition 4.7. ∎

Summarizing, we have

THEOREM 4.12. *Let $\mathfrak{A}[\tau]$ be a fully representable *-topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Assume that $\mathfrak{A}[\tau]$ is a Fréchet space and the following conditions hold:*

- (rc) *Every linear functional ω which is continuous and $\mathfrak{B}$-positive is representable;*
- (sq) *for every $x \in \mathfrak{A}$, there exists a sequence $\{b_n\} \subset \mathfrak{B}$ such that $b_n \xrightarrow{\tau} x$ and the sequence $\{b_n^* b_n\}$ is increasing, in the sense of the order of $\mathfrak{A}^+(\mathfrak{B})$.*

Then $\mathfrak{A}$ is *-semisimple.

Proof. Assume, on the contrary, that there exists $x \in \mathfrak{A} \setminus \{0\}$ such that $\varphi(x, x) = 0$ for every $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$. If ω is continuous and $\mathfrak{B}$-positive, then by assumption it is representable; thus $\overline{\varphi_\omega}$, which is everywhere defined on $\mathfrak{A} \times \mathfrak{A}$, is continuous, by Corollary 4.11. Let $x = \lim_{n\to\infty} b_n$, with $b_n \in \mathfrak{B}$ and $\{b_n^* b_n\}$ increasing. Then

$$0 \le \lim_{n\to\infty} \omega(b_n^* b_n) = \lim_{n\to\infty} \varphi_\omega(b_n, b_n) = \overline{\varphi_\omega}(x, x) = 0.$$

Hence $\omega(b_n^* b_n) = 0$ for every $n \in \mathbb{N}$. But this contradicts Theorem 4.2. ∎

As we have seen in Example 4.6, condition (rc) is not fulfilled in general. To get an example where this condition is satisfied, it is enough to replace in Example 4.6 the normed partial *-algebra $L^1(I)$ with $L^2(I)$ (which is fully representable, as shown in [7]). It is easily seen that both conditions (rc) and (sq) are satisfied in this case. It has been known for a long time that this partial *-algebra is *-semisimple [6].

5. Bounded elements in *-semisimple partial *-algebras. *-Semisimple topological partial *-algebras are characterized by the existence of a sufficient family of ips-forms. This fact was used in [4] and [7] to derive a number of properties that we want to revisit in the current larger framework.

5.1. $\mathcal{M}$-bounded elements. First we adapt to the present case the definition of $\mathcal{M}$-bounded elements given in [4, Def. 4.9] for an $\mathfrak{A}_0$-regular topological partial *-algebra.

DEFINITION 5.1. Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and a sufficient family $\mathcal{M}$ of continuous ips-forms with core $\mathfrak{B}$. An element $x \in \mathfrak{A}$ is called *$\mathcal{M}$-bounded* if there exists $\gamma_x > 0$ such that

$$|\varphi(xa, b)| \le \gamma_x \, \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \quad \forall \varphi \in \mathcal{M},\ a, b \in \mathfrak{B}.$$

A useful characterization of $\mathcal{M}$-bounded elements is given by the following proposition, whose proof is similar to that of [4, Proposition 4.10].

PROPOSITION 5.2. *Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. Then an element $x \in \mathfrak{A}$ is $\mathcal{M}$-bounded if, and only if, there exists $\gamma_x \in \mathbb{R}$ such that $\varphi(xa, xa) \le \gamma_x^2 \varphi(a, a)$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$.*

If x, y are $\mathcal{M}$-bounded elements and their weak product $x \,\square\, y$ exists, then $x \,\square\, y$ is also $\mathcal{M}$-bounded.

LEMMA 5.3. *Let an $\mathcal{M}$-bounded element $x \in \mathfrak{A}$ have a strong inverse x^{-1}. Then $\pi(x)$ has a strong inverse for every quasi-symmetric *-representation π.*

Proof. Let $x \in \mathfrak{A}$ with strong inverse x^{-1}, i.e., $x \bullet x^{-1} = x^{-1} \bullet x = e$. Let π be a *-representation with $\pi(e) = I$. Then

$$I = \pi(e) = \pi(x \bullet x^{-1}) = \pi(xx^{-1}) = \pi(x) \,\square\, \pi(x^{-1}) = \pi(x) \circ \pi(x^{-1}).$$

It follows that the strong inverse $\pi(x)^{-1} := \pi(x^{-1})$ of $\pi(x)$ exists. ∎

Given $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, we denote by $\rho_\circ^{\mathcal{D}}(X)$ the set of all complex numbers λ such that $X - \lambda I_{\mathcal{D}}$ has a strong bounded inverse [3, Section 3] and by $\sigma_\circ^{\mathcal{D}}(X) := \mathbb{C} \setminus \rho_\circ^{\mathcal{D}}(X)$ the corresponding spectrum of X.

If π is a *-representation of $\mathfrak{A}$, from [3, Proposition 3.9] it follows that $\sigma_\circ^{\mathcal{D}}(\pi(x)) = \sigma(\overline{\pi(x)})$. If, in particular, π is a quasi-symmetric *-representation, we can conclude, by Lemma 5.3, that $\rho^{\mathcal{M}}(x) \subseteq \rho(\overline{\pi(x)}) = \rho_\circ^{\mathcal{D}}(\pi(x))$, where $\rho^{\mathcal{M}}(x)$ denotes the set of complex numbers λ such that the strong inverse $(x - \lambda e)^{-1}$ exists as an $\mathcal{M}$-bounded element of $\mathfrak{A}$ [4, Def. 4.28]. Hence,

$$\sigma(\overline{\pi(x)}) \subseteq \sigma^{\mathcal{M}}(x). \tag{6}$$

Exactly as for partial *-algebras of operators, there is here a natural distinction between hermitian elements x of $\mathfrak{A}$ (i.e. $x = x^*$) and self-adjoint elements (hermitian and with real spectrum).

DEFINITION 5.4. The element $x \in \mathfrak{A}$ is called *$\mathcal{M}$-self-adjoint* if it is hermitian and $\sigma^{\mathcal{M}}(x) \subseteq \mathbb{R}$.

PROPOSITION 5.5. *If $x \in \mathfrak{A}$ is $\mathcal{M}$-self-adjoint, then for every quasi-symmetric $\pi \in \mathrm{Rep}_c(\mathfrak{A})$, the operator $\pi(x)$ is essentially self-adjoint.*

Proof. If $x \in \mathfrak{A}$ is $\mathcal{M}$-self-adjoint, then, for every $\pi \in \mathrm{Rep}_c(\mathfrak{A})$, the operator $\pi(x)$ is symmetric and $\sigma^{\mathcal{M}}(x) \subseteq \mathbb{R}$. By (6) it follows that $\overline{\pi(x)}$ is self-adjoint, hence $\pi(x)$ is essentially self-adjoint. ∎

5.2. Order bounded elements

5.2.1. *Order structure.* Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. If $\mathfrak{A}[\tau]$ is *-semisimple, there is a natural order on $\mathfrak{A}$ defined by the family $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ or by any sufficient subfamily $\mathcal{M}$ of $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$, and this order can be used to define a different notion of *boundedness* of an element $x \in \mathfrak{A}$ (see [7, 9, 13]).

DEFINITION 5.6. Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. A subset $\mathfrak{K}$ of $\mathfrak{A}_h := \{x \in \mathfrak{A} : x = x^*\}$ is called a $\mathfrak{B}$-*admissible wedge* if

1. $e \in \mathfrak{K}$ if $\mathfrak{A}$ has a unit e;
2. $x + y \in \mathfrak{K}$ for all $x, y \in \mathfrak{K}$;
3. $\lambda x \in \mathfrak{K}$ for all $x \in \mathfrak{K}$ and $\lambda \geq 0$;
4. $(a^*x)a = a^*(xa) =: a^*xa \in \mathfrak{K}$ for all $x \in \mathfrak{K}$ and $a \in \mathfrak{B}$.

As usual, $\mathfrak{K}$ leads to the definition of an order on the real vector space $\mathfrak{A}_h$ by $x \leq y \Leftrightarrow y - x \in \mathfrak{K}$.

In the rest of this section, we will suppose that the partial *-algebras under consideration are *semi-associative*. Under this assumption, the first equality in part 4 of Definition 5.6 is automatically satisfied.

Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. We put again

$$\mathfrak{B}^{(2)} = \Big\{ \sum_{k=1}^{n} x_k^* x_k : x_k \in \mathfrak{B},\, n \in \mathbb{N} \Big\}.$$

If $\mathfrak{B}$ is a *-algebra, this is nothing but the set (wedge) of positive elements of $\mathfrak{B}$. The $\mathfrak{B}$-*strongly positive* elements of $\mathfrak{A}$ are then defined as the elements of $\mathfrak{A}^+(\mathfrak{B}) := \overline{\mathfrak{B}^{(2)}}^{\tau}$, already defined in Section 4. Since $\mathfrak{A}$ is semi-associative, the set $\mathfrak{A}^+(\mathfrak{B})$ of $\mathfrak{B}$-strongly positive elements is a $\mathfrak{B}$-admissible wedge.

We also define

$$\mathfrak{A}^+_{\rm alg} = \Big\{ \sum_{k=1}^{n} x_k^* x_k : x_k \in R\mathfrak{A},\, n \in \mathbb{N} \Big\},$$

the set (wedge) of positive elements of $\mathfrak{A}$, and we put $\mathfrak{A}^+_{\rm top} := \overline{\mathfrak{A}^+_{\rm alg}}^{\tau}$. The semi-associativity implies that $R\mathfrak{A} \cdot R\mathfrak{A} \subseteq R\mathfrak{A}$ and so $\mathfrak{A}^+_{\rm top}$ is $R\mathfrak{A}$-admissible.

Let $\mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$. An element $x \in \mathfrak{A}$ is called $\mathcal{M}$-*positive* if

$$\varphi(xa, a) \geq 0, \quad \forall \varphi \in \mathcal{M},\, a \in \mathfrak{B}.$$

An $\mathcal{M}$-positive element is automatically hermitian. Indeed, if $\varphi(xa, a) \geq 0$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$, then $\varphi(a, x^*a) = \varphi(xa, a) \geq 0$ and $\varphi(x^*a, a) \geq 0$; hence $\varphi((x - x^*)a, a) = 0$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$. By Lemma 3.7(iii), it follows that $x = x^*$.

We denote by $\mathfrak{A}^+_{\mathcal{M}}$ the set of all $\mathcal{M}$-positive elements. Clearly $\mathfrak{A}^+_{\mathcal{M}}$ is a $\mathfrak{B}$-admissible wedge.

Proposition 5.7. *The following inclusions hold:*

$$\mathfrak{A}^+(\mathfrak{B}) \subseteq \mathfrak{A}^+_{\rm top} \subseteq \mathfrak{A}^+_{\mathcal{M}}, \quad \forall \mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A}). \tag{7}$$

Proof. We only prove the second inclusion. Let $x \in \mathfrak{A}^+_{\rm top}$. Then $x = \lim_\alpha b_\alpha$, where $b_\alpha = \sum_{i=1}^n c^*_{\alpha,i} c_{\alpha,i}$ with $c_{\alpha,i} \in R\mathfrak{A}$. Thus, by ($\mathsf{ips}_4$),

$$\begin{aligned}\varphi(xa,a) &= \lim_\alpha \varphi(b_\alpha a, a) = \lim_\alpha \varphi\Big(\sum_i (c^*_{\alpha,i} c_{\alpha,i})a, a\Big) \\ &= \lim_\alpha \sum_i \varphi((c^*_{\alpha,i} c_{\alpha,i})a, a) = \lim_\alpha \sum_i \varphi(c_{\alpha,i}a, c_{\alpha,i}a) \geq 0. \ \blacksquare\end{aligned}$$

Of course, one expects that under certain conditions the reverse inclusions hold, that is, that the three sets in (7) actually coincide. A partial answer is given in Corollary 5.16.

Example 5.8. We give here two examples where the wedges considered above coincide.

(1) The first example, very elementary, is obtained by considering the space $L^p(X)$, $p \geq 2$. Indeed, it is easily seen that the $\mathcal{M}$-positivity of a function f simply means that $f(t) \geq 0$ a.e. in X (here $\mathcal{M}$ is the family of ips-forms defined in Example 3.9). On the other hand, it is well-known that such a function can be approximated in norm by a sequence of nonnegative functions of $L^\infty(X)$.

(2) Let T be a self-adjoint operator with dense domain $D(T)$ and denote by $E(\cdot)$ the spectral measure of T. We consider the space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ where $\mathcal{D} := \mathcal{D}^\infty(T) = \bigcap_{n\in\mathbb{N}} D(T^n)$. We prove that if $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is endowed with the topology t_{s^*} and $\mathcal{M}$ is the family of ips-forms defined in Example 3.8, then every $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ which is $\mathcal{M}$-positive, i.e., $\langle X\xi \,|\, \xi\rangle \geq 0$ for every $\xi \in \mathcal{D}$, is the t_{s^*}-limit of elements of $\mathcal{L}^\dagger_b(\mathcal{D})^{(2)}$. Indeed, if Δ, Δ' are bounded Borel subsets of the real line, then $E(\Delta^{(\prime)})\xi \in \mathcal{D}$ for every $\xi \in \mathcal{H}$. This implies that $E(\Delta')YE(\Delta)$ is a bounded operator in $\mathcal{H}$ for every $Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and its restriction to $\mathcal{D}$ belongs to $\mathcal{L}^\dagger_b(\mathcal{D})$. Put $\Delta_N = (-N, N]$, $N \in \mathbb{N}$, and $\delta_m = (m, m+1]$, $m \in \mathbb{Z}$. The $\mathcal{M}$-positivity of X implies that $E(\Delta_N)XE(\Delta_N) = B^*_N B_N$ for some bounded operator B_N. Then, observing that $\sum_{m\in\mathbb{Z}} E(\delta_m) = I$, in strong or strong* sense, we obtain

$$\begin{aligned}E(\Delta_N)XE(\Delta_N) &= B^*_N B_N = E(\Delta_N)B^*_N B_N E(\Delta_N) \\ &= E(\Delta_N)B^*_N\Big(\sum_{m\in\mathbb{Z}} E(\delta_m)\Big)B_N E(\Delta_N) \\ &= \sum_{m\in\mathbb{Z}} (E(\Delta_N)B^*_N E(\delta_m))(E(\delta_m)B_N E(\Delta_N)).\end{aligned}$$

This proves that $E(\varDelta_N)XE(\varDelta_N)$ belongs to the t_{s^*}-closure of $\mathcal{L}_b^\dagger(\mathcal{D})^{(2)}$. Now, if we let $N \to \infty$, we easily get $\|X\xi - E(\varDelta_N)XE(\varDelta_N)\xi\| \to 0$ and so the statement is proved.

An improvement of Theorem 4.2 is provided by the following

Corollary 5.9. *Let $\mathcal{M}$ be sufficient. Then, for every $x \in \mathfrak{A}^+_{\mathcal{M}}$, $x \neq 0$, there exists $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ with the properties*

(a) $\omega(y) \geq 0$ *for all* $y \in \mathfrak{A}^+_{\mathcal{M}}$;
(b) $\omega(x) > 0$.

Proof. By Proposition 5.7, if $x \in \mathfrak{A}^+_{\mathcal{M}}$, $x \neq 0$, there exist $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$ such that $\varphi(xa, a) > 0$. Hence the linear functional $\omega(y) := \varphi(ya, a)$ has the desired properties. ∎

Proposition 5.10. *Let the family $\mathcal{M}$ be sufficient. Then $\mathfrak{A}^+_{\mathcal{M}}$ is a cone, i.e., $\mathfrak{A}^+_{\mathcal{M}} \cap (-\mathfrak{A}^+_{\mathcal{M}}) = \{0\}$.*

Proof. If $x \in \mathfrak{A}^+_{\mathcal{M}} \cap (-\mathfrak{A}^+_{\mathcal{M}})$, then $\varphi(xa, a) \geq 0$ and $\varphi((-x)a, a) \geq 0$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$. Hence $\varphi(xa, a) = 0$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$. The sufficiency of $\mathcal{M}$ then implies $x = 0$. ∎

Remark 5.11. The fact that $\mathfrak{A}^+_{\mathcal{M}}$ is a cone automatically implies that $\mathfrak{A}^+(\mathfrak{B})$ is a cone too.

The following statement shows that $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$-positivity is exactly what is needed if we want the order to be preserved under any continuous *-representation. A partially equivalent statement is given in [7, Proposition 3.1]. To lighten notation, we put $\mathfrak{A}^+_{\mathcal{P}} := \mathfrak{A}^+_{\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})}$.

Proposition 5.12. *Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Then the element $x \in \mathfrak{A}$ belongs to $\mathfrak{A}^+_{\mathcal{P}}$ if and only if the operator $\pi(x)$ is positive for every (τ, t_s)-continuous *-representation π with $\pi(e) = I_{\mathcal{D}(\pi)}$.*

Proof. Let $x \in \mathfrak{A}^+_{\mathcal{P}}$ and let π be a (τ, t_s)-continuous *-representation of $\mathfrak{A}$ with $\pi(e) = I_{\mathcal{D}(\pi)}$. The sesquilinear form φ^ξ_π defined by

$$\varphi^\xi_\pi(x, y) := \langle \pi(x)\xi \mid \pi(y)\xi\rangle, \qquad x, y \in \mathfrak{A},$$

is a continuous ips-form, as shown in the proof of Proposition 3.2. Then

$$\varphi^\xi_\pi(xa, a) = \langle \pi(xa)\xi \mid \pi(a)\xi\rangle = \langle (\pi(x) \,\square\, \pi(a))\xi \mid \pi(a)\xi\rangle;$$

in particular, for $a = e$, $\langle \pi(x)\xi \mid \xi\rangle \geq 0$.

Conversely, let $\varphi \in \mathcal{P}$ and π_φ be the corresponding GNS representation. Then, as remarked in the proof of Proposition 3.2, π_φ is (τ, t_s)-continuous. We have, for every $a \in \mathfrak{B}$,

$$\varphi(xa, a) = \langle \pi_\varphi(x)\lambda_\varphi(a) \mid \lambda_\varphi(a)\rangle \geq 0,$$

i.e., $x \in \mathfrak{A}^+_{\mathcal{P}}$. ∎

Proposition 5.13. *Let $\mathfrak{A}$ be a fully-representable *-topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Assume that $\mathfrak{A}[\tau]$ is a Fréchet space. Then the following statements are equivalent:*

(i) $x \in \mathfrak{A}^{+}_{\mathcal{P}}$.
(ii) $\omega(x) \geq 0$ *for all* $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$.

Proof. (i)⇒(ii): If $x \in \mathfrak{A}^{+}_{\mathcal{P}}$, then $\varphi(xa, a) \geq 0$ for all $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ and $a \in \mathfrak{A}_0$. If $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$, by the assumptions and by Proposition 4.10 it follows that $\overline{\varphi_\omega}$ is an everywhere defined ips-form and thus, by Corollary 4.11, it is continuous. Hence,

$$\omega(a^* x a) = \overline{\varphi_\omega}(xa, a) \geq 0, \qquad \forall a \in \mathfrak{B}.$$

For $a = e$, we get $\omega(x) \geq 0$.

(ii)⇒(i): If $\omega(x) \geq 0$ for all $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$, then this also holds for every linear functional ω^a_φ, $a \in \mathfrak{B}$, defined by $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ as in Proposition 4.7. Then

$$\varphi(xa, a) = \omega^a_\varphi(x) \geq 0, \qquad \forall \varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A}).$$

By definition, this means that $x \in \mathfrak{A}^{+}_{\mathcal{P}}$. ∎

In complete analogy with Proposition 3.9 of [7], one can prove the following

Proposition 5.14. *Let $\mathfrak{A}[\tau]$ be a *-semisimple *-topological partial *-algebra with multiplication core $\mathfrak{B}$. Assume that the following condition* (P) *holds:*

(P) *if $y \in \mathfrak{A}$ and $\omega(a^* y a) \geq 0$ for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ and $a \in \mathfrak{A}_0$, then $y \in \mathfrak{A}^{+}(\mathfrak{B})$.*

Then, for each $x \in \mathfrak{A}$, the following statements are equivalent:

(i) $x \in \mathfrak{A}^{+}(\mathfrak{B})$.
(ii) $\omega(x) \geq 0$ *for every* $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$.
(iii) $\pi(x) \geq 0$ *for every* (τ, t_w)*-continuous *-representation* π *of* $\mathfrak{A}$.

Remark 5.15. In [7, Proposition 3.9] it was required that the family $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ of continuous linear functionals does not annihilate positive elements. This is always true for *-semisimple partial *-algebras, because of Proposition 5.10 above.

The previous propositions allow us to compare the different cones defined so far.

Corollary 5.16. *Under the assumptions of Propositions* 5.13 *and* 5.14*, one has* $\mathfrak{A}^{+}(\mathfrak{B}) = \mathfrak{A}^{+}_{\mathcal{P}}$.

5.2.2. *Order bounded elements.* Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. As we have seen in Section 5.2.1, $\mathfrak{A}[\tau]$ has several natural orders, all related to the topology τ. Each of them can be used to define *bounded* elements. We begin in a purely algebraic way, starting from an arbitrary $\mathfrak{B}$-admissible cone $\mathfrak{K}$.

Let $x \in \mathfrak{A}$; put $\Re(x) = (x + x^*)/2$, $\Im(x) = (x - x^*)/2i$. Then $\Re(x), \Im(x) \in \mathfrak{A}_h$ (the set of self-adjoint elements of $\mathfrak{A}$) and $x = \Re(x) + i\Im(x)$.

Definition 5.17. An element $x \in \mathfrak{A}$ is called $\mathfrak{K}$-*bounded* if there exists $\gamma \geq 0$ such that

$$\pm\Re(x) \leq \gamma e, \qquad \pm\Im(x) \leq \gamma e.$$

We denote by $\mathfrak{A}_b(\mathfrak{K})$ the family of $\mathfrak{K}$-bounded elements.

The following statements are easily checked.

1. $\alpha x + \beta y \in \mathfrak{A}_b(\mathfrak{K})$ for all $x, y \in \mathfrak{A}_b(\mathfrak{K})$ and $\alpha, \beta \in \mathbb{C}$.
2. $x \in \mathfrak{A}_b(\mathfrak{K}) \Leftrightarrow x^* \in \mathfrak{A}_b(\mathfrak{K})$.

Remark 5.18. If $\mathfrak{A}$ is a *-algebra, then, as shown in [9, Lemma 2.1], one also has

3. $x, y \in \mathfrak{A}_b(\mathfrak{K}) \Rightarrow xy \in \mathfrak{A}_b(\mathfrak{K})$.
4. $a \in \mathfrak{A}_b(\mathfrak{K}) \Leftrightarrow aa^* \in \mathfrak{A}_b(\mathfrak{K})$.

These statements do not hold in general when $\mathfrak{A}$ is a partial *-algebra. They are true, of course, for elements of $\mathfrak{B}$.

For $x \in \mathfrak{A}_h$, put

$$\|x\|_b := \inf\{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}.$$

This is a seminorm on the real vector space $(\mathfrak{A}_b(\mathfrak{K}))_h$.

Lemma 5.19. *Let $\mathcal{M}$ be sufficient. If $\mathfrak{K} = \mathfrak{A}^+_{\mathcal{M}}$, then $\|\cdot\|_b$ is a norm on $(\mathfrak{A}_b(\mathcal{M}))_h$.*

Proof. By Proposition 5.10, $\mathfrak{A}^+_{\mathcal{M}}$ is a cone. Put $E = \{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}$. If $\inf E = 0$, then, for every $\epsilon > 0$, there exists $\gamma_\epsilon \in E$ such that $\gamma_\epsilon < \epsilon$. This implies that $-\epsilon e \leq x \leq \epsilon e$. If $\varphi \in \mathcal{M}$, we get $-\epsilon\varphi(a, a) \leq \varphi(xa, a) \leq \epsilon\varphi(a, a)$ for every $a \in \mathfrak{B}$. Hence, $\varphi(xa, a) = 0$. By the sufficiency of $\mathcal{M}$, it follows that $x = 0$. ∎

Let $\mathfrak{A}[\tau]$ be a *-semisimple topological partial *-algebra with multiplication core $\mathfrak{B}$. We can then specify the wedge $\mathfrak{K}$ as one of those defined above. Take first $\mathfrak{K} = \mathfrak{A}^+_{\mathcal{M}}$, where $\mathcal{M} = \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ is the sufficient family of all continuous ips-forms with core $\mathfrak{B}$. For simplicity, we write again $\mathcal{P} := \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$, hence $\mathfrak{A}^+_{\mathcal{P}} := \mathfrak{A}^+_{\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})}$ and $\mathfrak{A}_b(\mathcal{P}) := \mathfrak{A}_b(\mathcal{P}_{\mathfrak{B}}(\mathfrak{A}))$.

Proposition 5.20. *If $x \in \mathfrak{A}_b(\mathcal{P})$, then $\pi(x)$ is a bounded operator for every (τ, t_s)-continuous *-representation of $\mathfrak{A}$. Moreover, if $x = x^*$, then $\|\pi(x)\| \le \|x\|_b$.*

Proof. This follows easily from Proposition 5.12 and the definitions. ∎

The following theorem generalizes [7, Theorem 5.5].

Theorem 5.21. *Let $\mathfrak{A}[\tau]$ be a fully representable, semi-associative *-topological partial *-algebra, with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Assume that $\mathfrak{A}[\tau]$ is a Fréchet space. Then the following statements are equivalent:*

(i) $x \in \mathfrak{A}_b(\mathcal{P})$.

(ii) *There exists $\gamma_x > 0$ such that*

$$|\omega(a^*xa)| \le \gamma_x \omega(a^*a), \qquad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}),\ \forall a \in \mathfrak{B}.$$

(iii) *There exists $\gamma_x > 0$ such that*

$$|\omega(b^*xa)| \le \gamma_x \omega(a^*a)^{1/2} \omega(b^*b)^{1/2}, \qquad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}),\ \forall a, b \in \mathfrak{B}.$$

Proof. It is sufficient to consider the case $x = x^*$.

(i)⇒(iii): If $x = x^* \in \mathfrak{A}_b(\mathcal{P})$, there exists $\gamma > 0$ such that $-\gamma e \le x \le \gamma e$, or, equivalently,

$$-\gamma\varphi(a, a) \le \varphi(xa, a) \le \gamma\varphi(a, a), \qquad \forall \varphi \in \mathcal{P},\ a \in \mathfrak{B}.$$

Since $\mathfrak{A}$ is fully representable, $D(\overline{\varphi_\omega}) = \mathfrak{A}$ and, by Corollary 4.11, it is a continuous ips-form with core $\mathfrak{B}$. Thus, as seen in the proof of Proposition 3.2, $\pi_{\overline{\varphi_\omega}}$ is (τ, t_s)-continuous. Hence, by Proposition 5.20, $\pi_{\overline{\varphi_\omega}}(x)$ is bounded and $\|\pi_{\overline{\varphi_\omega}}(x)\| \le \|x\|_b$. Therefore,

$$\begin{aligned} |\omega(b^*xa)| &= |\overline{\varphi_\omega(xa, b)}| \le \overline{\varphi_\omega}(xa, xa)^{1/2} \varphi_\omega(b, b)^{1/2} \\ &= \|\pi_{\overline{\varphi_\omega}}(x)\lambda_{\overline{\varphi_\omega}}(a)\|\, \varphi_\omega(b, b)^{1/2} \le \|x\|_b \gamma_x \omega(a^*a)^{1/2} \omega(b^*b)^{1/2}. \end{aligned}$$

(iii)⇒(ii): Obvious.

(ii)⇒(i): Assume that there exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \le \gamma_x \omega(a^*a), \qquad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}),\ a \in \mathfrak{B}. \tag{8}$$

Define

$$\widetilde{\gamma} := \sup\{|\omega(a^*xa)| : \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0,\ \omega(a^*a) = 1\}.$$

Let $\varphi \in \mathcal{P}$ and $a \in \mathfrak{B}$. By Proposition 4.7, the linear functional ω_φ^a defined by $\omega_\varphi^a(x) = \varphi(xa, a)$, $x \in \mathfrak{A}$, is continuous and representable. If $\varphi(a, a) = 0$, then, by (8), $\varphi(xa, a) = 0$. If $\varphi(a, a) > 0$, we get

$$\varphi((\widetilde{\gamma}e \pm x)a, a) = \widetilde{\gamma}\varphi(a, a) \pm \varphi(xa, a) = \varphi(a, a)(\widetilde{\gamma} \pm \varphi(xu, u)) \ge 0,$$

where $u = a\varphi(a,a)^{-1/2}$. Hence, by the arbitrariness of φ and a, we have $x \in \mathfrak{A}_b(\mathcal{P})$. ■

We can now compare the notion of order bounded element with that of $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$-bounded element given in Definition 5.1.

THEOREM 5.22. *Let $\mathfrak{A}[\tau]$ be a *-semisimple topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. For $x \in \mathfrak{A}$, the following statements are equivalent:*

(i) *x is $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$-bounded.*
(ii) *$x \in \mathfrak{A}_b(\mathcal{P})$.*
(iii) *$\pi(x)$ is bounded for every $\pi \in \mathrm{Rep}_c(\mathfrak{A})$, and*

$$\sup\{\|\overline{\pi(x)}\| : \pi \in \mathrm{Rep}_c(\mathfrak{A})\} < \infty.$$

Proof. It is sufficient to consider the case $x = x^*$.

(i)⇒(ii): If $x = x^*$ is $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$-bounded, we have, for some $\gamma > 0$,

$$-\gamma\varphi(a,a) \le \varphi(xa,a) \le \gamma\varphi(a,a), \qquad \forall \varphi \in \mathcal{P},\ a \in \mathfrak{B}.$$

This means that $-\gamma e \le x \le \gamma e$ in the sense of the order induced by $\mathfrak{A}^+_{\mathcal{P}}$. Hence $x \in \mathfrak{A}_b(\mathcal{P})$.

(ii)⇒(iii): Let $\pi \in \mathrm{Rep}_c(\mathfrak{A})$ and $\xi \in \mathcal{D}(\pi)$. Define φ^ξ_π as in the proof of Proposition 5.12. Then $\varphi^\xi_\pi \in \mathcal{P}$. Hence by (ii), $|\varphi^\xi_\pi(xa,a)| \le \gamma_x \varphi^\xi_\pi(a,a)$ for some $\gamma_x > 0$ which depends on x only. In other words, $|\langle \pi(x)\xi \,|\, \xi\rangle| \le \gamma_x\|\xi\|^2$. This in turn easily implies that $|\langle \pi(x)\xi \,|\, \eta\rangle| \le \gamma_x\|\xi\|\,\|\eta\|$ for every $\xi, \eta \in \mathcal{D}(\pi)$. Hence $\pi(x)$ is bounded and $\|\overline{\pi(x)}\| \le \gamma_x$.

(iii)⇒(i): Put $\gamma_x := \sup\{\|\overline{\pi(x)}\| : \pi \in \mathrm{Rep}_c(\mathfrak{A})\}$. Then

$$|\langle \pi(x)\xi \,|\, \xi\rangle| \le \|\pi(x)\xi\|\,\|\xi\| \le \gamma_x\|\xi\|^2, \qquad \forall \xi \in \mathcal{D}_\pi.$$

This holds, in particular, for the GNS representation π_φ associated to any $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$, since π_φ is (τ, t_s)-continuous. Hence, for every $a \in \mathfrak{B}$, we get

$$|\varphi(xa,a)| = |\langle \pi_\varphi(x)\lambda_\varphi(a) \,|\, \lambda_\varphi(a)\rangle| \le \gamma_x\|\lambda_\varphi(a)\|^2 = \gamma_x\varphi(a,a).$$

Using the polarization identity, one finally gets

$$|\varphi(xa,b)| \le \gamma_x\, \varphi(a,a)^{1/2}\varphi(b,b)^{1/2}, \qquad \forall \varphi \in \mathcal{P},\ a, b \in \mathfrak{B}.$$

This proves that x is $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$-bounded. ■

Theorem 5.22 shows that, under the assumptions we have made, order boundedness is nothing but the $\mathcal{M}$-boundedness studied in [4]. So all results proved there apply to the present situation (in particular those concerning the structure of the topological partial *-algebra under consideration and its spectral properties). Clearly, the crucial assumption is the existence of sufficiently many continuous ips-forms, that is, the *-semisimplicity.

EXAMPLE 5.23. In particular, Theorem 5.22 shows that, in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ (see Example 3.8 for notation), bounded elements defined by $\mathcal{M}$ and those defined by the order coincide and (as expected) the family of bounded elements is $\mathcal{L}_b^\dagger(\mathcal{D},\mathcal{H})$. Of course, one could get this result directly, using well-known properties of operators.

Also in the case of L^p-spaces ($p > 2$) considered in Example 3.9, one deduces that the two notions of boundedness coincide and the bounded part is exactly $L^\infty(X)$, as can also be proved by elementary arguments.

So far we have considered the order boundedness defined by the cone $\mathfrak{A}^+_{\mathcal{P}}$, but other choices are possible. For instance we may consider the order induced by $\mathfrak{A}^+(\mathfrak{B})$. It is clear that, if $x \in \mathfrak{A}_b(\mathfrak{A}^+(\mathfrak{B}))$, then $x \in \mathfrak{A}_b(\mathcal{P})$. On the other hand, if $x \in \mathfrak{A}_b(\mathcal{P})$ and the assumptions of Theorem 5.21 hold, there exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \le \gamma_x\omega(a^*a), \qquad \forall\omega \in \mathcal{R}_c(\mathfrak{A},\mathfrak{B}),\ \forall a \in \mathfrak{B}.$$

Hence, if condition (P) holds too, we can conclude, by adapting the argument used in the proof of Theorem 5.21, that $x \in \mathfrak{A}_b(\mathfrak{A}^+(\mathfrak{B}))$. We leave a deeper analysis of the general question to future papers.

6. Concluding remarks. As we have discussed in the Introduction, the notion of bounded element for a topological partial *-algebra plays an important role in the whole discussion. We have at hand two different notions, one ($\mathcal{M}$-boundedness) based on a sufficient family of ips-forms, and the other (order boundedness) based on some $\mathfrak{B}$-admissible wedge, where $\mathfrak{B}$ is a multiplication core. Both seem very reasonable definitions and, as we have seen, they can be compared in many occasions. In the framework of (topological) *-algebras, it is even possible that every element is order bounded (see examples in [10, Section 5]) ([3]). The analogous problem for partial *-algebras is open (in other words we do not know if there exist topological partial *-algebras where every element is order bounded) and we conjecture that a *complete* topological partial *-algebra $\mathfrak{A}$ whose elements are all bounded is necessarily an algebra. This is certainly true in the case where $\mathcal{M}$-boundedness is considered, where $\mathcal{M}$ is a well-behaved family of ips-forms in the sense of Definition 4.26 of [4]. Indeed, as shown there (Proposition 4.27), under these assumptions the set of $\mathcal{M}$-bounded elements is a C*-algebra. The same, of course, holds true in the situation considered in Theorem 5.22 above, if the family $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ is well-behaved. However, the general question is open.

([3]) The terminology adopted in that paper comes from algebraic geometry, so that an admissible cone is called there a *quadratic module*.

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