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FUNCTIONAL ANALYSIS AND NONLINEAR DIFFERENTIAL EQUATIONS[1]

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1. The aim of this paper is to study the nonlinear differential equation

$$Ex = Nx \tag{1}$$

where N is a nonlinear operator in a real Hilbert space S, and E is a linear differential operator in S with preassigned linear homogeneous boundary conditions. The idea is to reduce the problem to a finite dimensional setting and this technique has been used by several authors. We use here a method due to Cesari [**4**]. This method has been extensively developed in the existence analysis of differential equations by Cesari, Hale, Locker, Mawhin and others. For a detailed bibliography one is referred to Cesari [**5**].

In this paper, by applying results from the theory of monotone operators, we show that, under suitable monotonicity hypotheses on N, the equation $Ex = Nx$ can be solved. In the present short presentation we restrict ourselves to the simplest hypotheses on E, N and S, even though the results obtained here hold under more general conditions.

2. Let S be the direct sum of the subspaces S_0 and S_1 and let $P:S \to S_0$ be a projection operator with null space S_1, and $H:S_1 \to S_1$ a linear operator such that (h_1) $H(I - P)Ex = (I - P)x$, x belonging to the domain of E. If y is a solution of (1), then $Ey = Ny$ implies $H(I - P)Ey = H(I - P)Ny$. Hence, $(I - P)y = H(I - P)Ny$; and finally

$$y = Py + H(I - P)Ny. \tag{2}$$

Thus, any solution of (1) is a solution of (2). If we also have that (h_2) $EPx = PEx$ and (h_3) $EH(I - P)Nx = (I - P)Nx$, then from (2) we derive

$$Ey = EPy + EH(I - P)Ny = PEy + (I - P)Ny.$$

Hence, $Ey - Ny = P(Ey - Ny)$. Thus, any solution y of (2) is a solution of (1) if and only if y satisfies

$$P(Ey - Ny) = 0. \tag{3}$$

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Thus, under hypotheses (h_1), (h_2), and (h_3), we have

THEOREM. *An element y in S is a solution of* (1) *if and only if y is a solution of equations* (2) *and* (3).

Equations (2) and (3) are called the *auxiliary* and *bifurcation* equations respectively. Note that, if S is a separable Hilbert space with norm $\|x\|^2 = \langle x \cdot x \rangle$ and $(\phi_1, \phi_2, \ldots)$ as an orthonormal basis, and we consider S_0 as spanned by $\{\phi_1, \phi_2, \ldots, \phi_m\}$, then (3) reduces to the finite system of equations $(Ey - Ny) \cdot \phi_i = 0, i = 1, 2, \ldots, m$.

3. Let us assume that the associated linear problem $Ex + \lambda x = 0$ (with preassigned linear homogeneous boundary conditions) has a countable system of eigenvalues λ_i and eigenfunctions ϕ_i such that $\lambda_i \leqq \lambda_{i+1}$, $\lambda_i \to +\infty$ as $i \to \infty$ and $\{\phi_i\}$ is a complete orthonormal system in the Hilbert space $S = L_2(A)$ of all square integrable functions $x(\alpha)$, $\alpha \in A$.

Any element $x \in S$ can be written as $\sum c_i\phi_i$.

Let $Px = \sum_1^m c_i\phi_i$ and $Ex = -\sum_1^\infty c_i\lambda_i\phi_i$. Thus,

$$(I - P)x = \sum_{m+1}^{\infty} c_i\phi_i, \qquad (I - P)x \in S_1,$$

and for m such that $\lambda_{m+1} > 0$, let $H: S_1 \to S_1$ be defined by $H(I - P)x = -\sum_{m+1}^\infty c_i\lambda_i^{-1}\phi_i$.

It can be easily seen that $H(I - P)Ex = (I - P)x$, $EPx = PEx$, $EH(I - P)x = (I - P)x$.

For $x = \sum c_i\phi_i$, we have

$$\begin{aligned}\langle -H(I - P)x, x\rangle &= \sum_{m+1}^{\infty} c_i^2\lambda_i^{-1} \geqq \lambda_{m+1} \sum_{m+1}^{\infty} c_i^2\lambda_i^{-2} \\ &= \lambda_{m+1} \|-H(I - P)x\|^2.\end{aligned}$$

Hence, the operator $-H(I - P)$ is a linear, monotone operator. Since it is bounded, it is maximal monotone.

We now use the Theorem above to solve (1). To this end, we have to solve (2) and (3) respectively. Let us first consider the auxiliary equation (2), i.e., $y = Py + H(I - P)Ny$. Let x^* be any element of S_0 and consider the equation

$$y - H(I - P)Ny = x^*. \tag{4}$$

This equation is of the type $u + LNu = x^*$, where L is a (linear) maximal monotone operator; it has been studied by Browder [**2**], Brezis [**1**], Kolodner [**7**] and several others, where N is assumed to satisfy suitable monotonicity hypotheses.

In view of the fact that $\langle -H(I - P)x, x\rangle \geqq \lambda_{m+1} \|-H(I - P)x\|^2$ and applying the result of Hess [6], we conclude that (4) has always a unique solution y^* for each $x^* \in S_0$, provided N is also hemicontinuous.

We now proceed to consider the bifurcation equation (3). Thus we have to solve the equation $PNy^* = PEy^*$, where y^* is the solution of (4) corresponding to $x^* \in S_0$. But $PEy^* = EPy^* = Ex^*$ and thus equation (3) reduces to

$$PN[I - H(I - P)N]^{-1}x^* - Ex^* = 0.$$

Let $M = N[I - H(I - P)N]^{-1}$. And let $u = Ma$, $v = Mb$, where $a, b \in S_0$. Then, $u = Np$, $v = Nq$, where $p = (I - H(I - P)N)^{-1}a$ and $q = (I - H(I - P)N)^{-1}b$. Thus

$$\begin{aligned}\langle u - v, a - b\rangle &= \langle Np - Nq, a - b\rangle \\ &= \langle Np - Nq, p - H(I - P)Np - q + H(I - P)Nq\rangle \\ &= \langle Np - Nq, p - q\rangle \\ &\quad + \langle Np - Nq, -H(I - P)Np + H(I - P)Nq\rangle.\end{aligned}$$

The first term on the right hand is $\geqq 0$ because N is monotone and the second is so because $-H(I - P)$ is monotone. Hence,

$$PM = PN[I - H(I - P)N]^{-1}$$

treated as an operator from S_0 to S_0 is monotone, for if $a, b \in S_0$, then $\langle PMa - PMb, a - b\rangle = \langle Ma - Mb, a - b\rangle$. Further, if $a \in S_0$, then the equation

$$a = \{I + PN[I - H(I - P)N]^{-1}\}x \tag{5}$$

reduces to $a = x + PNp$, where $p = [I - H(I - P)N]^{-1}x$, or

$$p - H(I - P)Np + PNp = a. \tag{6}$$

By arguing as before it can be shown that this equation is solvable for p, and thus it follows from (6) that we can find $x \in S_0$ such that (5) is solvable. Hence, $PN[I - H(I - P)N]^{-1}$ is maximal monotone over S_0, a finite dimensional space.

Thus we are reduced to an equation in the finite dimensional space S_0 of the form $Mx^* - Ex^* = 0$ where M is maximal monotone. If $\langle Ex^*, x^*\rangle \leqq 0$, as is the case when all the λ_i's are $\geqq 0$, then the above equation is solvable. If, however, E has a finite number of negative eigenvalues, then one can proceed in several ways. Thus if $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_m \leqq 0$, then one can apply Browder [3, p. 21] and conclude: If $[I - H(I - P)N]^{-1}$ is continuous and

$$\langle Nx_1 - Nx_2, x_1 - x_2\rangle \geqq c\|x_1 - x_2\|^2, \qquad c > -\lambda_1,$$

then the bifurcation equation is solvable.

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