

**FUNCTIONAL CENTRAL LIMIT THEOREMS FOR  
 STRICTLY STATIONARY PROCESSES SATISFYING  
 THE STRONG MIXING CONDITION**

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**1. Summary.**

The object of this paper is to prove the functional central limit theorems for strictly stationary processes satisfying the strong mixing condition under the same assumptions in Ibragimov [3]. The results generalize those of Davydov [2].

**2. Main results.**

Let  $\{\xi_n; n=0, \pm 1, \pm 2, \dots\}$  be a strictly stationary process with  $E\xi_j=0$ , satisfying the strong mixing (s. m.) condition, i.e.,

$$(1) \quad \sup_{A \in \mathfrak{M}_{-\infty}^a, B \in \mathfrak{M}_{a+s}^\infty} |P(AB) - P(A)P(B)| = \alpha(s) \rightarrow 0 \quad (s \rightarrow \infty),$$

where  $\mathfrak{M}_a^b$  denotes the  $\sigma$ -algebra generated by  $\{\xi_j; j=a, \dots, b\}$ . Write  $S_n = \xi_1 + \dots + \xi_n$  and  $\sigma^2 = E\xi_0^2 + 2 \sum_{j=1}^\infty E\xi_0 \xi_j$ . Let  $D = D[0, 1]$  be the space of functions  $x$  on  $[0, 1]$  that are right-continuous and have left-hand limits, and let  $\mathcal{D}$  be the  $\sigma$ -field of Borel sets for the Skorokhod topology (cf. [1]). When  $0 < \sigma < \infty$ , we define random elements  $X_n(t)$  of  $D$  by

$$(2) \quad X_n(t, \omega) = \frac{1}{\sigma \sqrt{n}} S_{[nt]}(\omega), \quad 0 \leq t \leq 1; n=1, 2, \dots$$

The following theorems imply that functional central limit theorems hold under the same conditions of theorems 1.6 and 1.7 in [3] which assure the validity of central limit theorems.

**THEOREM 1.** *If  $\xi_j$ 's are bounded, i.e.,  $|\xi_j| < C < \infty$  with probability one and if*

$$(3) \quad \sum_{n=1}^\infty \alpha(n) < \infty \quad \text{and} \quad \alpha(n) \leq \frac{M}{n \log n},$$

*then  $\sigma^2 < \infty$ . If  $\sigma > 0$  and if  $X_n$  is defined by (2), then the distribution of  $X_n$  converges weakly to Wiener measure  $W$  on  $(D, \mathcal{D})$ .*

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THEOREM 2. *If  $E|\xi_j|^{2+\delta} < \infty$  for some  $\delta > 0$  and if*

$$(4) \quad \sum_{n=1}^{\infty} \{\alpha(n)\}^{\delta/2+\delta} < \infty,$$

*then  $\sigma^2 < \infty$ . If  $\sigma > 0$ , then the distribution of  $X_n$  converges weakly to Wiener measure  $W$  on  $(D, \mathcal{D})$ .*

**3. Proof of theorem 1.**

The first half of theorem 1 is theorem 1.6 in [3]. To prove the latter half, it suffices to show that the finite dimensional distributions of  $X_n$  converges weakly to those of  $W$  and that the sequence  $\{X_n\}$  is tight. The convergence of the finite dimensional distributions is easily obtained by the method in [1] or [2].

To prove tightness, it is enough to show (cf. [1]) that for any  $\epsilon > 0$  there exist a  $\lambda > 1$  and an integer  $n_0$  such that

$$(5) \quad P\left\{\max_{i \leq n} |S_i| \geq 3\lambda\sigma\sqrt{n}\right\} \leq \frac{\epsilon}{\lambda^2} \quad (n \geq n_0).$$

For any integer  $n (\geq 2)$ , put  $p = \lfloor n^{1/2} \log^{-3/8} n \rfloor$  and  $k = \lfloor n/p \rfloor$ . Since  $\xi_j$ 's are bounded, so for all sufficiently large  $n$

$$(6) \quad P\{|\xi_1| + \dots + |\xi_{2p}| \geq \lambda\sigma\sqrt{n}\} = 0.$$

Given  $\epsilon > 0$ , choose  $\lambda (> 1)$  so that

$$(7) \quad P\{|S_i| > \lambda\sigma\sqrt{i}\} \leq \frac{\epsilon}{3\lambda^2} \quad \text{for all } i,$$

which is possible because of uniform integrability of  $\{S_n^2/n\}$  (cf. theorem 5.4, [1]).

If  $E_j = \{\max_{i < j} |S_i| < 3\lambda\sigma\sqrt{n} \subseteq |S_j|\}$ , then

$$(8) \quad \begin{aligned} & P\left(\max_{i \leq n} |S_i| \geq 3\lambda\sigma\sqrt{n}\right) \\ & \leq P(|S_n| \geq \lambda\sigma\sqrt{n}) + P\left(\bigcup_{j=1}^n [E_j \cap \{|S_n - S_j| \geq 2\lambda\sigma\sqrt{n}\}]\right) \\ & \leq P(|S_n| \geq \lambda\sigma\sqrt{n}) \\ & \quad + \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p [E_{ip+j} \cap \{|S_n - S_{ip+j}| \geq 2\lambda\sigma\sqrt{n}\}]\right) \\ & \quad + \sum_{j=(k-1)p+1}^n P(|S_n - S_j| \geq 2\lambda\sigma\sqrt{n}) \end{aligned}$$

$$\begin{aligned}
 &\leq P(|S_n| \geq \lambda\sigma\sqrt{n}) \\
 &\quad + \sum_{i=0}^{k-2} \left\{ P\left(\bigcup_{j=1}^p [E_{i,p+j} \cap \{|S_n - S_{(i+2)p}| \geq \lambda\sigma\sqrt{n}\}]\right) \right. \\
 &\quad + \sum_{j=1}^p P(|S_{(i+2)p} - S_{i,p+j}| \geq \lambda\sigma\sqrt{n}) \\
 &\quad \left. + \sum_{j=(k-1)p+1}^n P(|\xi_1| + \dots + |\xi_{n-j}| \geq 2\lambda\sigma\sqrt{n}) \right\} \\
 &\leq P(|S_n| \geq \lambda\sigma\sqrt{n}) \\
 &\quad + \sum_{i=0}^{k-2} P\left(\left[\bigcup_{j=1}^p E_{i,p+j}\right] \cap \{|S_n - S_{(i+2)p}| \geq \lambda\sigma\sqrt{n}\}\right) \\
 &\quad + 2n P(|\xi_1| + \dots + |\xi_{2p}| \geq \lambda\sigma\sqrt{n}).
 \end{aligned}$$

As  $\bigcup_{j=1}^p E_{i,p+j} \in \mathfrak{M}_{(i+2)p}^{(i+1)p}$  and  $\{|S_n - S_{(i+2)p}| \geq \lambda\sigma\sqrt{n}\} \in \mathfrak{M}_{(i+2)p}^\infty$ , so we have, using (7),

$$\begin{aligned}
 &\sum_{i=0}^{k-2} P\left(\left[\bigcup_{j=1}^p E_{i,p+j}\right] \cap \{|S_n - S_{(i+2)p}| \geq \lambda\sigma\sqrt{n}\}\right) \\
 &\leq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p E_{i,p+j}\right) P(|S_n - S_{(i+2)p}| \geq \lambda\sigma\sqrt{n}) + k\alpha(p) \\
 (9) \quad &\leq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p E_{i,p+j}\right) P(|S_{n-(i+2)p}| \geq \lambda\sigma\sqrt{n-(i+2)p}) + k\alpha(p) \\
 &\leq \frac{\varepsilon}{3\lambda^2} + k\alpha(p).
 \end{aligned}$$

Furthermore

$$(10) \quad \lim_{n \rightarrow \infty} k\alpha(p) \leq M \lim_{n \rightarrow \infty} \frac{n}{[n^{1/2} \log^{-3/8} n]^2 \log [n^{1/2} \log^{-3/8} n]} = 0.$$

Hence, combining (6) and (7), we have (5). Thus the proof is completed.

#### 4. Proof of theorem 2.

The first half is theorem 1.7 in [3]. We proceed as the proof of theorem 1. Since the convergence of the finite dimensional distributions is easily proved (cf. [2]), we need only to verify the tightness of  $\{X_n\}$ .

If  $f_N(x) = x$  for  $|x| \leq N$  and  $=0$  for  $|x| > N$ , then the process  $\{f_N(\xi_j)\}$  satisfies the s.m. condition (1) at least with the same function  $\alpha(n)$ . Put  $N = n^{1/2(1+\delta)}$  and  $d_n^2 = (1/n)E[\sum_{j=1}^n (f_N(\xi_j) - Ef_N(\xi_j))]^2$ . It is obvious that  $d_n^2 \rightarrow \sigma^2$  ( $n \rightarrow \infty$ ). Let  $\bar{f}_N(x) = x - f_N(x)$ . Then

$$\begin{aligned}
 & P\left(\max_{l \leq n} |S_l| \geq 2\lambda\sigma\sqrt{n}\right) \\
 & \leq P\left(\max_{l \leq n} \left| \sum_{j=1}^l (f_N(\xi_j) - Ef_N(\xi_j)) \right| \geq \lambda\sigma\sqrt{n}\right) \\
 (11) \quad & + P\left(\max_{l \leq n} \left| \sum_{j=1}^l (\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)) \right| \geq \lambda\sigma\sqrt{n}\right) \\
 & \leq P\left(\max_{l \leq n} \left| \sum_{j=1}^l (f_N(\xi_j) - Ef_N(\xi_j)) \right| \geq \lambda\sigma\sqrt{n}\right) \\
 & + P\left(\sum_{j=1}^n |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \geq \lambda\sigma\sqrt{n}\right).
 \end{aligned}$$

Since  $\alpha(n)$  is monotone decreasing, it follows from (4) that  $\alpha(n) = o(n^{-(2+\delta)/\delta})$ . Thus if we put  $p = \lceil n^{\delta/2(1+\delta)} \rceil$  and  $k = \lfloor n/p \rfloor$ , then

$$\begin{aligned}
 k\alpha(p) & \sim k\alpha(p^{-(2+\delta)/\delta}) \\
 (12) \quad & \sim n^{(2+\delta)/2(1+\delta)} o(n^{-(2+\delta)/2(1+\delta)}) \\
 & \rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

Now, for all sufficiently large  $\lambda$

$$(13) \quad \sum_{j=1}^p |f_N(\xi_j) - Ef_N(\xi_j)| \leq 2pN < \frac{1}{2} \lambda\sigma\sqrt{n}$$

with probability one, and so

$$(14) \quad P\left(\sum_{j=1}^p |f_N(\xi_j) - Ef_N(\xi_j)| \geq \lambda\sigma\sqrt{n}\right) = 0$$

for all sufficiently large  $\lambda$ . Thus, applying the method of the proof of theorem 1 to this case, we obtain that for any  $\varepsilon > 0$  there exists a  $\lambda_0$  such that

$$(15) \quad P\left(\max_{l \leq n} \left| \sum_{j=1}^l (f_N(\xi_j) - Ef_N(\xi_j)) \right| \geq \lambda\sigma\sqrt{n}\right) \leq \frac{\varepsilon}{2\lambda^2} \quad (\lambda \geq \lambda_0).$$

Next, we shall estimate  $P(\sum_{j=1}^n |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \geq \lambda\sigma\sqrt{n})$ . Using the inequality in the corollary to lemma 2.1 in [2] and Minkowsky's inequality,

$$\begin{aligned}
 & E|\bar{f}_N(\xi_0) - E\bar{f}_N(\xi_0)| \cdot |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \\
 & \leq E|\bar{f}_N(\xi_0) - E\bar{f}_N(\xi_0)| \cdot E|\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \\
 (16) \quad & + 12\{E|\bar{f}_N(\xi_0) - E\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)} \{\alpha(j)\}^{\delta/(2+\delta)}
 \end{aligned}$$

$$\begin{aligned} &\leq 4\{E|\bar{f}_N(\xi_0)|\}^2 \\ &\quad + 48\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)} \\ &\leq (4/N^{2(1+\delta)})\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^2 \\ &\quad + 48\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)} \\ &= (4/n)\gamma_N^2 + 48\gamma_N^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)} \end{aligned}$$

for  $j=1, 2, \dots, n$ , where  $\gamma_N = E|\bar{f}_N(\xi_0)|^{2+\delta}$ . Therefore

$$\begin{aligned} &\frac{1}{\sigma_n^2} E\left(\sum_{j=1}^n |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)|\right)^2 \\ &\leq \frac{1}{\sigma^2} \left\{ E|\bar{f}_N(\xi_0) - E\bar{f}_N(\xi_0)|^2 \right. \\ (17) \quad &\quad \left. + 2 \sum_{j=1}^n E|\bar{f}_N(\xi_0) - E\bar{f}_N(\xi_0)| \cdot |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \right\} \\ &\leq \frac{1}{\sigma^2} \left\{ \gamma_N/N^\delta + 8\gamma_N^2 + 96\gamma_N^{\delta/(2+\delta)} \sum_{j=1}^{\infty} \{\alpha(j)\}^{\delta/(2+\delta)} \right\}. \end{aligned}$$

As  $\sum_{j=1}^{\infty} \{\alpha(j)\}^{\delta/(2+\delta)} < \infty$ , so the last part of the above inequality tends to zero when  $n \rightarrow \infty$ . Consequently, for all  $n$  sufficiently large

$$(18) \quad P\left(\sum_{j=1}^n |\bar{f}_N(\xi_j) - E\bar{f}_N(\xi_j)| \geq \lambda\sigma\sqrt{n}\right) \leq \frac{\epsilon}{2\lambda^2}.$$

From (15) and (18) we deduce that for any  $\epsilon > 0$ , there exist a  $\lambda$  and an  $n_0$  such that

$$(19) \quad P\left(\max_{i \leq n} |S_i| \geq 2\lambda\sigma\sqrt{n}\right) \leq \frac{\epsilon}{\lambda^2} \quad (n \geq n_0),$$

which implies the tightness of  $\{X_n\}$ . Thus we have the theorem.

### 5. Randomly selected partial sums.

For each  $n$ , let  $\nu_n$  be a positive integer-valued random variable defined on the same probability space as the  $\xi_n$ . As in [1] define  $Y_n$  by

$$(20) \quad Y_n(t, \omega) = \frac{1}{\sigma\sqrt{\nu_n(\omega)}} S_{[\nu_n(\omega)t]}(\omega) = X_{\nu_n(\omega)}(t, \omega), \quad (0 \leq t \leq 1).$$

**THEOREM 3.** *Suppose that the hypotheses of theorem 1 (or 2) are satisfied and that*

$$(21) \quad \frac{\nu_n}{a_n} \xrightarrow{P} \theta,$$

where  $\theta$  is a positive random variable and  $a_n \rightarrow \infty$ . If  $\sigma > 0$ , then the distribution of  $Y_n$  converges weakly to the distribution of  $W$ .

*Proof.* The proof of theorem 17.2 in [1] can be carried over to this case in exactly the same way.

**6. Functions of processes satisfying the s. m. condition.**

As in [3], let  $\{\xi_j; j=0, \pm 1, \pm 2, \dots\}$  be strictly stationary and strong mixing. Let  $f$  be a measurable mapping from the space of doubly infinite sequences  $(\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$  of real numbers into the real line. Define random variables

$$(22) \quad f_j = f(\dots, \xi_{j-1}, \xi_j, \xi_{j+1}, \dots),$$

where  $\xi_j$  occupies the 0-th place in the argument of  $f$ . It is obvious that  $\{f_j\}$  is strictly stationary but need not be strong mixing. We shall obtain limit theorems for  $\{f_j\}$  under the analogous assumptions in [3].

Write  $S_n = f_1 + \dots + f_n$ ,  $\sigma^2 = Ef_0^2 + 2 \sum_{j=1}^{\infty} Ef_0 f_j$ . When  $0 < \sigma^2 < \infty$ , define  $Z_n$  by

$$(23) \quad Z_n(t, \omega) = \frac{1}{\sigma \sqrt{n}} S_{[nt]}(\omega), \quad 0 \leq t \leq 1.$$

**THEOREM 4.** *Let the stationary process  $\{\xi_j\}$  satisfy the s. m. condition (1), let the random variable  $f$  be measurable with respect to  $\mathfrak{M}_{-\infty}^{\infty}$ , and let the process  $\{f_j\}$  be obtained from  $\{\xi_j\}$  as described above. Suppose that the following conditions are satisfied.*

1.  $Ef = 0$  and  $|f| < C < \infty$  with probability one,
2.  $\sum_{k=1}^{\infty} \{E|f - E\{f|\mathfrak{M}_{-k}^k\}|^2\}^{1/2} < \infty$ ,
3.  $\sum_{k=1}^{\infty} \alpha(k) < \infty$  and  $\alpha(k) \leq \frac{M}{k \log k}$ .

Then  $\sigma^2 < \infty$ . If  $\sigma > 0$  and  $Z_n$  is defined by (23), then the distribution of  $Z_n$  converges weakly to Wiener measure  $W$  on  $(D, \mathcal{D})$ .

*Proof.* The first part is a result in [3]. Since the convergence of finite dimensional distributions can be proved by the method used in the proof of theorem 21.1 in [1], it is enough to prove the tightness.

The proof is in the same line as that of theorem 21.1 in [1]. Let  $p = [n^{1/2} \log^{-3/8} n]$  and  $k = [n/p]$ . Define  $U_i = E\{S_{i-2p} | \mathfrak{M}_{-\infty}^{i-2p}\}$  and  $V_i = E\{S_n - S_{i+2p} | \mathfrak{M}_{i+2p}^{\infty}\}$ . In these definitions we adopt the conventions that  $S_{i-2p} = 0$  if  $i < 2p$  and  $S_n - S_{i+2p} = 0$  if  $i + 2p > n$ . If we put

$$(24) \quad \mu(p) = \sum_{k=p}^{\infty} \{E|f_0 - E\{f_0|\mathfrak{M}_k^k\}|^2\}^{1/2},$$

then for all  $k$  and  $i$

$$(25) \quad E|S_k - E\{S_k|\mathfrak{M}_{-\infty}^{k+p}\}|^2 \leq \mu^2(p),$$

$$(26) \quad E|U_i - S_i|^2 \leq 2ES_{2p}^2 + 2\mu^2(p)$$

and

$$(27) \quad E|V_i - (S_n - S_i)|^2 \leq 2ES_{2p}^2 + 2\mu^2(p).$$

Since  $|f_0| < C$  with probability one, so

$$(28) \quad P\left(|f_1| + \dots + |f_{2p}| \geq \frac{1}{2} \lambda \sigma \sqrt{n}\right) = 0$$

for all sufficiently large  $n$ . Thus, for all  $i$

$$(29) \quad \begin{aligned} & P(|S_i - U_i| \geq \lambda \sigma \sqrt{n}) \\ & \leq P\left(|S_{i-2p} - E\{S_{i-2p}|\mathfrak{M}_{-\infty}^{i-p}\}| \geq \frac{1}{2} \lambda \sigma \sqrt{n}\right) \\ & \quad + P\left(|f_1| + \dots + |f_{2p}| \geq \frac{1}{2} \lambda \sigma \sqrt{n}\right) \\ & \leq \frac{4\mu^2(p)}{\lambda^2 \sigma^2 n} \end{aligned}$$

and similarly for all  $i$

$$(30) \quad P(|V_i - (S_n - S_i)| \geq \lambda \sigma \sqrt{n}) \leq \frac{4\mu^2(p)}{\lambda^2 \sigma^2 n}.$$

Since  $\{S_n^2/n\}$  is uniformly integrable (cf. the proof of theorem 21.1, [1]), there is a  $\lambda > 1$  such that

$$(31) \quad P(|S_k| \geq \lambda \sigma \sqrt{k}) \leq \frac{\varepsilon}{3\lambda^2}$$

for all  $k$ . By (29)

$$(32) \quad P\left(\max_{i \leq n} |S_i| \geq 6\lambda \sigma \sqrt{n}\right) \leq P\left(\max_{i \leq n} |U_i| \geq 5\lambda \sigma \sqrt{n}\right) + \frac{4\mu^2(p)}{\lambda^2 \sigma^2 n}.$$

Let  $E_j = \{\max_{i < j} |U_i| < 5\lambda \sigma \sqrt{n} \leq |U_j|\}$ . As  $E_j \in \mathfrak{M}_{-\infty}^{j-2p}$  and  $V_{j+2p}$  is measurable with respect to  $\mathfrak{M}_{j+3p}^{\infty}$ , so using (30) and (31),

$$\begin{aligned}
 & P\left(\bigcup_{j=1}^{n-1} [E_j \cap \{|V_j| \geq 2\lambda\sigma \sqrt{n}\}]\right) \\
 & \leq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p [E_{i,p+j} \cap \{|V_{(i+2)p}| \geq \lambda\sigma \sqrt{n}\}]\right) \\
 & \quad + nP(|f_1| + \dots + |f_{2p}| \geq \lambda\sigma \sqrt{n}) \\
 (33) \quad & = \sum_{i=0}^{k-2} P\left(\left[\bigcup_{j=1}^p E_{i,p+j}\right] \cap \{|V_{(i+2)p}| \geq \lambda\sigma \sqrt{n}\}\right) \\
 & \leq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p E_{i,p+j}\right)P(|V_{(i+2)p}| \geq \lambda\sigma \sqrt{n}) + k\alpha(p) \\
 & \leq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^p E_{i,p+j}\right)\left\{P(|S_{n-(i+2)p}| \geq \lambda\sigma \sqrt{n-(i+2)p}) + \frac{4\mu^2(p)}{\lambda^2\sigma^2n}\right\} + k\alpha(p) \\
 & \leq \frac{\varepsilon}{3\lambda^2} + \frac{4\mu^2(p)}{\lambda^2\sigma^2n} + k\alpha(p).
 \end{aligned}$$

Accordingly, from (29), (30) and (33),

$$\begin{aligned}
 & P\left(\max_{i \leq n} |U_i| \geq 5\lambda\sigma \sqrt{n}\right) \\
 & \leq P(|S_n| \geq \lambda\sigma \sqrt{n}) + P\left(\bigcup_{j=1}^{n-1} [E_j \cap \{|S_n - U_j| \geq 4\lambda\sigma \sqrt{n}\}]\right) \\
 & \leq P(|S_n| \geq \lambda\sigma \sqrt{n}) + \sum_{j=1}^{n-1} P(|S_n - S_j - V_j| \geq \lambda\sigma \sqrt{n}) \\
 (34) \quad & \quad + P\left(\bigcup_{j=1}^{n-1} [E_j \cap \{|V_j| \geq 2\lambda\sigma \sqrt{n}\}]\right) + \sum_{j=1}^{n-1} P(|S_j - U_j| \geq \lambda\sigma \sqrt{n}) \\
 & \leq \frac{\varepsilon}{3\lambda^2} + \frac{4\mu^2(p)}{\lambda^2\sigma^2} \\
 & \quad + \left(\frac{\varepsilon}{3\lambda^2} + \frac{4\mu^2(p)}{\lambda^2\sigma^2n} + k\alpha(p)\right) + \frac{4\mu^2(p)}{\lambda^2\sigma^2} \\
 & = \frac{\varepsilon}{3\lambda^2} + 4\left(2 + \frac{1}{n}\right)\frac{\mu^2(p)}{\lambda^2\sigma^2} + k\alpha(p)
 \end{aligned}$$

and so from (32) and (34)

$$(35) \quad P\left(\max_{i \leq n} |S_i| \geq 6\lambda\sigma \sqrt{n}\right) \leq \frac{2\varepsilon}{3\lambda^2} + 4\left(3 + \frac{1}{n}\right)\frac{\mu^2(p)}{\lambda^2\sigma^2} + k\alpha(p).$$

Since  $4(3+n^{-1})\mu^2(p)/\lambda^2\sigma^2 \rightarrow 0$  and  $k\alpha(p) \rightarrow 0$  as  $n \rightarrow \infty$ , we have



$$(36) \quad P\left(\max_{i \leq n} |S_i| \geq 6\lambda\sigma \sqrt{n}\right) \leq \frac{\varepsilon}{\lambda^2}$$

for all sufficiently large  $n$ . This completes the proof.

Using the methods of proofs of theorems 2 and 4, we have the following

**THEOREM 5.** *Suppose that the following conditions are satisfied:*

1.  $Ef=0$  and  $E|f|^{2+\delta} < \infty$  for some  $\delta > 0$ ,
2.  $\sum_{k=1}^{\infty} \{E|f - E\{f|\mathfrak{M}_{-k}^k\}|^2\}^{1/2} < \infty$ ,
3.  $\sum_{k=1}^{\infty} \{\alpha(k)\}^{\delta/(2+\delta)} < \infty$ .

Then  $\sigma^2 < \infty$ . If  $\sigma > 0$ , then the distribution of  $Z_n$ , defined by (23), converges weakly to Wiener measure  $W$  on  $(D, \mathcal{D})$ .

*Proof.* As before, it suffices to prove the tightness of  $\{Z_n\}$ . Define  $g_N(f_0) = f_0$  ( $|f_0| \leq N$ ),  $= 0$  ( $|f_0| > N$ ), and  $\bar{g}_N(f_0) = f_0 - g_N(f_0)$ . Then

$$(37) \quad \begin{aligned} & P\left(\max_{j \leq n} |S_j| \geq 2\lambda\sigma \sqrt{n}\right) \\ & \leq P\left(\max_{j \leq n} \left| \sum_{i=1}^j (g_N(f_i) - E g_N(f_i)) \right| \geq \lambda\sigma \sqrt{n}\right) \\ & \quad + P\left(\max_{j \leq n} \left| \sum_{i=1}^j (\bar{g}_N(f_i) - E \bar{g}_N(f_i)) \right| \geq \lambda\sigma \sqrt{n}\right) \\ & \leq P\left(\max_{j \leq n} \left| \sum_{i=1}^j (g_N(f_i) - E g_N(f_i)) \right| \geq \lambda\sigma \sqrt{n}\right) \\ & \quad + P\left(\sum_{i=1}^n |\bar{g}_N(f_i) - E \bar{g}_N(f_i)| \geq \lambda\sigma \sqrt{n}\right). \end{aligned}$$

Let  $p = [n^{\delta/2(1+\delta)}]$ ,  $k = [n/p]$  and  $N = n^{1/2(1+\delta)}$ . Define  $\bar{\xi}_j^{(s)} = E\{\bar{g}_N(f_j) - E\bar{g}_N(f_j) | \mathfrak{M}_{-s+j}^{s+j}\}$  and  $\bar{\eta}_j^{(s)} = \bar{g}_N(f_j) - E\bar{g}_N(f_j) - \bar{\xi}_j^{(s)}$ . Then, for  $j > 2s$ ,  $\bar{\xi}_j^{(s)}$  is measurable with respect to  $\mathfrak{M}_{j-s}^{j+s}$ , and  $\bar{\xi}_0^{(s)}$  is measurable with respect to  $\mathfrak{M}_{-s}^s$ . Hence

$$(38) \quad \begin{aligned} & E|\bar{\xi}_0^{(s)}| \cdot |\bar{\xi}_j^{(s)}| \\ & \leq \{E|\bar{\xi}_0^{(s)}|\}^2 + 8\{E|\bar{\xi}_0^{(s)}|^{2+\delta}\}^{2/(2+\delta)} \{\alpha(j-2s)\}^{\delta/(2+\delta)} \\ & \leq \frac{4}{N^{2(1+\delta)}} \{E|\bar{g}_N(f_0)|^{2+\delta}\}^2 \\ & \quad + 32\{E|\bar{g}_N(f_0)|^{(2+\delta)}\}^{2/(2+\delta)} \cdot \{\alpha(j-2s)\}^{\delta/(2+\delta)}. \end{aligned}$$

Moreover

$$\begin{aligned}
 (39) \quad & E|\bar{\xi}_i^{(s)}| \cdot |\bar{\eta}_j^{(s)}| \leq \{E|\bar{\eta}_j^{(s)}|^{(2+\delta)/(1+\delta)}\}^{(1+\delta)/(2+\delta)} \cdot \{E|\bar{\xi}_i^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \\
 & \leq \{E|\bar{\xi}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \{E|\bar{\eta}_0^{(s)}|^2\}^{1/2}
 \end{aligned}$$

and

$$\begin{aligned}
 (40) \quad & E|\bar{\eta}_i^{(s)}| \cdot |\eta_j^{(s)}| \\
 & \leq \{E|\bar{\eta}_0^{(s)}|^{(2+\delta)/(1+\delta)}\}^{(1+\delta)/(2+\delta)} \{E|\bar{\eta}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \\
 & \leq \{E|\bar{\eta}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \{E|\bar{\eta}_0^{(s)}|^2\}^{1/2}.
 \end{aligned}$$

Using (38), (39) and (40), we obtain

$$\begin{aligned}
 & \sum_{j=1}^n E|\bar{g}_N(f_0) - E\bar{g}_N(f_0)| \cdot |\bar{g}_N(f_j) - E\bar{g}_N(f_j)| \\
 & = \sum_{j=1}^n E|\bar{\xi}_0^{([j/3])} + \bar{\eta}_0^{([j/3])}| \cdot |\bar{\xi}_j^{([j/3])} + \bar{\eta}_j^{([j/3])}| \\
 & \leq \sum_{j=1}^n \{E|\bar{\xi}_0^{([j/3])}| \cdot |\bar{\xi}_j^{([j/3])}| + E|\bar{\xi}_0^{([j/3])}| \cdot |\bar{\eta}_j^{([j/3])}| \\
 & \quad + E|\bar{\eta}_0^{([j/3])}| \cdot |\bar{\xi}_j^{([j/3])}| + E|\bar{\eta}_0^{([j/3])}| \cdot |\bar{\eta}_j^{([j/3])}|\} \\
 & \leq \sum_{j=1}^n \left[ \frac{4}{N^{2(1+\delta)}} \{E|\bar{g}_N(f_0)|^{2+\delta}\}^2 \right. \\
 & \quad + 32\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{2/(2+\delta)} \left\{ \alpha\left(\left[\frac{j}{3}\right]\right) \right\}^{\delta/(2+\delta)} \\
 & \quad + 2\{E|\bar{\xi}_0^{([j/3])}|^{2+\delta}\}^{1/(2+\delta)} \cdot \{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2} \\
 & \quad \left. + \{E|\bar{\eta}_0^{([j/3])}|^{2+\delta}\}^{1/(2+\delta)} \cdot \{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2} \right] \\
 & \leq 4\{E|\bar{g}_N(f_0)|^{2+\delta}\}^2 \\
 & \quad + 32\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{2/(2+\delta)} \sum_{j=1}^{\infty} \left\{ \alpha\left(\left[\frac{j}{3}\right]\right) \right\}^{\delta/(2+\delta)} \\
 & \quad + 8\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{1/(2+\delta)} \sum_{j=1}^{\infty} \{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2} \\
 & \leq 4\gamma_N^2 + C_1\gamma_N^{2/(2+\delta)} + C_2\gamma_N^{1/(2+\delta)},
 \end{aligned}
 \tag{41}$$

where  $\gamma_N = E|\bar{g}_N(f_0)|^{2+\delta} \rightarrow 0$  ( $n \rightarrow \infty$ ). Hence we have

$$\begin{aligned}
 & P\left(\sum_{j=1}^n |\bar{g}_N(f_j) - E\bar{g}_N(f_j)| \geq \lambda\sigma \sqrt{n}\right) \\
 & \leq \frac{1}{\lambda^2\sigma^2 n} E\left\{\sum_{j=1}^n |\bar{g}_N(f_j) - E\bar{g}_N(f_j)|\right\}^2 \\
 & \leq \frac{1}{\lambda^2\sigma^2} \left\{E|\bar{g}_N(f_0) - E\bar{g}_N(f_0)|^2\right. \\
 (42) \quad & \left. + 2\sum_{j=1}^n E|\bar{g}_N(f_0) - E\bar{g}_N(f_0)| \cdot |\bar{g}_N(f_j) - E\bar{g}_N(f_j)|\right\} \\
 & \leq \frac{1}{\lambda^2\sigma^2} \left\{\frac{\gamma_N}{N^\delta} + 8\gamma_N^2 + 2C_1\gamma_N^{2/(2+\delta)} + 2C_2\gamma_N^{1/(2+\delta)}\right\} \\
 & \leq \frac{\varepsilon}{2\lambda^2}
 \end{aligned}$$

for all sufficiently large  $n$ . By the same argument as in the proof of theorem 4, we have

$$(43) \quad P\left(\max_{j \leq n} \left|\sum_{i=1}^j (g_N(f_i) - E g_N(f_i))\right| \geq \lambda\sigma \sqrt{n}\right) \leq \frac{\varepsilon}{2\lambda^2}$$

for all sufficiently large  $n$ . Thus it follows from (37), (42) and (43) that

$$(44) \quad P\left(\max_{j \leq n} |S_j| \geq 2\lambda\sigma \sqrt{n}\right) \leq \frac{\varepsilon}{2\lambda^2},$$

which implies the tightness of  $\{Z_n\}$ . The proof is completed.

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