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Abstract

The purpose of this paper is to present an introduction to a class of functional differential equations presently being studied vigorously by myself and some of my colleagues. The class includes difference equations, differential-difference equations as well as retarded functional differential equations; that is, those systems in which the derivative of the state of the system at a given time depends only upon the state of the system for previous values of time. If the solutions of our system have enough smoothness properties, then they satisfy equations for which the derivative of the state at a given time depend both upon the state and the derivative of the state for previous values of time; that is, neutral functional differential equations. The advantage in the approach seems to be the unification that is provided as well as the fact that a geometric theory becomes more feasible.

I. Difference and Differential-Difference Equations.

In this section, we discuss the fundamental classes of difference and differential-difference equations together with the information that is required to define a solution of such equations.

1. Difference equations. Let R^m be m-dimensional real Euclidean space. A difference equation is a relation of the form

$$x(t) = g(t,x(t-1),\dots,x(t-m), \tag{1.1}$$

where x is an n-vector, m is a positive integer and g is a continuous function on R^{nm+1}. Suppose $\sigma, x_1, \dots, x_m$ are given real numbers. A reasonable initial value problem for (1.1) is to specify that $x(\sigma-1) = x_1, \dots, x(\sigma-m) = x_m$ and determine $x(\sigma+k)$, $k = 0,1,2,\dots$ from relation (1.1). A solution of (1.1) is uniquely determined by the initial data $(\sigma, x_1, \dots, x_m)$ and depends continuously upon the initial data.

Since difference equations are not the main concern of this paper, we do not dwell upon the properties of solutions and refer the reader to the references [1,2]. It is instructive on the other hand to consider another class of difference equations which generally are not discussed in much detail. Consider the equation

$$x(t) = h(t,x(t-r_1),\dots,x(t-r_m)) \tag{1.2}$$

where h is continuous on R^{nm+1}, $0 < r_1 < \dots < r_m$, $m > 1$, are such that at least one of the ratios r_j/r_k is irrational. In such a situation, it is no longer possible to specify real numbers $(\sigma, x_1, \dots, x_m)$ and find a function $x(t)$ satisfying (1.2) for any $t > \sigma$ which also satisfies $x(\sigma-r_1) = x_1, \dots, x(\sigma-r_m) = x_m$. A reasonable initial value problem for (1.2) can be specified in the following manner. Suppose φ is a given continuous n-vector function on $[\sigma-r_m, \sigma]$ with

$$\varphi(\sigma) = h(\sigma, \varphi(\sigma-r_1), \dots, \varphi(\sigma-r_m)).$$

A solution of (1.2) through the point (σ, φ) is defined to be a continuous function defined on an interval $[\sigma-r_m, \sigma+A)$, $A > 0$, which coincides with φ on $[\sigma-r_m, \sigma]$ and satisfies (1.2) on $[\sigma, \sigma+A)$. The basic properties of these equations are discussed in [3].

2. Differential-difference equations. For simplicity in notation, we discuss in this section only a differential-difference equation (DDE) of the form

$$\dot{x}(t) = f(t,x(t),x(t-r),\dot{x}(t-r),x(t+\nu)) \tag{2.1}$$

where $r > 0$, $\nu > 0$ are given constants, $\dot{x}(t) = dx(t)/dt$ and x is an n-vector. Much more general situations will arise in the subsequent discussion. Even for (2.1), it is not clear how to phrase a reasonable problem for which there will exist a solution. To get a better feel for the difficulties involved, we consider special cases of (2.1) and as a consequence obtain a classification of DDE's. For more details, see [4].

By far the simplest type are the retarded differential-difference equations (RDDE)

$$\dot{x}(t) = f(t,x(t),x(t-r)), \tag{2.2}$$

where $r > 0$ and f is continuous on R^{2n+1}. Since a specification of x at t and $t - r$ uniquely determines $\dot{x}(t)$, a reasonable initial value problem for (2.2) is the following. Suppose σ is a given real number and φ is a continuous n-vector function on $[\sigma-r,\sigma]$. A function x will be said to be a solution of (2.2) through (σ,φ) if x is continuous on $[\sigma-r,\sigma+A)$, $A > 0$, coincides with φ on $[\sigma-r,\sigma]$, $\dot{x}(t)$ is continuous on $(\sigma,\sigma+A)$ and satisfies (2.2) on $(\sigma,\sigma+A)$.

Using the equivalent integral equation

$$(2.3) \qquad \begin{aligned} x(t) &= \varphi(t), \quad \sigma - r \le t \le \sigma, \\ x(t) &= \varphi(\sigma) + \int_\sigma^t f(s,x(s),x(s-r))ds, \quad t \ge \sigma, \end{aligned}$$

and proceeding as in ordinary differential equations, one can prove the local existence of a solution x through (σ,φ). With further

restrictions on f one obtains uniqueness and continuous dependence of the solution on the initial data (σ,φ).

The following observation is very important. Even though φ is only continuous, the solution through (σ,φ) is continuously differentiable for $t > \sigma$. This indicates that the natural evolution of the solution of system (2.2) is in the direction of increasing t. For a given φ, there may not exist any solution of (2.3) on an interval $[\sigma-r-A,\sigma]$, $A > 0$, which coincides with φ on $[\sigma-r,\sigma]$. In fact, if such a solution x exists, then there is an $\varepsilon > 0$ such that $\dot{\varphi}(t)$ must exist for $\sigma - \varepsilon < t \le \sigma$ and

$$\dot{\varphi}(t) = f(t,\varphi(t),x(t-r)), \quad \sigma - \varepsilon < t \le \sigma.$$

Therefore, for an arbitrary continuous function φ, there can be no solution in the direction of decreasing t. Furthermore, even if $\dot{\varphi}(t)$ is continuous for $\sigma - r \le t \le \sigma$, there will not be a solution unless

$$\dot{\varphi}(\sigma) = f(\sigma,\varphi(\sigma),\varphi(\sigma-r)). \tag{2.4}$$

The relation (2.4) may not even be enough since a solution of (2.2) for decreasing t will involve the determination of $x(t-r)$ from (2.2) as a function of $\dot{x}(t),x(t)$.

A differential-difference equation of neutral type (NDDE) is a relation of the form

$$\dot{x}(t) = f(t,x(t),x(t-r),\dot{x}(t-r)) \tag{2.5}$$

where $r > 0$ and f is continuous on R^{3n+1}. Since a specification of x at t and $x,\dot{x}$ at $t - r$ uniquely determines $\dot{x}(t)$, it is natural to specify the following initial value problem. Suppose σ is a given real number and φ is an n-vector function on $[\sigma-r,\sigma]$ which is continuous together with its first derivative. A solution of (2.5) through (σ,φ) is a function x defined on an interval $[\sigma-r,\sigma+A)$, $A > 0$, which coincides with φ on $[\sigma-r,\sigma]$, has a continuous first derivative except at the points $\sigma + kr$ for all $k = 0,1,2,\ldots$ for which $\sigma + kr$ belongs to $[\sigma,\sigma+A)$. A theory for (2.5) along this line is developed in [4].

One shortcoming of the above definition of the initial value problem is that it cannot be generalized to the situation in which there is general dependence of $\dot{x}(t)$ upon values of $x(s)$ for $s \leq t$. Even in (2.5), and even more so when r depends on t, there are great difficulties in discussing the dependence of solutions upon the initial data (σ,φ). Other objections arise if one tries to develop a geometric theory for (2.5) in the same spirit as in ordinary differential equations. There have been many papers devoted to the formulation of the initial value problem and the reader may consult [5,6] for references.

A very significant contribution to this question was made by Driver [7] who gave a formulation which recently has also been generalized by Melvin in his Ph.D. dissertation at Brown. We only illustrate the ideas for equation (2.5). Suppose φ is a given absolutely continuous function on $[\sigma-r,\sigma]$. A solution of (2.5) through (σ,φ) is an absolutely continuous function defined on an interval $[\sigma-r,\sigma+A)$, $A > 0$, coinciding with φ on $[\sigma-r,\sigma]$ and satisfying (2.5) almost everywhere on $[\sigma,\sigma+A)$. Of course, in order for this initial value problem to make sense, the function f must satisfy the following property. If x is any given absolutely continuous function on $[\sigma-r,\sigma+A)$ and

$$F(t) = f(t,x(t),x(t-r),\dot{x}(t-r)), \qquad \sigma \leq t < \sigma + A,$$

then the function F must be locally integrable on $[\sigma,\sigma+A)$. A satisfactory function f is

$$f(t,x,y,z) = g(t,x,y)z + h(t,x,y). \tag{2.6}$$

If f satisfies (2.6) and the initial value problem is defined as above, then a theory of existence, uniqueness and continuous dependence on the initial data is developed in [7].

As an alternative way to look at a special type of NDDE which occurs frequently in the applications, consider the equation

$$\dot{x}(t) = g(t,x(t-r))\dot{x}(t-r) + h(t,x(t),x(t-r)) \tag{2.7}$$

where g,h are continuous functions of their arguments and $g(t,x)$ has a continuous first derivative in t. If

$$G(t,x) = \int_0^x g(t,s)ds \tag{2.8}$$

then equation (2.7) can be written as

$$\frac{d}{dt}[x(t) - G(t,x(t-r))] = H(t,x(t),x(t-r)) \tag{2.9}$$

where $H(t,x,y) = h(t,x,y) - \partial G(t,y)/\partial t$. It is now possible to pose the following initial value problem for (2.9). Suppose φ is a given continuous n-vector function on $[\sigma-r,\sigma]$. A solution of (2.9) through (σ,φ) is a continuous function x defined on $[\sigma-r,\sigma+A)$, $A > 0$, coinciding with φ on $[\sigma-r,\sigma]$ such that the function $x(t) - G(t,x(t-r))$, not $x(t)$, is continuously differentiable on $(\sigma,\sigma+A)$ and satisfies (2.9) on $(\sigma,\sigma+A)$. A theory in this direction was initiated in [8], [9] and will receive more attention later. It is interesting to note that a discussion of (2.9) in this setting includes the RDDE (2.2) [take $G = 0$ in (2.9)] without the necessity of imposing additional smoothness conditions on the initial function φ.

In a NDDE and in contrast to RDDE, there is no reason to suspect that the solution enjoys any more smoothness properties than the

initial data. Also, it is conceivable in certain situations that a solution can be defined for decreasing t. To illustrate this point, consider the simple NDDE

$$\dot{x}(t) = f(t,x(t),x(t-r)) + \dot{x}(t-r). \tag{2.10}$$

We have already discussed a solution through (σ,φ) for increasing values of t. On the other hand if we write (2.10) as

$$\dot{x}(t-r) = \dot{x}(t) - f(t,x(t),x(t-r))$$

then the same procedure will define solutions for values of $t < \sigma$. Since there is generally no preference for integrating in any particular direction, this is probably the reason for the term neutral.

A differential-difference equation of advanced type (ADDE) is a relation of the form

$$\dot{x}(t) = f(t,x(t),x(t+r)) \tag{2.11}$$

where $r > 0$. Since this equation is the same as a RDDE with t replaced by $-t$, any solution of (2.11) which is defined for increasing t must correspond to initial data satisfying special conditions since it is the same as integrating a RDDE for decreasing t.

An example of a differential-difference equation of mixed type (MDDE) is a relation of the form

$$\dot{x}(t) = f(t,x(t),x(t-r),x(t+r)) \tag{2.12}$$

where $r > 0$. Very little is known about such equations and it certainly appears that a reasonable formulation for the existence of a solution should be in terms of boundary values at two given points.

As a final remark on DDE, one cannot avoid noticing the analogue between the above classification and the classification of second order partial differential equations. In fact, a RDDE has properties suggestive of a parabolic equation, a NDDE those of an hyperbolic equation and a MDDE those of an elliptic equation. Actually, these analogues are more than superficial because certain problems in parabolic equations can be reduced to a RDDE (see [10]) and some in hyperbolic to NDDE (see [11]). The relation mentioned with elliptic problems seems plausible.

II. Functional Differential Equations

In this section we generalize the concept of DDE and give some of the basic properties of the equations of retarded and neutral type.

3. Definition of the equation. Suppose $r \geq 0$ is a given real number, $R = (-\infty,\infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C([a,b],E^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into E^n with the topology of uniform convergence. If $[a,b] = [-r,0]$, let $C = C([-r,0],E^n)$ and designate the norm of an element φ in C by $|\varphi| = \sup_{-r\leq\theta\leq 0}|\varphi(\theta)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If $\sigma \in R$, $A \geq 0$ and $x \in C([\sigma-r,\sigma+A],E^n)$, then for each t in $[\sigma,\sigma+A]$, let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. The symbol Ω will always denote an open set in $R \times C$.

If $D,f\colon \Omega \to E^n$ are continuous, then a functional differential equation (FDE) is a relation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t). \tag{3.1}$$

A function x is said to be a solution of (3.1) if there are $\sigma \in R$, $A > 0$, such that $x \in C([\sigma-r,\sigma+A),E^n)$, $(t,x_t) \in \Omega$, $D(t,x_t)$ is continuously differentiable and satisfies (3.1) on $(\sigma,\sigma+A)$. It is not required that $x(t)$ be differentiable on $(\sigma,\sigma+A)$. A solution of (3.1) through $(\sigma,\varphi) \in \Omega$ is a solution $x = x(\sigma,\varphi)$ of (3.1) on $[\sigma-r,\sigma+A)$ such that $x_\sigma = \varphi$.

Equation (3.1) is very general and includes ordinary differential equations $[r = 0,\ D(t,\varphi) = \varphi(0)]$ as well as the following:

$$(3.2) \qquad \frac{d}{dt} x(t) = f(t,x_t)$$

$$(3.3) \qquad \frac{d}{dt} [x(t) - bx(t-r)] = f(t,x_t)$$

$$(3.4) \qquad \frac{d}{dt} x(t-r) = f(t,x_t)$$

$$(3.5) \qquad \frac{d}{dt} x(t - \frac{r}{2}) = f(t,x_t).$$

Much more general equations are included in (3.1) and a complete classification in the spirit of Section 2 is not available. However, equation (3.2) is called a retarded functional differential equation (RFDE) $[D(t,\varphi) = \varphi(0)]$ and equations (3.3) - (3.5) are respectively special cases of neutral (NFDE), advanced (AFDE) and mixed (MFDE) functional differential equations. Difference equations are also included in (3.1) by taking $f = 0$ and considering only initial (σ,φ) for which $D(\sigma,\varphi) = 0$.

From the above definition a FDE is a triple (D,f,r) and it is clear that the basic problem is to determine the behavior of the solutions of (3.1) on (D,f,r). To be more specific, suppose (D,f,r) are required to lie in some linear topological space. Given a certain property of the solutions of (3.1) for a given (D,f,r), is this property preserved for the solutions of equations (3.1) corresponding to those triples in a neighborhood of (D,f,r)? Most of the papers in the literature deal precisely with such questions.

Without more conditions on D in (3.1), it seems hopeless at the present time to obtain very general results. In fact, even the initial value problem for (3.1) will not have a solution since it includes (3.4) and (3.5), special cases of which were discussed in some detail in Section 2. Our first objective, therefore, is to impose additional restrictions on D in order for the initial value problem to be well defined.

<u>Definition 3.1</u>. Suppose $\Omega \subset R \times C$ is open, $D: \Omega \to E^n$ is continuous, $D(t,\varphi)$ has a continuous Frechet derivative $D_\varphi(t,\varphi)$ with respect to φ on Ω and

$$(3.6) \qquad D_\varphi(t,\varphi)\psi = \int_{-r}^{0} [d_\theta \mu(t,\varphi,\theta)]\psi(\theta)$$

for $(t,\varphi) \in \Omega$, $\psi \in C$, where $\mu(t,\varphi,\theta)$ is an $n \times n$ matrix function of bounded variation in $\theta \in [-r,0]$. For any β in $[-r,0]$, we say D is <u>atomic at β on Ω</u>, if

$$\det A(t,\varphi,\beta) \neq 0$$

$$(3.7) \qquad A(t,\varphi,\beta) = \mu(t,\varphi,\beta^+) - \mu(t,\varphi,\beta^-)$$

$$\left| \int_{\beta-s}^{\beta+s} [d_\theta \mu(t,\varphi,\theta)]\psi(\theta) - A(t,\varphi,\beta)\psi(\beta) \right| \leq \gamma(t,\varphi,s,\beta)|\psi|$$

for $(t,\varphi) \in \Omega$, $\psi \in C$, $s \geq 0$, where $A(t,\varphi,\beta)$ is continuous in (t,φ) and $\gamma(t,\varphi,s,\beta)$ is continuous in (t,φ,s), $\gamma(t,\varphi,0,\beta) = 0$.

Definition 3.2. A neutral functional differential equation (NFDE) is a system (3.1) for which D is atomic at zero on Ω. The system is autonomous if D, f are independent of t.

A RFDE corresponds to a NFDE with $D(t,\varphi) = \varphi(0)$. System (2.9) is a NFDE. Another very special case of a NFDE is the case in which $D(t,\varphi)$ is linear in φ and satisfies

$$D(t,\varphi) = \int_{-r}^{0} [d_\theta \mu(t,\theta)]\varphi(\theta)$$

$$\det B(t) \neq 0, \quad B(t) = \mu(t,0) - \mu(t,0^-) \tag{3.8}$$

$$\left| \int_{-s}^{0} [d_\theta \mu(t,\theta)]\varphi(\theta) - B(t)\varphi(0) \right| \leq \gamma(t,s)|\varphi|$$

for $(t,\varphi) \in \Omega$, $s \geq 0$, B, γ continuous, $\gamma(t,0) = 0$. A special case of this latter situation is (3.2) and the system

$$\frac{d}{dt}[x(t) - \alpha(t)x(h(t))] = -\dot{\alpha}(t)x(h(t)) \tag{3.9}$$

where $h(t) < t$, $\dot{h}(t) > 0$, $\ddot{h}(t)$ are continuous for $t \geq 0$ and $\alpha = 1/\dot{h}$. If x is a solution of (3.9) which has a derivative almost everywhere, then x will satisfy the equation $\dot{x}(t) = \dot{x}(h(t))$. These examples should indicate the conditions that are imposed by

defining a NFDE in the above manner. A more general formulation is contained in [9].

4. Basic properties of solutions. The following results are proved in [12].

Theorem 4.1. If (3.1) is a NFDE on Ω, then, for any $(\sigma,\varphi) \in \Omega$, there is a solution of (3.1) through (σ,φ).

Theorem 4.2. Suppose (3.1) is a NFDE on Ω and for any closed bounded set W in Ω with a δ-neighborhood of W in Ω, f maps W into a bounded set, D, D_φ are uniformly continuous on W and D is uniformly atomic at zero on W. If x is a noncontinuable solution of (3.1) on $[\sigma-r,b)$, then there is a t' in $[\sigma,b)$ such that $(t',x_{t'}) \notin W$. If $D(t,\varphi)$ is linear in φ, then $(t,x_t) \notin W$ for $t' \leq t < b$.

Theorem 4.3. Suppose (3.1) is a NFDE on Ω, $A(t,\varphi,0)$, D are uniformly continuous on closed bounded subsets of Ω and the solution $x(\sigma,\varphi)$ of (3.1) through (σ,φ) is unique, then $x(\sigma,\varphi)(t)$ is continuous in (σ,φ,t) in its domain of definition.

Results on the continuous dependence of a solution of (3.1) on (D,f,r) are also contained in [12]. Even for RFDE, the development of the field proceeded until the early 1950's by considering the solution $x(\sigma,\varphi)$ of (3.1) through (σ,φ) for a given value of t as a mapping from C into E^n. At that time, Krasovskii pointed out that the appropriate setting was to consider the solution at time t as a mapping from C into C, the mapping being defined by $x_t(\sigma,\varphi)$. More precisely, the converse theorems of Liapunov do not hold if the discussion is restricted to E^n, but do hold if the interpretation is in C (see [13, 14]). The presentation in this paper also assumes that the state space is C.

In the following, we always suppose the solution $x(\sigma,\varphi)(t)$ of the NFDE (3.11) through (σ,φ) is continuous in (t,σ,φ) and defined for all $t \geq \sigma$, $(\sigma,\varphi) \in R \times C$. Define the map $T(t,\sigma): C \to C$, $t \geq \sigma$, by

$$T(t,\sigma)\varphi = x_t(\sigma,\varphi). \tag{4.1}$$

Due to the fact that there is in general no reason to suspect the solution of a NFDE is any smoother than the initial data, it suggests that the mapping $T(t,\sigma)$ defined by (4.1) will be a homomorphism provided one can give sufficient conditions for the existence of a solution for decreasing t. This is actually the case and as a consequence of [12], we have

<u>Theorem 4.4.</u> Suppose D is atomic at 0 and $-r$ on Ω. If $D(t,\varphi)$, $A(t,\varphi,0)$, $A(t,\varphi,-r)$ are uniformly continuous on closed bounded subsets of Ω and $f(t,\varphi)$ is locally lipschitzian in φ,

then the mapping $T(t,\sigma)$ is a homeomorphism.

If $D(t,\varphi) = \varphi(0)$; that is (3.1) is a RFDE and $f(t,\varphi)$ is atomic at $-r$ on Ω, then the mapping $T(t,\sigma)$ is one-to-one.

For $r > 0$, the map $T(t,\sigma)$ corresponding to a RFDE can never be a homeomorphism. In fact, a simple application of the Arzela-Ascoli lemma implies the following results.

<u>Lemma 4.1.</u> For RFDE, the map $T(t,\sigma)$ is locally completely continuous for $t \geq \sigma + r$; that is, $T(t,\sigma)$ is continuous and for any $t \geq \sigma + r$, $\varphi \in C$, there is a neighborhood $V(t,\sigma,\varphi)$ of φ such that $T(t,\sigma)V(t,\sigma,\varphi)$ is precompact in C.

<u>Lemma 4.2.</u> For RFDE, if $f: R \times C \to E^n$ takes bounded sets into bounded sets and $T(t,\sigma)$ takes bounded sets into bounded sets, then $T(t,\sigma)$ is completely continuous for $t \geq \sigma + r$.

<u>Lemma 4.3.</u> For RFDE, if $f: R \times C \to E^n$ takes bounded sets into bounded sets and, for a given φ, $\{T(t,\sigma)\varphi, t \geq \sigma\}$ is bounded, then $\{T(t,\sigma)\varphi, t \geq \sigma\}$ belongs to a compact set of C.

Lemma 4.3 implies in particular that a bounded orbit of an autonomous RFDE has a nonempty ω-limit set which is compact, connected and invariant, a result very important in the study of stability of RFDE (see [15], [16]). Many other geometric properties of the solutions of a RFDE have been developed using C as the state space and we refer to [17-25] for results and references.

The fact that the solution operator $T(t,\sigma)$ of a RFDE is completely continuous allows one to apply sophisticated mathematical machinery to the study of the properties of solutions. Theorem 4.4 shows that no such nice property holds for the map $T(t,\sigma)$ of a NFDE. However, there is a large class of NFDE which seem to be important in the applications and such that an analogue of Lemma 4.3 holds and the map $T(t,\sigma)$ for t sufficiently large has the fixed point property; that is, if $T(t,\sigma)$ for t sufficiently large maps a closed bounded convex subset U of C into U, then there is a fixed point in U. This class is described in detail in the next section.

5. Stable operators. For simplicity, we suppose throughout this section that $D: C \to E^n$ is linear, continuous, atomic at zero and consider the NFDE

$$\frac{d}{dt} D(x_t) = f(t,x_t) \tag{5.1}$$

where $f: R \times C \to E^n$ is continuous and takes bounded sets into bounded sets. The following concept was introduced in [26].

Definition 5.1. D is said to be stable if there are $K > 0$, $\alpha > 0$ such that the solution $x = x(\varphi)$ of the homogeneous functional equation

$$D(x_t) = 0, \quad x_0 = \varphi, \quad D(\varphi) = 0$$

satisfies

$$|x_t(\varphi)| \leq Ke^{-\alpha t}|\varphi|, \quad t \geq 0.$$

If $D(\varphi) = \varphi(0)$, then D is stable; that is, RFDE correspond to a stable D. If $D(\varphi) = \sum_{k=1}^{N} A_k\varphi(-\tau_k)$ with the A_k constant $n \times n$ matrices, $\tau_j > 0$, τ_j/τ_k rational if $N > 1$, and all roots of the equation

$$\det [I - \sum_{k=1}^{N} A_k\rho^{-\tau_k}] = 0$$

have modulii less than one, then D is stable.

Lemma 5.1 [26]. If D is stable and (5.1) is autonomous, then every bounded orbit of (5.1) belongs to a compact subset of C.

Lemma 5.2 [Hale, unpublished]. If D is stable, $T(t,\sigma)$ is defined by (4.1) and maps closed bounded sets into bounded sets, then there is a $\beta > 0$ such that

$$T(t,\sigma) = T_1(t,\sigma) + T_2(t,\sigma)$$

where $T_1(t,\sigma)$ is a contraction and $T_2(t,\sigma)$ is completely continuous for $t > \beta$.

[illegible] seems to open up the possibility of many new applications in [illegible], say to the existence of periodic solutions of autonomous or nonautonomous equations (5.1).

Lemma 5.1 and stable operators D have been exploited in a systematic manner in the investigation of stability by means of Liapunov functionals (see [26, 28]). Suppose D is stable, L is linear and F, G as well as their first Fréchet derivatives vanish at zero. A detailed discussion of the behavior of the solutions of

$$\frac{d}{dt}[D(x_t) - G(x_t)] = L(x_t) + F(x_t) \tag{5.2}$$

near zero is contained in [29], [30].

6. <u>Behavior of solutions of RFDE.</u> In this section, we give many specific examples of RFDE in order to contrast the behavior with ordinary differential equations.

<u>Remark 6.1.</u> Two distinct solutions of (5.2) considered in E^n may intersect an infinite number of times. In fact, consider the scalar equation

$$\dot{x}(t) = x(t-\pi/2)$$

which has the solutions $x = \sin t$, $x = \cos t$.

If $x(\sigma,\varphi)$ is the solution of (5.2) through (σ,φ), let us define the <u>trajectory through (σ,φ)</u> as the set $\bigcup_{t\geq\sigma}(t, x_t(\sigma,\varphi))$ in $R \times C$. Uniqueness of solutions of (5.2) implies that if $x_\tau(\sigma,\varphi) = x_\tau(\sigma,\psi)$ for some $\tau > \sigma$ then $x_t(\sigma,\varphi) = x_t(\sigma,\psi)$ for $t \geq \tau$; that is, that part of the trajectories defined by taking $t \geq \tau$ coincide. That uniqueness does not necessarily hold in the direction of decreasing t is contained in

<u>Remark 6.2. The operator $T(t,\sigma)$ defined in (4.1) need not be one-to-one.</u> In fact, consider the scalar RFDE

$$\dot{x}(t) = -\alpha x(t-1)[1-x^2(t)] \tag{6.1}$$

where α is a constant. Equation (6.1) has the solution $x(t) = 1$ for all t in $(-\infty,\infty)$. On the other hand, if $\varphi \in C$, $\varphi(0) = 1$ then $x(0,\varphi)(t) = 1$ for all $t \geq 0$. Therefore, for all such initial values, $x_t(0,\varphi)$, $t \geq 1$, is the constant function 1. A translate of a subspace of C of codimension one is mapped by $T(t,0)$, $t \geq 0$, into a point.

The fact that the map $T(t,\sigma)$ need not be one-to-one is very disturbing. Sufficient conditions for one-to-oneness were given in Theorem 4.4, but it is instructive to look at the general situation in a little more detail. Suppose $\Omega = R \times C$ and all solutions $x(\sigma,\varphi)$ of (5.2) are defined on $[\sigma-r,\infty)$. <u>We say $(\sigma,\varphi) \in R \times C$ is equivalent to $(\sigma,\psi) \in R \times C$</u>, $(\sigma,\varphi) \sim (\sigma,\psi)$ if there is a $\tau \geq \sigma$

such that $x_\tau(\sigma,\varphi) = x_\tau(\sigma,\psi)$; that is (σ,φ) is equivalent to (σ,ψ) if the trajectories through (σ,φ) and (σ,ψ) have a point in common. It is easy to see that "~" is an equivalence relation and the space C is decomposed into equivalence classes $\{V_\alpha\}$ for each fixed σ. If $T(t,\sigma)$ is one-to-one, then each equivalence class consists of a single point; namely, the initial value φ at σ. For each equivalence class V_α choose a representative element $\varphi^{\sigma,\alpha}$ and let

$$W(\sigma) = \bigcup_\alpha \varphi^{\sigma,\alpha}. \tag{6.2}$$

From the point of view of the qualitative theory of functional differential equations, the set $W(\sigma)$ is very interesting since it is a maximal set on which the map $T(t,\sigma)$ is one-to-one. However, it seems to be very difficult to say much about the properties of $W(\sigma)$. In fact, without some more precise description of the manner in which $\varphi^{\sigma,\alpha}$ is chosen from V_α, one cannot hope to discuss such topological properties of $W(\sigma)$ as connectedness. For example, consider the scalar equation

$$\dot{x}(t) = 0$$

considered as a functional differential equation with lag $r > 0$. If $C_a = \{\varphi \in C: \varphi(0) = a\}$, then $\varphi \in C_a$ implies $x_t(\sigma,\varphi)$ is the constant function a for $t \geq \sigma + r$. Therefore, the equivalence

classes V_α are the sets C_α, $-\infty < \alpha < \infty$, for each σ. An arbitrary choice of $\varphi^{\sigma,\alpha}$ leads to a very uninteresting set $W(\sigma)$. On the other hand, $W(\sigma)$ consisting of all the constant functions is certainly the set that is of interest for the equation. In a general situation, we know nothing about the "appropriate" choice of $\varphi^{\sigma,\alpha}$. The following examples are given to indicate some of the other difficulties involved.

Remark 6.3. For autonomous linear equations, $W(0)$ is completely determined in a finite time interval and can be chosen as a linear subspace of C. In fact, for an autonomous linear equation, D. Henry [31] has shown there is a number τ such that if $x_t(0,\varphi) = x_t(0,\psi)$ for $t \geq t_0$, then $t_0 \leq \tau$; that is, the equivalence classes V_α are completely determined in the interval $[0,\tau]$. Let $T(t,0) = T(t)$ and consider the set $S = \{\varphi \in C: T(t)\varphi = 0, t \geq \tau\}$. This is a closed linear subspace of C invariant under $T(t)$. The set S admits projection in C (continuous?), $C = S \oplus U$ where U is also invariant under $T(t)$. Furthermore, $T(t)$ is one-to-one on U. Thus, we can take $W(0) = U$ and each element of U corresponds to one of the equivalence classes V_α.

Remark 6.4. For nonlinear equations, the equivalence classes V_α may involve the consideration of trajectories which have a point in common after any preassigned times. The following example is due to A. Hausrath. For $\beta > 0$, $r = 1$, consider the scalar equation

$$\dot{x}(t) = \beta[|x_t| - x(t)].$$

For a given φ in $C = C([-1,0],R)$, there is a unique solution $x = x(\varphi,\beta)(t)$ of this equation through $(0,\varphi)$ which is continuous in $(\varphi,\beta.t)$.

If $\varphi(0) \geq 0$, $\varphi \neq 0$, then $x(\varphi,\beta)(t)$ is a positive constant for $t \geq 1$. In fact, since $\dot{x}(t) \geq 0$, it follows that $|x_t| = x(t)$ for $t \geq 1$ and uniqueness implies $x(t)$ is a constant $\geq \varphi(0)$ for $t \geq 1$. Also, if $\varphi(0) = 0$, then $\varphi \neq 0 \Longrightarrow \dot{x}(0) > 0$ and $x(t) > 0$ for $t \geq 1$. Therefore, for any positive constant function, the corresponding equivalence class contains more than one element. Also, the above argument and the autonomous nature of the equation show that the equivalence class corresponding to the constant function zero contains only zero.

If $\varphi(0) < 0$, then it is clear that $x(\varphi,\beta)(t)$ approaches a constant as $t \to \infty$. If $x(\varphi,\beta)(t)$, $\varphi(0) < 0$, has a zero $z(\varphi,\beta)$, it must be simple and, therefore, $z(\varphi,\beta)$ is continuous in φ,β. For any $\beta > 0$, there exists a $\varphi \in C$ such that $z(\varphi,\beta)$ exists. In fact, let $\varphi \in C$, $\varphi(0) = -1$, $\varphi(\theta) = -\gamma$, $\gamma > 1$, $-1 \leq \theta \leq -1/2$ and let $\varphi(\theta)$ be a monotone increasing function for $-1/2 \leq \theta \leq 0$. As long as $x(t) \leq 0$ and $0 \leq t \leq 1/2$, we have $|x_t| = \gamma$ and

$$\dot{x}(t) = \beta[\gamma - x(t)] \geq \beta\gamma.$$

Therefore, $x(t) \geq \beta\gamma t - 1$. For $\beta\gamma/2 > 1$, it follows that x must

have a zero.

The closed subset $C_{-1} = \{\varphi \in C\colon \varphi(0) = -1\}$ can be written as $C_{-1} = C^o_{-1} \cup C^n_{-1}$ where $C^o_{-1} = \{\varphi \in C_{-1}\colon z(\varphi,\beta)$ exists$\}$, $C^n_{-1} = \{\varphi \in C_{-1}\colon z(\varphi,\beta)$ does not exist$\}$. Since $z(\varphi,\beta)$ is continuous, the set C^o_{-1} is open and, therefore, C^n_{-1} is closed. For any $\varphi \in C^n_{-1}$, $x(\varphi,\beta)(t) \to 0$ monotonicity as $t \to \infty$. Therefore, if C^n_{-1} is not empty, then there is a sequence $\varphi_j \in C^o_{-1}$, $\varphi_j \to \varphi \in C^n_{-1}$ as $j \to \infty$ and $z(\varphi_j,\beta) \to \infty$ as $j \to \infty$.

There is a $\beta_o > 0$ such that C^n_{-1} is not empty. In fact, choose $\beta_o > 0$ less than or equal to that value of β for which the equation $\lambda + \beta = -\beta e^{-\lambda}$ has a real root λ_o of multiplicity two. For this β_o, the equation $\lambda + \beta = -\beta e^{-\lambda}$ has two real negative roots. If λ_o is one of these roots, then $x(t) = -e^{-\lambda_o t}$ is a solution of the above equation with initial value $\varphi_o(\theta) = -e^{-\lambda_o \theta}$, $-r \leq \theta \leq 0$, $\varphi_o \in C_{-1}$. Therefore, C^n_{-1} is not empty.

With β_o as above, it follows that $\delta(\beta_o) \overset{\text{def}}{=} \sup_{\varphi \in C^o_{-1}} z(\varphi,\beta_o) = \infty$. Therefore, using the fact that our original equation is positive homogeneous of degree 1 in x, it follows that for any positive constants a, t_o, there exists a $\varphi \in C$, such $x(\varphi,\beta_o)(t) = a$, $t \geq t_o$, $x(\varphi,\beta_o)(t) < a$ for $0 \leq t \leq t_o$. This proves the assertion in the remark.

In [31], it is shown that linear autonomous equations have the property that no two distinct solutions can exist on $(-\infty,\infty)$ and coincide on $[0,\infty)$. The following remark asserts this statement is false for nonlinear equations.

Remark 6.5. There may be two distinct solutions of a RFDE defined on $(-\infty,\infty)$ and yet they coincide on $[0,\infty)$. The following example is due to A. Hausrath. Let $r = 1$, $f(s) = 0$, $0 \le s \le 1$, $f(s) = -3\left(\sqrt[3]{s} - 1\right)^2$, $s > 1$, and consider the equation

$$\dot{x}(t) = f(|x_t|).$$

The function $x \equiv 0$ is a solution of this equation on $(-\infty,\infty)$. Also, the function $x(t) = -t^3$, $t < 0$, $= 0$, $t \ge 0$ is also a solution. In fact, since $x \le 1$ for $t \ge -1$, it is clear that x satisfies the equation for $t \ge 0$. Since x is monotone decreasing for $t \le 0$, $|x_t| = x(t-1) = -(t-1)^3$ and $\dot{x}(t) = -3t^2$. It is easy to verify that $-3t^2 = f((1-t)^3)$ for $t < 0$.

Remark 6.6. The map $T(t,\sigma)$ is locally bounded for any $t \ge \sigma$; that is, for any $t \ge \sigma$, $\varphi \in C$, there is a neighborhood $V(t,\sigma,\varphi)$ of φ such that $T(t,\sigma)V(t,\sigma,\varphi)$ is bounded. This is an immediate consequence of the continuity of $T(t,\sigma)\varphi$ in φ. The fact that $T(t,\sigma)\varphi$ is continuous in t,σ,φ actually implies the following stronger result: For any $T > 0$, $\sigma \in R$, $\varphi \in C$, $\varepsilon > 0$, there is a neighborhood $V(\varepsilon,\sigma,\varphi,T)$ of φ such that

$$|T(t,\sigma)\psi - T(t,\sigma)\varphi| < \varepsilon, \quad \sigma \le t \le \sigma + T, \ \psi \in V(\varepsilon,\sigma,\varphi,T).$$

Remark 6.7. $T(t,\sigma)$ may not take closed bounded sets of C

into bounded sets of C. The following example is due to K. Hannsgen. Suppose $r = 1/4$, $C = C([-r,0],R)$ and consider the equation

$$(6.3) \qquad \dot{x}(t) = f(t,x_t) \overset{\text{def}}{=} x^2(t) - \int_{\min\{t-r,0\}}^{0} |x(s)|\,ds.$$

It is clear that f takes closed bounded sets into bounded sets and is even locally Lipschitzian. If $B = \{\varphi \in C: |\varphi| \le 1\}$ and $x(b)$ is the solution of (6.3), then $x(b)$ is always ≥ -1. Also, $\dot{x}(b)(0^+) < 1$ for all $b \in B$ and, thus, there is a $\sigma > 0$, independent of b such that $x(b)(\sigma) < (1-\sigma)^{-1}$. If $y(t,\sigma,x(b)(\sigma))$, $y(\sigma,\sigma,x(b)(\sigma)) = x(b)(\sigma)$ is the solution of $\dot{y}(t) = y^2(t)$, then $-1 \le x(b)(t) \le y(t,\sigma,x(b)(\sigma)) < (1-t)^{-1}$, $\sigma \le t \le r$. Thus, $x(b)(t)$ exists for $-r \le t \le r$ and $x(b)(r) < (1-r)^{-1}$ for all $b \in B$. For $t \ge r$, $\dot{x}(b)(t) = x^2(b)(t)$ and the fact that $x(b)(r) < (1-r)^{-1}$ implies $x(b)(t)$ exists for $-r \le t \le 1$.

If we show that for any $\varepsilon > 0$, there is a $b \in B$ such that $x(b)(r) > (1-r)^{-1} - \varepsilon$, then the set $x(B)(1)$ is not bounded. To show this, suppose $\varepsilon > 0$ is given, $C = |1-r|^{-1}$, $M = 2Cre^{2Cr} + 1$. Choose $b \in B$ so that $b(0) = 1$, $\int_{t-r}^{0} (b(t))dt < \varepsilon/M$ and let $y(t) = y(t,0,1)$, $y(0,0,1) = 1$, be the solution of $\dot{y}(t) = y^2(t)$ and $x(t) = x(b)(t)$. If $\psi(t) = y(t) - x(t)$ for $0 < t < r$, then $\psi(t) \ge 0$ and $\dot{\psi}(t) \le 2C\psi(t) + \varepsilon/M$. Since $\psi(0) = 0$, one thus obtains $\psi(r) \le \varepsilon$. This shows that $x(r) = y(r) - \psi(r) = (1-r)^{-1} - \psi(r) \ge (1-r)^{-1} - \varepsilon$ and proves the general assertion made above.

Remark 6.8. There are functional differential equations for which there is a $t_o > 0$ with $T(t,\sigma)C = \{0\}$ for all $t \geq t_o$. The following example is taken from [32]. Consider the equation

$$\dot{x}(t) = -\alpha(t)x(t-1)$$

where

$$\alpha(t) = 2\sin^2 \pi t, \qquad t \in [2n, 2n+1]$$
$$= 0 \quad , \qquad t \in (2n-1, 2n)$$

for each integer n. For any $\sigma \in R$, $\varphi \in C$, we show $T(t,\sigma)\varphi = 0$, $t \geq \sigma + 4$. In fact, if N is the smallest odd integer such that $N \geq \sigma$, then $x(t) = x(N)$, $t \in [N, N+1]$ and

$$\dot{x}(t) = -\alpha(t)x(N), \qquad t \in [N+1, N+2].$$

Thus,

$$x(N+2) = x(N)[1-2\int_{N+1}^{N+2} \sin^2 \pi s ds] = 0.$$

Therefore, $x(t) = 0$ for $t \in [N+2, N+3]$ and $x(t) = 0$ for $t \geq N + 2$.

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13. ABSTRACT

The purpose of this paper is to present an introduction to a class of functional differential equations presently being studied vigorously by myself and some of my colleagues. The class includes difference equations, differential difference equations as well as retarded functional differential equations; that is, those systems in which the derivative of the state of the system at a given time depends only upon the state of the system for previous values of time. If the solutions of our system have enough smoothness properties, then they satisfy equations for which the derivative of the state at a given time depend both upon the state and the derivative of the state for previous values of time; that is, neutral functional differential equations. The advantage in the approach seems to be the unification that is provided as well as the fact that a geometric theory becomes more feasible.

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