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Functional equations for Mahler measures of genus-one curves

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In this paper we will establish functional equations for Mahler measures of families of genus-one two-variable polynomials. These families were previously studied by Beauville, and their Mahler measures were considered by Boyd, Rodriguez Villegas, Bertin, Zagier, and Stienstra. Our functional equations allow us to prove identities between Mahler measures that were conjectured by Boyd. As a corollary, we also establish some new transformations for hypergeometric functions.

1. History and introduction

The goal of this paper is to establish identities between the logarithmic Mahler measures of polynomials with zero varieties corresponding to genus-one curves. Recall that the logarithmic Mahler measure (which we shall henceforth simply refer to as the Mahler measure) of an n-variable Laurent polynomial $P(x_1, x_2, \ldots, x_n)$ is defined by

$$\mathrm{m}\left(P(x_1, \ldots, x_n)\right) = \int_0^1 \cdots \int_0^1 \log \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right) \right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.$$

Many difficult questions surround the special functions defined by Mahler measures of elliptic curves.

The first example of the Mahler measure of a genus-one curve was studied in [Boyd 1998; Deninger 1997]. Boyd found that

$$\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} L'(E, 0), \tag{1-1}$$

where E denotes the elliptic curve of conductor 15 that is the projective closure of $1 + x + 1/x + y + 1/y = 0$. As usual, $L(E, s)$ is its L-function, and the question mark above the equals sign indicates numerical equality verified up to 28 decimal places.

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Deninger [1997] gave an interesting interpretation of this formula. He obtained the Mahler measure by evaluating the Bloch regulator of an element $\{x, y\}$ from a certain K-group. In other words, the Mahler measure is given by a value of an Eisenstein–Kronecker series. Therefore Bloch's and Beilinson's conjectures predict that

$$\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big)=cL'(E,0),$$

where c is some rational number. Let us add that, even if Beilinson's conjectures were known to be true, this would not suffice to prove equality (1-1), since we still would not know the height of the rational number c.

This picture applies to other situations as well. Boyd [1998] performed extensive numerical computations within the family of polynomials $k+x+1/x+y+1/y$, as well as within some other genus-one families. Boyd's numerical searches led him to conjecture identities such as

$$\mathrm{m}\Big(5+x+\frac{1}{x}+y+\frac{1}{y}\Big)\stackrel{?}{=}6\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big),$$

$$\mathrm{m}\Big(8+x+\frac{1}{x}+y+\frac{1}{y}\Big)\stackrel{?}{=}4\mathrm{m}\Big(2+x+\frac{1}{x}+y+\frac{1}{y}\Big).$$

Boyd conjectured conditions predicting when formulas like (1-1) should exist for the Mahler measures of polynomials with integral coefficients. This was further studied by Rodriguez Villegas [1999], who interpreted these conditions in the context of Bloch's and Beilinson's conjectures. He also used modular forms to express the Mahler measures as Kronecker–Eisenstein series in more general cases. In turn, this allowed him to prove some equalities such as

$$\mathrm{m}\Big(4\sqrt{2}+x+\frac{1}{x}+y+\frac{1}{y}\Big)=L'\big(E_{4\sqrt{2}},0\big), \tag{1-2}$$

$$\mathrm{m}\Big(3\sqrt{2}+x+\frac{1}{x}+y+\frac{1}{y}\Big)=qL'\big(E_{3\sqrt{2}},0\big), \tag{1-3}$$

where q is a rational number that is (numerically) equal to $5/2$. The first equality can be proved using the fact that the corresponding elliptic curve has complex multiplication, and therefore the conjectures are known for this case due to Bloch [2000]. The second equality depends on the fact that one has the modular curve $X_0(24)$, and the conjectures then follow from a result of Beilinson.

Rodriguez Villegas [2002] subsequently used the relationship between Mahler measures and regulators to prove a conjecture of Boyd [1998]:

$$\mathrm{m}(y^2+2xy+y-x^3-2x^2-x)=\tfrac{5}{7}\mathrm{m}(y^2+4xy+y-x^3+x^2).$$

He proved this identity without actually expressing the Mahler measures in terms of L-series. Bertin [2004] has also proved similar identities using these ideas.

Although the conjecture in (1-1) remains open, we will in fact prove two of Boyd's other conjectures this paper.

Theorem 1.1. *Assume that $q = 5/2$ in (1-3). Then*

$$\mathrm{m}\Big(2 + x + \frac{1}{x} + y + \frac{1}{y}\Big) = L'\big(E_{3\sqrt{2}}, 0\big), \tag{1-4}$$

$$\mathrm{m}\Big(8 + x + \frac{1}{x} + y + \frac{1}{y}\Big) = 4L'\big(E_{3\sqrt{2}}, 0\big). \tag{1-5}$$

Our proof of this combines two interesting functional equations for the function

$$m(k) := \mathrm{m}\Big(k + x + \frac{1}{x} + y + \frac{1}{y}\Big).$$

Kurokawa and Ochiai [2005] recently proved the first functional equation, which says that, if $k \in \mathbb{R}\backslash\{0\}$,

$$m(4k^2) + m\Big(\frac{4}{k^2}\Big) = 2m\Big(2\Big(k + \frac{1}{k}\Big)\Big). \tag{1-6}$$

In Section 3 we use regulators to give a new proof of Equation (1-6). We will also prove a second functional equation in Section 2.1 using q-series. In particular, if k is nonzero and $|k| < 1$,

$$m\Big(2\Big(k + \frac{1}{k}\Big)\Big) + m\Big(2\Big(\mathrm{i}k + \frac{1}{\mathrm{i}k}\Big)\Big) = m\Big(\frac{4}{k^2}\Big). \tag{1-7}$$

Theorem 1.1 follows from setting $k = 1/\sqrt{2}$ in both identities, and then showing that $5m(\mathrm{i}\sqrt{2}) = 3m(3\sqrt{2})$. We have proved this final equality in Section 3.6.

This paper is divided into two sections of roughly equal length. In Section 2 we prove more identities like (1-7), which arise from expanding Mahler measures in q-series. In particular, we look at identities for four special functions defined by the Mahler measures of genus-one curves (see Equations (2-1) through (2-4) for notation). Equation (2-14) is undoubtedly the most important result in this part of the paper, since it implies that infinitely many identities like (1-7) exist. Sections 2.1 and 2.2 are mostly devoted to transforming special cases of (2-14) into interesting identities between the Mahler measures of rational polynomials. While the theorems in those subsections rely heavily on Ramanujan's theory of modular equations to alternative bases, we have attempted to maximize readability by eliminating q-series manipulation wherever possible. Finally, we have devoted Section 2.3 to proving some useful computational formulas. As a corollary we establish several new transformations for hypergeometric functions, including

$$\sum_{n=0}^{\infty}\left(\frac{k(1-k)^2}{(1+k)^2}\right)^n \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j} = \frac{(1+k)^2}{\sqrt{(1+k^2)\big((1-k-k^2)^2-5k^2\big)}} \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5(1+k-k^2)}{(1+k^2)^2\big((1-k-k^2)^2-5k^2\big)^2}\right). \tag{1-8}$$

We have devoted Section 3 to further studying the relationship between Mahler measures and regulators. We show how to recover the Mahler measure q-series expansions and the Kronecker–Eisenstein series directly from Bloch's formula for the regulator. This in turn shows that the Mahler measure identities can be viewed as consequences of functional identities for the elliptic dilogarithm.

Many of the identities in this paper can be interpreted from both a regulator perspective and from a q-series perspective. The advantage of the q-series approach is that it simplifies the process of finding new identities. The fundamental result in Section 2, Equation (2-14), follows easily from the Mahler measure q-series expansions. Unfortunately the q-series approach does not provide an easy way to explain identities like (1-6). Unlike most of the other formulas in Section 2, Kurokawa's and Ochiai's result *does not* follow from (2-14). An advantage of the regulator approach, is that it enables us to construct proofs of both (1-6) and (1-7) from a unified perspective. Additionally, the regulator approach seem to provide the only way to prove the final step in Theorem 1.1, namely to show that $5m(\mathrm{i}\sqrt{2}) = 3m(3\sqrt{2})$. Thus, a complete view of this subject matter should incorporate both regulator and q-series perspectives.

2. Mahler measures and q-series

We will consider four important functions defined by Mahler measures:

$$\mu(t) = \mathrm{m}\left(\frac{4}{\sqrt{t}}+x+\frac{1}{x}+y+\frac{1}{y}\right), \tag{2-1}$$

$$n(t) = \mathrm{m}\left(x^3+y^3+1-\frac{3}{t^{1/3}}xy\right), \tag{2-2}$$

$$g(t) = \mathrm{m}\left((x+y)(x+1)(y+1)-\frac{1}{t}xy\right), \tag{2-3}$$

$$r(t) = \mathrm{m}\left((x+y+1)(x+1)(y+1)-\frac{1}{t}xy\right). \tag{2-4}$$

Throughout Section 2 we will use the notation $\mu(t) = m(4/\sqrt{t})$ for convenience. Recall from [Rodriguez Villegas 1999] and [Stienstra 2006] that each of these functions has a simple q-series expansion when t is parameterized correctly. To summarize, if we let $(x;q)_\infty = (1-x)(1-xq)\left(1-xq^2\right)\ldots$, and

$$M(q) = 16q\frac{(q;q)_\infty^8\left(q^4;q^4\right)_\infty^{16}}{\left(q^2;q^2\right)_\infty^{24}}, \tag{2-5}$$

$$N(q) = \frac{27q\left(q^3;q^3\right)_\infty^{12}}{(q;q)_\infty^{12} + 27q\left(q^3;q^3\right)_\infty^{12}}, \tag{2-6}$$

$$G(q) = q^{1/3}\frac{\left(q;q^2\right)_\infty}{\left(q^3;q^6\right)_\infty^3}, \tag{2-7}$$

$$R(q) = q^{1/5}\frac{\left(q;q^5\right)_\infty\left(q^4;q^5\right)_\infty}{\left(q^2;q^5\right)_\infty\left(q^3;q^5\right)_\infty}, \tag{2-8}$$

then for $|q|$ sufficiently small,

$$\mu(M(q)) = -\operatorname{Re}\left(\frac{1}{2}\log(q) + 2\sum_{j=1}^{\infty} j\chi_{-4}(j)\log(1-q^j)\right), \tag{2-9}$$

$$n(N(q)) = -\operatorname{Re}\left(\frac{1}{3}\log(q) + 3\sum_{j=1}^{\infty} j\chi_{-3}(j)\log(1-q^j)\right), \tag{2-10}$$

$$g(G^3(q)) = -\operatorname{Re}\left(\log(q) + \sum_{j=1}^{\infty}(-1)^{j-1} j\chi_{-3}(j)\log(1-q^j)\right), \tag{2-11}$$

$$r(R^5(q)) = -\operatorname{Re}\left(\log(q) + \sum_{j=1}^{\infty} j\operatorname{Re}\left((2-\mathrm{i})\chi_r(j)\right)\log(1-q^j)\right). \tag{2-12}$$

In particular, $\chi_{-3}(j)$ and $\chi_{-4}(j)$ are the usual Dirichlet characters, and $\chi_r(j)$ is the character of conductor five with $\chi_r(2) = \mathrm{i}$. We have used the notation $G(q)$ and $R(q)$, as opposed to something like $\tilde{G}(q) = G^3(q)$, in order to preserve Ramanujan's notation. As usual, $G(q)$ corresponds to Ramanujan's cubic continued fraction, and $R(q)$ corresponds to the Rogers–Ramanujan continued fraction [Andrews and Berndt 2005].

The first important application of the q-series expansions is that they can be used to calculate the Mahler measures numerically. For example, we can calculate $\mu(1/10)$ with Equation (2-9), provided that we can first determine a value of q for which $M(q) = 1/10$. Fortunately, the theory of elliptic functions shows that if

$\alpha = M(q)$, then

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right). \tag{2-13}$$

Using Equation (2-13) we easily compute $q = .01975\ldots$, and it follows that $\mu(1/10) = 2.524718\ldots$ The function defined in Equation (2-13) is called the *elliptic nome*, and is sometimes denoted by $q_2(\alpha)$. Theorem 2.6 provides similarly explicit inversion formulas for Equations (2-5) through (2-8).

The second, and perhaps more significant fact that follows from these q-series, is that linear dependencies exist between the Mahler measures. In particular, if

$$f(q) \in \left\{\mu(M(q)), n(N(q)), g(G^3(q)), r(R^5(q))\right\},$$

then for an appropriate prime p

$$\sum_{j=0}^{p-1} f\left(\mathrm{e}^{2\pi \mathrm{i} j/p} q\right) = (1 + p^2\chi(p)) f(q^p) - p\chi(p) f(q^{p^2}), \tag{2-14}$$

where $\chi(j)$ is the character from the relevant q-series. The prime p satisfies the restriction that $p \neq 2$ when $f(q) = g(G^3(q))$, and $p \not\equiv 2, 3 \pmod 5$ when $f(q) = r\left(R^5(q)\right)$. The astute reader will immediately recognize that (2-14) is essentially a Hecke eigenvalue equation. A careful analysis of the exceptional case that occurs when $p = 2$ and $f(q) = g(G^3(q))$ leads to the important and surprising inverse relation:

$$\begin{aligned} 3n(N(q)) &= g(G^3(q)) - 8g(G^3(-q)) + 4g(G^3(q^2)), \\ 3g(G^3(q)) &= n(N(q)) + 4n(N(q^2)). \end{aligned} \tag{2-15}$$

In the next two subsections we discuss methods for transforming (2-14) and (2-15) into so-called functional equations.

2.1. ***Functional equations from modular equations.*** Since the primary goal of this paper is to find relations between the Mahler measures of *rational* (or at least algebraic) polynomials, we will require modular equations to simplify our results. For example, consider (2-14) when $f(q) = \mu(M(q))$ and $p = 2$:

$$\mu(M(q)) + \mu(M(-q)) = \mu(M(q^2)). \tag{2-16}$$

For our purposes, Equation (2-16) is only interesting if $M(q)$, $M(-q)$, and $M(q^2)$ are all simultaneously algebraic. Fortunately, it turns out that $M(q)$ and $M(q^2)$ (hence also $M(-q)$ and $M(q^2)$) satisfy a well known polynomial relation.

Definition 2.1. Suppose that $F(q) \in \{M(q), N(q), G(q), R(q)\}$. An n–th degree modular equation is an algebraic relation between $F(q)$ and $F(q^n)$.

We will not need to derive any new modular equations in this paper. Berndt proved virtually all of the necessary modular equations while editing Ramanujan's notebooks; see [Andrews and Berndt 2005; Berndt 1989; 1991; 1998]. Ramanujan seems to have arrived at most of his modular equations through complicated q-series manipulations (of course this is speculation since he did not write down any proofs!). Modular equations involving $M(q)$ correspond to the classical modular equations [Berndt 1991], relations for $N(q)$ correspond to Ramanujan's signature three modular equations [Berndt 1998], and most of the known modular equations for $G(q)$ and $R(q)$ appear in [Andrews and Berndt 2005].

Now we can finish simplifying Equation (2-16). Since the classical second-degree modular equation shows that whenever $|q| < 1$,

$$\frac{4M(q^2)}{\left(1+M(q^2)\right)^2} = \left(\frac{M(q)}{M(q)-2}\right)^2,$$

we easily obtain the parameterizations:

$$M(q) = \frac{4k^2}{(1+k^2)^2}, \quad M(-q) = \frac{-4k^2}{(1-k^2)^2}, \quad \text{and} \quad M(q^2) = k^4.$$

Substituting these parametric formulas into Equation (2-16) yields:

Theorem 2.2. *The following identity holds whenever* $|k| < 1$:

$$\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right) = \mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right) + \mathrm{m}\left(2\mathrm{i}\left(k-\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

We need to make a few remarks about working with modular equations before proving the main theorem in this section. Suppose that for some algebraic function $P(X, Y)$:

$$P(F(q), F(q^p)) = 0,$$

where $F(q) \in \{M(q), N(q), G(q), R(q)\}$. By an elementary change of variables $q \to \mathrm{e}^{2\pi \mathrm{i} j/p} q$, it follows that $P(F(\mathrm{e}^{2\pi \mathrm{i} j/p} q), F(q^p)) = 0$ for every $j \in \{0, 1, \dots, p-1\}$. If $P(X, Y)$ is symmetric in X and Y, it also follows that $P(F(q^{p^2}), F(q^p))$ vanishes. Therefore, if $P(X, Y)$ is sufficiently simple (for example a symmetric genus-zero polynomial), we can find simultaneous parameterizations for $F(q^p)$, $F(q^{p^2})$, and $F(\mathrm{e}^{2\pi \mathrm{i} j/p} q)$ for all j. In such an instance, (2-14) reduces to an interesting functional equation for one of the four Mahler measures $\mu(t)$, $n(t)$, $g(t)$, $r(t)$. Five basic functional equations follow from applying these ideas to (2-14).

Theorem 2.3. *For* $|k| < 1$ *and* $k \neq 0$, *we have*

$$\mu\left(\frac{4k^2}{(1+k^2)^2}\right) + \mu\left(\frac{-4k^2}{(1-k^2)^2}\right) = \mu(k^4). \tag{2-17}$$

The following identities hold for $|u|$ *sufficiently small but nonzero*:

$$n\left(\frac{27u(1+u)^4}{2(1+4u+u^2)^3}\right) + n\left(-\frac{27u(1+u)}{2(1-2u-2u^2)^3}\right) = 2n\left(\frac{27u^4(1+u)}{2(2+2u-u^2)^3}\right) - 3n\left(\frac{27u^2(1+u)^2}{4(1+u+u^2)^3}\right). \tag{2-18}$$

If $\zeta_3 = e^{2\pi i/3}$ *and* $Y(t) = 1 - \left(\frac{1-t}{1+2t}\right)^3$, *then*

$$n(u^3) = \sum_{j=0}^{2} n(Y(\zeta_3^j u)). \tag{2-19}$$

If $\zeta_3 = e^{2\pi i/3}$ *and* $Y(t) = t\left(\frac{1-t+t^2}{1+2t+4t^2}\right)$, *then*

$$g(u^3) = \sum_{j=0}^{2} g(Y(\zeta_3^j u)). \tag{2-20}$$

If $\zeta_5 = e^{2\pi i/5}$ *and* $Y(t) = t\left(\frac{1-2t+4t^2-3t^3+t^4}{1+3t+4t^2+2t^3+t^4}\right)$, *then*

$$r(u^5) = \sum_{j=0}^{4} r(Y(\zeta_5^j u)). \tag{2-21}$$

Proof. We have already sketched a proof of (2-17) in the discussion preceding Theorem 2.2.

Proving (2-18) requires the second-degree modular equation from Ramanujan's theory of signature 3. If $\beta = N(q^2)$ and α is either $N(q)$, $N(-q)$, or $N(q^4)$, then

$$27\alpha\beta(1-\alpha)(1-\beta) - (\alpha+\beta-2\alpha\beta)^3 = 0. \tag{2-22}$$

If we choose u so that $N(q^2) = 27u^2(1+u)^2/(4(1+u+u^2)^3)$, we can use (2-22) to verify easily that

$$N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}, \qquad N(-q) = -\frac{27u(1+u)}{2(1-2u-2u^2)^3},$$
$$N(q^4) = \frac{27u^4(1+u)}{2(2+2u-u^2)^3}.$$

The proof of (2-18) follows from applying these parameterizations to (2-14) when $f(q) = n(N(q))$, and $p = 2$.

The proof of (2-19) requires Ramanujan's third-degree, signature 3 modular equation. In particular, if $\alpha = N(q)$ and $\beta = N(q^3)$, then

$$\alpha = 1 - \left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3 = Y\left(\beta^{1/3}\right). \tag{2-23}$$

Since $N^{1/3}(q^3) = q \times \{\text{power series in } q^3\}$, a short computation shows that for all $j \in \{0, 1, 2\}$, we have $N(\zeta_3^j q) = Y(\zeta_3^j N^{1/3}(q^3))$. Choosing u such that $N(q^3) = u^3$, we must have $N(\zeta_3^j q) = Y(\zeta_3^j u)$. Equation (2-19) follows from applying these parametric formulas to (2-14) when $f(q) = n(N(q))$, and $p = 3$.

Since the proofs of Equations (2-20) and (2-21) rely on similar arguments to the proof of (2-19), we will simply state the prerequisite modular equations. In particular, (2-20) follows from Ramanujan's third-degree modular equation for the cubic continued fraction. If $\alpha = G(q)$ and $\beta = G\left(q^3\right)$, then

$$\alpha^3 = \beta\left(\frac{1-\beta+\beta^2}{1+2\beta+4\beta^2}\right). \tag{2-24}$$

Similarly, (2-21) follows from the fifth-degree modular equation for the Rogers–Ramanujan continued fraction. In particular, if $\alpha = R(q)$ and $\beta = R\left(q^5\right)$,

$$\alpha^5 = \beta\left(\frac{1-2\beta+4\beta^2-3\beta^3+\beta^4}{1+3\beta+4\beta^2+2\beta^3+\beta^4}\right). \tag{2-25}$$

□

The functional equations in Theorem 2.3 only hold in restricted subsets of $\mathbb{C}$. To explain this phenomenon we will go back to (2-14). As a general rule, we have to restrict q to values for which *none* of the Mahler measure integrals in (2-14) vanish on the unit torus. In other words, we can only consider the set of q's for which each term in (2-14) can be calculated from the appropriate q-series. Next, we may need to further restrict the domain of q depending on where the relevant parametric formulas hold. For example, parameterizations such as $N(q) = 27u(1+u)^4/(2(1+4u+u^2)^3)$ and $N(q^2) = 27u^2(1+u)^2/(4(1+u+u^2)^3)$ hold for $|q|$ sufficiently small, but fail when q is close to 1. After determining the domain of q, we can calculate the domain of u by solving a parametric equation to express u in terms of a q-series.

Theorem 2.4. *For $|p|$ sufficiently small but nonzero*,

$$3g(p) = n\left(\frac{27p}{(1+4p)^3}\right) + 4n\left(\frac{27p^2}{(1-2p)^3}\right). \tag{2-26}$$

Furthermore, for $|u|$ *sufficiently small but nonzero,*

$$3n\left(\frac{27u(1+u)^4}{2(1+4u+u^2)^3}\right) = g\left(\frac{u}{2(1+u)^2}\right) - 8g\left(-\frac{u(1+u)}{2}\right) + 4g\left(\frac{u^2}{4(1+u)}\right). \tag{2-27}$$

Proof. We will prove (2-27) first. Recall that (2-15) shows that

$$3n(N(q)) = g(G^3(q)) - 8g\left(G^3(-q)\right) + 4g(G^3(q^2)).$$

Suppose that $q = q_2(u(2+u)^3/(1+2u)^3)$, where $q_2(\alpha)$ is the elliptic nome. Classical eta function inversion formulas (which we omit) show that for $|u|$ sufficiently small: $G^3(q) = u/(2(1+u)^2)$, $G^3(-q) = -u(1+u)/2$, $G^3(q^2) = u^2/(4(1+u))$, $N(q) = 27u(1+u)^4/(2(1+4u+u^2)^3)$, and $N(q^2) = 27u^2(1+u)^2/(4(1+u+u^2)^3)$.

To prove (2-26) first recall that

$$3g(G^3(q)) = n(N(q)) + 4n\big(N(q^2)\big).$$

If we let $p = u/(2(1+u)^2)$, then it follows that $G^3(q) = p$, $N(q) = 27p/(1+4p)^3$, and $N(q^2) = 27p^2/(1+2p)^3$. □

Theorem 2.4 shows that $g(t)$ and $n(t)$ are essentially interchangeable. In Section 2.3 we will use (2-26) to derive an extremely useful formula for calculating $g(t)$ numerically.

2.2. ***Identities arising from higher modular equations.*** The functional equations presented in Section 2.1 are not the only interesting formulas that follow from (2-14). Rather those results represent the subset of functional equations in which every Mahler measure depends on a rational argument (possibly in a cyclotomic field). If we consider the higher modular equations, then we can establish formulas involving the Mahler measures of the modular polynomials themselves. Equation (2-31) is the simplest formula in this class of results.

Consider (2-14) when $p = 3$ and $f(q) = \mu(M(q))$:

$$\sum_{j=0}^{2} \mu\big(M(\zeta_3^j q)\big) = -8\mu\left(M(q^3)\right) + 3\mu\left(M(q^9)\right). \tag{2-28}$$

By the third-degree modular equation, if $\alpha \in \left\{M(q), M(\zeta_3 q), M(\zeta_3^2 q), M(q^9)\right\}$ and $\beta = M(q^3)$, then

$$G_3(\alpha,\beta) := (\alpha^2+\beta^2+6\alpha\beta)^2 - 16\alpha\beta\,(4(1+\alpha\beta)-3(\alpha+\beta))^2 = 0. \tag{2-29}$$

Since $G_3(\alpha,\beta) = 0$ defines a curve with genus greater than zero, it is impossible to find simultaneous rational parameterizations for all four zeros in α. For example, if we let $\beta = M(q^3) = p(2+p)^3/(1+2p)^3$, then we can obtain the rational expression $M\left(q^9\right) = p^3(2+p)/(1+2p)$, and three messy formulas involving radicals for the other zeros. Despite this difficulty, Equation (2-28) still reduces to an interesting formula if we recall the factorization

$$G_3\left(\alpha, M(q^3)\right) = \left(\alpha - M(q^9)\right)\prod_{j=0}^{2}\left(\alpha - M(\zeta_3^j q)\right), \tag{2-30}$$

and then use the fact that Mahler measure satisfies $\mathrm{m}(P) + \mathrm{m}(Q) = \mathrm{m}\,(PQ)$.

Theorem 2.5. *If $G_3(\alpha, \beta)$ is as defined in (2-29), then for $|p|$ sufficiently small but nonzero,*

$$\begin{aligned}\mathrm{m}\left(G_3\left(\frac{(x+x^{-1})^2(y+y^{-1})^2}{16}, \frac{1}{p}\left(\frac{1+2p}{2+p}\right)^3\right)\right)\\ = -16\log(2) - 16\mu\left(p\left(\frac{2+p}{1+2p}\right)^3\right) + 8\mu\left(p^3\left(\frac{2+p}{1+2p}\right)\right).\end{aligned} \tag{2-31}$$

Proof. First notice that from the elementary properties of Mahler's measure

$$\mu(t) = \frac{1}{2}\mathrm{m}\left(\frac{16}{(x+x^{-1})^2(y+y^{-1})^2} - t\right) - \frac{1}{2}\log|t|.$$

Applying this identity to (2-28) and appealing to (2-30) yields

$$\begin{aligned}\mathrm{m}\left(G_3\left(\frac{16}{(x+x^{-1})^2(y+y^{-1})^2}, M(q^3)\right)\right)\\ = \log\left|M(q)M(\zeta_3 q)M(\zeta_3^2 q)M(q^9)\right| - 16\mu(M(q^3)) + 8\mu(M(q^9)).\end{aligned}$$

Elementary q-product manipulations show that

$$M^4(q^3) = M(q)M(\zeta_3 q)M(\zeta_3^2 q)M(q^9),$$

and since $\alpha^4\beta^4 G_3(1/\alpha, 1/\beta) = G_3(\alpha, \beta)$, we obtain

$$\mathrm{m}\left(G_3\left(\frac{(x+x^{-1})^2(y+y^{-1})^2}{16}, \frac{1}{M(q^3)}\right)\right) = -16\log 2 - 16\mu(M(q^3)) + 8\mu(M(q^9)).$$

Finally, if we choose p so that $M(q^3) = p((2+p)/(1+2p))^3$, then $M(q^9) = p^3((2+p)/(1+2p))$, and the theorem follows. □

Although we completely eliminated the q-series expressions from (2-31), this is not necessarily desirable (or even possible) in more complicated examples. Take the identity involving resultants which follows from (2-14) (and some manipulation) when $p=11$ and $f(q)=r\left(R^5(q)\right)$:

$$\mathrm{m}\left(\operatorname{Res}_z\left(z^5-\frac{xy}{(x+1)(y+1)(x+y+1)},\, P\big(z,\, R^5(q)\big)\right)\right)$$
$$=-12\mathrm{m}(1+x+y)+12\log|R^5(q)|+122r(R^5(q))-11r(R^5(q^{11})). \quad (2\text{-}32)$$

In this formula $P(u, v)$ is the polynomial

$$P(u,v)=uv(1-11v^5-v^{10})(1-11u^5-u^{10})-(u-v)^{12},$$

which also satisfies $P(R(q), R(q^{11}))=0$ [Rogers 1920]. Even if rational parameterizations existed for $R(q)$ and $R(q^{11})$, substituting such formulas into (2-32) would probably just make the identity prohibitively complicated.

2.3. ***Computationally useful formulas and a few related hypergeometric transformations.*** While many methods exist for numerically calculating each of the four Mahler measures $\{\mu(t), n(t), g(t), r(t)\}$, two simple and efficient methods are directly related to the material discussed so far.

The first computational method relies on the q-series expansions. For example, we can calculate $\mu(\alpha)$ with Equation (2-9), provided that a value of q exists for which $M(q)=\alpha$. Amazingly, the elliptic nome function, defined in Equation (2-13), furnishes a value of q whenever $|\alpha|<1$. Similar inversion formulas exist for all of the q-products in Equations (2-5) through (2-8). Suppose that for $j\in\{2, 3, 4, 6\}$

$$q_j(\alpha)=\exp\left(-\frac{\pi}{\sin(\pi/j)}\frac{{}_2F_1(1/j, 1-1/j; 1; 1-\alpha)}{{}_2F_1(1/j, 1-1/j; 1; \alpha)}\right), \quad (2\text{-}33)$$

then we have the following theorem:

Theorem 2.6. *With α and q appropriately restricted, the following table gives inversion formulas for Equations (2-5) through (2-8):*

α	q	α	q
$M(q)$	$q_2(\alpha)$	$G(q)$	$q_2\left(\frac{u(2+u)^3}{(1+2u)^3}\right)$ *with* $\alpha^3=\frac{u}{2(1+u)^2}$
$N(q)$	$q_3(\alpha)$	$R(q)$	$q_4\left(\frac{64k(1+k-k^2)^5}{(1+k^2)^2((1+11k-k^2)^2-125k^2)^2}\right)$ *with* $\alpha^5=\frac{k(1-k)^2}{(1+k)^2}$

For example, if $|q|<1$ and $\alpha=M(q)$, then $q=q_2(\alpha)$.

Proof. The inversion formulas for $M(q)$ and $G(q)$ follow from classical eta function identities, and the inversion formula for $N(q)$ follows from eta function identities in Ramanujan's theory of signature three.

The inversion formula for $R(q)$ seems to be new, so we will prove it. Let us suppose that $\alpha = R(q)$ and $k = R(q)R^2(q^2)$, where q is fixed. A formula of Ramanujan [Andrews and Berndt 2005] shows that $\alpha^5 = k(1-k)^2/(1+k)^2$, which establishes the second part of the formula. Now suppose that $q = q_2(\alpha_2)$, where $\alpha_2 = M(q)$. A classical identity shows that

$$q\,(-q;q)_\infty^{24} = \frac{\alpha_2}{16(1-\alpha_2)^2},$$

and comparing this to Ramanujan's identity

$$q\,(-q;q)_\infty^{24} = \left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5,$$

we deduce that

$$\frac{\alpha_2}{(1-\alpha_2)^2} = 16\left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5. \tag{2-34}$$

Now recall that the theory of the signature 4 elliptic nome shows that

$$q = q_2(\alpha_2) = q_4\left(\frac{4\alpha_2}{(1+\alpha_2)^2}\right) = q_4\left(\frac{4\alpha_2/(1-\alpha_2)^2}{1+4\alpha_2/(1-\alpha_2)^2}\right).$$

Substituting (2-34) into this final result yields

$$q = q_4\left(\frac{64k(1+k-k^2)^5}{(1+k^2)^2\big((1+11k-k^2)^2-125k^2\big)^2}\right),$$

which completes the proof. □

The second method for calculating the four Mahler measures, $\mu(t)$, $n(t)$, $g(t)$, and $r(t)$ depends on reformulating them in terms of hypergeometric functions. For example, Rodriguez Villegas [1999] proved the formula

$$\mu(t) = -\frac{1}{2}\,\mathrm{Re}\left(\log(t/16) + \int_0^t \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;u\right)-1}{u}\,\mathrm{d}u\right).$$

Translated into the language of generalized hypergeometric functions, this becomes

$$\mu(t) = -\,\mathrm{Re}\left(\frac{t}{8}\,{}_4F_3\left({\textstyle\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};t\right) + \frac{1}{2}\log(t/16)\right). \tag{2-35}$$

He also proved a formula for $n(t)$ which is equivalent to

$$n(t) = -\operatorname{Re}\left(\frac{2t}{27}{}_4F_3\left(\substack{\frac{4}{3},\frac{5}{3},1,1\\2,2,2};\, t\right) + \frac{1}{3}\log(t/27)\right). \tag{2-36}$$

Formulas like (2-35) and (2-36) hold obvious appeal. From a computational perspective they are useful because most mathematics programs have routines for calculating generalized hypergeometric functions. For example, when $|t| < 1$ the Taylor series for the ${}_4F_3$ function easily gives better numerical accuracy than the Mahler measure integrals. Combining Equation (2-36) with (2-26) also yields a useful formula for calculating $g(t)$ whenever $|t|$ is sufficiently small:

$$g(t) = -\operatorname{Re}\left(\frac{2t}{(1+4t)^3}{}_4F_3\left(\substack{\frac{4}{3},\frac{5}{3},1,1\\2,2,2};\, \frac{27t}{(1+4t)^3}\right)\right.$$
$$\left. + \frac{8t^2}{(1-2t)^3}{}_4F_3\left(\substack{\frac{4}{3},\frac{5}{3},1,1\\2,2,2};\, \frac{27t^2}{(1-2t)^3}\right) + \log\left(\frac{t^3}{(1+4t)(1-2t)^4}\right)\right). \tag{2-37}$$

So far we have been unable to find a similar expression for $r(t)$.

Open Problem 1. Express $r(t)$ in terms of generalized hypergeometric functions.

Besides their computational importance, identities like (2-35) allow for a reformulation of Boyd's conjectures in the language of hypergeometric functions. For example, the conjecture

$$\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} L'(E, 0),$$

where E is an elliptic curve with conductor 15, becomes

$$L'(E, 0) \stackrel{?}{=} -2\operatorname{Re}\left({}_4F_3\left(\substack{\frac{3}{2},\frac{3}{2},1,1\\2,2,2};\, 16\right)\right).$$

A proof of this identity would represent an important addition to the vast literature concerning transformations and evaluations of generalized hypergeometric functions.

In the remainder of this section we will apply our results to deduce a few interesting hypergeometric transformations. For example, differentiating (2-37) leads to an interesting corollary:

Corollary 2.7. *For $|t|$ sufficiently small,*

$$\omega(t) := \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^3 = \frac{1}{1-2t}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27t^2}{(1-2t)^3}\right), \tag{2-38}$$

and furthermore

$$\omega\left(\frac{p}{2(1+p)^2}\right) = (1+p)\omega\left(\frac{p^2}{4(1+p)}\right), \tag{2-39}$$

whenever $|p|$ is sufficiently small.

Proof. We can prove (2-38) by differentiating each side of (2-37), and then by appealing to Stienstra's formulas [2006]. A second possible proof follows from showing that both sides of (2-38) satisfy the same differential equation.

The shortest proof of (2-39) follows from a formula of Zagier [Stienstra 2006]:

$$\omega(G^3(q)) = \prod_{n=0}^{\infty} \frac{(1-q^{2n})(1-q^{3n})^6}{(1-q^n)^2(1-q^{6n})^3}.$$

First use Zagier's identity to verify that $G^2(q)\omega(G^3(q)) = G(q^2)\omega(G^3(q^2))$, and then apply the parameterizations for $G^3(q)$ and $G^3(q^2)$ from 2.4. □

We will also make a few remarks about the derivative of $r(t)$. Stienstra has shown that

$$r(t) = -\operatorname{Re}\left(\log t + \int_0^t \frac{\phi(u)-1}{u}\,\mathrm{d}u\right), \tag{2-40}$$

where $\phi(t)$ is defined by

$$\phi(t) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{2-41}$$

Even though we have not discovered a formula for $r(t)$ involving hypergeometric functions, we can still express $\phi(t)$ in terms of the hypergeometric function.

Theorem 2.8. *Let $\phi(t)$ be defined by (2-41). For $|k|$ sufficiently small,*

$$\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right) = \frac{(1+k)^2}{\sqrt{(1+k^2)\big((1-k-k^2)^2-5k^2\big)}} \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5(1+k-k^2)}{(1+k^2)^2\big((1-k-k^2)^2-5k^2\big)^2}\right), \tag{2-42}$$

$$\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right) = \frac{(1-k)}{\sqrt{(1+k^2)\big((1+11k-k^2)^2-125k^2\big)}} \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k(1+k-k^2)^5}{(1+k^2)^2\big((1+11k-k^2)^2-125k^2\big)^2}\right). \tag{2-43}$$

Furthermore, $\phi(t)$ satisfies the functional equation

$$\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right)=\frac{1-k}{(1+k)^2}\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right). \tag{2-44}$$

Proof. We prove (2-44) first. A result from [Verrill 2001] shows that

$$\phi^2\left(R^5(q)\right)=\frac{q}{R^5(q)}\frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty}. \tag{2-45}$$

Combining (2-45) with the trivial formula $(q^2,q^2)_\infty=(q;q)_\infty(-q;q)_\infty$, we get

$$\frac{\phi^2\left(R^5(q)\right)}{\phi^2\left(R^5(q^2)\right)}=\frac{R^5(q^2)}{R^5(q)}\frac{\left\{q^{1/24}\left(-q;q\right)_\infty\right\}}{\left\{q^{5/24}\left(-q^5;q^5\right)_\infty\right\}^5}. \tag{2-46}$$

We will apply four of Ramanujan's formulas to finish the proof. If $k=R(q)R^2(q^2)$, we have for $|q|$ sufficiently small (see [Andrews and Berndt 2005])

$$R^5(q)=k\left(\frac{1-k}{1+k}\right)^2, \tag{2-47}$$

$$R^5(q^2)=k^2\left(\frac{1+k}{1-k}\right), \tag{2-48}$$

$$q^{1/24}\left(-q;q\right)_\infty=\left(\frac{k}{1-k^2}\right)^{1/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{5/24}, \tag{2-49}$$

$$q^{5/24}\left(-q^5;q^5\right)_\infty=\left(\frac{k}{1-k^2}\right)^{5/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{1/24}. \tag{2-50}$$

Equation (2-44) follows immediately from substituting these parametric formulas into (2-46).

Next we prove (2-42). Combining Equation (2-47) with Entry 3.2.15 in [Andrews and Berndt 2005], we easily obtain

$$q^{5/24}\left(q^5;q^5\right)_\infty=\left(\frac{k(1-k^2)^2}{\left(1+k-k^2\right)\left(1-4k-k^2\right)^2}\right)^{1/6}q^{1/24}\left(q;q\right)_\infty. \tag{2-51}$$

Now we evaluate the eta product $q^{1/24}\left(q;q\right)_\infty$. Recall that if $q=q_4(z)$, then

$$q^{1/24}(q;q)_\infty=2^{-1/4}z^{1/24}(1-z)^{1/12}\sqrt{{}_2F_1\left(\tfrac14,\tfrac34;1;z\right)}.$$

In Theorem 2.6 we showed that if $k=R(q)R^2(q^2)$ then

$$q=q_4\left(\frac{64k(1+k-k^2)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right);$$

hence

$$q^{1/24}(q;q)_\infty = \left(\frac{k(1-k^2)^2(1+k-k^2)^5(1-4k-k^2)^{10}}{(1+k^2)^6\big((1+11k-k^2)^2-125k^2\big)^6}\right)^{1/24} \times \sqrt{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k(1+k-k^2)^5}{(1+k^2)^2\big((1+11k-k^2)^2-125k^2\big)^2}\right)}. \quad (2\text{-}52)$$

Substituting (2-52), (2-51), and (2-47) into (2-45) completes the proof of (2-42). The proof of (2-43) also follows from an extremely similar argument. □

We conclude this section by recording a few formulas which do not appear in [Andrews and Berndt 2005], but which were probably known to Ramanujan. We point out that Maier obtained several results along these lines in [Maier 2006]. The functional equation for $\phi(t)$ (after substituting $z = k/(1-k^2)$) implies a new hypergeometric transformation:

$$\sqrt{\frac{(1+11z)^2-125z^2}{(1-z)^2-5z^2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1,\frac{64z^5(1+z)}{(1+4z^2)((1-z)^2-5z^2)^2}\right) = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64z(1+z)^5}{(1+4z^2)\big((1+11z)^2-125z^2\big)^2}\right). \quad (2\text{-}53)$$

Perhaps not surprisingly, we can also use the arguments in this section to deduce that

$$q_4^5\left(\frac{64z(1+z)^5}{(1+4z^2)\big((1+11z)^2-125z^2\big)^2}\right) = q_4\left(\frac{64z^5(1+z)}{(1+4z^2)\big((1-z)^2-5z^2\big)^2}\right), \quad (2\text{-}54)$$

which implies a rational parametrization for the fifth-degree modular equation in Ramanujan's theory of signature 4.

3. A regulator explanation

Now we will reinterpret our identities in terms of the regulators of elliptic curves. The elliptic curves in question are defined by the zero varieties of the polynomials whose Mahler measure we studied. First we explain the relationship between Mahler measures and regulators. Then we use regulators to deduce formulas involving Kronecker–Eisenstein series, including Equations (2-9), (2-10), (2-11), and (2-12).

We will follow some of the ideas from [Rodriguez Villegas 2002].

3.1. ***The elliptic regulator.*** Let F be a field. By Matsumoto's Theorem, $K_2(F)$ is generated by the symbols $\{a,b\}$ for $a,b\in F^*$, which satisfy the bilinearity relations

$\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\}$ and $\{a, b_1b_2\} = \{a, b_1\}\{a, b_2\}$, and the Steinberg relation $\{a, 1-a\} = 1$.

Recall that for a field F, with discrete valuation v, and maximal ideal $\mathcal{M}$, the tame symbol is given by

$$(x, y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \bmod \mathcal{M}$$

(see [Rodriguez Villegas 1999]). Note that this symbol is trivial if $v(x) = v(y) = 0$. In the case when $F = \mathbb{Q}(E)$ (from now on E denotes an elliptic curve), a valuation is determined by the order of the rational functions at each point $S \in E(\overline{\mathbb{Q}})$. We will denote the valuation determined by a point $S \in E(\overline{\mathbb{Q}})$ by v_S.

The tame symbol is then a map $K_2(\mathbb{Q}(E)) \to \mathbb{Q}(S)^*$.

We have

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{S \in E(\overline{\mathbb{Q}})} \mathbb{Q}Q(S)^* \times \mathbb{Q},$$

where the last arrow corresponds to the coproduct of tame symbols.

Therefore an element $\{x, y\} \in K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ can be seen as an element in $K_2(E) \otimes \mathbb{Q}$ whenever $(x, y)_{v_S} = 1$ for all $S \in E(\overline{\mathbb{Q}})$. All of the families considered in this paper are tempered according to [Rodriguez Villegas 1999], and therefore they satisfy the triviality of tame symbols.

The regulator map (defined by Beilinson, after work of Bloch) is given by

$$\mathrm{r} : K_2(E) \to H^1(E, \mathbb{R})$$
$$\{x, y\} \mapsto \left\{\gamma \to \int_\gamma \eta(x, y)\right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$, and

$$\eta(x, y) := \log|x| \,\mathrm{d} \arg y - \log|y| \,\mathrm{d} \arg x.$$

Here we think of $H^1(E, \mathbb{R})$ as the dual of $H_1(E, \mathbb{Z})$. The regulator is well defined because $\eta(x, 1-x) = \mathrm{d}D(x)$, where

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z) \log|z|$$

is the Bloch–Wigner dilogarithm.

In terms of the general formulation of Beilinson's conjectures this definition is not completely correct. One needs to go a step further and consider $K_2(\mathcal{E})$, where $\mathcal{E}$ is a Néron model of E over $\mathbb{Z}$. In particular, $K_2(\mathcal{E})$ is a subgroup of $K_2(E)$. It seems (see [Rodriguez Villegas 1999]) that a power of $\{x, y\}$ always lies in $K_2(\mathcal{E})$.

Assume that E is defined over $\mathbb{R}$. Because of the way that complex conjugation acts on η, the regulator map is trivial for the classes in $H_1(E, \mathbb{Z})^+$. In particular, these cycles remain invariant under complex conjugation. Therefore it suffices to consider the regulator as a function on $H_1(E,\mathbb{Z})^-$.

We write $E(\mathbb{C})\cong\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, where τ is in the upper half-plane. Then $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}\cong\mathbb{C}^*/q^{\mathbb{Z}}$, where z mod $\Lambda=\mathbb{Z}+\tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$. Bloch [2000] defines the regulator function in terms of a Kronecker–Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right)=\frac{y_\tau^2}{\pi}\sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}, \tag{3-1}$$

where y_τ is the imaginary part of τ.

Let $J(z)=\log|z|\log|1-z|$, and let

$$D(x)=\mathrm{Im}(\mathrm{Li}_2(x))+\arg(1-x)\log|x|$$

be the Bloch–Wigner dilogarithm.

Consider the function

$$J_\tau(z)=\sum_{n=0}^{\infty}J(zq^n)-\sum_{n=1}^{\infty}J(z^{-1}q^n)+\frac{1}{3}\log^2|q|B_3\left(\frac{\log|z|}{\log|q|}\right) \tag{3-2}$$

on $E(\mathbb{C})\cong\mathbb{C}^*/q^{\mathbb{Z}}$, where $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$ is the third Bernoulli polynomial. If we recall that the elliptic dilogarithm is defined by

$$D_\tau(z):=\sum_{n\in\mathbb{Z}}D(zq^n), \tag{3-3}$$

then the regulator function (see [Bloch 2000]) is given by

$$R_\tau=D_\tau-\mathrm{i}J_\tau. \tag{3-4}$$

By linearity, R_τ extends to divisors with support in $E(\mathbb{C})$. Let x and y be nonconstant functions on E with divisors

$$(x)=\sum m_i(a_i),\qquad (y)=\sum n_j(b_j).$$

Following [Bloch 2000] and the notation in [Rodriguez Villegas 1999], we recall the diamond operation $\mathbb{C}(E)^*\otimes\mathbb{C}(E)^*\to\mathbb{Z}[E(\mathbb{C})]^-$

$$(x)\diamond(y)=\sum m_in_j(a_i-b_j).$$

Here $\mathbb{Z}[E(\mathbb{C})]^-$ means that $[-P]\sim-[P]$.

Because R_τ is an odd function, we obtain a map

$$\mathbb{Z}[E(\mathbb{C})]^-\to\mathbb{R}.$$

Theorem 3.1 [Beĭlinson 1980]. *Let $E/\mathbb{R}$ be an elliptic curve, x, y nonconstant functions in $\mathbb{C}(E)$, and $\omega \in \Omega^1$. Then*

$$\int_{E(\mathbb{C})} \bar{\omega} \wedge \eta(x, y) = \Omega_0 R_\tau((x) \diamond (y)),$$

where Ω_0 is the real period.

Although a more general version of Beilinson's Theorem exists for elliptic curves defined over the complex numbers, the above version has a simpler formulation.

Corollary 3.2 (after an idea of Deninger). *If x and y are nonconstant functions in $\mathbb{C}(E)$ with trivial tame symbols, then*

$$-\int_\gamma \eta(x, y) = \operatorname{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right), \quad \textit{where } \Omega = \int_\gamma \omega.$$

Proof. Notice that $\mathrm{i}\eta(x, y)$ is an element of the two-dimensional vector space $H^2_{\mathscr{D}}(E(\mathbb{C}), \mathbb{R}(2))$ generated by ω and $\bar{\omega}$. Then we may write

$$\mathrm{i}\eta(x, y) = \alpha[\omega] + \beta[\bar{\omega}],$$

from which we obtain

$$\int_\gamma \mathrm{i}\eta(x, y) = \alpha\Omega + \beta\bar{\Omega}.$$

On the other hand, we have

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x, y) \wedge \bar{\omega} = \alpha \int_{E(\mathbb{C})} \omega \wedge \bar{\omega} = \alpha \mathrm{i} 2\Omega_0^2 y_\tau,$$

and

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x, y) \wedge \omega = -\beta \mathrm{i} 2\Omega_0^2 y_\tau.$$

By Beilinson's Theorem

$$\int_\gamma \mathrm{i}\eta(x, y) = -\frac{R_\tau((x) \diamond (y))\Omega}{2\Omega_0 y_\tau} + \frac{\overline{R_\tau((x) \diamond (y))}\bar{\Omega}}{2\Omega_0 y_\tau},$$

and the statement follows. □

3.2. ***Regulators and Mahler measure.*** From now on, we will set $k = 4/\sqrt{t}$ in the first family (2-1).

Rodriguez Villegas [1999] proved that if $P_k(x, y) = k + x + 1/x + y + 1/y$ does not intersect the torus $\mathbb{T}^2$, then

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} \mathrm{r}(\{x, y\})(\gamma). \tag{3-5}$$

Here the $\sim_{\mathbb{Z}}$ stands for "up to an integer", and γ is a closed path that avoids the poles and zeros of x and y. In particular, γ generates the subgroup $H_1(E,\mathbb{Z})^-$ of $H_1(E,\mathbb{Z})$ where conjugation acts by -1.

We would like to use this property, however we need to exercise caution. In particular, $P_k(x, y)$ intersects the torus whenever $|k| \le 4$ and $k \in \mathbb{R}$. Let us recall the idea behind the proof of (3-5) for the special case of $P_k(x, y)$. Writing

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

we have

$$m(k) = \mathrm{m}(yP_k(x,y)) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\bigl(\log^+|y_{(1)}(x)| + \log^+|y_{(2)}(x)|\bigr)\frac{\mathrm{d}x}{x}.$$

This last equality follows from applying Jensen's formula with respect to the variable y. When the polynomial does not intersect the torus, we may omit the $+$ sign on the logarithm, since each $y_{(i)}(x)$ is always inside or outside the unit circle. Indeed, there is always a branch inside the unit circle and a branch outside. It follows that

$$m(k) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi}\int_{\mathbb{T}^1}\eta(x,y), \tag{3-6}$$

where $\mathbb{T}^1$ is interpreted as a cycle in the homology of the elliptic curve defined by $P_k(x, y) = 0$, namely $H_1(E,\mathbb{Z})$.

If $k \in [-4, 4]$, then we may also assume that $k > 0$ since this particular Mahler measure does not depend on the sign of k. The equation

$$k + x + \frac{1}{x} + y + \frac{1}{y} = 0$$

certainly has solutions when $(x, y) \in \mathbb{T}^2$. However, for $|x| = 1$ and k real, the number $k + x + 1/x$ is real, and therefore $y + 1/y$ must be real. This forces two possibilities: either y is real or $|y| = 1$. Let $x = \mathrm{e}^{\mathrm{i}\theta}$, then for $-\pi \le \theta \le \pi$ we have

$$-k - 2\cos\theta = y + \frac{1}{y}. \tag{3-7}$$

The limiting case occurs when $|k + 2\cos\theta| = 2$. Since we have assumed that k is positive, this condition becomes $k + 2\cos\theta = 2$, which implies that $y = -1$. When $k + 2\cos\theta > 2$ one solution for y, say, $y_{(1)}$, becomes a negative number less than -1, thus $|y_{(1)}| > 1$ (the other solution $y_{(2)}$ is such that $|y_{(2)}| < 1$). When $k+2\cos\theta < 2$, y_i lies inside the unit circle and never reaches 1. What is important is that $|y_{(1)}| \ge 1$ and $|y_{(2)}| \le 1$, so we can still write (3-6) even if there is a nontrivial intersection with the torus.

3.3. ***Functional identities for the regulator.*** We recall a result by Bloch [2000] on the modularity of R_τ:

Proposition 3.3. *Take* $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *and define* $\tau' = \dfrac{\alpha\tau+\beta}{\gamma\tau+\delta}$. *If we set*

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix},$$

then

$$R_{\tau'}(\mathrm{e}^{2\pi\mathrm{i}(a'+b'\tau')}) = \frac{1}{\gamma\bar{\tau}+\delta} R_\tau(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}). \tag{3-8}$$

We will need to use some functional equations for J_τ. Recall the trivial property

$$J(z) = p \sum_{x^p=z} J(x). \tag{3-9}$$

Proposition 3.4. *Let* p *be an odd prime, let* $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, *and let* $q_j = \mathrm{e}^{2\pi\mathrm{i}(\tau+j)/p}$ *for* $j \in \{0, 1, \ldots, p-1\}$. *Suppose that* $(N, k) = 1$, *and* $p \equiv \pm 1$ *or* $0 \pmod N$. *Then*

$$(1 + \chi_{-N}(p)p^2) J_{N\tau}(q^k) = \sum_{j=0}^{p-1} p J_{N(\tau+j)/p}(q_j^k) + \chi_{-N}(p) J_{Np\tau}(q^{pk}), \tag{3-10}$$

and for any z,

$$(\chi_{-N}(p) + p^2) J_{N\tau}(z) = \sum_{j=0}^{p-1} p J_{N(\tau+j)/p}(z) + \chi_{-N}(p) J_{Np\tau}(z). \tag{3-11}$$

Proof. Notice that

$$\sum_{j=0}^{p-1} J_{N(\tau+j)/p}(q_j^k) = \sum_{n=0}^{\infty}\sum_{j=0}^{p-1} J(q_j^{Nn+k}) - \sum_{n=1}^{\infty}\sum_{j=0}^{p-1} J(q_j^{Nn-k}) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).$$

By (3-9) this equals

$$\begin{aligned}
&\sum_{\substack{n=0\\ p\nmid Nn+k}}^{\infty} \frac{1}{p} J(q^{Nn+k}) - \sum_{\substack{n=1\\ p\nmid Nn-k}}^{\infty} \frac{1}{p} J(q^{Nn-k})\\
&\qquad + \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} p J(q^{(Nn+k)/p}) - \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} p J(q^{(Nn-k)/p}) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right)\\
&= \sum_{n=0}^{\infty} \frac{1}{p} J(q^{Nn+k}) - \sum_{n=1}^{\infty} \frac{1}{p} J(q^{Nn-k}) - \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} \frac{1}{p} J(q^{Nn+k}) + \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} \frac{1}{p} J(q^{Nn-k})\\
&\qquad + \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} p J(q^{(Nn+k)/p}) - \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} p J(q^{(Nn-k)/p}) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).
\end{aligned}$$

Upon rearranging, this expression becomes

$$\frac{1}{p}J_{N\tau}(q^k)-\frac{4\pi^2y_\tau^2N^2}{3p}B_3\Big(\frac{k}{N}\Big)$$
$$-\sum_{\substack{n=0\\ p|Nn+k}}^{\infty}\frac{1}{p}J((q^p)^{(Nn+k)/p})+\sum_{\substack{n=1\\ p|Nn-k}}^{\infty}\frac{1}{p}J((q^p)^{(Nn-k)/p})$$
$$+\sum_{\substack{n=0\\ p|Nn+k}}^{\infty}pJ(q^{(Nn+k)/p})-\sum_{\substack{n=1\\ p|Nn-k}}^{\infty}pJ(q^{(Nn-k)/p})+\frac{4\pi^2y_\tau^2N^2}{3p}B_3\Big(\frac{k}{N}\Big),$$

or again

$$\frac{1}{p}J_{N\tau}(q^k)-\frac{\chi_{-N}(p)}{p}J_{Np\tau}(q^{pk})+\chi_{-N}(p)pJ_{N\tau}(q^k).$$

This proves the assertion.

The second equality follows in a similar fashion. □

It is possible to prove analogous identities for D_τ and R_τ.

Proposition 3.5. $J_{(2\mu+1)/2}(\mathrm{e}^{\pi\mathrm{i}\mu})=J_{2\mu}(\mathrm{e}^{\pi\mathrm{i}\mu})-J_{2\mu}(-\mathrm{e}^{\pi\mathrm{i}\mu})$.

Proof. Let $z=\mathrm{e}^{\pi\mathrm{i}\mu}$. then

$$\begin{aligned}J_{2\mu}(z)-J_{2\mu}(-z)&\\&=J(z)-J(-z)+\sum_{n=1}^{\infty}\big(J(zq^n)-J(-zq^n)-J(z^{-1}q^n)+J(-z^{-1}q^n)\big)\\&=\sum_{n=0}^{\infty}\big(J(\mathrm{e}^{\pi\mathrm{i}\mu(4n+1)})-J(-\mathrm{e}^{\pi\mathrm{i}\mu(4n+1)})-J(\mathrm{e}^{\pi\mathrm{i}\mu(4n+3)})+J(-\mathrm{e}^{\pi\mathrm{i}\mu(4n+3)})\big).\end{aligned}$$

On the other hand,

$$J_{(2\mu+1)/2}(z)=\sum_{n=0}^{\infty}\big(J((-1)^n\mathrm{e}^{\pi\mathrm{i}\mu(2n+1)})-J((-1)^{n+1}\mathrm{e}^{\pi\mathrm{i}\mu(2n+1)})\big),$$

which proves the equality. □

3.4. ***The first family.*** First we write the equation

$$x+\frac{1}{x}+y+\frac{1}{y}+k=0$$

in Weierstrass form. Consider the rational transformation

$$X=\frac{k+x+y}{x+y}=-\frac{1}{xy},\qquad Y=\frac{k(y-x)(k+x+y)}{2(x+y)^2}=\frac{(y-x)\Big(1+\frac{1}{xy}\Big)}{2xy},$$

which leads to

$$Y^2 = X\big(X^2 + (\tfrac{1}{4}k^2 - 2)X + 1\big).$$

It is useful to state the inverse transformation:

$$x = \frac{kX - 2Y}{2X(X-1)}, \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

Notice that E_k contains a torsion point of order 4 over $\mathbb{Q}(k)$, namely $P = (1, k/2)$. Indeed, this family is the modular elliptic surface associated to $\Gamma_0(4)$.

We can show that $2P = (0,0)$, and $3P = (1, -k/2)$.

Now we have

$$\begin{aligned}
(X) &= 2(2P) - 2O,\\
(x) &= (2(P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\
&= (P) - (2P) - (3P) + O,\\
(y) &= (2(3P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\
&= -(P) - (2P) + (3P) + O.
\end{aligned}$$

Computing the diamond operation between the divisors of x and y yields

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

Now assume that $k \in \mathbb{R}$ and $k > 4$. We will choose an orientation for the curve and compute the real period. Because P is a point of order 4 and $\int_0^1 \omega$ is real, we may assume that P corresponds to $3\Omega_0/4$.

The next step is to understand the cycle $|x| = 1$ as an element of $H_1(E, \mathbb{Z})$. We would like to compute the value of $\Omega = \int_\gamma \omega$. First recall that

$$\omega = \frac{\mathrm{d}X}{2Y} = \frac{\mathrm{d}x}{x(y - y^{-1})}.$$

When $k > 4$, consider conjugation of ω. This sends x to x^{-1} and $\mathrm{d}x/x$ and $-\mathrm{d}x/x$. There is no intersection with the torus, so y remains invariant. Therefore we conclude that Ω is the complex period, and $\Omega/\Omega_0 = \tau$, where τ is purely imaginary.

Therefore, for k real and $|k| > 4$,

$$m(k) = \frac{4}{\pi}\,\mathrm{Im}\Big(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\Big).$$

Now take $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. By Proposition 3.3

$$R_\tau(-\mathrm{i}) = R_\tau(\mathrm{e}^{-2\pi\mathrm{i}/4}) = \bar{\tau} R_{-1/\tau}(\mathrm{e}^{-2\pi\mathrm{i}/(4\tau)}),$$

therefore
$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-1/\tau}(\mathrm{e}^{-2\pi\mathrm{i}/(4\tau)}).$$
If we let $\mu = -1/(4\tau)$, then for $k \in \mathbb{R}$ we obtain
$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}(\mathrm{e}^{2\pi\mathrm{i}\mu}) = \mathrm{Im}\left(\frac{1}{\pi y_\mu} R_{4\mu}(\mathrm{e}^{2\pi\mathrm{i}\mu})\right)$$
$$= \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+4\mu n)^2(m+4\bar{\mu}n)}\right),$$
thus recovering a result of Rodriguez Villegas. We can extend this result to all $k \in \mathbb{C}$, by arguing that both $m(k)$ and $-(1/(\pi y_\mu))J_{4\mu}(\mathrm{e}^{2\pi\mathrm{i}\mu})$ are the real parts of holomorphic functions that coincide at infinitely many points; see [Rodriguez Villegas 1996].

Now we show how to deduce (1-7) and (1-6). Applying (3-10) with $N = 4$, $k = 1$, and $p = 2$, we have
$$J_{4\mu}(q) = 2J_{2\mu}(q_0) + 2J_{2(\mu+1)}(q_1),$$
which translates into
$$\frac{1}{y_{4\mu}} J_{4\mu}(\mathrm{e}^{2\pi\mathrm{i}\mu}) = \frac{1}{y_{2\mu}} J_{2\mu}(\mathrm{e}^{\pi\mathrm{i}\mu}) + \frac{1}{y_{2\mu}} J_{2\mu}(-\mathrm{e}^{\pi\mathrm{i}\mu}).$$
This is the content of (1-7). Setting $\tau = -1/(2\mu)$, we may also write
$$D_{\tau/2}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + D_\tau(-\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\tau}). \tag{3-12}$$
Next we use Proposition 3.5:
$$J_{(2\mu+1)/2}(\mathrm{e}^{\pi\mathrm{i}\mu}) = J_{2\mu}(\mathrm{e}^{\pi\mathrm{i}\mu}) - J_{2\mu}(-\mathrm{e}^{\pi\mathrm{i}\mu}),$$
which translates into
$$\frac{1}{y_{(2\mu+1)/2}} J_{(2\mu+1)/2}(\mathrm{e}^{\pi\mathrm{i}\mu}) = \frac{2}{y_{2\mu}} J_{2\mu}(\mathrm{e}^{\pi\mathrm{i}\mu}) - \frac{2}{y_{2\mu}} J_{2\mu}(-\mathrm{e}^{\pi\mathrm{i}\mu}).$$
Setting $\tau = -1/(2\mu)$, and using $\left(\begin{smallmatrix}1 & 0\\ -2 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ on the left-hand side, we have
$$D_{(\tau-1)/2}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - D_\tau(-\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\tau}). \tag{3-13}$$
Combining Equations (3-12) and (3-13), we see that
$$2D_\tau(-\mathrm{i}) = D_{\tau/2}(-\mathrm{i}) + D_{(\tau-1)/2}(-\mathrm{i}).$$
This is the content of (1-6).

Similarly, we may deduce (2-14) from (3-10) when $k = 1$, $N = 4$, and p is an odd prime.

3.5. ***A direct approach.*** It is also possible to prove (1-6) and (1-7) directly, without considering the μ-parametrization or the explicit form of the regulator.

For those formulas, it is easy to explicitly write the isogenies at the level of the Weierstrass models. By using the well-known isogeny of degree 2

$$\phi : \{E : y^2 = x(x^2+ax+b)\} \to \{\widehat{E} : \hat{y}^2 = \hat{x}(\hat{x}^2 - 2a\hat{x} + (a^2-4b))\}$$

given by (see for example [Cassels 1991; Silverman 1992])

$$(x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(b-x^2)}{x^2}\right),$$

where we require that $a^2 - 4b \neq 0$, we find

$$\phi_1 : E_{2n+(2/n)} \to E_{4n^2}, \quad (X, Y) \mapsto \left(\frac{X(n^2X+1)}{X+n^2}, -\frac{n^3Y(X^2+2n^2X+1)}{(X+n^2)^2}\right),$$

$$\phi_2 : E_{2n+(2/n)} \to E_{4/n^2}, \quad (X, Y) \mapsto \left(\frac{X(X+n^2)}{n^2X+1}, -\frac{Y(n^2X^2+2X+n^2)}{n(n^2X+1)^2}\right).$$

Write x_1, y_1, X_1, Y_1 for the rational functions and r_1 for the regulator in E_{4n^2}, and $x_2, y_2, X_2, Y_2, \mathrm{r}_2$ for the corresponding objects in E_{4/n^2}. It follows that

$$\begin{aligned}\pm m(4n^2) &= \mathrm{r}_1(\{x_1, y_1\}) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1, y_1) = \frac{1}{4\pi}\int_{|X|=1} \eta(x_1\circ\phi_1, y_1\circ\phi_1)\\ &= \tfrac{1}{2}\,\mathrm{r}(\{x_1\circ\phi_1, y_1\circ\phi_1\}),\end{aligned}$$

where the factor of 2 follows from the degree of the isogeny. Similarly, we find

$$\pm m\left(\frac{4}{n^2}\right) = \mathrm{r}_2(\{x_2, y_2\}) = \tfrac{1}{2}\,\mathrm{r}(\{x_2\circ\phi_2, y_2\circ\phi_2\}).$$

Now we need to compare the values of

$$\mathrm{r}(\{x_1\circ\phi_1, y_1\circ\phi_1\}), \quad \mathrm{r}(\{x_2\circ\phi_2, y_2\circ\phi_2\}), \quad \text{and} \quad \mathrm{r}(\{x, y\}).$$

Recall that $(x)\diamond(y) = 8(P)$, where $P = (1, k/2)$. When $k = 2(n+1/n)$, we will also consider the point $Q = (-1/n^2, 0)$, which has order 2 (then $P+Q = (-1, n-1/n)$, $2P+Q = (-n^2, 0)$, etc).

Let P now denote the point in $E_{2n+(2/n)}$, and let P_1 denote the corresponding point in E_{4n^2}. We have the following table:

$$\phi_1 : \quad \begin{array}{rcl} 3P,\ P+Q & \to & P_1,\\ 2P,\ \ Q & \to & 2P_1,\\ P,\ 3P+Q & \to & 3P_1,\\ O_0,\ 2P+Q & \to & O_1.\end{array}$$

Using this table, and the divisors (x_1) and (y_1) in E_{4n^2}, we can compute $(x_1 \circ \phi_1) \diamond (y_1 \circ \phi_1)$. We find that

$$(x_1 \circ \phi_1) \diamond (y_1 \circ \phi_1) = -16(P) + 16(P+Q),$$

and similarly

$$(x_2 \circ \phi_2) \diamond (y_2 \circ \phi_2) = -16(P) - 16(P+Q).$$

These computations show that

$$\tfrac{1}{2}\,\mathrm{r}_0(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}) + \tfrac{1}{2}\,\mathrm{r}_0(\{x_2 \circ \phi_2, y_2 \circ \phi_2\}) = 2\,\mathrm{r}_0(\{x_0, y_0\}), \tag{3-14}$$

and therefore

$$\mathrm{r}_1(\{x_1, y_1\}) + \mathrm{r}_2(\{x_2, y_2\}) = 2\,\mathrm{r}_0(\{x_0, y_0\}). \tag{3-15}$$

We can conclude the proof of (1-6) by inspecting signs.

To prove (1-7), it is necessary to use the isomorphism ϕ from (3-16).

3.6. *Relations among $m(2)$, $m(8)$, $m(3\sqrt{2})$, and $m(\mathrm{i}\sqrt{2})$.* Setting $n = 1/\sqrt{2}$ in (1-7), we obtain

$$m(3\sqrt{2}) + m(\mathrm{i}\sqrt{2}) = m(8).$$

Doing the same in (1-6), we find that

$$m(2) + m(8) = 2m(3\sqrt{2}).$$

In this section we will establish the identity

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}),$$

from which we can deduce expressions for $m(2)$ and $m(8)$.

Consider the functions f and $1-f$, where $f = (\sqrt{2}Y - X)/2 \in \mathbb{C}(E_{3\sqrt{2}})$. Their divisors are

$$\left(\frac{\sqrt{2}Y - X}{2}\right) = (2P) + 2(P+Q) - 3O,$$

$$\left(1 - \frac{\sqrt{2}Y - X}{2}\right) = (P) + (Q) + (3P+Q) - 3O.$$

The diamond operation yields

$$(f) \diamond (1-f) = 6(P) - 10(P+Q).$$

But $(f) \diamond (1-f)$ is trivial in K-theory, hence

$$6(P) \sim 10(P+Q).$$

Now consider the isomorphism

$$\phi : E_{2n+(2/n)} \to E_{2(\mathrm{i}n+1/\mathrm{i}n)}, \quad (X, Y) \mapsto (-X, \mathrm{i}Y). \tag{3-16}$$

This isomorphism implies that

$$\mathrm{r}_{\mathrm{i}\sqrt{2}}(\{x, y\}) = \mathrm{r}_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\}).$$

But we know that

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P+Q).$$

This implies

$$6\,\mathrm{r}_{3\sqrt{2}}(\{x, y\}) = 10\,\mathrm{r}_{\mathrm{i}\sqrt{2}}(\{x, y\}) \quad \text{and} \quad 3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

We conclude that

$$m(8) = \tfrac{8}{5}m(3\sqrt{2}), \quad m(2) = \tfrac{2}{5}m(3\sqrt{2}),$$

and finally

$$m(8) = 4m(2).$$

3.7. ***The Hesse family.*** We will now sketch the case of the Hesse family:

$$x^3 + y^3 + 1 - \frac{3}{t^{1/3}}xy.$$

This family corresponds to $\Gamma_0(3)$. The diamond operation yields

$$(x) \diamond (y) = 9(P) + 9(A) + 9(B), \tag{3-17}$$

where P is a point of order 3, defined over $\mathbb{Q}(t^{1/3})$, and A, B are points of order 3 such that $A + B + P = O$.

For $0 < t < 1$, we have

$$n(t) = \frac{9}{2\pi}\,\mathrm{Im}\left(\frac{\tau}{y_\tau}\Big(R_\tau(\mathrm{e}^{4\pi\mathrm{i}/3}) + R_\tau(\mathrm{e}^{4\pi\mathrm{i}(1+\tau)/3}) + R_\tau(\mathrm{e}^{2\pi\mathrm{i}(2+\tau)/3})\Big)\right).$$

If we let $\mu = -1/\tau$, we obtain, after several steps,

$$n(t) = \mathrm{Re}\left(\frac{27\sqrt{3}y_\mu}{4\pi^2}\sum_{k,n}{}' \frac{\chi_{-3}(n)}{(3\mu k+n)^2(3\bar{\mu}k+n)}\right).$$

As in the previous example, this result may be extended to the complement of κ (the set of t where the polynomial intersects the torus) by comparing holomorphic functions.

3.8. ***The $\mathbf{\Gamma_0^0(6)}$ example.*** We will now sketch a treatment of Stienstra's example [2006]:

$$(x+1)(y+1)(x+y) - \frac{1}{t}xy.$$

Applying the diamond operation, we have

$$(x) \diamond (y) = -6(P) - 6(2P),$$

where P is a point of order 6.

For t small, one can write

$$g(t)=\frac{3}{\pi}\,\mathrm{Im}\big(\frac{\tau}{y_\tau}R_\tau(\xi_6^{-1})+R_\tau(\xi_3^{-1})\big).$$

Eventually, one reaches the expression for $g(t)$ found in [Stienstra 2006]:

$$\mathrm{Re}\left(\frac{36y_\mu}{\pi^2}\sum_{m,n}{}'\frac{\chi_{-3}(m)}{(m+6\mu n)^2(m+6\bar{\mu}n)}\right)+\mathrm{Re}\left(\frac{9y_\mu}{2\pi^2}\sum_{m,n}{}'\frac{\chi_{-3}(m)}{(m+3\mu n)^2(m+3\bar{\mu}n)}\right),$$

3.9. ***The $\Gamma_0^0(5)$ example.*** Our final example is

$$(x+y+1)(x+1)(y+1)-\frac{1}{t}xy.$$

Applying the diamond operation, we find that

$$(x)\diamond(y)=10(P)+5(2P),$$

where P is a torsion point of order 5.

For $t>0$,

$$r(t)=\frac{5}{2\pi}\,\mathrm{Im}\left(\frac{\tau}{y_\tau}\big(2R_\tau(\mathrm{e}^{8\pi\mathrm{i}/5})+R_\tau(\mathrm{e}^{6\pi\mathrm{i}/5})\big)\right).$$

Finally,

$$r(t)=-\,\mathrm{Re}\left(\frac{25\mathrm{i}y_\mu}{4\pi^2}\sum_{m,n}{}'\frac{2(\zeta_5^m-\zeta_5^{-m})+\zeta_5^{2m}-\zeta_5^{-2m}}{(m+5\mu n)^2(m+5\bar{\mu}n)}\right).$$

In conclusion, we see that the modular structure comes from the form of the regulator function, and the functional identities are consequences of the functional identities of the elliptic dilogarithm.

4. Conclusion

We have used both regulator and q-series methods to prove a variety of identities between the Mahler measures of genus-one polynomials. We will conclude this paper with a final open problem.

Open Problem 2. How do you characterize all the functional equations of $\mu(t)$?

We have seen that there are identities like (1-6), stating that

$$2\mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right)$$
$$=\mathrm{m}\left(4k^2+x+\frac{1}{x}+y+\frac{1}{y}\right)+\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

While this formula does not follow from (2-14), it can be proved with regulators.

Indeed, the last section showed us that we can obtain functional identities for the Mahler measures by looking at functional equations for the elliptic dilogarithm.

Now, understanding these identities is a very hard problem. To give an idea of the dimensions of this problem, we note that (3-10) corresponds to the integration of an identity for the Hecke operator T_p. This suggests that more identities will follow from looking at the general operator T_n. And this is just the beginning of the story…

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