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FUNCTIONAL LARGE DEVIATION PRINCIPLES FOR FIRST-PASSAGE-TIME PROCESSES

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We apply an extended contraction principle and superexponential convergence in probability to show that a functional large deviation principle for a sequence of stochastic processes implies a corresponding functional large deviation principle for an associated sequence of first-passage-time or inverse processes. Large deviation principles are established for both inverse processes and centered inverse processes, based on corresponding results for the original process. We apply these results to obtain functional large deviation principles for renewal processes and superpositions of independent renewal processes.

1. Introduction. In this paper we investigate how a (functional) large deviation principle (LDP) for a sequence of stochastic processes can be used to deduce a corresponding (functional) LDP for an associated sequence of first-passage-time or inverse processes. Given a real-valued stochastic process $X \equiv (X(t),\ t \geq 0)$ with sample paths that are unbounded above and satisfy $X(0) \geq 0$, the associated *inverse process* is defined by

$$X^{-1}(t) \equiv \inf\{s > 0\colon X(s) > t\}, \qquad t \geq 0. \tag{1.1}$$

(We use $\equiv$ to denote a definition.)

Previous papers [6, 17, 25, 27, 28] have shown how convergence in distribution in the function space $D \equiv D[0,\infty)$ with one of the Skorohod [22] topologies of $\{X_n,\ n \geq 1\}$, where $X_n \equiv (X_n(t),\ t \geq 0)$, is related to that of $\{X_n^{-1},\ n \geq 1\}$. Those papers also show how convergence in distribution of the sequences of centered processes $\{c_n(X_n - e),\ n \geq 1\}$ and $\{c_n(e - X_n^{-1}),\ n \geq 1\}$ are related, where $c_n \to \infty$ as $n \to \infty$ and e is the identity function; that is, $e(t) = t,\ t \geq 0$.

Our purpose here is to show that these results have fairly direct analogs in the large deviations context, with the contraction principle playing the role of the continuous mapping theorem and an extended contraction principle in [13] and [18], Section 2 (also see [2] and [24]), playing the role of extensions of the continuous mapping theorem in Theorem 5.5 of [1] and on page 68 of [28]. The general theme of relating LDP's to weak convergence is discussed by Puhalskii [13–16, 18, 19]. This paper extends [3] and [18]. Glynn and Whitt [3] established some corresponding relations between one-dimensional LDP's

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for inverse processes. Section 3 of [18] dealt with functional LDP's for inverse processes for the weak topology and established functional LDP's for renewal processes. This paper corrects functional LDP's for inverse processes corresponding to Theorems 3.2 and 6.1 here that had been given in a preliminary draft of [3].

Here is how the present paper is organized. In Section 2 we discuss function space topologies and restrictions on the limit functions under which the inverse map (1.1) is continuous. In Section 3 we present LDP's based directly on these continuity properties. In Section 4 we establish preliminary results about superexponential convergence in probability, which plays the role with LDP's that ordinary convergence in probability plays with weak convergence, as in Theorem 4.1 of [1]. In Section 5 we use the results about superexponential convergence in probability to obtain LDP's for centered first-passage-time processes. In Sections 6 and 7 we apply these results to obtain LDP's for renewal processes and superpositions of renewal processes. The LDP's for centered processes are established in a triangular array setting.

The results in this paper are applied in [21]. There functional LDP's are established for waiting times and departure times in single-server queues with unlimited waiting space. (The results here and in [21] are briefly summarized in [20].) Just as for the heavy-traffic diffusion limits in [5], the results for inverse processes help establish large deviation principles for processes stemming from the basic network operations of superposition, splitting and departure. We illustrate this phenomenon here by our treatment of superposition processes. More generally, the large deviation principles are important for determining the probabilities of rare events in the queueing model, such as hitting times of high levels. The large deviation principles also are intimately connected to the asymptotics of steady-state tail probabilities in the queueing model; for example, see [4] and Section 6 of [18].

We close this introduction by giving an illustrative application. Suppose that $(N(t),\ t \geq 0)$ is a counting process such that $t^{-1}N(t) \to c > 0$ as $t \to \infty$, and we want to approximate the probability $P(N(t) \geq at,\ N(2t) \leq (a+b)t)$ for large t, where $a > c > b$. Thus we are considering the probability that N has unusually large values in the interval $[0, t]$ and unusually small values in the interval $[t, 2t]$. By the methods here, it may be possible to show that $\{(n^{-1}N(nt),\ t \geq 0),\ n \geq 1\}$ obeys an LDP in function space with rate function

$$I(x) = \int_0^\infty \lambda(\dot{x}(t))\,dt$$

for absolutely continuous x with $x(0) = 0$ and $I(x) = \infty$ otherwise, where λ is a (nonnegative convex) "local" rate function on $\mathbb{R}$ with $\lambda(c) = 0$. Then we may apply the contraction principle with this LDP to deduce that

$$\limsup_{t\to\infty} t^{-1} \log P(t^{-1}N(t) \geq a,\ t^{-1}N(2t) \leq a+b) \leq -(\lambda(a) + \lambda(b))$$

and

$$\liminf_{t\to\infty} t^{-1}\log P(t^{-1}N(t) > a,\ t^{-1}N(2t) < a+b) \geq -(\lambda(a)+\lambda(b)),$$

which supports the rough approximation

$$P(t^{-1}N(t) \geq a,\ t^{-1}N(2t) \leq a+b) \approx \exp(-t(\lambda(a)+\lambda(b)))$$

for large t.

2. Functions and topologies on D and its subsets. This section largely follows Section 7 of [28], but there will be a few changes. In particular, there will be a correction for treating the inverse function with the M_1 topology. (This correction is relevant for convergence in distribution as well as for LDP's.) Let D be the space of all right-continuous real-valued functions $x \equiv (x(t),\ t \geq 0)$ on the nonnegative half line $[0,\infty)$ with left limits everywhere in $(0,\infty)$. Let E be the subset of functions x in D that are unbounded above and satisfy $x(0) \geq 0$. Let $D^{\uparrow}$ be the subset of nondecreasing functions in D and let $E^{\uparrow} = E \cap D^{\uparrow}$.

We are primarily interested in the *inverse function*, defined for any $x \in E$ by (1.1); there x is a sample path of X. Also define the *supremum function* for any $x \in D$ by

$$x^{\uparrow}(t) \equiv \sup\{x(s)\colon 0 \leq s \leq t\}, \qquad t \geq 0. \tag{2.1}$$

Obviously, if $x \in E$, then $x^{-1} \in E^{\uparrow}$ and $x^{\uparrow} \in E^{\uparrow}$.

We consider the Skorohod [22] J_1 and M_1 topologies on D and a minor modification of the M_1 topology denoted by M_1'. For these topologies, D is metrizable as a separable metric space. Let D have the Borel σ-field induced by its topology. For the topologies we consider, the Borel σ-fields coincide with the usual Kolmogorov σ-field generated by the finite-dimensional projection maps. The J_1 topology is quite familiar; it is as in [1], [7], [28]. Recall that the M_1 topology on D is defined in terms of the *completed graph*

$$\begin{aligned}\Gamma(x) \equiv \big\{&(u,t) \in R \times R_+\backslash\{0\}\colon u \in [x(t-)\wedge x(t),\ x(t-)\vee x(t)]\big\}\\ &\cup \{(x(0),0)\},\end{aligned} \tag{2.2}$$

where $x(t-)$ denotes the left limit of x at t, $\wedge$ denotes the minimum and $\vee$ denotes the maximum; see [12], [22], [27], [28]. We will call a pair of continuous functions $(u,t) \equiv ((u(s),t(s)),\ s \geq 0)$ such that $t(s)$ is nondecreasing with $t(0)=0$ a *parameterization* of $\Gamma(x)$ if $\Gamma(x) = \cup\{(u(s),t(s))\colon s \geq 0\}$. A sequence $\{x_n,\ n \geq 1\}$ in D converges to x in $D(M_1)$ if there exist parameterizations (u_n,t_n) of x_n, $n \geq 1$, and (u,t) of x such that

$$\sup_{s\leq T}\{|u_n(s)-u(s)| + |t_n(s)-t(s)|\} \to 0 \quad \text{as } n \to \infty \tag{2.3}$$

for all $T > 0$.

What we would like is for the inverse function in (1.1) to be continuous on E, but we must impose constraints when we work with the Skorohod [22] J_1

and M_1 topologies with domain extended from $[0, 1]$ to $[0, \infty)$. We have the following result.

LEMMA 2.1. (a) *The supremum function is continuous in the* J_1 *and* M_1 *topologies*. (b) *The inverse function in* (1.1) *is measurable on* E, *is continuous in the* M_1 *topology at those* x *for which* $x^{-1}(0) = 0$, *and is continuous in the* J_1 *topology at each strictly increasing* x.

Part (a) is in Section 6 of [28]. The J_1 result in part (b) and the need for the J_1 condition are given on page 82 of [28]. However, the M_1 condition is missing in [27] and Theorem 7.1 of [28]. To see that the M_1 condition is needed, let $x_n(t) = t/n$, $0 \le t < 1$, and $x_n(t) = t - 1$, $t \ge 1$. Clearly $x_n \to x$ (M_1) where $x(t) = 0$, $0 \le t < 1$ and $x(t) = t - 1$, $t \ge 1$. However, $x_n^{-1}(0) = 0 \not\to x^{-1}(0) = 1$ as $n \to \infty$, so that $x_n^{-1} \not\to x^{-1}$ (M_1) as $n \to \infty$.

With the M_1 condition added, the M_1 continuity proof is as in [27]. We look at the inverse as the composition of the inverse and supremum maps. Hence, it suffices to consider the inverse map on $E^{\uparrow}$. Continuity is established by noting that, given the M_1 condition, each parameterization (u, t) of x can serve as a parametric representation of x^{-1} when the roles of u and t are switched.

Another approach to the problem of the continuity of the inverse mapping on D is to change the topology instead of adding the extra condition. In [18] the continuity for the weak topology was proved. Here we use a weaker topology, which we call M_1' and which is defined in the same way as M_1, except that we change $\Gamma(x)$ to

$$\Gamma'(x) \equiv \{(u, t) \in R \times R_+ \colon u \in [x(t-) \wedge x(t), x(t-) \vee x(t)]\}, \tag{2.4}$$

where $x(0-) = 0$. Stated another way, $\Gamma'(x)$ is the extended graph $\Gamma(x)$ complemented by adding the segment $[0, x(0)]$ if $x(0) \ge 0$ or $[x(0), 0]$ if $x(0) \le 0$. We say that $x_n \to x$ in $D(M_1')$ if (2.3) holds for some parameterizations of $\Gamma'(x_n)$ and $\Gamma'(x)$. More rigorously, $D(M_1')$ is a metric space with metric d' defined as follows. If $x \equiv (x(t),\ t \ge 0)$ and $y \equiv (y(t),\ t \ge 0)$ are elements of D, let

$$\begin{aligned} d_k'(x, y) \equiv \inf\Big\{ \sup_{s \le k} \{ & |u(s) g_k(t(p)) - v(s) g_k(r(s))| \\ & + |t(s) g_k(t(s)) - r(s) g_k(t(s))| \} \Big\}, \end{aligned} \tag{2.5}$$

where $g_k(t)$ equals 1 for t less than k, equals 0 for t greater than $k + 1$ and is a linear interpolation between k and $k + 1$, and $(u(s), t(s),\ s \ge 0)$ and $(v(s), r(s),\ s \ge 0)$ are parameterizations of x and y, respectively, and the infimum is taken over all the parameterizations. Then

$$d'(x, y) \equiv \sum_{k=1}^{\infty} \frac{d_k'(x, y) \wedge 1}{2^k}. \tag{2.6}$$

metrizes M_1'. It is not difficult to show that (D, d') is a separable metric space and d' induces the Kolmogorov σ-field. In addition, the M_1 topology is stronger than the M_1' topology; that is, overall the topologies are ordered by $J_1 \to M_1 \to M_1'$. Convergence $x_n \to x$ is equivalent for all three topologies at continuous x with $x(0) = 0$. Moreover, on $E^{\uparrow}$ with the M_1' topology $x_n \to x$ is equivalent to pointwise convergence $x_n(t) \to x(t)$ at all continuity points except possibly for $t = 0$. The following is the key lemma.

LEMMA 2.2. *The supremum function in* (2.1) *on* D *and the inverse function in* (1.1) *on* E *are continuous in the* M_1' *topology*.

PROOF. The argument for the supremum function is straightforward. The claim for the inverse function follows since if (u, t) is a Γ'-parameterization of $x \in E^{\uparrow}$, then (t, u) is a Γ'-parameterization of x^{-1}. □

We will need another basic lemma. Let e be the identity function; $e(t) = t$ for $t \geq 0$. Let c be any real number.

LEMMA 2.3. *If* $x \in E^{\uparrow}$, *then*

$$d'(c(x - e),\ c(e - x^{-1})) \leq d'(x, e).$$

PROOF. For any k and $\varepsilon > 0$, let $(u(s),\ t(s),\ s > 0)$ and $(t'(s),\ t'(s),\ s \geq 0)$ be Γ'-parameterizations of x and e, respectively, so that

$$\sup_{s \leq k}\{|u(s) - t'(s)| + |t(s) - t'(s)|\} \leq d_k'(x,\ e) + \varepsilon.$$

Note that $(c(u(s) - t(s)),\ t(s),\ s \geq 0)$ is a Γ'-parameterization of $c(x - e)$. Moreover, since $x \in E^{\uparrow}$, $(c(u(s) - t(s)),\ u(s),\ s \geq 0)$ is a Γ'-parameterization of $c(e - x^{-1})$. Using these parameterizations, we see that, for any c,

$$\begin{aligned} d_k'(c(x - e),\ c(e - x^{-1})) &\leq \sup_{s \leq k}\{|u(s) - t(s)|\} \\ &\leq \sup_{s \leq k}\{|u(s) - t'(s)| + |t(s) - t'(s)|\} \\ &\leq d_k'(x, e) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, $d_k'(c(x-e),\ c(e-x^{-1})) \leq d_k'(x, e)$ for each k, from which the conclusion follows.

3. Initial large deviation conclusions. We now draw large deviation conclusions from the continuity properties in Section 2. Recall that all spaces we consider are separable metric spaces. Following Varadhan [23, 24], we say that a function $I(x)$ defined on a metric space S and taking values in $[0, \infty]$ is a *rate function* if the sets $\{x \in S\colon I(x) \leq a\}$ are compact for all $a \geq 0$, and a sequence $\{P_n,\ n \geq 1\}$ of probability measures on the Borel σ-field of S (or a sequence of random elements $\{X_n,\ n \geq 1\}$ with values in S and distributions P_n) obeys the LDP with the rate function I if

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n(F) \leq -\inf_{x \in F} I(x)$$

for all closed $F \subset S$, and

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(G) \geq -\inf_{x\in G} I(x)$$

for all open $G \subset S$.

We establish new LDP's from previously established ones by applying the contraction principle [23, 24] or an extension [13] and [18], Section 2. Here are statements: the *contraction principle* states that if $\{X_n,\ n \geq 1\}$ obeys an LDP with rate function I and if f is continuous, then $\{f(X_n), n \geq 1\}$ obeys an LDP with rate function

$$I'(y) \equiv \inf_{x:\ f(x)=y} I(x). \tag{3.1}$$

Our *extended contraction principle* states that if $\{X_n,\ n \geq 1\}$ obeys an LDP with rate function I, if $\{f_n,\ n \geq 1\}$ is a sequence of measurable functions, if the function f is continuous in restriction to the sets $\{x\colon I(x) \leq a\},\ a \geq 0$, and if $f_n(x_n) \to f(x)$ as $n \to \infty$ for all x_n for which $x_n \to x$ as $n \to \infty$ for all x for which $I(x) < \infty$, then $\{f_n(X_n),\ n \geq 1\}$ obeys an LDP with rate function (3.1). (This statement is actually a consequence of a more general result in [18]; see Theorem 2.1 and following Remarks 1 and 2 there.) An important special case is $f_n = f$, as in the contraction principle, where f is continuous at each x with $I(x) < \infty$. In either case, if in addition f is a bijection, then we can write $I'(y) = I(f^{-1}(y))$. The applications here illustrate the importance of the extended contraction principle. We consider both single functions that are not continuous everywhere and sequences of functions.

The next three theorems follow immediately from Lemmas 2.1 and 2.2 and the contraction principle or its extension.

THEOREM 3.1. *If $\{X_n,\ n \geq 1\}$ obeys the LDP in $E(J_1)$ with rate function I_X, then $\{X_n^{\uparrow},\ n \geq 1\}$ obeys the LDP in $E^{\uparrow}(J_1)$ with rate function*

$$I_{X^{\uparrow}}(x) \equiv \inf_{\substack{y\in E:\\ x=y^{\uparrow}}} \{I_X(y)\}, \qquad x \in E^{\uparrow}. \tag{3.2}$$

If in addition $I_X(x) = \infty$ whenever x is not strictly increasing, then $\{X_n^{-1},\ n \geq 1\}$ obeys the LDP in $E^{\uparrow}(J_1)$ with rate function

$$I_{X^{-1}}(x) \equiv \inf_{\substack{y\in E\\ y:\ y^{-1}=x}} \{I_X(y)\} = I_{X^{\uparrow}}(x^{-1}), \qquad x \in E^{\uparrow}. \tag{3.3}$$

[As a consequence, $I_{X^{-1}}(x) = \infty$ if x is not continuous.]

THEOREM 3.2. *If $\{X_n,\ n \geq 1\}$ obeys the LDP in $E(M_1)$ with rate function I_X, then $\{X_n^{\uparrow},\ n \geq 1\}$ obeys the LDP in $E^{\uparrow}(M_1)$ with rate function $I_{X^{\uparrow}}$ in* (3.2). *If $I_X(x) = \infty$ whenever $x^{-1}(0) > 0$, then $\{X_n^{-1},\ n \geq 1\}$ obeys the LDP in $E^{\uparrow}(M_1)$ with rate function $I_{X^{-1}}$ in* (3.3). *[As a consequence $I_{X^{-1}}(x) = \infty$ if $x(0) > 0$].*

THEOREM 3.3. *If* $\{X_n,\ n \geq 1\}$ *obeys the LDP in* $E(M_1')$ *with rate function* I_X, *then* $\{X_n^{\uparrow},\ n \geq 1\}$ *obeys the LDP in* $E^{\uparrow}(M_1')$ *with rate function* $I_{X^{\uparrow}}$ *in* (3.2) *and* $\{X_n^{-1},\ n \geq 1\}$ *obeys the LDP in* $E^{\uparrow}(M_1')$ *with rate function* $I_{X^{-1}}$ *in* (3.3).

In (3.3) we have used the fact that the inverse map is a bijection on $E^{\uparrow}$ in order to write $I_{X^{-1}}(x) = I_{X^{\uparrow}}(x^{-1})$.

REMARK 3.1. It may be convenient to establish an LDP for an inverse process by applying Theorems 3.2 or 3.3 for the M_1 or M_1' topology instead of the stronger J_1 topology, but the LDP extends to the stronger J_1 topology from M_1 or M_1' if the rate function $I_{X^{-1}}(x)$ is infinite at discontinuous x and for M_1' if in addition the rate function $I_{X^{-1}}(x) = \infty$ when $x(0) \neq 0$. This is because convergence $x_n \to x$ for continuous x with $x(0) = 0$ is equivalent for the three topologies and we can apply the extended contraction principle to the identity maps. Indeed, the LDP extends to the stronger uniform topology, under which D is nonseparable, provided that X_n^{-1} remains a bonafide random element. However, in general this need not be the case since the Borel σ-field is richer than the Kolmogorov σ-field; see [1], Section 18. Indeed, measurability with respect to the Borel σ-field associated with the uniform topology fails even for the Poisson process. Thus, for general LDP's on D it is often important to work with topologies like the Skorohod topologies, for which the Borel and Kolmogorov σ-fields coincide.

In many (but not all) cases, the rate functions $I_X(x)$ for X_n and $I_{X^{-1}}(x)$ for X_n^{-1} have the form

$$I_X(x) = \int_0^{\infty} \lambda_X(\dot{x}(t))\, dt \tag{3.4}$$

if x is absolutely continuous and $x(0) = 0$ and $I_X(x) = \infty$ otherwise, and

$$I_{X^{-1}}(x) = \int_0^{\infty} \lambda_{X^{-1}}(\dot{x}(t))\, dt \tag{3.5}$$

if x is absolutely continuous and $x(0) = 0$ and $I_{X^{-1}}(x) = \infty$ otherwise, where λ_X and $\lambda_{X^{-1}}$ are convex local rate functions on $\mathbb{R}$. We can then apply Theorems 3.1–3.3 to deduce the relation between the rate functions λ_X and $\lambda_{X^{-1}}$ on $\mathbb{R}$, which is consistent with what was established directly by [3].

THEOREM 3.4. *If* $\{X_n, n \geq 1\}$ *obeys the LDP in* $E^{\uparrow}$ *for one of the topologies* J_1, M_1 *or* M_1' *with the rate function* I_X *satisfying* (3.4), *where* $\lambda_X(0) = \infty$, *then* $\{X_n^{-1},\ n \geq 1\}$ *obeys the LDP in* $E^{\uparrow}(J_1)$ *with the rate function* $I_{X^{-1}}$ *from* (3.5), *where*

$$\lambda_{X^{-1}}(z) = z\lambda_X(1/z),\ \lambda_{X^{-1}}(0) = \infty. \tag{3.6}$$

If the function λ_X (*respectively*, $\lambda_{X^{-1}}$) *is convex downwards, then the sequence of random variables* $\{X_n(1),\ n \geq 1\}$ (*respectively*, $\{X_n^{-1}(1),\ n \geq 1\}$) *obeys the LDP in* $\mathbb{R}$ *with rate function* λ_X (*respectively*, $\lambda_{X^{-1}}$).

PROOF. Since $\lambda_X(0) = \infty$, $I_X(x) = \infty$ if x^{-1} is not absolutely continuous (as follows from Lemma 3.6 in [18]). By Remark 3.1, the LDP for $\{X_n,\ n \geq 1\}$ holds in $E^{\uparrow}(J_1)$. By Theorem 3.1 $\{X_n^{-1},\ n \geq 1\}$ obeys the LDP in $E^{\uparrow}(J_1)$ with rate function $I_{X^{-1}}$ which, for absolutely continuous x and $x^{-1} = y$, is given by

$$\begin{aligned} I_{X^{-1}}(x) = I_X(x^{-1}) = I_X(y) &= \int_0^\infty \lambda_X(\dot{y}(s))\,ds \\ &= \int_0^\infty \lambda_X(\dot{y}(x(s)))\dot{x}(s)\,ds \\ &= \int_0^\infty \lambda_{X^{-1}}(\dot{x}(s))\,ds \quad \text{since } y \circ x = e \end{aligned}$$

by performing a change of variables. Hence (3.6) holds. As indicated above, we next apply the extended contraction principle with the projection map defined by $\pi(x) = x(1)$ to obtain the LDP's in $\mathbb{R}$. Since λ_X is convex, the infimum over x such that $x(1) = z$ is attained at $\dot{x}(t) = z$ for $0 \leq t \leq 1$. The relation between λ_X and $\lambda_{X^{-1}}$ was established in [3]. □

It is important to note, however, that the rate functions I_X of $\{X_n,\ n \geq 1\}$ and $I_{X^{-1}}$ of $\{X_n^{-1},\ n \geq 1\}$ may involve functions with jumps, so that (3.4) and (3.5) need not hold; see [9], [11], [18] and Section 6 below. Then the connections to LDP's on $\mathbb{R}$ is more complicated, for example, because the projection map is not necessarily continuous. Functions with jumps may play a role in either I_X or $I_{X^{-1}}$ or both. However, Theorem 3.4 does apply to the renewal theory examples in Sections 6 and 7 under regularity conditions.

4. Superexponential convergence in probability. As a basis for establishing relations between LDP's for centered processes and associated centered inverse processes, paralleling Theorems 7.3–7.5 of [28], we establish some preliminary results about superexponential convergence in probability.

As in [16], we say that a sequence $\{X_n,\ n \geq 1\}$ of random elements of a metric space (S, ρ) *converges superexponentially in probability* to an element $x_0 \in S$ if, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P^{1/n}(\rho(X_n, x_0) > \varepsilon) = 0 \tag{4.1}$$

and we write $X_n \xrightarrow{P^{1/n}} x_0$. This mode of convergence plays a role in large deviations similar to the role convergence in probability plays in weak convergence. We collect some simple properties of superexponential convergence in probability in the following lemmas. The similarity with weak convergence should be evident; for example, see [1], [28]. In the following lemmas, S is the space D with any of the topologies J_1, M_1 or M_1'.

Parts of the next lemma can obviously be extended to general metric spaces; for example, see [15], [16]. Note the similarity of parts (b) and (c) to Theorems 4.1 and 4.4 of [1]. Part (c) follows directly from part (b) (also it is Lemma 3.3 in [16]).

LEMMA 4.1. (a) $X_n \stackrel{P^{1/n}}{\longrightarrow} x_0$ *if and only if* $\{X_n,\ n \ge 1\}$ *obeys the LDP with rate function*

$$\delta(x - x_0) \equiv \begin{cases} 0, & x = x_0, \\ \infty, & x \ne x_0. \end{cases} \tag{4.2}$$

(b) *If* $\{X_n, n \ge 1\}$ *obeys the LDP in* D *for one of the topologies* J_1, M_1 *and* M_1' *with rate function* I *and* $Y_n \stackrel{P^{1/n}}{\longrightarrow} y_0$, *then* $\{(X_n, Y_n),\ n \ge 1\}$ *obeys the LDP in* $D \times D$ *for the product topology with rate function* $I(x) + \delta(y - y_0)$, *and* $\{X_n + Y_n,\ n \ge 1\}$ *obeys the LDP in* D *for the same topology with rate function* $I(x - y_0),\ x \in D$.

(c) *If* $\{X_n,\ n \ge 1\}$ *obeys the LDP in* D *for metric* m *with rate function* I *and* $m(X_n, Y_n) \stackrel{P^{1/n}}{\longrightarrow} 0$, *then* $\{Y_n,\ n \ge 1\}$ *obeys the LDP in* D *for metric* m *with rate function* I.

LEMMA 4.2. (a) *Let* $x_0 \equiv (x_0(t),\ t \ge 0)$ *be continuous with* $x(0) = 0$ *if the topology is* M_1'. *Then* $X_n \stackrel{P^{1/n}}{\longrightarrow} x_0$ *if and only if*

$$\lim_{n\to\infty} P^{1/n}\Big(\sup_{t\le T} |X_n(t) - x_0(t)| > \varepsilon\Big) = 0$$

for all $\varepsilon > 0$ *and* $T > 0$.

(b) *If, for* $c_n \to \infty$, $\{c_n X_n,\ n \ge 1\}$ *obeys the LDP with some rate function, then* $X_n \stackrel{P^{1/n}}{\longrightarrow} \theta$ *as* $n \to \infty$, *where* $\theta(t) = 0,\ t \ge 0$.

(c) *If* X_n *has paths in* E *for* $n \ge 1$ *and* $X_n \stackrel{P^{1/n}}{\longrightarrow} e$, *then* $X_n^{-1} \stackrel{P^{1/n}}{\longrightarrow} e$.

PROOF. For part (a), we do the proof only for the J_1 topology; the proof for the other topologies is similar. By the triangle inequality,

$$\sup_{0\le t\le T} |X_n(t) - x_0(t)| \le \sup_{0\le t\le T} \{|X_n(t) - x_0(\lambda_n(t))| + |x_0(\lambda_n(t)) - x_0(t)|\},$$

where $\lambda_n(t)$, for $0 \le t \le T$, is any homeomorphism of $[0, T]$. Hence,

$$\sup_{0\le t\le T} |X_n(t) - x_0(t)| \le d_T(X_n, x_0) + w_{x_0}(d_T(X_n, x_0)),$$

where d_T is the Skorohod J_1 metric on $D[0, T]$ and $w_x(\delta)$ is the modulus of continuity of x as in [1], page 54. For any x_0 and ε, let δ be such that $w_{x_0}(\delta) \le \varepsilon$, which is possible because x_0 is continuous. Then

$$\begin{aligned}\{d_T(X_n, x_0) \le \delta \wedge \varepsilon\} &\subseteq \Big\{\sup_{0\le t\le T} |X_n(t) - x_0(t)| \le 2\varepsilon\Big\}\\ &\subseteq \{d_T(X_n, x_0) \le 2\varepsilon\},\end{aligned}$$

which implies the result. We also use the fact that a J_1 metric d on $D[0, \infty)$ can be related to the metric d_T on $D[0, T]$ for large T; see (2.2) of [28]; that is, for any $\varepsilon > 0$, $\{d(X_n, x_0) \le \varepsilon\} \subseteq \{d_T(X_n, x_0) \le 2\varepsilon\} \subseteq \{d(X_n, x_0) \le 3\varepsilon\}$ for T suitably large. For (b), note that by the definition of the LDP and using that $c_n \to \infty$, we have, for $\varepsilon > 0$, $A > 0$, and metric m (associated with one of the topologies J_1, M_1 or M_1'),

$$\limsup_{n\to\infty} P^{1/n}(m(X_n, \theta) > \varepsilon) \le \limsup_{n\to\infty} P^{1/n}(m(c_n X_n, \theta) \ge A\varepsilon)$$
$$\le \sup_{x:\ m(x,\theta)\ge A\varepsilon} \exp(-I(x)),$$

and the latter goes to 0 as $A \to \infty$ since the sets $\{x\colon I(x) \le a\}$, $a \ge 0$, are compact and hence bounded. For part (c), we apply Lemmas 2.1, 2.2 and part (a) of Lemma 4.1. □

We now discuss the composition map, denoted by $\circ$. Recall that if $x \equiv (x(t),\ t \ge 0) \in D$ and $y \equiv (y(t),\ t \ge 0) \in E^{\uparrow}$, then $x \circ y = (x(y(t)),\ t \ge 0) \in D$; see [28]. Note the similarity of this lemma with [1], page 145.

LEMMA 4.3. *Let $\{X_n,\ n \ge 1\}$ obey an LDP for one of the topologies J_1, M_1 and M_1' with rate function I, and let $\{Y_n,\ n \ge 1\}$ be a sequence of nonnegative processes with paths from $E^{\uparrow}$ such that $Y_n \xrightarrow{P^{1/n}} y_0$. If the topology is M, then let y_0 be continuous. If the topology is M_1', then let y_0 be continuous with $y_0(0) = 0$. If $I(x) = \infty$ for discontinuous x, or y_0 is continuous and strictly increasing, then $\{X_n \circ Y_n,\ n \ge 1\}$ obeys the LDP with rate function*

$$I'(z) \equiv \inf_{x:\ x\circ y_0 = z} I(x), \qquad z \in D.$$

PROOF. By Lemma 4.2, $\{(X_n, Y_n),\ n \ge 1\}$ obeys the LDP in $D \times D$ with $I(x)+\delta(y-y_0)$. The claim then follows by Theorem 3.1 in [28] and the extended contraction principle. An analog of Theorem 3.1 of [28] holds for M_1 and M_1' if the limit y there is continuous.

5. LDPs for centered processes. We now apply the lemmas in Section 4 to deduce the following results. For them, we assume that the processes X_n have paths in E and $c_n \to \infty$ as $n \to \infty$.

THEOREM 5.1. *If the sequence $\{c_n(X_n - e),\ n \ge 1\}$ obeys the LDP in D for the J_1 topology with rate function I such that $I(x) = \infty$ for functions x which have positive jumps or have $x(0) \ne 0$, then the sequence $\{c_n(e - X_n^{-1}),\ n \ge 1\}$ also obeys the LDP in D for the J_1 topology with the rate function I.*

PROOF. We follow the argument in [28], Theorem 7.3. By Lemma 4.2(b), $X_n \xrightarrow{P^{1/n}} e$ and by Lemma 4.2(c), $X_n^{-1} \xrightarrow{P^{1/n}} e$. Lemma 4.3 then implies that $\{c_n(X_n - e) \circ X_n^{-1},\ n \ge 1\}$ obeys the LDP with I. Since

$$c_n(e - X_n^{-1}) = c_n(X_n - e) \circ X_n^{-1} + c_n(e - X_n \circ X_n^{-1}),$$

by Lemma 4.1(b), the theorem would follow from

$$c_n(e - X_n \circ X_n^{-1}) \xrightarrow{P^{1/n}} 0,$$

which in turn, by Lemma 4.2(a), follows by

$$\sup_{t\le T} c_n|X_n\circ X_n^{-1}(t)-t| \xrightarrow{P^{1/n}} 0, \qquad T>0. \tag{5.1}$$

Since (cf. [28])

$$0\le \sup_{t\le T}(X_n\circ X_n^{-1}(t)-t)\le \sup_{t\le X_n^{-1}(T)}(\Delta X_n(t))^+,$$

where $\Delta x(t)$, for $x=(x(t),\ t\ge 0)$, denotes the jump of x at t with $\Delta x(0)=x(0)$, we have for $A>0$, $\varepsilon>0$, that

$$\begin{aligned} &P^{1/n}\Bigl(\sup_{t\le T} c_n|X_n\circ X_n^{-1}(t)-t|>\varepsilon\Bigr)\\ &\quad\le P^{1/n}(X_n^{-1}(T)>A)+P^{1/n}\Bigl(\sup_{t\le A} c_n(\Delta X_n(t))^+>\varepsilon\Bigr). \end{aligned} \tag{5.2}$$

Now it is not difficult to see that the function $x\to\sup_{t\le A}(\Delta x(t))^+$ is continuous in the J_1 topology at each x with no positive jumps; for example, see [8], Chapter 6, Section 2. Then, by the extended contraction principle and the LDP for $\{c_n(X_n-e),\ n\ge 1\}$,

$$\limsup_{n\to\infty} P^{1/n}\Bigl(\sup_{t\le A} c_n(\Delta X_n(t))^+\ge\varepsilon\Bigr)\le \sup_{x:\ \sup_{t\le A}(\Delta x(t))^+\ge\varepsilon}\exp(-I(x))=0,$$

proving that the second term on the right-hand side of (5.2) goes to 0 as $n\to\infty$. The first term goes to 0 as $n\to\infty$ and $A\to\infty$ since $X_n^{-1}\xrightarrow{P^{1/n}} e$. Hence, the limit (5.1) has been proved, so the theorem has been proved. □

In order to obtain a result paralleling Theorem 5.1 for the M_1' topology, we first establish a result for centered supremum processes.

THEOREM 5.2. *If the sequence $\{c_n(X_n-e),\ n\ge 1\}$ obeys the LDP in D for either M_1 or the M_1' topology with rate function I, then $\{c_n(X_n^{\uparrow}-e),\ n\ge 1\}$ obeys the LDP in $D^{\uparrow}$ for the same topology with rate function I.*

PROOF. We can use the extended contraction principle with the functions $f_n(y)=(y+c_ne)^{\uparrow}-c_ne$. Assume that $y_n=c_n(x_n-e)\to y\equiv x$. Since $f_n(y_n)=c_n(x_n^{\uparrow}-e)$, it suffices to show that $f_n(y_n)\to y$ whenever $y_n\to y$ in D for the M_1 or M_1' topology, which follows by Theorem 6.3(ii) in [28]. (The proof there needs changing when x has negative jumps.)

THEOREM 5.3. *If the sequence $\{c_n(X_n-e),\ n\ge 1\}$ obeys the LDP in D for the M_1' topology with rate function I, then the sequence $\{c_n(e-X_n^{-1}),\ n\ge 1\}$ obeys the LDP in D for the M_1' topology with the rate function I.*

PROOF. First apply Theorem 5.2 to see that it suffices to assume that $X_n\in E^{\uparrow}$ for each n. By Lemma 4.2(b), $d'(X_n,e)\xrightarrow{P^{1/n}} 0$, so that $d'(c_n(X_n-e),c_n(e-X_n^{-1}))\xrightarrow{P^{1/n}} 0$ by Lemma 2.3. Finally, apply Lemma 4.1(c).

The last result of the section is a straightforward extension of Theorems 5.1 and 5.3, but is quite useful in applications.

THEOREM 5.4. *If the sequence* $\{c_n(X_n - a_n e), n \geq 1\}$, *where* a_n *are real numbers converging to* $a > 0$, *obeys the LDP in* D *for the* M_1' *topology with rate function* $I(x)$, *then the sequence* $\{(c_n(X_n - a_n e), c_n(X_n^{-1} - a_n^{-1}e)), n \geq 1\}$ *obeys the LDP in* $D \times D$ *for* (*the product topology associated with*) *the* M_1' *topology with rate function* $I_1(x, y) = I(x)$, *when* $y = -a^{-1}x \circ (a^{-1}e)$, *and* $I_1(x, y) = \infty$ *otherwise*.

If the LDP for $\{c_n(X_n - a_n e), n \geq 1\}$ *holds for the* J_1 *topology with* I *equal to infinity at functions* x *with positive jumps or with* $x(0) \neq 0$, *then the LDP for* $\{c_n(X_n - a_n e), c_n(X_n^{-1} - a_n^{-1}e)), n \geq 1\}$ *holds for* (*the product topology associated with*) *the* J_1 *topology with the rate function* I_1.

PROOF. Noting that $x^{-1} - a_n^{-1}e = -(e - (a_n^{-1}x)^{-1}) \circ (a_n^{-1}e)$, we have as in the preceding argument that $d'(c_n(X_n^{-1} - a_n^{-1}e), -c_n a_n^{-1}(X_n^{\uparrow} - a_n e) \circ (a_n^{-1}e)) \xrightarrow{P^{1/n}} 0$ so that in the statement of the theorem we can replace $\{(c_n(X_n - a_n e), c_n(X_n^{-1} - a_n^{-1}e)), n \geq 1\}$ by $\{(c_n(X_n - a_n e), -a_n^{-1} g_n(c_n(X_n - a_n e)) \circ (a_n^{-1}e)), n \geq 1\}$, where $g_n(x) = (x + c_n a_n e)^{\uparrow} - c_n a_n e$. The claim follows by Lemma 4.3 and the extended contraction principle since $g_n(x_n) \to x$ as $x_n \to x$ and $a_n \to a$.

On writing

$$c_n(a_n^{-1}e - X_n^{-1}) = c_n a_n^{-1}(X_n - a_n e) \circ X_n^{-1} + c_n a_n^{-1}(e - X_n \circ X_n^{-1}),$$

we can apply for the case of the J_1 topology the argument of the proof of Theorem 5.1. □

6. Large deviations for renewal processes. In this section, we apply the results of previous sections to derive LDP's for sequences of renewal processes. Corresponding results can be established for cases in which the i.i.d. condition is relaxed, drawing upon Zajic [29] and references therein. We first assume that the X_n are defined by

$$X_n(t) \equiv \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i, \tag{6.1}$$

where ξ_i, $i \geq 1$, are i.i.d., nonnegative, $E\xi_1 > 0$.

Let $(N(t),\ t \geq 0)$ be the renewal process with $\xi_1, \xi_2, \dots$ as the times between renewals; that is,

$$N(t) \equiv \max\left\{k \geq 1\colon \sum_{i=1}^{k} \xi_i \leq t\right\}, \qquad t \geq 0, \tag{6.2}$$

with $N(t) = 0$ if $\xi_1 > t$, and let $N_n \equiv (N(nt)/n,\ t \geq 0)$, $n \geq 1$. We are going to derive the LDP for the sequence $\{N_n,\ n \geq 1\}$. This will be done by reducing this problem to the LDP for $\{X_n,\ n \geq 1\}$ and by invoking earlier results for $\{X_n,\ n \geq 1\}$. For this purpose, note that

$$N(nt)/n = X_n^{-1}(t) - 1/n. \tag{6.3}$$

The following theorem is a version of Theorem 3.1 in [18]. Part (b) is the same. Part (a) is also equivalent, because the M_1' topology here and the weak topology in [18] coincide on $E^{\uparrow}$, since both correspond to pointwise convergence at all continuity points except 0.

THEOREM 6.1. *Assume that* $E\exp(\alpha\xi_1) < \infty$ *for some* $\alpha > 0$. *Let* $\alpha^* = \sup\{\alpha\colon E\exp(\alpha\xi_1) < \infty\}$. (a) *Then* $\{N_n,\ n \geq 1\}$ *obeys the LDP in* $E^{\uparrow}$ *for the* M_1' *topology with rate function*

$$I_N(x) \equiv \int_0^{\infty} \sup_{\alpha<\alpha^*}\{\alpha - \dot{x}_1^l(t)\log E\exp(\alpha\xi_1)\}\,dt - \log P(\xi_1 = 0)x_2^l(\infty), \tag{6.4}$$

where $x = x_1^l + x_2^l$ *is the Lebesgue decomposition of* x *with respect to Lebesgue measure*; x_1^l *is the absolutely continuous component with* $x_1^l(0) = 0$, x_2^l *is the singular component and* $\dot{x}_1^l(t)$ *is the derivative*. *In* (6.4) *it is assumed*, *that the product on the right-hand side is* 0 *if* $P(\xi_1 = 0) = 0$ *and* $x_2^l(\infty) = 0$.

(b) *If*, *in addition*, $P(\xi_1 = 0) = 0$, *then the LDP holds for the* J_1 *topology with* $I_N(x) = \infty$ *if* x *is not absolutely continuous*, *or* $x(0) \neq 0$.

PROOF. By [18], Lemma 3.2, $\{X_n,\ n \geq 1\}$ obeys the LDP on $E^{\uparrow}$ for the weak (and hence the M_1') topology with rate function

$$I_X(x) \equiv \int_0^{\infty} \sup_{\alpha<\alpha^*}\{\alpha\dot{x}_1^l(t) - \log E\exp(\alpha\xi_1)\}\,dt + \alpha^* x_2^l(\infty), \tag{6.5}$$

where $\infty \cdot 0 = 0$. (Note that if $\alpha^* = \infty$, then $I_X(x) = \infty$ whenever x is not absolutely continuous, but otherwise this is not the case.) The first part of the proof of part (a) is completed by applying Theorem 3.3, Lemma 4.3 and (6.3); see [18] for details. For part (b), note that the extra condition makes $I_N(x) = \infty$ for discontinuous x or if $x(0) \neq 0$. Then we use the fact that $x_n \to x(J_1)$ is equivalent to $x_n \to x(M_1')$ for continuous x with $x(0) = 0$. Hence, we can apply the extended contraction principle with the identity map to strengthen the topology, as noted in Section 3. □

REMARK 6.1. In the setting of Theorem 6.1 assume, in addition, that $E\exp(\alpha\xi_1) < \infty$ for all α and $P(\xi_1 = 0) = 0$. Then we can use Theorem 3.4 to get the familiar LDP's in R for the projections at $t = 1$. Under the assumptions, $I_X(x) = \infty$, when x is either not absolutely continuous, or $x(0) \neq 0$, or x is not strictly increasing for the rate function (6.5), so that the conditions of Theorem 3.4 are satisfied with the local rate functions in $\mathbb{R}$ being

$$\lambda_X(z) = \sup_{\alpha}\{\alpha z - \log E\exp(\alpha\xi_1)\}$$

and

$$\lambda_{X^{-1}}(z) = z\lambda_X(1/z) = \sup_{\alpha}\{\alpha - z\log E\exp(\alpha\xi_1)\}.$$

An application of the theorem provides the one-dimensional LDP's.

We now establish an LDP in D for centered renewal processes, which is a form of moderate deviations; see page 79 of [15]. Motivated by applications to queues in heavy traffic, we choose to work in the more general setting of triangular arrays. More specifically, we consider a sequence of renewal processes indexed by n and denote by $\xi_{n,i}$, $i \geq 1$, the times between renewals. We next define

$$N'_n(t) = \max\left(k \geq 1: \sum_{i=1}^{k} \xi_{n,i} \leq t\right), \tag{6.6}$$

$$N_n(t) = \frac{1}{a_n} N'_n(a_n t), \tag{6.7}$$

where $a_n/n \to \infty$. For completeness, we first state the result of Example 7.2 [15] on "the moderate deviation invariance principle" for partial sums of triangular arrays (prototypes for partial sums of r.v. are in [10] and [26], Theorem 4.4.3).

LEMMA 6.1. *Let* $\{\zeta_{n,i},\ i \geq 1\}, n \geq 1$, *be a triangular array of row-wise i.i.d. random variables with* $E\zeta_{n,1} = 0$, $\operatorname{Var} \zeta_{n,1} \to \sigma^2$. *Define*

$$Z_n(t) = \frac{1}{\sqrt{n a_n}} \sum_{i=1}^{\lfloor a_n t \rfloor} \zeta_{n,i}.$$

Let at least one of the following conditions hold:

(i) $(\log a_n)/n \to \infty$ *and* $\sup_n E|\zeta_{n,1}|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$,

(ii) *for some* $\beta \in (0,1]$, $a_n^{\beta}/n^{2-\beta} \to \infty$ *and* $\sup_n E\exp(\alpha|\zeta_{n,1}|^{\beta}) < \infty$ *for some* $\alpha > 0$.

Then $\{Z_n,\ n \geq 1\}$ *obeys the LDP in* D *for the* J_1 *topology with rate function*

$$I_X(x) \equiv \begin{cases} \dfrac{1}{2\sigma^2} \displaystyle\int_0^{\infty} \dot{x}(t)^2\,dt, & \text{if } x \text{ is absolutely continuous and } x(0) = 0,\\ \infty, & \text{otherwise}. \end{cases} \tag{6.8}$$

The proof is in [15]. (Note that the case $\beta = 1$ which is not included there is dealt with by the same argument.)

THEOREM 6.2. *Let* N_n *be defined by* (6.6) *and* (6.7). *Let* $\operatorname{Var} \xi_{n,1} \to \sigma^2$ *and* $E\xi_{n,1} \to \lambda^{-1}$ *as* $n \to \infty$. *Assume that at least one of the following conditions hold*:

(i) $(\log a_n)/n \to \infty$ *and* $\sup_n E\xi_{n,1}^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$,

(ii) *for some* $\beta \in (0,1]$, $a_n^{\beta}/n^{2-\beta} \to \infty$ *and* $\sup_n E\exp(\alpha\xi_{n,1}^{\beta}) < \infty$ *for some* $\alpha > 0$.

Then $\{\sqrt{a_n/n}(N_n - e(E\xi_{n,1})^{-1}),\ n \geq 1\}$ *obeys the LDP in* D *for the* J_1 *topology with rate function*

$$(6.9)\qquad I_N(x) \equiv \begin{cases} \dfrac{1}{2\sigma^2\lambda^3}\displaystyle\int_0^\infty \dot{x}(t)^2\,dt, & \text{if } x \text{ is absolutely continuous and } x(0)=0,\\[2ex] \infty, & \text{otherwise.}\end{cases}$$

PROOF. By Lemmas 6.1 and 4.1(b), the sequence $(\sqrt{a_n/n}(X_n - eE\xi_{n,1}),\ n \geq 1)$, where $X_n(t) = 1/a_n \sum_{i=1}^{\lfloor a_n t \rfloor} \xi_{n,i}$, obeys the LDP in D for the J_1 topology with rate function I_X from (6.8). The proof is completed by observing that, in analogy with (6.3), $N_n = X_n^{-1} - a_n^{-1}$ and applying Theorem 5.4 and Lemma 4.1(b) and the contraction principle.

COROLLARY. *Under the assumptions of Theorem* 6.2, $\{\sqrt{a_n/n}(N_n(1) - (E\xi_{n,1})^{-1}),\ n \geq 1\}$ *obeys the LDP in* $\mathbb{R}$ *with rate function*

$$\lambda_N(z) = \frac{z^2}{2\sigma^2\lambda^3}, \qquad z \in \mathbb{R}.$$

PROOF. Apply the extended contraction principle with the projection map. By (6.9), the resulting rate function is

$$\lambda_N(z) = \inf_{\substack{x \in D:\\ x(1)=z}} \{I_N(x)\} = \frac{z^2}{2\sigma^2\lambda^3}. \qquad \square$$

7. Superpositions of renewal processes. The results in Section 6 extend easily to superpositions of renewal functions provided that the component rate functions are finite only for continuous functions x. Otherwise, we have the difficulty that addition is not continuous on $D \times D$ [28]. However, from Theorem 6.1 we see that in general the rate functions can be finite for discontinuous x. We avoid this problem by making additional assumptions, as in Remark 6.1.

Let $\{\xi_i^j,\ i \geq 1\}$, $1 \leq j \leq k$, be k independent sequences of i.i.d. nonnegative random variables with $E\xi_1^j > 0$. Let $(N^j(t),\ t \geq 0)$, $1 \leq j \leq k$, be the associated k mutually independent renewal counting processes, defined as in (6.2), and let $N = N^1 + \cdots + N^k$. For each j, let X_n^j be the normalized partial sum process defined as in (6.1) and let X_n be the normalized partial sum process associated with the superposition process, defined by

$$(7.1)\qquad X_n(t) \equiv n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i, \qquad t \geq 0,$$

where ξ_i is the ith interval between points in the superposition process N. Let N_n^j and N_n be associated normalized counting processes; that is,

$$(7.2)\qquad N_n^j(t) \equiv n^{-1}N^j(nt) \quad \text{and} \quad N_n(t) \equiv n^{-1}N(nt), \qquad t \geq 0.$$

We will derive LDP's for $\{N_n,\ n \geq 1\}$ and $\{X_n,\ n \geq 1\}$. For this purpose, note that $N_n^j(t) = (X_n^j)^{-1}(t) - n^{-1}$, so that

$$N_n^j = (X_n^j)^{-1} - n^{-1}, \qquad N_n = X_n^{-1} - n^{-1}, \tag{7.3}$$

and

$$X_n = (N_n + n^{-1})^{-1}. \tag{7.4}$$

The following theorem extends Theorem 6.1.

THEOREM 7.1. *Assume that* $E\exp(\alpha\xi_1^j) < \infty$, $1 \leq j \leq k$, *for some* $\alpha > 0$. *Let* $\alpha_j^* \equiv \sup\{\alpha\colon E\exp(\alpha\xi_1^j) < \infty\}$, $1 \leq j \leq k$, *and* $\alpha^* = \sum_{j=1}^k \alpha_j^*$. *Also assume that* $P(\xi_1^j = 0) = 0$ *for all* j, $1 \leq j \leq k$. *Then the sequence* $\{N_n,\ n \geq 1\}$ *in* (7.2) *obeys the LDP in* $E^\uparrow$ *for the* J_1 *topology with rate function*

$$I_N(x) = \int_0^\infty \sup_{\alpha < \alpha^*}\{\alpha - \dot{x}(t)\psi(\alpha)\}\,dt \tag{7.5}$$

$$= \int_0^\infty \sup_{\alpha < \bar{\alpha}}\{\phi(\alpha) - \alpha\dot{x}(t)\}\,dt, \tag{7.6}$$

where

$$\psi(\alpha) = \phi^{-1}(\alpha), \qquad \phi(\alpha) = \sum_{j=1}^k \phi_j(\alpha), \qquad \phi_j(\alpha) = \psi_j^{-1}(\alpha), \tag{7.7}$$

$$\bar{\alpha} = \min_{1\leq j\leq k} \lim_{\alpha\uparrow\alpha_j^*} \psi_j(\alpha) \quad \text{with } \psi_j(\alpha) = \log E\exp(\alpha\xi_1^j), \tag{7.8}$$

if x *is absolutely continuous and* $x(0) = 0$, *while* $I_N(x) = \infty$ *otherwise. If in addition there is one* j *for which* $E\exp(\alpha\xi_1^j) < \infty$ *for all* α, *then* $\{X_n,\ n \geq 1\}$ *in* (7.1) *obeys the LDP in* $E^\uparrow$ *for the* J_1 *topology with rate function*

$$\begin{aligned} I_X(x) = I_N(x^{-1}) &= \int_0^\infty \sup_{\alpha\in R}\{\alpha\dot{x}(t) - \psi(\alpha)\}\,dt \\ &= \int_0^\infty \sup_{\alpha\in R}\{\dot{x}(t)\phi(\alpha) - \alpha\}\,dt, \end{aligned} \tag{7.9}$$

if x *is absolutely continuous with* $x(0) = 0$, *and* $I_X(x) = \infty$ *otherwise.*

PROOF. Since the normalized processes N_n^j are independent, by Theorem 6.1 and [9] the sequence $\{(N_n^1, N_n^2, \ldots, N_n^k),\ n \geq 1\}$ of random elements of $D([0,\infty), R)^k$ obeys the LDP for the J_1 topology with rate function

$$I(x_1, \ldots, x_k) \equiv \sum_{j=1}^k \int_0^\infty \sup_{\alpha < \alpha_j^*}\{\alpha - \dot{x}_j(t)\psi_j(\alpha)\}\,dt, \tag{7.10}$$

when $x_1, \ldots, x_k$ are absolutely continuous with respect to Lebesgue measure and $x_j(0) = 0$ for all j, while $I(x_1, \ldots, x_k) = \infty$ otherwise. We start working toward the first expression in (7.5).

By the extended contraction principle, the superposition N_n obeys the LDP for the J_1 topology with rate function $I_N(x)$, where $I_N(x) = \infty$ if x is not absolutely continuous or $x(0) \neq 0$. Using an argument as in [18] (including a minimax theorem on the third line), if x is absolutely continuous and $x(0) = 0$, then

$$
\begin{aligned}
I_N(x) &= \inf_{x_1+\cdots+x_k=x} \sum_{j=1}^{k} \int_0^\infty \sup_{\alpha<\alpha_j^*}\{\alpha - \dot{x}_j(t)\psi_j(\alpha)\}\,dt \\
&= \int_0^\infty \inf_{\sum_{j=1}^k \dot{x}_j(t)=\dot{x}(t)} \sup_{\substack{\alpha_j<\alpha_j^*,\\ j=1,\dots,k}} \left\{\sum_{j=1}^k \alpha_j - \sum_{j=1}^k \dot{x}_j(t)\psi_j(\alpha_j)\right\} dt \\
&= \int_0^\infty \sup_{\substack{\alpha_j<\alpha_j^*,\\ j=1,\dots,k}} \inf_{\sum_{j=1}^k \dot{x}_j(t)=\dot{x}(t)} \left\{\sum_{j=1}^k \alpha_j - \sum_{j=1}^k \dot{x}_j(t)\psi_j(\alpha_j)\right\} dt \\
&= \int_0^\infty \sup_{\substack{\alpha_j<\alpha_j^*,\\ j=1,\dots,k}} \left\{\sum_{j=1}^k \alpha_j - \dot{x}(t) \max_{j=1,\dots,k} \psi_j(\alpha_j)\right\} dt.
\end{aligned}
\tag{7.11}
$$

The required now follows since

$$
\psi(\alpha) = \inf\left\{\max_{1\le j\le k} \psi_j(\alpha_j)\colon \sum_{j=1}^k \alpha_j = \alpha,\ \alpha_j < \alpha_j^*\right\},
$$

with ∞ being the infimum over the empty set. [The infimum is attained at points α_j for which all the $\psi_j(\alpha_j)$ are equal, for if we have that $\psi_{j'}(\alpha_{j'}) > \psi_{j''}(\alpha_{j''})$ we can make $\alpha_{j'}$ smaller and $\alpha_{j''}$ larger keeping their sum unchanged.] The equality (7.6) is obvious.

Turning to X_n, we observe that $\alpha^* = \infty$ if there is a j for which $E\exp(\alpha\xi_1^j) < \infty$ for all $\alpha > 0$. Hence the local rate function in (7.5) is ∞ at 0, and an application of Theorem 3.4 completes the proof. The second equality again is obvious. □

We now establish an extension of Theorem 6.2.

THEOREM 7.2. *Let* $\{((N_n^j(t),\ t\ge 0),\ j=1,\dots,k),\ n\ge 1\}$, *be a sequence of k-tuples of independent renewal processes. Let* $\{\xi_{n,i}^j,\ i\ge 1\},\ j=1,\dots,k$ *denote their respective interrenewal times. Assume that* $\operatorname{Var}\xi_{n,1}^j \to \sigma_j^2$ *and* $E\xi_{n,1}^j \to \lambda_j^{-1}$ *for* $j=1,\dots,k$ *as* $n\to\infty$. *Assume that one of the conditions* (i) *or* (ii) *in Theorem* 6.2 *holds for all* j, $1\le j\le k$. *Then the sequence* $\{\sqrt{a_n/n}(N_n - e\sum_{j=1}^k (E\xi_{n,1}^j)^{-1}),\ n\ge 1\}$, *where* $N_n(t) = 1/a_n \sum_{j=1}^k N_n^j(a_n t)$, *obeys the LDP in* D *for the* J_1 *topology with rate function*

$$
I_N(x) \equiv \tfrac{1}{2}\left(\sum_{j=1}^k \sigma_j^2\lambda_j^3\right)^{-1} \int_0^\infty \dot{x}(t)^2\,dt \tag{7.12}
$$

for absolutely continuous x with $x(0) = 0$, and $I_N(x) = \infty$ otherwise. Moreover, if $\xi_{n,i}$, $i \geq 1$, are the times between events in the superposition process $\sum_{j=1}^k N_n^j$ and $X_n(t) = 1/a_n \sum_{i=1}^{\lfloor a_n t \rfloor} \xi_{n,i}$, then $\{\sqrt{a_n/n}(X_n - e(\sum_{j=1}^k (E\xi_{n,1}^j)^{-1})^{-1}),\ n \geq 1\}$ obeys the LDP in D for the J_1 topology with rate function

$$I_X(x) \equiv \tfrac{1}{2}\biggl(\sum_{j=1}^k \lambda_j\biggr)^3 \biggl(\sum_{j=1}^k \sigma_j^2 \lambda_j^3\biggr)^{-1} \int_0^\infty \dot{x}(t)^2\,dt \tag{7.13}$$

for absolutely continuous x with $x(0) = 0$, and $I_X(x) = \infty$ otherwise.

PROOF. By Theorem 6.2 and in analogy with the proof of Theorem 7.1, the sequence of processes $\{\sqrt{a_n/n}(N_n - e\sum_{j=1}^k (E\xi_{n,i}^j)^{-1}),\ n \geq 1\}$ obeys the LDP for the J_1 topology with rate function $I_N(x)$, which for absolutely continuous x has the form

$$\begin{aligned} I_N(x) &= \inf_{x_1+\cdots+x_k=x} \sum_{j=1}^k \frac{1}{2\sigma_j^2\lambda_j^3} \int_0^\infty \dot{x}_j(t)^2\,dt \\ &= \frac{1}{2}\int_0^\infty \inf_{\sum_{j=1}^k \dot{x}_j(t)=\dot{x}(t)} \sum_{j=1}^k \frac{1}{\sigma_j^2\lambda_j^3}\dot{x}_j(t)^2\,dt \\ &= \frac{1}{2}\biggl(\sum_{j=1}^k \sigma_j^2\lambda_j^3\biggr)^{-1} \int_0^\infty \dot{x}(t)^2\,dt, \end{aligned}$$

with the last line following from the Cauchy–Schwarz inequality. By Theorem 5.4 and Lemma 4.1(b), a corresponding limit holds for X_n. □

REMARK. The rate functions I_N in (7.12) and I_X in (7.13) have the form

$$I(x) = \frac{1}{2\gamma}\int_0^\infty \dot{x}(t)^2\,dt, \tag{7.14}$$

where γ is the asymptotic variance of the processes $\sum_{j=1}^k N_n^j$ and X_n, respectively; that is, for (7.12),

$$\gamma = \lim_{n\to\infty} a_n^{-1}\operatorname{Var} \sum_{j=1}^k N_n^j(a_n),$$

while for (7.13),

$$\gamma = \lim_{n\to\infty} a_n^{-1}\operatorname{Var}\sum_{i=1}^{a_n} \xi_{n,i};$$

that is, the constant γ is the same as appears in the central limit theorems. The form of the coefficient in the rate function as a limit variance is typical when one deals with quadratic rate functions (cf. [15], Corollaries 6.3, 6.4 and 6.7).

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