Functionals that do not attain their norm

María D. Acosta* Antonio Aizpuru Richard M. Aron
Francisco J. García–Pacheco

With warm wishes for a great friend and colleague, Jean Schmets, on the occasion of his 65^{th} *birthday*

Abstract

We study the set of non–norm–attaining functionals on a Banach space, giving a sufficient condition for the density of this set. We also find a large class of Banach spaces for which the set of norm–attaining functionals is (dense–) lineable. In addition, among other results, we provide a new proof of the fact that every real Banach space can be equivalently renormed so that the set of non–norm–attaining functionals is non–dense.

1 Introduction and background

Our primary focus in this article will be on the structure of the set of functionals on a real Banach space that do not attain their norm. One reason for this interest arises from geometrical considerations and the connection with the Banach–Mazur conjecture (see [11].) Namely, it was shown in [3] that *every transitive and separable Banach space in which the set of non–norm–attaining functionals is not dense is rotund.* Another motivation comes from an open problem concerning the *lineability* of the set $\mathsf{NA}\,(X)$ of norm–attaining functionals on a Banach space X. Specifically, it is unknown if $\mathsf{NA}\,(X)$ always contains an infinite dimensional, or even a $2-$dimensional

*The first author was partially supported by D.G.E.S., project no. BFM 2003-01681. Some of the research of the third author was done while he was a visitor in the Departamento de Análisis Matemático of the Universidad Complutense de Madrid, to which thanks are gratefully acknowledged.

2000 *Mathematics Subject Classification :* Primary 46B20, 46B03, 46B07.

Key words and phrases : norm–attaining functional, non–norm–attaining functional, non–density, density.

subspace. The paper in [6] is an excellent reference about this problem. As we will see, the complementary set, $X^*\backslash \mathsf{NA}(X)$, of non–norm–attaining functionals on X contains infinite dimensional closed subspaces in many cases. It appears to be open whether this property characterizes non–reflexive spaces X.

We recall the following relatively new concepts related to the "size" of subsets of Banach spaces. A subset M of a Banach space is said to be

1. *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector subspace;
2. *dense–lineable* if $M \cup \{0\}$ contains an infinite dimensional dense vector subspace;
3. *spaceable* ([16]) if $M \cup \{0\}$ contains an infinite dimensional closed vector subspace.

In order to have a better perspective of these new concepts, we refer the reader to the papers in [4], [5], [8], [9], [10], [15], and [16], where it is proved that several pathological properties occur more often than one might expect in the sense described in the definitions above.

2 Results on concrete spaces

In this section we study the lineability and spaceability properties of the set of non–norm–attaining functionals on some particular Banach spaces. We begin with spaces of continuous functions.

Theorem 2.1. *Let K be an infinite compact Hausdorff topological space. Then, $\mathsf{NA}(\mathcal{C}(K))$ is lineable.*

Proof. Let us consider the vector space V of all regular Borel measures μ, with bounded variation on K, such that there exists a finite set $L \subset K$ with $|\mu|(K \setminus L) = 0$. (Observe that V is no other than the vector space generated by $\{\delta_t : t \in K\}$.) Also notice that if $\mu \in V$ and $\|\mu\| = 1$, then $\sum_{l \in L} |\mu(\{l\})| = 1$. Now, define the continuous function $f : L \longrightarrow [-1, 1]$ as

$$f(l) = \begin{cases} \frac{|\mu(\{l\})|}{\mu(\{l\})} & \text{if } \mu(\{l\}) \neq 0, \\ 0 & \text{if } \mu(\{l\}) = 0, \end{cases}$$

for all $l \in L$. Since K is normal and L is closed, we deduce by Urysohn's lemma that there exists a continuous extension $\tilde{f} : K \longrightarrow [-1, 1]$ of f. To finish, we have that $\|\tilde{f}\| = 1$ and

$$\int_K \tilde{f} d\mu = \sum_{l \in L} f(l)\, \mu(\{l\}) = \sum_{l \in L} |\mu(\{l\})| = 1. \qquad \blacksquare$$

Remark 2.2. Observe that there are some infinite compact Hausdorff topological spaces K for which $\mathsf{NA}(\mathcal{C}(K))$ is not spaceable, in view of [6, Proposition 2.20].

Lemma 2.3. *Let K be an infinite compact Hausdorff topological space. Then, there exists a sequence $(l_j)_{j \in \mathbb{N}} \subseteq K$ such that $l_i \neq l_j$ if $i \neq j$, and verifying one of the following conditions:*

1. $(l_j)_{j\in\mathbb{N}}$ *converges to some point* $\infty \in K$.

2. $\{l_j : j \in \mathbb{N}\}$ *does not contain isolated points.*

Proof. Firstly, note that if K is scattered then we deduce (see [18]) that K is sequentially compact. Therefore, we can assume that K is not scattered. It is known (see [18]) that $K = P \cup D$ where P is closed and perfect (the *perfect kernel* of K) and D is open and scattered (the *scattered kernel* of K.) (Note that this decomposition holds not only for compact Hausdorff spaces, but for any topological space.) Now, there exists a sequence $(l_j)_{j\in\mathbb{N}} \subseteq P$ such that $l_i \neq l_j$ for $i \neq j$ and the set $\{l_j : j \in \mathbb{N}\}$ does not contain isolated points. ■

Lemma 2.4. *There exists an infinite dimensional vector subspace M of ℓ_1 such that every $(\alpha_i)_{\in\mathbb{N}} \in M \setminus \{0\}$ verifies one of the following conditions:*

1. $\{j \in \mathbb{N} : \alpha_j > 0\}$ *is non–empty and finite and* $\{j \in \mathbb{N} : \alpha_j < 0\}$ *is infinite;*

2. $\{j \in \mathbb{N} : \alpha_j > 0\}$ *is infinite and* $\{j \in \mathbb{N} : \alpha_j < 0\}$ *is non–empty and finite.*

Proof. Let us consider the following elements of ℓ_1:

$$\begin{cases} y_1 = \left(1, -1, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \dots\right) \\ y_2 = \left(0, 0, 1, -1, \frac{1}{3^5}, \frac{1}{3^6}, \frac{1}{3^7}, \frac{1}{3^8}, \dots\right) \\ y_3 = \left(0, 0, 0, 0, 1, -1, \frac{1}{4^7}, \frac{1}{4^8}, \dots\right) \\ \vdots \\ y_k = \left(0, \dots, 0, 1, -1, \frac{1}{(k+1)^{2k+1}}, \frac{1}{(k+1)^{2k+2}}, \dots\right) \\ \vdots \end{cases}$$

We will take $M = \operatorname{span}\{y_k : k \in \mathbb{N}\}$. Indeed, assume that $0 \neq y \in M$ and take $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ not all zero so that $y = \lambda_1 y_1 + \cdots + \lambda_k y_k$. Then, y has the form

$$\left(\lambda_1, -\lambda_1, \frac{\lambda_1}{2^3} + \lambda_2, \frac{\lambda_1}{2^4} - \lambda_2, \dots, \frac{\lambda_1}{2^{2k+1}} + \frac{\lambda_2}{3^{2k+1}} + \cdots + \frac{\lambda_k}{(k+1)^{2k+1}}, \dots\right). \qquad (2.1)$$

Therefore, from equation (2.1) we can see that M is the desired vector space. ■

We note that in the above proof, if $y \in \overline{\operatorname{span}}\{y_k : k \in \mathbb{N}\} \setminus \operatorname{span}\{y_k : k \in \mathbb{N}\}$, then y has the form

$$\left(\lambda_1, -\lambda_1, \frac{\lambda_1}{2^3} + \lambda_2, \frac{\lambda_1}{2^4} - \lambda_2, \frac{\lambda_1}{2^5} + \frac{\lambda_2}{3^5} + \lambda_3, \frac{\lambda_1}{2^6} + \frac{\lambda_2}{3^6} - \lambda_3, \dots\right); \qquad (2.2)$$

which shows that the closure of M does not serve our purposes.

Theorem 2.5. *Let K be an infinite compact Hausdorff topological space. Then, $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$ is lineable. If, in addition, K possesses a non–trivial convergent sequence, then $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$ is spaceable.*

Proof. By taking into consideration Lemma 2.3, we will firstly assume the existence in K of a non–trivial sequence with no isolated points. So, let $(l_j)_{j\in\mathbb{N}} \subseteq K$ be a sequence free of isolated points and verifying that $l_i \neq l_j$ if $i \neq j$. Let us consider in $\mathcal{C}(K)^*$ the infinite dimensional vector subspace $W = \left\{\sum_{j=1}^{\infty} \alpha_j \delta_{l_j} : (\alpha_j)_{j\in\mathbb{N}} \in M\right\}$, where M is the infinite dimensional vector subspace constructed in Lemma 2.4. Observe that W is linearly isometric to M. Let $\mu \in W$ with $\|\mu\| = 1$, so that μ is of the form $\sum_{j=1}^{\infty} \alpha_j \delta_{l_j}$ with $(\alpha_j)_{j\in\mathbb{N}} \in M$. Since $\|\mu\| = 1$, we have that $\sum_{j=1}^{\infty} |\alpha_j| = 1$. We can assume, without loss of generality, that $A = \{j \in \mathbb{N} : \alpha_j < 0\}$ is non–empty and finite and $B = \{j \in \mathbb{N} : \alpha_j > 0\}$ is infinite. Let $f \in \mathcal{C}(K)$ with $\|f\| = \int_K f d\mu = 1$. Then, $\sum_{j=1}^{\infty} \alpha_j f(l_j) = 1$, and from here we deduce that, if $j \in A$ then $f(l_j) < 0$, and if $j \in B$ then $f(l_j) > 0$. However, by taking into account that l_k is a cluster point of $\{l_j : j \in \mathbb{N}\}$ for every $k \in A$, we have that there must be an infinite amount of natural numbers $k \in \mathbb{N}$ with $f(l_k) < 0$, which is a contradiction. Therefore, each non–zero measure $\mu \in W$ is, in fact, in $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$.

Secondly, we will suppose the existence in K of a non–trivial convergent sequence. So, let $L = (l_j)_{j\in\mathbb{N}} \subseteq K$ be a convergent sequence to some point $\infty \in K$ and verifying that $l_i \neq l_j$ if $i \neq j$. We can set $\mathbb{N} = \bigcup_{k=1}^{\infty} N_k$ where the N_k's are pairwise disjoint containing infinitely many even numbers and infinitely many odd numbers. Thus, we can write $L = \bigcup_{k=1}^{\infty} L_k$ where $l \in L_k$ if and only if $l = l_j$ for some $j \in N_k$. Now choose, for each $k \in \mathbb{N}$, any regular Borel measure μ_k with bounded variation on K, such that $|\mu_k|(K \setminus L_k) = 0$ and the sign of each $\mu_k(\{l_j\})$ is $(-1)^j$, for $j \in N_k$. Let $0 \neq \mu \in W := \overline{\operatorname{span}}\{\mu_k : k \in \mathbb{N}\}$ and consider a sequence $(\sum_{k=1}^{\infty} \lambda_k^n \mu_k)_{n\in\mathbb{N}}$ converging to μ, where $(\lambda_k^n)_{k\in\mathbb{N}} \in c_{00}$ for every $n \in \mathbb{N}$. Since

$$\left(\sum_{k=1}^{\infty} \lambda_k^n \mu_k\right)(\{l\}) \longrightarrow \mu(\{l\}) \text{ as } n \to \infty$$

for every $l \in L$, we deduce that $|\mu|(K \setminus L) = 0$. Now, since $\mu \neq 0$, there exists $l_0 \in L$ such that $\mu(\{l_0\}) \neq 0$. There also exists $k_0 \in \mathbb{N}$ such that $l_0 \in L_{k_0}$ and so $(\sum_{k=1}^{\infty} \lambda_k^n \mu_k)(\{l_0\}) = \lambda_{k_0}^n \mu_{k_0}(\{l_0\})$ for every $n \in \mathbb{N}$, which means that

$$\lambda_{k_0}^n \longrightarrow \lambda_{k_0} := \frac{\mu(\{l_0\})}{\mu_{k_0}(\{l_0\})} \text{ as } n \to \infty.$$

Then, if $l \in L_{k_0}$ we have that $\mu(\{l\}) = \lambda_{k_0} \mu_{k_0}(\{l\})$. Therefore, the signs of $\mu(\{l\})$ are alternating for $l \in L_{k_0}$. Finally, assume that $\|\mu\| = 1$. Then $\sum_{l\in L} |\mu(\{l\})| = 1$. Now if $f \in \mathcal{C}(K)$, $\|f\| = 1$, and $\int_K f d\mu = 1$, we have that $\sum_{l\in L} f(l) \mu(\{l\}) = 1$, which means that $f(l) = \frac{|\mu(\{l\})|}{\mu(\{l\})}$ for all $l \in L$ such that $\mu(\{l\}) \neq 0$. However, this is impossible because $(f(l_j))_{j\in\mathbb{N}}$ converges to $f(\infty)$. Therefore, each non–zero measure $\mu \in W$ is, in fact, in $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$. ■

We now turn to spaces of integrable functions.

Lemma 2.6. *Let (Ω, μ) be a measure space such that μ is σ–finite. Then, an element $x^* \in \mathsf{L}_1(\mu)^* \equiv \mathsf{L}_\infty(\mu)$ attains its norm if and only if there is a measurable set A with $\mu(A) > 0$ satisfying that $|x^*(t)| = \|x^*\|$ for all $t \in A$.*

Proof. First, if some subset A of positive measure satisfies the above condition, then the function $\theta : \Omega \longrightarrow \mathbb{K}$ given by

$$\theta(t) = \begin{cases} \frac{|x^*(t)|}{x^*(t)} & \text{if } t \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is measurable and clearly satisfies that $\|\theta\|_\infty \leq 1$. Since μ is σ–finite, there is a measurable subset $B \subset A$ such that $0 < \mu(B) < +\infty$. Thus the element $g := \theta \frac{\chi_B}{\mu(B)}$ is a function in the unit ball of $\mathsf{L}_1(\mu)$ and satisfies the inequality

$$\|x^*\|_\infty = \int_\Omega x^* g d\mu \leq \int_\Omega |x^* g| d\mu \leq \int_\Omega \|x^*\|_\infty |g| \, d\mu = \|x^*\|_\infty. \tag{2.3}$$

Conversely, assume that $x^* \neq 0$ attains its norm at $g \in S_{\mathsf{L}_1(\mu)}$. Then we have that both x^* and g verify equation (2.3). Hence we obtain that $|x^*(t)g(t)| = \|x^*\|_\infty |g(t)|$ almost everywhere. If we set $C = \{t \in \Omega : g(t) \neq 0\}$, then C is measurable and $\mu(C) > 0$. Therefore, there is a subset $Z \subset C$ such that $\mu(Z) = 0$ and $|x^*(t)| = \|x^*\|_\infty$ for all $t \in A := C \setminus Z$. ∎

Theorem 2.7. *Assume that* (Ω, μ) *is a measure space such that* $\mathsf{L}_1(\mu)$ *is infinite dimensional and* μ *is* σ*–finite. Then* $\mathsf{L}_1(\mu)^* \backslash \mathsf{NA}(\mathsf{L}_1(\mu))$ *is spaceable. If, in addition, the measure* μ *is atomless, then* $\mathsf{L}_1(\mu)^* \setminus \mathsf{NA}(\mathsf{L}_1(\mu))$ *is dense in* $\mathsf{L}_1(\mu)^*$.

Proof. Let us first show the spaceability of $\mathsf{L}_1(\mu)^* \backslash \mathsf{NA}(\mathsf{L}_1(\mu))$. Since $\mathsf{L}_1(\mu)$ is infinite dimensional, there exists a countable family $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint measurable sets such that, for each $n \in \mathbb{N}$, $\mu(A_n) > 0$ and $A_n = \bigcup_{k=1}^\infty B_n^k$, where $(B_n^k)_{k \in \mathbb{N}}$ is a sequence of pairwise disjoint measurable sets of positive measure. Next, consider the vector subspace of $\mathsf{L}_\infty(\mu)$ defined as

$$M := \left\{ \sum_{n=1}^\infty x(n) \left(\sum_{k=1}^\infty \alpha(k) \chi_{B_n^k} \right) : x \in \ell_\infty \right\},$$

where $\alpha := (\alpha(k))_{k \in \mathbb{N}}$ is a fixed strictly increasing sequence of real positive numbers such that $\|\alpha\|_\infty = 1$. If $x^* = \sum_{n=1}^\infty x(n) \left(\sum_{k=1}^\infty \alpha(k) \chi_{B_n^k} \right)$ is an element of M, then

$$\begin{aligned} \|x^*\|_\infty &= \sup \{ \|x^* \chi_{A_n}\|_\infty : n \in \mathbb{N} \} \\ &= \sup \left\{ \|x^* \chi_{B_n^k}\|_\infty : n, k \in \mathbb{N} \right\} \\ &= \sup \{ |x(n)\alpha(k)| : n, k \in \mathbb{N} \} \\ &= \sup \{ |x(n)| : n \in \mathbb{N} \} \\ &= \|x\|_\infty. \end{aligned}$$

For $0 \neq x^* \in M$ and $t \in \Omega$, we clearly have that $x^*(t) = 0$ for all $t \notin \bigcup_{n \in \mathbb{N}} A_n$. If $t \in A_n$ for some n, then there is k such that $x \in B_n^k$ and so we have that $|x^*(t)| = |x(n)\alpha(k)| < \|x\|_\infty = \|x^*\|_\infty$. Therefore for every $t \in \Omega$, we have that $|x^*(t)| < \|x^*\|_\infty$, and thus x^* cannot attain its norm on $L_1(\mu)$.

We now turn to density of $\mathsf{L}_1(\mu)^* \setminus \mathsf{NA}(\mathsf{L}_1(\mu))$. We take $x^* \in \mathsf{L}_\infty(\mu)$ and $r > 0$. We will show that the ball centered at x^* of radius r contains a functional that does not attain its norm. We can clearly assume that x^* attains its norm and $x^* \neq 0$, and hence there is a measurable set $A \subset \Omega$ with $\mu(A) > 0$ such that $|x^*(t)| = \|x^*\|_\infty$ for all $t \in A$. Since μ is atomless, we can write $A = \bigcup_{n\in\mathbb{N}} A_n$, where $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets such that $\mu(A_n) > 0$ for each $n \in \mathbb{N}$. Next, we choose a strictly increasing convergent sequence $(r_n)_{n\in\mathbb{N}}$ of real numbers such that $r_n \geq 1$ for all $n \in \mathbb{N}$ and whose limit l satisfies $l \leq 1 + \frac{r}{\|x^*\|_\infty}$. We will denote by y^* the element of $\mathsf{L}_\infty(\mu)$ given by

$$y^* = \sum_{n=1}^{\infty} r_n \chi_{A_n} x^* + (1 - \chi_A)x^*,$$

where the above convergence is pointwise. Then

$$\begin{aligned}\|y^*\|_\infty &= \max\{\|y^*\chi_A\|_\infty, \|(1-\chi_A)y^*\|_\infty\} \\ &\leq \max\{\sup\{r_n : n \in \mathbb{N}\}\|x^*\|_\infty, \|x^*\|_\infty\} \\ &= l\|x^*\|_\infty.\end{aligned}$$

In fact, $r_n\|x^*\|_\infty = \|r_n\chi_{A_n}x^*\|_\infty \leq \|y^*\|_\infty$ for all $n \in \mathbb{N}$, so $l\|x^*\|_\infty \leq \|y^*\|_\infty$, and we have that $\|y^*\|_\infty = l\|x^*\|_\infty$. Since the sequence $(r_n)_{n\in\mathbb{N}}$ is strictly increasing and since the essential supremum is not attained at any measurable set with positive measure, y^* does not attain its norm at $\mathsf{L}_1(\mu)$. Also,

$$\begin{aligned}\|y^* - x^*\|_\infty &= \left\|\sum_{n=1}^{\infty}(r_n - 1)\chi_{A_n}x^*\right\|_\infty \\ &\leq \sup\{r_n - 1 : n \in \mathbb{N}\}\|x^*\|_\infty \\ &\leq \|x^*\|_\infty(l-1) \\ &\leq r.\end{aligned}$$

As a consequence, we have checked that $y^* \in x^* + r\mathsf{B}_{\mathsf{L}_1(\mu)}$ and y^* does not attain its norm. ∎

Remark 2.8. Note that if the measure μ has an atom $A \subset \Omega$ of finite measure, then for all $x^* \in \mathsf{L}_\infty(\mu)$ such that $\|x^* - \chi_A\|_\infty < \frac{1}{2}$, we have that x^* is norm–attaining. As a consequence, $\mathsf{L}_1(\mu)^* \setminus \mathsf{NA}(\mathsf{L}_1(\mu))$ is not dense.

3 Results on general spaces

Our aim in this section is to try to generalize the previous results to a general class of Banach spaces. In much of this section, we will make use of various classical notions, such as smoothness, rotundity, etc., from the geometry of Banach spaces. Background information can be obtained in [12] and [13]. We also refer the reader to [1], [2], and [7] for a wider perspective of these concepts.

3.1 (Dense–)lineability of $\mathsf{NA}(X)$

In this subsection we obtain sufficient conditions for the set $\mathsf{NA}(X)$ to be (dense–) lineable. These conditions also allow us to extend our earlier work to a larger class of Banach spaces by means of renormings.

Theorem 3.1. *Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$.*

1. *If $(e_n)_{n\in\mathbb{N}}$ is monotone, then $\mathsf{NA}(X)$ is lineable.*
2. *If $(e_n)_{n\in\mathbb{N}}$ is monotone and shrinking, then $\mathsf{NA}(X)$ is dense–lineable.*

Proof.

1. We will show that every non–zero vector in

$$M = \left\{ \sum_{k=1}^{n} \alpha_k e_k^* : \alpha_k \in \mathbb{K} \text{ for } 1 \leq k \leq n \text{ and } n \in \mathbb{N} \right\}$$

is norm–attaining. For each n, let P_n be the canonical projection onto $[e_1, ..., e_n]$. Let us take $x^* := \sum_{k=1}^{n} \alpha_k e_k^* \in M$. Since $P_n^t(x^*) = x^*$, then

$$\begin{aligned} \|x^*\| &= \sup \{|x^*(x)| : x \in \mathsf{B}_X\} \\ &= \sup \left\{|P_n^t(x^*)(x)| : x \in \mathsf{B}_X\right\} \\ &= \sup \{|x^*(P_n x)| : x \in \mathsf{B}_X\}. \end{aligned}$$

Since the basis is monotone, $P_n(\mathsf{B}_X)$ is the closed unit ball of $[e_1, \ldots, e_n]$ and so x^* attains its norm.

2. Note that, if $(e_n)_{n\in\mathbb{N}}$ is shrinking as well, then the sequence of biorthogonal functionals $(e_n^*)_{n\in\mathbb{N}}$ in X^* is a Schauder basis for the dual X^*, and hence M is dense in X^*. ∎

Corollary 3.2. *Let X be a Banach space. If X possesses an infinite dimensional complemented subspace Y with a Schauder basis, then X can be equivalently renormed so that $\mathsf{NA}(X)$ is lineable.*

Proof. Since Y has a basis, it can be equivalently renormed such that the Schauder basis is monotone (see [14]). Since $X = Y \oplus Z$, then we can use the euclidean norm in the above decomposition. With such a norm we have that $\mathsf{NA}(Y) \subset \mathsf{NA}(X)$, and hence $\mathsf{NA}(X)$ is lineable. ∎

3.2 Non–lineability of $\mathsf{NA}(X)$

In this subsection we concentrate on the non–lineability of $\mathsf{NA}(X)$. As we will see, the hypothesis of smoothness will be crucial for the development of this subsection, and everything here will be based upon the following fact.

Remark 3.3. Let X be a smooth Banach space. If $x^* \neq y^* \in X^* \setminus \{0\}$ are so that $\|x^*\| = \|y^*\| = \left\|\frac{x^*+y^*}{2}\right\|$, then $x^* + y^*$ cannot be norm–attaining. As a consequence, the complement in S_{X^*} of the set of extreme points of B_{X^*} is always contained in $X^* \setminus \mathsf{NA}(X)$.

Theorem 3.4. *Let X be a smooth Banach space. If Y is a vector subspace contained in $\mathsf{NA}(X)$, then Y must be rotund.*

Proof. Assume that Y is not rotund. Let then $x^* \neq y^* \in Y \setminus \{0\}$ are so that $\|x^*\| = \|y^*\| = \left\|\frac{x^*+y^*}{2}\right\|$. Clearly, $x^* + y^* \in Y$, but according to the previous remark, $x^* + y^*$ cannot be norm–attaining. ■

Remark 3.5. The previous theorem provides us with a method of trying to find Banach spaces whose set of norm–attaining functionals is not lineable. Intuitively, from the geometry of spaces like c_0 or ℓ_1, one could think that these spaces cannot possess rotund subspaces of dimension strictly greater than one. According to this, a possible way to obtain the non–lineability of $\mathsf{NA}\,(X)$ is to renorm conveniently c_0 to make it smooth and obtain the desired property for ℓ_1 to not have such rotund subspaces.

3.3 Density of $X^* \setminus \mathsf{NA}\,(X)$

Obtaining dense–lineability or spaceability of the set of non–norm–attaining functionals seems to be difficult for general spaces. Here, we will try to find some sufficient conditions for the set $X^* \setminus \mathsf{NA}\,(X)$ to be dense.

Theorem 3.6. *Let X be a Banach space. Then,* $\mathrm{cl}\,(\mathsf{S}_X \setminus \mathrm{ext}\,(\mathsf{B}_X))$ *contains the set* $\bigcup \{C \subseteq \mathsf{S}_X : C \text{ is a face of } \mathsf{B}_X \text{ and } \mathrm{diam}\,(C) > 0\}$.

Proof. Let C be a non-trivial face of S_X and $x \in C$. Then there is $y \in C\setminus\{x\}$ and so the segment between x and y lies in $C \subset S_X$. To conclude the proof, it suffices to observe that for every $t \in (0,1)$, $tx + (1-t)\,y$ is not an extreme point of B_X. ■

Corollary 3.7. *Let X be a smooth Banach space. If* $\mathrm{cl}\,((X^* \setminus \mathsf{NA}\,(X)) \cap \mathsf{S}_{X^*})$ *contains all rotund points of* B_{X^*}*, then* $(X^* \setminus \mathsf{NA}\,(X)) \cap \mathsf{S}_{X^*}$ *is dense in* S_{X^*}.

Proof. According to Remark 3.3, $(X^* \setminus \mathsf{NA}\,(X)) \cap \mathsf{S}_{X^*} \supseteq \mathsf{S}_{X^*} \setminus \mathrm{ext}\,(\mathsf{B}_{X^*})$. Therefore $\mathrm{cl}\,((X^* \setminus \mathsf{NA}\,(X)) \cap \mathsf{S}_{X^*}) \supseteq \mathrm{cl}\,(\mathsf{S}_{X^*} \setminus \mathrm{ext}\,(\mathsf{B}_{X^*}))$. In view of Theorem 3.6, $\mathrm{cl}\,(\mathsf{S}_{X^*} \setminus \mathrm{ext}\,(\mathsf{B}_{X^*}))$ contains all the elements in S_{X^*} which are not rotund points. So the stated result follows from the assumption. ■

Remark 3.8. The assumption of smoothness in the above result cannot be dropped. In fact, there are spaces with a dense set of smooth points in the unit sphere and satisfying the second assumption that we imposed, but not the conclusion. For instance, for $X = \ell_1$, the unit sphere of X^* has no rotund points and S_X has a dense set of smooth points. However, as we already remarked, $\mathsf{NA}(\ell_1)$ has non–empty interior and so $(\ell_\infty \setminus \mathsf{NA}(\ell_1)) \cap \mathsf{S}_{\ell_\infty}$ is not dense in S_{ℓ_∞}.

On the other hand, Corollary 3.7 yields a strong consequence related to the Banach–Mazur conjecture. Recall that a Banach space is said to be *transitive* if for any two points in the unit sphere there exists a surjective linear isometry of the space mapping one point into the other. In 1932, Mazur conjectured that all transitive and separable Banach spaces are Hilbert. Reference [11] contains an excellent description of all the background related to this old open problem.

Theorem 3.9. *Let X be a transitive and separable Banach space. Assume that* $(X^* \setminus \mathsf{NA}\,(X)) \cap \mathsf{S}_{X^*}$ *is not dense in* S_{X^*}*. Then X is rotund and* $\mathsf{J}_X\,(x)$ *is a rotund point of* B_{X^*} *for every* $x \in \mathsf{S}_X$*. In particular, the set of rotund points of* B_{X^*} *is dense in* S_{X^*}.

Proof. Let us recall that every transitive and separable Banach space is smooth (see [11] and [19]) and that, given any $x \in S_X$, by $\mathsf{J}_X(x)$ we mean the unique functional in S_{X^*} verifying that $\mathsf{J}_X(x)(x) = \|x\| = \|\mathsf{J}_X(x)\|$. Now, in [3] it is shown that X is rotund. Finally, if no $x \in \mathsf{S}_X$ is such that $\mathsf{J}_X(x)$ is a rotund point of B_{X^*} then, according to Corollary 3.7, we deduce that $(X^* \setminus \mathsf{NA}(X)) \cap \mathsf{S}_{X^*}$ is dense in S_{X^*}. Therefore, there must exist at least one $x \in \mathsf{S}_X$ such that $\mathsf{J}_X(x)$ is a rotund point of B_{X^*}. Then, if $y \in \mathsf{S}_X$ we can find a surjective linear isometry $T : X \longrightarrow X$ mapping y into x. Thus, $T^*(\mathsf{J}_X(x)) = \mathsf{J}_X(y)$ and $\mathsf{J}_X(y)$ is also a rotund point of B_{X^*}. ■

3.4 Non–density of $X^* \setminus \mathsf{NA}(X)$

In the final part of this paper we provide another proof of the fact that every real Banach space can be equivalently renormed so that the set of non–norm–attaining functionals is non–dense (see [1] and [17].) Motivated by the technique of proof that is offered here, we conjecture that we can equivalently renorm any Banach space X so that $X^* \setminus \mathsf{NA}(X)$ is nowhere dense. We will start with the following theorem, which gives a sufficient condition for the non–density of the set of non–norm–attaining functionals.

A smooth face is a face with non-empty interior relative to the unit sphere.

Theorem 3.10. *Let X be a Banach space. Then,* $\mathrm{cl}((X^* \setminus \mathsf{NA}(X)) \cap \mathsf{S}_{X^*})$ *is contained in the set* $\bigcup\{\mathrm{bd}_{\mathsf{S}_{X^*}}(C) : C$ *is a face of* $\mathsf{B}_{X^*}\}$. *In particular, if* B_{X^*} *possesses a smooth face, then* $(X^* \setminus \mathsf{NA}(X)) \cap \mathsf{S}_{X^*}$ *cannot be dense in* S_{X^*}.

Proof. Let C be any smooth face of B_{X^*} and take any $x^* \in \mathrm{int}_{\mathsf{S}_{X^*}}(C)$. By the Bishop–Phelps theorem, there is a norm–attaining functional $y^* \in \mathrm{int}_{\mathsf{S}_{X^*}}(C)$. Let $x \in \mathsf{S}_X$ so that $y^*(x) = 1$. Now, $\mathrm{int}_{\mathsf{S}_{X^*}}(C)$ is contained in the set of all smooth points of B_{X^*}. Since $\mathrm{int}_{\mathsf{S}_{X^*}}(C)$ is a convex set contained in S_{X^*}, we deduce that $x(x^*) = 1$; that is, $x^*(x) = 1$. As a consequence, every element in $\mathrm{int}_{\mathsf{S}_{X^*}}(C)$ is a norm–attaining functional. Finally, on the one hand, given any two distinct faces C and D of B_{X^*}, $\mathrm{int}_{\mathsf{S}_{X^*}}(C) \cap \mathrm{int}_{\mathsf{S}_{X^*}}(D) = \varnothing$, and on the other hand, given any face C of B_{X^*} with $\mathrm{int}_{\mathsf{S}_{X^*}}(C) \neq \varnothing$, $\mathrm{cl}(\mathrm{int}_{\mathsf{S}_{X^*}}(C)) = C$. Therefore, the sets $\bigcup\{\mathrm{bd}_{\mathsf{S}_{X^*}}(C) : C \text{ is a face of } \mathsf{B}_{X^*}\}$ and $\bigcup\{\mathrm{int}_{\mathsf{S}_{X^*}}(C) : C \text{ is a face of } \mathsf{B}_{X^*}\}$ are disjoint. Since their union is S_{X^*}, we have that $\bigcup\{\mathrm{bd}_{\mathsf{S}_{X^*}}(C) : C \text{ is a face of } \mathsf{B}_{X^*}\}$ is closed. ■

Theorem 3.10 leads us to the idea that, for non–density of the set $X^* \setminus \mathsf{NA}(X)$, it is sufficient to find an equivalent norm on X so that B_X has a smooth face. For this, the following result, which explains the relation between the faces of $\mathsf{B}_{X \bigoplus_\infty Y}$ in terms of the faces of B_X and B_Y, will be helpful.

Theorem 3.11. *Let X and Y be Banach spaces. Then:*

1. *If C is a maximal face of $\mathsf{B}_{X \oplus_\infty Y}$, then C is either of the form $\mathsf{B}_X \times D$, where D is a maximal face of B_Y, or of the form $E \times \mathsf{B}_Y$, where E is a maximal face of B_X. Conversely, for D and E maximal faces of B_Y and B_X respectively, both $\mathsf{B}_X \times D$ and $E \times \mathsf{B}_Y$ are maximal faces of $\mathsf{B}_{X \oplus_\infty Y}$.*

2. *If C is a smooth face of $\mathsf{B}_{X\oplus_\infty Y}$, then C is either of the form $\mathsf{B}_X \times D$, where D is a smooth face of B_Y, or of the form $E \times \mathsf{B}_Y$, where E is a smooth face of B_X. Conversely, for D and E smooth faces of B_Y and B_X respectively, both $\mathsf{B}_X \times D$ and $E \times \mathsf{B}_Y$ are smooth faces of $\mathsf{B}_{X\oplus_\infty Y}$.*

Proof.

1. By the Hahn-Banach separation Theorem, if Z is a Banach space and C is a face of S_Z, then there is $z^* \in \mathsf{S}_{Z^*}$ such that $C \subset \{z \in \mathsf{S}_Z : z^*(z) = 1\}$. If C is a maximal face, then in fact $C = \{z \in \mathsf{S}_Z : z^*(z) = 1\}$. Now, the set $\{z^* \in \mathsf{B}_{Z^*} : z^*(z) = 1, \text{ for all } z \in C\}$ is a w^*-closed convex set of the dual unit ball, and so, by the Krein–Milman Theorem, it contains some extreme point. So, we can assume that z^* is an extreme point of B_{Z^*}. If $Z = X \oplus_\infty Y$, then $Z^* = X^* \oplus_1 Y^*$ and so $z^* = x^*$ for some $x^* \in \mathsf{S}_{X^*}$ or $z^* = y^*$ for some $y^* \in \mathsf{S}_{Y^*}$. In the first case $C = E \times \mathsf{B}_Y$, where E is a face of the unit ball of X. Since C is maximal, then E is maximal too. In the second case, we can use the same argument and $C = \mathsf{B}_X \times D$, for some maximal face D of S_Y. Conversely, if E is a maximal face of S_X, then $E \times \mathsf{B}_Y$ is also a maximal face of $\mathsf{S}_{X\oplus_\infty Y}$.
2. As above, let us start by assuming that C is a smooth face of $\mathsf{B}_{X\oplus_\infty Y}$. Since C is maximal, C is either of the form $\mathsf{B}_X \times D$, where D is a maximal face of B_Y, or of the form $E \times \mathsf{B}_Y$, where E is a maximal face of B_X. Since projections are open, the result follows. Conversely, both $\mathsf{B}_X \times D$ and $E \times \mathsf{B}_Y$ are maximal faces of $\mathsf{B}_{X\oplus_\infty Y}$. Moreover, $\operatorname{int}_{\mathsf{S}_{X\oplus_\infty Y}}(\mathsf{B}_X \times D) = B_X \times \operatorname{int}_{\mathsf{S}_Y}(D)$ and $\operatorname{int}_{\mathsf{S}_{X\oplus_\infty Y}}(E \times \mathsf{B}_Y) = \operatorname{int}_{\mathsf{S}_X}(E) \times \mathsf{B}_Y$, so they are smooth faces. ∎

Theorem 3.12. *Any real Banach space can be equivalently renormed so that its new unit ball possesses a smooth face.*

Proof. Let $x^* \in \mathsf{S}_{X^*}$ be a norm–attaining functional and consider a point $a \in \mathsf{S}_X$ so that $x^*(a) = 1$. Now, it suffices to consider the equivalent norm on X given by $\|x\|_\infty = \max\{|x^*(x)|, \|x - x^*(x)a\|\}$ for every $x \in X$. Notice that $(X, \|\cdot\|_\infty) = \ker(x^*) \oplus_\infty \mathbb{R}a$. Finally, we apply Theorem 3.11. ∎

Remark 3.13. Note that the idea used in Theorem 3.12 does not work for the complex case. Indeed, the unit ball of $\mathbb{C}$ does not possess any smooth face. Even more, Theorem 3.11 shows that $\mathbb{C} \oplus_\infty \mathbb{C}$ does not have any smooth face either. This motivates the following question: *Does there exist a renorming of $\mathbb{C} \oplus_\infty \mathbb{C}$ so that the new unit ball has a smooth face?* We think that no unit ball of any complex space possesses a smooth face.

Corollary 3.14. *Let X be a real Banach space. Then X can be equivalently renormed so that the set of non–norm–attaining functionals is not dense.*

Proof. Let $a \in \mathsf{S}_X$ and $a^* \in \mathsf{S}_{X^*}$ with $a^*(a) = 1$ and consider the equivalent norm on X given by $\|x\|_1 = |a^*(x)| + \|x - a^*(x)a\|$ for every $x \in X$. Notice that $(X, \|\cdot\|_1) = \ker(a^*) \oplus_1 \mathbb{R}a$. The dual norm of $\|\cdot\|_1$ is $\|\cdot\|_1^* = \|\cdot\|_\infty$, where $\|x^*\|_\infty = \max\{|a(x^*)|, \|x^* - a(x^*)a^*\|\}$ for every $x^* \in X^*$. Again, $(X^*, \|\cdot\|_\infty) = \ker(a) \oplus_\infty \mathbb{R}a^*$. Finally, by applying first Theorem 3.12 and then Theorem 3.10, we deduce the result. ∎

Remark 3.15. In [1, Proposition 1] it is proved that, if the unit ball of a Banach space X possesses a strong vertex point, then the functional associated to it belongs to the interior of $\mathsf{NA}\,(X)$. From this arises the following question: *What relation does there exist between a smooth face and the property of being a strong vertex of the functional taking the value* 1 *at this smooth face?*

Added in proof: The fourth author has recently proved that every real Banach space with separable dual has a renorming that makes the set of non-norm-attaining functionals nowhere dense. These results will appear in a forthcoming paper.

References

[1] M.D. Acosta and M.R. Galán, *New characterizations of the reflexivity in terms of the set of norm–attaining functionals*, Canad. Math. Bull. **41** (1998), 279–289.

[2] A. Aizpuru and F.J. García–Pacheco, *Some questions about rotundity and renormings in Banach spaces*, J. Aust. Math. Soc. **79** (2005), 131–140.

[3] A. Aizpuru and F.J. García–Pacheco, *Rotundity in transitive and separable Banach spaces*, Preprint.

[4] R.M. Aron, V.I. Gurariy and J.B. Seoane–Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. **133** (2005), 795–803.

[5] R.M. Aron, D. Pérez–García and J.B. Seoane–Sepúlveda, *Algebrability of the set of non–convergent Fourier series*, Studia Math. **175** (2006), 83-90.

[6] P. Bandyopadhyay and G. Godefroy, *Linear structure in the set of norm–attaining functionals on a Banach space*, Preprint.

[7] P. Bandyopadhyay and B.–L. Lin, *Some properties related to nested sequence of balls in Banach spaces*, Taiwanese J. Math. **5** (2001), 19–34.

[8] F. Bayart, *Topological and algebraic generecity of divergence and of universality,* Studia Math. **167** (2005), 161-181.

[9] F. Bayart, *Linearity of sets of strange functions,* Mich. Math. J. **53** (2005), 291-303.

[10] F. Bayart and L. Quarta, *Algebras in sets of queer functions,* Israel J. Math. (to appear).

[11] J. Becerra–Guerrero and A. Rodríguez–Palacios, *Transitivity of the norm on Banach spaces*, Extracta Math. **17** (2002), 1–58.

[12] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64**, New York, 1993.

[13] J. Diestel, *Geometry of Banach spaces–selected topics,* Lecture Notes in Mathematics **485**, Springer–Verlag, Berlin–New York, 1975.

[14] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, Springer–Verlag, New York, 1984.

[15] F.J. García–Pacheco, N. Palmberg and J.B. Seoane–Sepúlveda, *Lineability and algebrability of pathological phenomena in Analysis*, to appear in J. Math. Anal. Appl.

[16] V.I. Gurariy and L. Quarta, *On lineability of sets of continuous functions,* J. Math. Anal. Appl. **294** (2004), 62-72.

[17] M. Jiménez–Sevilla and J.P. Moreno, *A note on norm–attaining functionals*, Proc. Amer. Math. Soc. **126** (1998), 1989–1997.

[18] K. Kuratowski, *Topology Vol. I*, Academic Press, New York–London, 1966.

[19] S. Mazur, *Über konvexe mengen in linearem normieten Räumen*, Studia Math. **4** (1933), 70–84.

Universidad de Granada,
Departamento de Análisis Matemático,
Granada, 18071, Spain
email: dacosta@ugr.es

Departamento de Matématicas,
Universidad de Cádiz, Puerto Real,
Cádiz, 11510, Spain
email: antonio.aizpuru@uca.es

Department of Mathematical Sciences,
Kent State University, Kent,
Ohio, 44242, United States
email: aron@math.kent.edu, fgarcia@math.kent.edu