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FUNCTIONS MONOTONE CLOSE TO BOUNDARY

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Abstract. Functions which are monotone close to boundary are defined. Some oscillation estimates are given for these functions in Orlicz classes. Criteria for monotonicity close to boundary are obtained.

1. Main results. Let $D \subset \boldsymbol{R}^2$ be a domain in the Euclidean plane. By $\tilde{\partial} D$ we denote the boundary of D in the extended plane $\tilde{\boldsymbol{R}}^2 = \boldsymbol{R}^2 \cup \{\infty\}$. For a subdomain $\Delta \subset D$, we set

$$\partial' \Delta = \tilde{\partial} \Delta \setminus \tilde{\partial} D \quad \text{and} \quad \partial'' \Delta = \tilde{\partial} \Delta \cap \tilde{\partial} D \,.$$

For an arbitrary function $f : D \to \boldsymbol{R}$ and $A \subset \bar{D}$, $A \neq \emptyset$, we put

$$\operatorname{osc}(f, A) = \sup_{a,b \in A} \limsup_{a_j \to a, b_j \to b} (f(a_j) - f(b_j)) \,,$$

where the lim sup is taken over all sequences $a_j \to a$, $b_j \to b$ of points in D. Let Γ be a subset of $\tilde{\partial} D$. A continuous function $f : D \to \boldsymbol{R}$ is called *monotone close to Γ* if for every subdomain $\Delta \subset D$ with $\partial'' \Delta \subset \Gamma$,

$$\operatorname{osc}(f, \Delta) \leq \operatorname{osc}(f, \partial' \Delta) \,, \tag{1.1}$$

see Martio et al. [10].

If $\Gamma = \emptyset$, then we have the well-known class of monotone functions in the sense of Lebesgue. If $\Gamma = \tilde{\partial} D$, then every function, monotone close to Γ, is a constant. This is evident, since, choosing $\Delta = D \setminus \{x_0\}$ where $x_0 \in D$ is an arbitrary point, it follows from (1.1) that

$$\operatorname{osc}(f, \Delta) \leq \operatorname{osc}(f, \{x_0\}) = 0 \,.$$

For another generalization of monotonicity in the sense of Lebesgue, see Manfredi [8].

For $a, b \in D$ we let

$$\rho_D(a, b) = \inf_{\gamma} \operatorname{diam} \gamma \,,$$

where the infimum is taken over all arcs $\gamma \subset D$ joining a and b. The quantity $\rho_D(a, b)$ is called the *inner distance* between a and b. Clearly, ρ_D defines a metric in D. For arbitrary sets $A, B \subset \boldsymbol{R}^2$ we let

$$\operatorname{dist}(A, B) = \sup_{x \in A} \operatorname{dist}(x, B) \,.$$

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For a set $\varGamma \subset \partial D$, $D \neq \emptyset$, and points $a, b \in D$ we set

$$\delta_D(a, b; \varGamma) = \limsup \operatorname{dist}(\gamma_k, \partial D \setminus \varGamma)\,,$$

where lim sup is taken over all sequences $\{\gamma_k\}$ of arcs $\gamma_k \subset D$ joining a and b such that $\operatorname{diam} \gamma_k \to \rho_D(a, b)$.

We will employ functions with Sobolev derivatives in some Orlicz classes, see [4, Chapter I]. A function $\varPhi : \boldsymbol{R} \to \boldsymbol{R}_+$ is called an *N-function* if it admits the representation

$$\varPhi(\tau) = \int_0^{|\tau|} p(t)dt\,, \tag{1.2}$$

where $p : (0, \infty) \to (0, \infty)$ is a positive non-decreasing function, continuous to the right, such that

$$p(0+) = 0\,, \quad p(\infty) = \lim_{t\to\infty} p(t) = \infty\,. \tag{1.3}$$

Let $D \subset \boldsymbol{R}^2$ be an open set. Let $\varPhi$ be an N-function. Recall that a function $f : D \to \boldsymbol{R}$ is *ACL* (absolutely continuous on lines) if for each cube Q, $\bar{Q} \subset D$, and for $j = 1, 2$ and for all z in the projection of Q into $\{x_j = 0\}$, except a one-dimensional set of measure zero, $t \mapsto f(z + te_j)$, $z + te_j \in Q$, is absolutely continuous. We say that a continuous function $f : D \to \boldsymbol{R}$ belongs to $ACL^{\varPhi}(D)$ if f is *ACL* and

$$\int_D \varPhi(|\nabla f|)dx_1dx_2 < \infty\,.$$

Here ∇f stands for the formal gradient (f'_{x_1}, f'_{x_2}), which exists almost everywhere in D. A continuous function f belongs to ACL^p if $f \in ACL^{\varPhi}$ with $\varPhi = t^p$, $p \geq 1$. The class ACL^2 coincides with the class *BL* introduced in 1906 by Levi [7] (see also Nikodym [13], Deny and Lions [2], Lelong-Ferrand [6], and Suvorov [14]). For N-functions $\varPhi$ of the general form, see Miklyukov [11], Kruglikov and Miklyukov [5] and Astala et al. [1]. The boundary behavior of monotone ACL^n functions has been studied by these authors and others (see, e.g., Manfredi and Villamor [9] and Mizuta [12] and references therein).

Our main result yields the following inequality for functions monotone close to boundary.

THEOREM 1.4. *Let D be a subdomain of $\boldsymbol{R}^2$, $\varGamma \subset \partial D$ an open Jordan arc, and $f \in ACL^{\varPhi}(D)$. If f is monotone close to $\varGamma$, then for every pair of points $a, b \in D$ with*

$$\rho_D(a, b) < \delta_D(a, b; \varGamma)\,,$$

the following estimate holds:

$$|f(a) - f(b)| \leq \kappa_0(\rho_D(a, b); \delta_D(a, b; \varGamma), \varPhi, I)\,, \tag{1.5}$$

where

$$I = \int_D \varPhi(|\nabla f|)dx_1dx_2\,,$$

and κ_0 is the function defined by (3.1).

Suppose that D is convex. Then $\rho_D(a, b) = |b - a|$ and

$$\delta_D(a, b; \Gamma) \leq 2 \min\{\operatorname{dist}(a, \partial D \setminus \Gamma), \operatorname{dist}(b, \partial D \setminus \Gamma)\}$$

provided that the condition

$$\rho_D(a, b) \leq \frac{1}{2}\delta_D(a, b; \Gamma)$$

holds. If D is convex and Φ satisfies some additional conditions, then the estimate (1.5) implies that f has a continuous extension to Γ. Specifically, if D is convex, then we suppose that for all $a \geq 1$ and all $\tau > 0$,

$$\Phi(a\tau) \leq c_\Phi a^2 \Phi(\tau) \tag{1.6}$$

with some constant c_Φ independent of a and τ, and that

$$\int_1^\infty \frac{\Phi(\tau)}{\tau^3} d\tau = \infty\,. \tag{1.7}$$

COROLLARY 1.8. *If in the situation of Theorem* 1.4, *D is convex and the N-function Φ satisfies* (1.6) *and* (1.7), *then f has a continuous extension to Γ.*

In what follows we give two sufficient conditions for monotonicity close to boundary.

THEOREM 1.9. *Let Φ and Ψ be arbitrary mutually complementary N-functions. Let $D \subset \boldsymbol{R}^2$ be a subdomain of $\boldsymbol{R}^2$, $\Gamma \subset \partial D$ and $f \in ACL^\Phi(D)$. Suppose that there exists a vector field $A(x) : D \to \boldsymbol{R}^2$ of the class $L^\Psi(D)$ such that for almost all $x \in D$ at which $\nabla f(x) \neq 0$,*

$$\sum_{i=1}^{2} f'_{x_i}(x) A_i(x) > 0\,. \tag{1.10}$$

If for every subdomain $\Delta \subset D$ such that $\partial'\Delta$ is locally rectifiable and $\partial''\Delta \subset \Gamma$, and for every $\phi \in ACL^\Phi(\Delta) \cap C(\Delta \cup \partial'\Delta)$

$$\int_{\partial'\Delta} \phi \langle A(x), \mathbf{n}\rangle |dx| = \int_\Delta \sum_{i=1}^{2} \phi'_{x_i} A_i(x) dx_1 dx_2\,, \tag{1.11}$$

then f is monotone close to Γ.

Because the assumptions in Theorem 1.9 are rather complicated, we illustrate the theorem with an example.

EXAMPLE 1.12. Let $D \subset \boldsymbol{R}^2$ be a bounded subdomain with smooth boundary and let $\Gamma \subset \partial D$ be an open proper subarc. Let f be a $C^1(D \cup \Gamma)$ solution of the equation

$$\operatorname{div}(|\nabla f|^{p-2} \nabla f) = 0\,, \quad p \geq 1\,.$$

Suppose that the normal derivative $\partial f/\partial n|_\Gamma = 0$. If we now choose

$$A(x) = \nabla f(x) |\nabla f(x)|^{p-2}\,,$$

then

$$\sum_{i=1}^{2} f'_{x_i}(x)A_i(x) = |\nabla f|^p$$

and the assumption (1.10) holds. Next, for every function ϕ and every subdomain Δ as in Theorem 1.9, we have

$$\begin{aligned}\int_{\partial'\Delta} \phi\langle A(x), \mathbf{n}\rangle|dx| &= \int_{\partial\Delta} \phi|\nabla f|^{p-2}\langle\nabla f(x), \mathbf{n}\rangle|dx| \\ &= \int_{\Delta} \phi \operatorname{div}|\nabla f|^{p-2}\nabla f\, dx_1 dx_2 + \int_{\Delta} |\nabla f|^{p-2}\langle\nabla\phi, \nabla f\rangle dx_1 dx_2 \\ &= \int_{\Delta} \sum_{i=1}^{2} \phi'_{x_i} A_i(x) dx_1 dx_2 .\end{aligned}$$

This formula implies (1.11) and by Theorem 1.9, f is monotone close to Γ.

In order to state the next result, let $\Gamma \subset \partial\tilde{D}$ and let $h : D \to (0, \infty)$ be a locally Lipschitz function such that $\lim_{x\to\Gamma} h(x) = 0$ and

$$0 < h_0 \le \operatorname*{ess\,inf}_D |\nabla h(x)| \le \operatorname*{ess\,sup}_D |\nabla h(x)| \le h_1 < \infty , \tag{1.13}$$

where h_0 and h_1 are some constants. We let $E_t = \{x \in D\ ;\ h(x) = t\}$.

THEOREM 1.14. *Let $f \in ACL^p_{\mathrm{loc}}(D)$, $p > 1$, be a bounded function. Suppose that there exists a vector field*

$$A = (A_1, A_2) : D \to \boldsymbol{R}^2 , \quad A \in L^q(D) , \quad q = p/(p-1)$$

such that

$$\begin{aligned}\sigma_1(x)|\nabla f(x)|^p &\le \sum_{i=1}^{2} f'_{x_i}(x)A_i(x) , \\ \left(\sum_{i=1}^{2} A_i^2(x)\right)^{1/2} &\le \sigma_2(x)|\nabla f(x)|^{p-1}\end{aligned} \tag{1.15}$$

for some continuous functions $\sigma_j : D \to (0, \infty)$, $j = 1, 2$, and that (1.11) *holds for all functions $\phi \in ACL^p(D)$,* $\operatorname{supp}\phi \subset\subset D$, *and for all subdomains $\Delta \subset\subset D$ with locally rectifiable boundaries. If*

$$\int_0 dt \left(\int_{D\cap E_t} \frac{\sigma_2^p}{\sigma_1^{p-1}} d\mathcal{H}^1(E_t)\right)^{1/(1-p)} = \infty , \tag{1.16}$$

then f is monotone close to Γ. In particular, if $\Gamma = \tilde{\partial} D$, then $f \equiv$ constant.

Let γ be a simple open Jordan arc lying in the upper half-plane with endpoints $(0, 0)$ and $(a, 0)$ on the x_1-axis. We set

$$\Gamma = \{(x_1, x_2)\ ;\ 0 \le x_1 \le a, x_2 = 0\} ,$$

and denote by D the subdomain of $\boldsymbol{R}^2$ lying between γ and Γ. Choosing $h(x) = x_2$ in Theorem 1.14, we obtain the following result.

COROLLARY 1.17. *Let f be as in Theorem* 1.14. *Suppose that there exists a vector field*

$$A = (A_1, A_2) : D \to \boldsymbol{R}^2 , \quad A \in L^q(D), \quad q = p/(p-1) ,$$

satisfying

$$\tag{1.18} \begin{gathered} \sigma_1(x_2)|\nabla f(x)|^p \leq \sum_{i=1}^{2} f'_{x_i}(x) A_i(x) , \\ \left(\sum_{i=1}^{2} A_i^2(x) \right)^{1/2} \leq \sigma_2(x_2)|\nabla f(x)|^{p-1} \end{gathered}$$

with some continuous functions $\sigma_1, \sigma_2 > 0$. *Suppose also that for all functions* $\phi \in ACL^p(D)$, $\operatorname{supp}\phi \subset\subset D$, *and all subdomains* $\Delta \subset\subset D$ *with locally rectifiable boundaries, the relation* (1.11) *holds. If*

$$\tag{1.19} \int_0 \frac{\sigma_1(t)}{\sigma_2^{p/(p-1)}(t)} dt = \infty ,$$

then f is monotone close to Γ.

EXAMPLE 1.20. Let D be as in Corollary 1.17 and let f be a C^1 solution in D of an equation

$$\operatorname{div}(\sigma|\nabla f|^{p-2}\nabla f) = 0 , \quad p > 1 ,$$

with a continuous weight function $\sigma(x) = \sigma(x_2)$. Setting

$$A_i(x) = \sigma|\nabla f|^{p-2} f'_{x_i} ,$$

we find

$$\sum_{i=1}^{2} f'_{x_i} A_i = \sigma|\nabla f|^p \quad \text{and} \quad |A| = \sigma|\nabla f|^{p-1} .$$

Thus, (1.18) holds with $\sigma_1 = \sigma_2 = \sigma$. The assumption (1.19) takes the form

$$\int_0 \sigma^{-1/(p-1)}(t) dt = \infty ,$$

which guarantees that f is monotone close to Γ.

2. Length and area principle. Each N-function is convex. The first condition in (1.3) means that

$$\tag{2.1} \lim_{\tau \to 0} \frac{\Phi(\tau)}{\tau} = 0 ,$$

and the second condition means that

$$\tag{2.2} \lim_{\tau \to \infty} \frac{\Phi(\tau)}{\tau} = \infty .$$

For every non-decreasing function $p : (0,\infty) \to (0,\infty)$, which is continuous to the right and satisfies (1.3), we find another function $q : (0,\infty) \to (0,\infty)$, $s \geq 0$, such that

$$q(s) = \sup_{p(t)\leq s} t\,.$$

It is easy to see that $q(s)$ also satisfies (1.3). The pair of N-functions

$$\Phi(\tau) = \int_0^{|\tau|} p(t)dt \quad \text{and} \quad \Psi(\tau) = \int_0^{|\tau|} q(s)ds$$

are called *mutually complementary*.

We give several estimates for ACL^{Φ}-functions, which are needed for studying functions monotone close to boundary. For this purpose we first recall Orlicz classes. Fix a Lebesgue measurable set $A \subset \boldsymbol{R}^2$ and an arbitrary N-function Φ. Then the *Orlicz class* $L^{\Phi}(A)$ is defined to be the set of Lebesgue measurable functions u in A such that

$$I(u, A; \Phi) \equiv \int_A \Phi[u(x)]dx_1dx_2 < \infty\,.$$

In applications the functions

$$\Phi(\tau) = \tau^2 \ln^{\alpha}(e+\tau)$$

play an important role, and the classes $L^{\Phi}(A)$ associated with this particular Φ are called *Zygmund classes* [1].

Let $\Phi(\tau)$ and $\Psi(\tau)$ be mutually complementary N-functions. By $\tilde{L}^{\Phi}(A)$ we denote the set of the functions $u(x)$ satisfying

$$\left|\int_A u(x)v(x)dx_1dx_2\right| < \infty \quad \text{for all } v(x) \in L^{\Psi}(A)\,.$$

Next, we set

$$\|u\|_{\Phi,A} \equiv \sup_{I(v,A;\Psi)\leq 1} \left|\int_A u(x)v(x)dx_1dx_2\right|. \tag{2.3}$$

After introducing equivalent norms, the class $\tilde{L}^{\Phi}(A)$ is converted to a linear norm space (see [4, Chapter II]). Namely, the definition (2.3) of the norm implies that:

(1) $\|u\|_{\Phi,A} = 0$ if and only if $u(x) = 0$ almost everywhere on A;
(2) $\|\alpha u\|_{\Phi,A} = |\alpha|\|u\|_{\Phi,A}$;
(3) $\|u_1 + u_2\|_{\Phi,A} \leq \|u_1\|_{\Phi,A} + \|u_2\|_{\Phi,A}$.

For the next lemma we refer the reader to [4].

LEMMA 2.4. (i) *For each* $u \in L^{\Phi}(A)$,

$$\|u\|_{\Phi,A} \leq \int_A \Phi(u)dx_1dx_2 + 1\,. \tag{2.5}$$

In particular, $L^{\Phi}(A) \subset \tilde{L}^{\Phi}(A)$.

(ii) *If* $\|u\|_{\Phi,A} \leq 1$, *then* $u \in L^{\Phi}(A)$ *and*

$$\int_A \Phi(u)dx_1dx_2 \leq \|u\|_{\Phi,A}\,. \tag{2.6}$$

In particular,

$$\int_A \Phi\left[\frac{u(x)}{\|u\|_{\Phi,A}}\right] dx_1 dx_2 \le 1 .$$

(iii) *For every pair of functions $u \in \tilde{L}^{\Phi}(A)$ and $v \in \tilde{L}^{\Psi}(A)$,*

$$\left|\int_A u(x)v(x)dx_1dx_2\right| \le \|u\|_{\Phi,A}\|v\|_{\Psi,A} . \tag{2.7}$$

LEMMA 2.8. *Suppose that a domain $D \subset \boldsymbol{R}^2$ is of finite area, Φ is an N-function and $f \in ACL^{\Phi}(D)$. Then $f \in ACL^1(D)$.*

PROOF. Since Φ satisfies (2.2), there exists $q > 0$ such that

$$\frac{\tau}{\Phi(\tau)} \le 1 \quad \text{for all } \tau \ge q .$$

Writing

$$D_1 = \{x \in D\,;\, |\nabla f(x)| \le q\}, \quad D_2 = \{x \in D\,;\, |\nabla f(x)| > q\} ,$$

we have

$$\begin{aligned}\int_D |\nabla f| dx_1dx_2 &= \int_{D_1} |\nabla f| dx_1dx_2 + \int_{D_2} |\nabla f| dx_1dx_2 \\ &\le q\int_{D_1} dx_1dx_2 + \int_{D_2} \frac{|\nabla f|}{\Phi(|\nabla f|)} \Phi(|\nabla f|)dx_1dx_2 \\ &\le q\mathcal{H}^2(D_1) + C\int_{D_2} \Phi(|\nabla f|)dx_1dx_2 ,\end{aligned}$$

where $\mathcal{H}^2$ is the two-dimensional Lebesgue measure and

$$C = \sup_{x\in D_2} \frac{|\nabla f(x)|}{\Phi(|\nabla f(x)|)} \le 1 .$$

Hence, we obtain

$$\begin{aligned}\int_D |\nabla f| dx_1dx_2 &\le q\mathcal{H}^2(D_1) + \int_{D_2} \Phi(|\nabla f|)dx_1dx_2 \\ &\le q\mathcal{H}^2(D) + \int_{D} \Phi(|\nabla f|)dx_1dx_2\end{aligned}$$

and f is ACL^1 as required. □

EXAMPLE 2.9. The functions $\Phi(\tau) = \tau^2$ and $\Phi(\tau) = \tau^2/\ln(e + |\tau|)$ are typical examples of N-functions satisfying (1.6) and (1.7). This is evident for $\Phi(\tau) = \tau^2$. We verify the necessary conditions for $\Phi(\tau) = \tau^2/\ln(e + |\tau|)$.

First, we observe that this function satisfies (1.3). Indeed, the derivative of $\Phi(\tau)$ has the form

$$\Phi'(\tau) = \tau\frac{2\ln(e+\tau) - \tau/(e+\tau)}{\ln^2(e+\tau)} , \quad \tau > 0 .$$

Clearly, $\Phi'(0+) = 0$. Next, we note that

$$\ln(e+\tau) > 1 \quad \text{for} \quad \tau > 0\,.$$

Hence, for all $\tau > 0$,

$$\begin{aligned}\Phi'(\tau) &= \tau\frac{2(e+\tau)\ln(e+\tau)-\tau}{(e+\tau)\ln^2(e+\tau)} > \tau\frac{2(e+\tau)-\tau}{(e+\tau)\ln^2(e+\tau)}\\ &= \tau\frac{2e+\tau}{(e+\tau)\ln^2(e+\tau)} > 0\,,\end{aligned}$$

from which we see

$$\Phi'(\infty) = \lim_{\tau\to\infty}\Phi'(\tau) = \infty\,.$$

Secondly, we verify (1.6). For every $a \geq 1$ we have

$$\Phi(a\tau) = a^2\tau^2/\ln(e+a|\tau|) \leq a^2\tau^2/\ln(e+|\tau|) = a^2\Phi(\tau)\,.$$

The property (1.7) is evident.

We denote by $S_D(a,r)$ the intersection of the circle $S(a,r)$ and the set $D \subset \boldsymbol{R}^2$. Fix ε and ε_0 such that $0 < \varepsilon < \varepsilon_0 < \infty$. Suppose that for all $\tau \in (\varepsilon, \varepsilon_0)$ the intersection $S_D(a,\tau) \neq \emptyset$. We set

$$D_{\varepsilon,\varepsilon_0} = \bigcup_{\varepsilon<\tau<\varepsilon_0} S_D(a,\tau)\,, \quad l(a,r) = \mathcal{H}^1(S_D(a,r))\,.$$

Let $\sigma_1, \sigma_2, \ldots$ be the components of $S_D(a,r)$. Then $S_D(a,r) = \bigcup_i \sigma_i$ and for a function $f : D \to \boldsymbol{R}$ we set

$$W_f(a,r) = \sum_i \operatorname{osc}(f,\sigma_i)\,.$$

We prove the following version of the Length and Area Principle for ACL^Φ-functions. A closely related class of functions was considered in [5].

THEOREM 2.10. *Let Φ be an N-function. Then for all $f \in ACL^\Phi(D)$ and all $0 < \varepsilon < \varepsilon_0 < \infty$,*

$$\int_\varepsilon^{\varepsilon_0} \Phi\left(\frac{W_f(a,\tau)}{l(a,\tau)}\right) l(a,\tau)\,d\tau \leq I(D_{\varepsilon,\varepsilon_0})\,, \tag{2.11}$$

where

$$I(D_{\varepsilon,\varepsilon_0}) = \int_{D_{\varepsilon,\varepsilon_0}} \Phi(|\nabla f|)\,dx_1 dx_2\,.$$

PROOF. We may assume that $a = 0$. Lemma 2.8 shows that a function $f \in ACL^\Phi$ belongs to $ACL^1_{\mathrm{loc}}(D)$. Let (r,θ) be polar coordinates in $\boldsymbol{R}^2$ and set $f^* = f(r,\theta)$. Now $f^* \in ACL^1_{\mathrm{loc}}(D)$ and, in particular, f^* is absolutely continuous on each line segment $\alpha \leq \theta \leq \beta$ on $S_D(0,\tau)$ for almost all τ. Hence, for almost all $\tau \in (\varepsilon, \varepsilon_0)$, we have

$$W_f(0,\tau) = \sum_i \operatorname{osc}(f,\sigma_i) \leq \sum_i \int_{\sigma_i} |\nabla f||dx| \leq \int_{S_D(0,\tau)} |\nabla f||dx|\,.$$

Recall that σ_i are the components of $S_D(0,\tau)$.

Jensen's integral inequality yields

$$\Phi\left(\frac{\int_E a(x)d\mathcal{H}^1}{\mathcal{H}^1(E)}\right) \le \frac{1}{\mathcal{H}^1(E)}\int_E \Phi(a(x))d\mathcal{H}^1 \tag{2.12}$$

for an arbitrary N-function Φ and every integrable function a [4, (8.2)]. Using (2.12) for almost all $\tau \in (\varepsilon, \varepsilon_0)$, we obtain

$$\Phi\left(\frac{W_f(0,\tau)}{l(0,\tau)}\right) \le \Phi\left(\frac{1}{l(0,\tau)}\int_{S_D(0,\tau)} |\nabla f||dx|\right) \le \frac{1}{l(0,\tau)}\int_{S_D(0,\tau)} \Phi\left(|\nabla f|\right)|dx|\,.$$

The functions on both sides of this inequality are measurable with respect to $\tau \in (\varepsilon, \varepsilon_0)$. Multiplying this inequality by $l(0,\tau)$ and integrating over $(\varepsilon, \varepsilon_0)$, we obtain

$$\int_\varepsilon^{\varepsilon_0} \Phi\left(\frac{W_f(0,\tau)}{l(0,\tau)}\right) l(0,\tau)d\tau \le \int_\varepsilon^{\varepsilon_0} d\tau \int_{S_D(0,\tau)} \Phi(|\nabla f|)|dx| = \int_{D_{\varepsilon,\varepsilon_0}} \Phi(|\nabla f|)dx_1dx_2\,.$$

□

The following might be the most useful corollary of Theorem 2.10.

COROLLARY 2.13. *For every N-function Φ and $f \in ACL^\Phi(D)$,*

$$\int_\varepsilon^{\varepsilon_0} \Phi\left(\frac{W_f(a,\tau)}{2\pi\tau}\right)\tau d\tau \le \frac{1}{2\pi} I(D_{\varepsilon,\varepsilon_0})\,. \tag{2.14}$$

PROOF. Every N-function Φ satisfies

$$\Phi(\alpha\tau) \le \alpha\Phi(\tau) \tag{2.15}$$

for all $\tau \ge 0$ and $\alpha \in [0,1]$. Indeed, write Φ in the form $\Phi(\tau) = \int_0^\tau p(t)dt$ as in (1.2). Then

$$\alpha\Phi(\tau) = \int_0^\tau p(t)d(\alpha t) \ge \int_0^\tau p(\alpha t)d(\alpha t) = \int_0^{\alpha\tau} p(t)dt = \Phi(\alpha\tau)$$

as required. Let $W \ge 0$. From (2.15) we conclude that

$$\Phi\left(\frac{W}{2\pi\tau}\right) = \Phi\left(\frac{W}{l(a,\tau)}\frac{l(a,\tau)}{2\pi\tau}\right) \le \Phi\left(\frac{W}{l(a,\tau)}\right)\frac{l(a,\tau)}{2\pi\tau}\,.$$

Hence,

$$\Phi\left(\frac{W_f(a,\tau)}{2\pi\tau}\right)2\pi\tau \le \Phi\left(\frac{W_f(a,\tau)}{l(a,\tau)}\right)l(a,\tau)\,,$$

and the desired conclusion follows from (2.11). □

3. The function k_0. Here we study the inequality (2.14). For an N-function Φ and an interval $(\varepsilon, \varepsilon_0)$ with $0 < \varepsilon < \varepsilon_0$, we set

$$\kappa_0(\varepsilon; \varepsilon_0, \Phi, I) = \sup\left\{\kappa\ ;\ \int_\varepsilon^{\varepsilon_0} \Phi\left(\frac{\kappa}{2\pi\tau}\right)\tau d\tau \le \frac{1}{2\pi}I\right\}, \tag{3.1}$$

where

$$I = \int_D \Phi(|\nabla f|)dx_1dx_2\,.$$

It follows from (2.1) and (2.2) that the function

$$F(\kappa) = \int_{\varepsilon}^{\varepsilon_0} \Phi\left(\frac{\kappa}{2\pi\tau}\right) \tau\, d\tau$$

is continuous and strictly monotone. Moreover,

$$F(0) = 0 \quad \text{and} \quad F(\infty) = \lim_{\kappa\to\infty} F(\kappa) = \infty\,.$$

Now $\kappa_0(\varepsilon) = \kappa_0(\varepsilon; \varepsilon_0, \Phi, I)$ is the unique positive root of the equation

$$F(\kappa) = (I/(2\pi))\,. \tag{3.2}$$

The function $F(\kappa)$ has the following properties.

(i) *For every N-function Φ the function $\kappa_0(\varepsilon) = \kappa_0(\varepsilon; \varepsilon_0, \Phi, I)$ is continuous and strictly increasing on* $(0, \varepsilon_0)$. *Moreover,* κ_0 *depends continuously on* ε_0.

The proof is evident.

(ii) *If an N-function Φ satisfies* (1.7), *then* $\lim_{\varepsilon\to 0} \kappa_0(\varepsilon) = 0$.

Indeed, suppose that there exists a number $m > 0$ such that for all sufficiently small $\varepsilon > 0$,

$$\kappa_0(\varepsilon) \geq m > 0\,.$$

Then by the definition (3.2) of $\kappa_0(\varepsilon)$ and the monotonicity of Φ, we have

$$\int_{\varepsilon}^{\varepsilon_0} \Phi\left(\frac{m}{2\pi\tau}\right) \tau\, d\tau \leq \frac{1}{2\pi} I\,.$$

Substituting $t = m/2\pi\tau$ in this inequality, we find

$$\left(\frac{m}{2\pi}\right)^2 \int_{m/2\pi\varepsilon_0}^{m/2\pi\varepsilon} \frac{\Phi(t)}{t^3} dt \leq \frac{1}{2\pi} I \quad \text{for all } 0 < \varepsilon < \varepsilon_0\,.$$

Letting $\varepsilon \to 0+$ we obtain a contradiction to (1.7).

(iii) *If an N-function Φ satisfies* (1.6), *then*

$$\liminf_{\varepsilon\to 0+} \kappa_0(\varepsilon) \bigg/ \left(\ln\frac{1}{\varepsilon}\right)^{-\beta} = \infty \quad \textit{for every } \beta > 1\,.$$

Indeed, suppose that this is not true; that is, for some sequence $\varepsilon_i \to 0$ of positive numbers the inequality

$$\kappa_0(\varepsilon_i) \leq c\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}$$

holds with some constant $c > 0$. From (3.2) we obtain

$$\int_{\varepsilon_i}^{\varepsilon_0} \Phi\left(\frac{\kappa_0(\varepsilon_i)}{2\pi\tau}\right) \tau\, d\tau = \frac{1}{2\pi} I \tag{3.3}$$

and hence

$$\int_{\varepsilon_i}^{\varepsilon_0} \Phi\left(\frac{c}{2\pi\tau}\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\right) \tau\, d\tau \geq \frac{1}{2\pi} I\,.$$

When $\varepsilon_0 \le 1$, from (1.6) and (2.15) we have

$$\frac{1}{2\pi} I \le \int_{\varepsilon_i}^{\varepsilon_0} \Phi\left(\frac{c}{2\pi\tau}\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\right)\tau\, d\tau \le c_\Phi \Phi\left(\frac{c}{2\pi}\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\right)\int_{\varepsilon_i}^{\varepsilon_0}\frac{d\tau}{\tau} d\tau$$
$$\le c_\Phi \left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\Phi\left(\frac{c}{2\pi}\right)\ln\frac{1}{\varepsilon_i} \le c_\Phi\left(\ln\frac{1}{\varepsilon_i}\right)^{1-\beta}\Phi\left(\frac{c}{2\pi}\right).$$

When $\varepsilon_0 > 1$, we also have

$$\frac{1}{2\pi} I \le \int_{\varepsilon_i}^{1} \Phi\left(\frac{c}{2\pi\tau}\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\right)\tau\, d\tau + \int_{1}^{\varepsilon_0} \Phi\left(\frac{c}{2\pi\tau}\left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\right)\tau\, d\tau$$
$$\le c_\Phi \Phi\left(\frac{c}{2\pi}\right)\left(\ln\frac{1}{\varepsilon_i}\right)^{1-\beta} + \left(\ln\frac{1}{\varepsilon_i}\right)^{-\beta}\int_1^{\varepsilon_0}\Phi\left(\frac{c}{2\pi\tau}\right)\tau\, d\tau\,.$$

For $\beta > 1$ we obtain a contradiction as $i \to \infty$.

(iv) We assume here that an N-function Φ satisfies (1.6) and (1.7). Then it is possible to find a majorant for $\kappa_0(\varepsilon)$. For this fix $0 < \varepsilon < \varepsilon_0$ and set $\kappa_0 = \kappa_0(\varepsilon)$. Substituting $t = \kappa_0/2\pi\tau$ in (3.3), we have

$$2\pi I = \kappa_0^2 \int_{\kappa_0/2\pi\varepsilon_0}^{\kappa_0/2\pi\varepsilon}\frac{\Phi(t)}{t^3} dt\,.$$

By property (ii) there exists a number $\alpha_1 = \alpha_1(\varepsilon_0)$ such that

$$\kappa_0(\varepsilon) \le 2\pi\varepsilon_0 \quad \text{for all } \ 0 < \varepsilon < \alpha_1\,,$$

and hence for all sufficiently small $\varepsilon > 0$,

$$2\pi I \ge \kappa_0^2 \int_1^{\kappa_0/2\pi\varepsilon}\frac{\Phi(t)}{t^3} dt\,.$$

By (iii), we see that

$$\liminf_{\varepsilon\to 0+}\frac{\kappa_0(\varepsilon)}{\sqrt{\varepsilon}} = \infty$$

and hence there exists a number $\alpha_2 = \alpha_2(\varepsilon_0)$ such that

$$\kappa_0(\varepsilon) \ge 2\pi\sqrt{\varepsilon} \quad \text{for all } \ 0 < \varepsilon < \alpha_2\,.$$

Therefore, for all sufficiently small $\varepsilon > 0$, we find

$$2\pi I \ge \kappa_0^2 \int_1^{1/\sqrt{\varepsilon}}\frac{\Phi(t)}{t^3} dt\,,$$

and arrive at the estimate

$$\kappa_0(\varepsilon, \varepsilon_0; \Phi, I) \le \left(2\pi I \Big/ \int_1^{1/\sqrt{\varepsilon}}\frac{\Phi(t)}{t^3} dt\right)^{1/2}, \tag{3.4}$$

which holds for all $0 < \varepsilon < \min\{\alpha_1(\varepsilon_0), \alpha_2(\varepsilon_0)\}$.

From (3.4) we obtain the following.

REMARK 3.5. For the N-functions Φ satisfying (1.6) and (1.7), and for all $\varepsilon > 0$ with $0 < \varepsilon < \min\{\alpha_1(\varepsilon_0), \alpha_2(\varepsilon_0)\}$, the inequality (2.14) can be written in the form

$$\inf_{\varepsilon<\tau<\varepsilon_0} W_f(a,\tau) \le \left(2\pi I \bigg/ \int_1^{1/\sqrt{\varepsilon}} \frac{\Phi(t)}{t^3} dt\right)^{1/2}, \tag{3.6}$$

where

$$I = \int_D \Phi(|\nabla f|) dx_1 dx_2 .$$

4. Proof of Theorem 1.4.

Fix a number $h > 0$ such that

$$\rho_D(a,b) + h < \delta_D(a,b;\Gamma) - h ,$$

and choose an arc $\gamma \subset D$ connecting a and b with

$$\operatorname{diam}\gamma < \rho_D(a,b) + h .$$

Let $x_0 \in \gamma$ and consider a family of circles $\{S(x_0,\tau)\}$, where $\tau \in [\varepsilon, \varepsilon_0]$ and

$$\varepsilon = \rho_D(a,b) + h, \quad \varepsilon_0 = \delta_D(a,b;\Gamma) - h .$$

By Corollary 2.13 and definition (3.1) we obtain

$$\inf_{\tau\in(\varepsilon,\varepsilon_0)} W_f(x_0,\tau) \le \kappa_0(\varepsilon;\varepsilon_0,\Phi,I) .$$

For every $\tau \in [\varepsilon, \varepsilon_0]$, each circle $S(x_0,\tau)$ separates points a, b from $\partial D \setminus \Gamma$. Hence, we can find a component σ of $S_D(x_0,\tau)$ separating a, b from $\partial D \setminus \Gamma$. Because $\tau \in [\varepsilon, \varepsilon_0]$, the ends of σ lie on Γ. We denote by D_σ the set of the points which are separated by σ from $\partial D \setminus \Gamma$.

The function f is monotone close to Γ and therefore,

$$\operatorname{osc}(f, D_\sigma) \le \operatorname{osc}(f, \partial' D_\sigma) .$$

Hence we have

$$\begin{aligned} |f(a) - f(b)| &\le \inf_{\tau\in(\varepsilon,\varepsilon_0)} \operatorname{osc}(f, D_\sigma) \le \inf_{\tau\in(\varepsilon,\varepsilon_0)} \operatorname{osc}(f, \partial' D_\sigma) \\ &\le \inf\nolimits_{\tau\in(\varepsilon,\varepsilon_0)} W_f(x_0,\tau) \le \kappa_0(\varepsilon;\varepsilon_0,\Phi,I) \\ &\le \kappa_0(\rho_D(a,b)+h;\delta_D(a,b;\Gamma)-h,\Phi,I) . \end{aligned}$$

Letting $h \to 0$, we obtain (1.5).

By (1.7) and by property (ii) in Section 3, we see that $\kappa_0(\rho_D(a,b)) \to 0$ as $\rho_D(a,b) \to 0$. This means that f can be continuously extended to Γ.

5. Proof of Theorem 1.9.

Fix a subdomain Δ of D with $\partial''\Delta \subset \Gamma$. First we prove that

$$\sup_{\Delta} f(x) = \sup_{\partial'\Delta} f(x) . \tag{5.1}$$

Suppose that this is not the case, that is, there exists a point $x_0 \in \Delta$ such that

$$f(x_0) > \sup_{\partial'\Delta} f(x) = M .$$

Choose $\varepsilon > M$ such that $f(x_0) > \varepsilon$. By Lemma 2.8 the function $f \in ACL^1(D)$ and, by [15, Theorem 5.4.4], for almost all ε the set $\{x \in \Delta\,;\ f(x) = \varepsilon\}$ is locally rectifiable. Fix a component $U, x_0 \in U$, of the set $\{x \in \Delta\,;\ f(x) > \varepsilon\}$. Without loss of generality, we may assume that $\partial' U$ is locally rectifiable. Using (1.11) with $\phi = f(x) - \varepsilon$, we write

$$\int_U \sum_{i=1}^{2} f'_{x_i} A_i(x) dx_1 dx_2 = \int_{\partial' U} (f - \varepsilon)\langle A(x), \mathbf{n}\rangle |dx| = 0\,.$$

(Since Φ and Ψ are mutually complementary, it follows from (2.5) and (2.7) that the left-hand integral and, hence, the right-hand integral exist.) From (1.10) it follows that

$$\nabla f(x) = 0 \quad \text{almost everywhere on } U\,,$$

and $f \equiv$ constant on U, which leads to a contradiction with the definition of the component $U, x_0 \in U$. Thus (5.1) follows.

Since $-f$ also satisfies (1.10), (5.1) yields

$$\inf_{\Delta} f(x) = \inf_{\partial'\Delta} f(x)\,. \tag{5.2}$$

Finally, (5.1) and (5.2) imply (1.1).

6. Proof of Theorem 1.14. Fix a subdomain Δ of D with $\partial''\Delta \subset \Gamma$. As in the proof of Theorem 1.9, it suffices to prove (5.1). Suppose that (5.1) is not true; that is, there is $x_0 \in \Delta$ such that

$$f(x_0) > M_0 \equiv \sup_{\partial'\Delta} f(x)\,.$$

As above, for some ε, $f(x_0) > \varepsilon > M_0$, we choose a component U of $\{x \in \Delta\,;\ f(x) > \varepsilon\}$ with a locally rectifiable boundary ∂U along which $f(x) - \varepsilon = 0$.

Fix numbers $0 < \delta' < \delta'' < h(x_0)$ and a non-negative Lipschitz function $\psi_0 : \boldsymbol{R}_+ \to \boldsymbol{R}$. Define $\psi : \boldsymbol{R}_+ \to \boldsymbol{R}$ by

$$\psi(\tau) = \begin{cases} 1 & \text{for } \delta'' < \tau < \infty\,, \\ \psi_0(\tau) & \text{for } \delta' \le \tau \le \delta''\,, \\ 0 & \text{for } 0 < \tau < \delta'\,. \end{cases}$$

Denote $\phi = \psi^p(f - \varepsilon)$ with $\psi = \psi(h(x))$ for $x \in U$ and $\phi \equiv 0$ for $x \in D \setminus U$. Clearly, $\phi \in ACL^p(D)$ and $\operatorname{supp}\phi \subset\subset D$. Applying (1.11) to ϕ, we have

$$\int_{\partial' U} \psi^p (f - \varepsilon)\langle A, \mathbf{n}\rangle |dx| = \int_U \psi^p \langle \nabla f, A\rangle dx_1 dx_2 + p\int_U \psi^{p-1}(f - \varepsilon)\langle \nabla\psi, A\rangle dx_1 dx_2\,.$$

Since the contour integral vanishes, we see that

$$\begin{aligned}\int_U \psi^p \langle \nabla f, A\rangle dx_1 dx_2 &= -p\int_U \psi^{p-1}(f - \varepsilon)\langle \nabla\psi, A\rangle dx_1 dx_2 \\ &\le p\int_U \psi^{p-1}|f - \varepsilon||\nabla\psi||A| dx_1 dx_2\,.\end{aligned}$$

Using (1.15), we then obtain

$$\begin{aligned}\int_U \psi^p\sigma_1|\nabla f|^p dx_1dx_2 &\\ &\le p\int_U \psi^{p-1}|f-\varepsilon||\nabla\psi|\sigma_2|\nabla f|^{p-1}dx_1dx_2\\ &\le pM\int_U \psi^{p-1}\frac{\sigma_1^{(p-1)/p}}{\sigma_1^{(p-1)/p}}|\nabla\psi|\sigma_2|\nabla f|^{p-1}dx_1dx_2\\ &\le pM\Big(\int_U \frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla\psi|^p dx_1dx_2\Big)^{1/p}\Big(\int_U \psi^p\sigma_1|\nabla f|^p dx_1dx_2\Big)^{(p-1)/p},\end{aligned}$$

where

$$M=\sup_{x\in U}|f(x)-\varepsilon|\,,$$

and hence

$$\int_U \psi^p\sigma_1|\nabla f|^p dx_1dx_2 \le p^pM^p\int_U \frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla\psi|^p dx_1dx_2\,. \tag{6.1}$$

Let

$$U(\delta'')=\{x\in U\ ;\ \delta''<h(x)\},\quad U(\delta',\delta'')=\{x\in U\ ;\ \delta'<h(x)<\delta''\}\,.$$

Since $\delta''<h(x_0)$ and $x_0\in U$, the set $U(\delta'')\ne\emptyset$. Noting the specific structure of ψ and using (6.1), we arrive at the inequality

$$\int_{U(\delta'')}\sigma_1|\nabla f|^p dx_1dx_2\le p^pM^p\int_{U(\delta',\delta'')}\frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla\psi_0|^p dx_1dx_2\,.$$

We have $|\nabla\psi_0| = |\psi_0'||\nabla h|$. By (1.13) together with the well-known co-area formula [3, Section 3.2], we find that

$$\int_{U(\delta',\delta'')}\frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla\psi_0|^p dx_1dx_2=\int_{\delta'}^{\delta''}|\psi_0'(\tau)|^p d\tau\int_{U\cap E_\tau}\frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla h|^{p-1}d\mathcal{H}^1(E_\tau)\,,$$

and then

$$\int_{U(\delta',\delta'')}\frac{\sigma_2^p}{\sigma_1^{p-1}}|\nabla\psi_0|^p dx_1dx_2\le h_1^{p-1}\int_{\delta'}^{\delta''}|\psi_0'(\tau)|^p d\tau\int_{U\cap E_\tau}\frac{\sigma_2^p}{\sigma_1^{p-1}}d\mathcal{H}^1(E_\tau)\,.$$

Thus we obtain

$$\int_{U(\delta'')}\sigma_1|\nabla f|^p dx_1dx_2\le p^pM^ph_1^{p-1}\int_{\delta'}^{\delta''}\xi(\tau)|\psi_0'(\tau)|^p d\tau\,, \tag{6.2}$$

where

$$\xi(\tau)=\int_{E_\tau}\frac{\sigma_2^p}{\sigma_1^{p-1}}d\mathcal{H}^1(E_\tau)\,.$$

We choose

$$\psi_0(\tau) = \int_{\delta'}^{\tau} \xi^{1/(1-p)}(t) \Bigg/ \int_{\delta'}^{\delta''} \xi^{1/(1-p)}(t)dt \quad \text{for } \delta' \le \tau \le \delta''.$$

Then

$$\int_{\delta'}^{\delta''} \xi(\tau)|\psi_0'(\tau)|^p d\tau = \left(\int_{\delta'}^{\delta''} \xi^{1/(1-p)}(\tau)d\tau \right)^{1-p},$$

and from (6.2) we have

$$\int_{U(\delta'')} \sigma_1 |\nabla f|^p dx_1\, dx_2 \le p^p M^p h_1^{p-1} \left(\int_{\delta'}^{\delta''} \xi^{1/(1-p)}(\tau)d\tau \right)^{1-p},$$

which holds for every $0 < \delta' < \delta''$. Letting $\delta' \to 0+$ and using (1.16), we obtain

$$\int_{U(\delta'')} \sigma_1 |\nabla f|^p dx_1 dx_2 = 0$$

and, in particular, $\nabla f \equiv 0$ on $U(\delta'')$. Since $\delta'' < h(x_0)$ is arbitrary, we see that $\nabla f \equiv 0$ on U, which means that $f \equiv \varepsilon$ on U. This is again a contradiction to the definition of U and the rest of the proof proceeds as in the end of the proof of Theorem 1.9.

7. Two examples. Let $D \subset \boldsymbol{R}^2$ be a domain with a Jordan boundary ∂D. *Let $\Gamma \subset \partial D$ be an open arc. Let $f : \bar{D} \to \boldsymbol{R}$ be a continuous function, monotone in the sense of Lebesgue in D, such that the restriction $f|_\Gamma$ has no points of strict local extremum*[1]. *Then f is monotone close to Γ.*

For the proof let Δ be a subdomain of D with $\partial''\Delta \subset \Gamma$. The function f, being monotone in the sense of Lebesgue, takes its maximum and minimum values in $\partial\Delta$. Hence there exist $x_1, x_2 \in \partial\Delta$ such that

$$\sup_{x\in\Delta} f(x) = f(x_1), \quad \inf_{x\in\Delta} f(x) = f(x_2).$$

If $x_1 \notin \Gamma$, then

$$\sup_{x\in\Delta} f(x) = \sup_{x\in\partial'\Delta} f(x). \tag{7.1}$$

For $x_1 \in \Gamma$ there are two possibilities (a) $x_1 \in \mathrm{Clo}(\partial'\Delta)$ and (b) $x_1 \notin \mathrm{Clo}(\partial'\Delta)$. By continuity, (7.1) holds in case (a). In case (b), there is an open neighborhood U of x_1 on Γ such that $U \subset \partial''\Delta$. Since f does not have a strict maximum on Γ, f takes greater values on $\partial\Delta$ than $f(x_1)$. This is a contradiction. Hence (7.1) always holds. The point x_2 can be handled similarly. Thus, we obtain

$$\mathrm{osc}(f, \Delta) = f(x_1) - f(x_2) = \sup_{x\in\partial'\Delta} f(x) - \inf_{x\in\partial'\Delta} f(x) = \mathrm{osc}(f, \partial'\Delta).$$

Consequently, the function f is monotone close to Γ.

There exist non-constant functions, monotone close to boundary, which do not have con-

[1] A continuous function $u : \Gamma \to \boldsymbol{R}$ has a strict local minimum (maximum) at a point $a \in \Gamma$ if there exists $\varepsilon > 0$ such that $u(a) < u(x)$ $(u(a) > u(x))$ for all $x \in \Gamma$, $0 < |x - a| < \varepsilon$.

tinuous extensions to the boundary.

Let $D = \{(x_1, x_2)\ ;\ x_2 > 0\}$ be the upper half-plane. Consider the function

$$f(x_1, x_2) = \sin \frac{1}{x_2}\,.$$

Clearly, f is monotone close to boundary $\Gamma = \{x = (x_1, x_2)\ ;\ x_2 = 0\}$, but it does not have a continuous extension to Γ.

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