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FUNCTIONS OF FINITE FRACTIONAL VARIATION AND THEIR APPLICATIONS TO FRACTIONAL IMPULSIVE EQUATIONS

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Abstract. We introduce a notion of a function of finite fractional variation and characterize such functions together with their weak σ-additive fractional derivatives. Next, we use these functions to study differential equations of fractional order, containing a σ-additive term—we prove existence and uniqueness of a solution as well as derive a Cauchy formula for the solution. We apply these results to impulsive equations, i.e. equations containing the Dirac measures.

Keywords: finite fractional variation; weak σ-additive fractional; derivative; fractional impulsive equation; Dirac measure; Cauchy formula

MSC 2010: 26A45, 34A37

1. Introduction

In the paper, we introduce a notion of a function of finite left variation of order $\alpha \in (0,1]$ and characterize such functions. Next, we define a weak σ-additive left derivative of order α of an integrable function and show that the class of functions possessing such derivative coincides with the set of functions of finite left variation of order α. In the last part, we apply the obtained results to fractional equations containing the weak σ-additive left derivative of order α of an integrable function, in particular—the Dirac measure. Analogous results can be derived for functions of finite right variation of order $\alpha \in (0,1]$. We deal only with the left case, so, in the next, we omit the term "left".

The study of impulsive equations was initiated in [16], [17], [18]. In the literature concerning impulsive equations and inclusions of the first order, two approaches are used. The first of them consists in prescribing the jumps of a solution to such an equation with the aid of the initial conditions ([1], [2], [3], [19], [23]). In this approach, by a solution of the problem one means a function which is piecewise differentiable. In the second approach, by a solution we mean a function of finite variation and satisfying the equation in a distributional sense. The jumps of a solution are enforced by the coefficients (measures) of the equation ([12], [25], [10], [27], [28], [13]). So far, impulsive systems of fractional order were investigated with the aid of the first method ([4], [26]—for equations with derivative in Caputo sense, [5], [21]—for equations with derivatives in Riemann-Liouville and Caputo sense, [11], [9]—for integro-differential equations with derivative in Caputo sense). We use the second approach. To the best of our knowledge this approach to fractional impulsive equations has not been applied by other authors.

In Section 2, we recall the notion of a weak integrable fractional derivative of an integrable function, of order $\alpha \in (0,1]$. Section 3 is devoted to the functions of finite fractional variation. In Section 4, we introduce the notion of a weak σ-additive fractional derivative of an integrable function, of order $\alpha \in (0,1]$, and characterize the class of functions possessing such derivative. Section 5 is devoted to the derivation of the existence and uniqueness theorem for a fractional equation containing a nonhomogeneous term being the weak σ-additive fractional derivative of an integrable function. We derive a Cauchy formula for the solution of such an equation, too. As particular cases, in Sections 6 and 7, we obtain the existence of unique solutions, together with the appropriate formulas, to impulsive equations, i.e. equations containing Dirac measures. In Appendix, for the convenience of the reader, we describe the tools used in the paper (cf. [20]).

2. α-absolutely continuous functions and weak integrable fractional derivative

Definition 2.1. We shall say that a function $x\colon [a,b] \to \mathbb{R}$ (defined a.e. on $[a,b]$) is absolutely continuous on $[a,b]$ if there exists a function $\widetilde{x}\colon [a,b] \to \mathbb{R}$ (defined everywhere on $[a,b]$) such that

$$x(t) = \widetilde{x}(t), \quad t \in [a,b] \text{ a.e.},$$

and for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{m} |\widetilde{x}(b_i) - \widetilde{x}(a_i)| < \varepsilon$$

for any collection $\{(a_i, b_i);\ i = 1, \ldots, k\}$, $k \in \mathbb{N}$, of disjoint open subintervals of $[a, b]$, such that

$$\sum_{i=1}^{m} |b_i - a_i| < \delta.$$

In such a case, we identify x with $\widetilde{x}$.

Let $\alpha > 0$, $\varphi \in L^1 = L^1([a, b], \mathbb{R})$. By the left Riemann-Liouville fractional integral of φ on the interval $[a, b]$ we mean ([22]) a function $I^{\alpha}_{a+}h$ given by

$$(I^{\alpha}_{a+}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a, b] \text{ a.e.,} \tag{2.1}$$

where Γ is the Euler function. One can show that the above integral exists and is finite a.e. on $[a, b]$. Additionally, we put $I^0_{a+}\varphi = \varphi$.

Let $\alpha \in (0, 1]$, $x \in L^1$. We say ([7], [15]) that x has the left Riemann-Liouville derivative $D^{\alpha}_{a+}x$ of order α if the integral $I^{1-\alpha}_{a+}x$ is absolutely continuous on $[a, b]$. We put in such a case $D^{\alpha}_{a+}x := \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}_{a+}x$.

Analogously, the function

$$(I^{\alpha}_{b-}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a, b] \text{ a.e.,}$$

is called the right Riemann-Liouville fractional integral of $\varphi \in L^1$ of order $\alpha > 0$ on the interval $[a, b]$. We put $I^0_{b-}\varphi = \varphi$. If $I^{1-\alpha}_{b-}x$ is absolutely continuous on $[a, b]$ for a function $x \in L^1$ and $\alpha \in (0, 1]$, then $D^{\alpha}_{b-}x := -\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}_{b-}x$ is called the right Riemann-Liouville derivative of x of order α.

One proves ([7]) that x possesses derivative $D^{\alpha}_{a+}x$ if and only if there exist a constant $c \in \mathbb{R}$ and a function $\varphi \in L^1$ such that

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a, b] \text{ a.e.}$$

The above representation is unique and can be treated as a fractional counterpart of the representation of an absolutely continuous function $x\colon\ [a, b] \to \mathbb{R}$, given by

$$x(t) = c + \int_a^t \varphi(\tau)\,\mathrm{d}\tau, \quad t \in [a, b] \ \text{ a.e.}$$

A function x possessing the left Riemann-Liouville derivative of order α will be called the left α-absolutely continuous function. In paper [15], it was shown that each left

α-absolutely continuous function x has the weak integrable left derivative of order α, i.e. there exists a unique function $g \in L^1$ such that

$$\int_a^b x(t) D^{\alpha}_{b-}\varphi(t)\,\mathrm{d}t = \int_a^b g(t)\varphi(t)\,\mathrm{d}t$$

for $\varphi \in C_{\mathrm{c}}^{\infty} = C_{\mathrm{c}}^{\infty}((a,b),\mathbb{R})$—the set of C^{∞} functions with compact supports contained in (a,b). Conversely, each function x that has weak integrable left derivative of order α is α-absolutely continuous. In such a case $g = D^{\alpha}_{a+}x$.

3. Functions of finite α-variation

Definition 3.1. We shall say that a function $x\colon [a,b] \to \mathbb{R}$ (defined a.e. on $[a,b]$) has finite variation on $[a,b]$ if there exists a function $\widetilde{x}\colon [a,b] \to \mathbb{R}$ (defined everywhere on $[a,b]$) such that

$$x(t) = \widetilde{x}(t), \quad t \in [a,b] \text{ a.e.},$$

$$\sup\Big\{\sum_{i=1}^{m} |\widetilde{x}(t_i) - \widetilde{x}(t_{i-1})|;\ a = t_0 < \ldots < t_m = b,\ m \in \mathbb{N}\Big\} < \infty. \tag{3.1}$$

We recall that a function $\widetilde{x}\colon [a,b] \to \mathbb{R}^n$ satisfying condition (3.1) has right limit $\widetilde{x}(t^+) = \lim\limits_{s\to t^+} \widetilde{x}(s)$ at any point $t \in [a,b)$, left limit $\widetilde{x}(t^-) = \lim\limits_{s\to t^-} \widetilde{x}(s)$ at any point $t \in (a,b]$ and the set of discontinuity points of $\widetilde{x}$ is at most denumerable.

Let $x\colon [a,b] \to \mathbb{R}$ be a function of finite variation. Consider the function

$$\overleftarrow{x}\colon [a,b] \ni t \longmapsto \begin{cases} \widetilde{x}(t^+) & \text{for } t \in [a,b), \\ \widetilde{x}(b^-) & \text{for } t = b \end{cases} \tag{3.2}$$

where $\widetilde{x}\colon [a,b] \to \mathbb{R}$ is a function satisfying the conditions given in Definition 3.1. In an elementary way one can prove

Lemma 3.2. *If $x\colon [a,b] \to \mathbb{R}^n$ is a function of finite variation, then*

(a) *$\overleftarrow{x}$ does not depend on the choice of the function $\widetilde{x}$ satisfying conditions given in Definition* 3.1*;*

(b) *$\overleftarrow{x} = x$ a.e. on $[a,b]$;*

(c) *$\overleftarrow{x}$ satisfies condition* (3.1)*;*

(d) *$\overleftarrow{x}(t) = \overleftarrow{x}(t^+)$ for any $t \in [a,b)$ and $\overleftarrow{x}(b) = \overleftarrow{x}(b^-)$;*

(e) *$\overleftarrow{x}$ is the unique function which is equal to x a.e. on $[a,b]$, satisfies* (3.1)*, is right-continuous on $[a,b)$ and left-continuous at $t = b$.*

In the sequel, we shall identify any function x of finite variation with its representant $\overleftarrow{x}$ and denote it by x.

It is clear that if $x\colon [a,b] \to \mathbb{R}$ is absolutely continuous, then $\overleftarrow{x}$ coincides with its unique representant $\widetilde{x}$ satisfying the conditions from Definition 2.1.

Now, we define the basic notion of the paper.

Definition 3.3. Let $\alpha \in (0,1]$ and $x \in L^1$. We shall say that x has finite fractional variation of order α (α-variation) on $[a,b]$ if the integral $I^{1-\alpha}_{a+}x$ has finite variation on $[a,b]$.

Of course, x has finite 1-variation if and only if it has finite variation.

Let us recall ([8]) that a function $x \in L^1$ has finite variation if and only if there exists a constant $C > 0$ such that

$$\left| \int_a^b x(t)\varphi'(t)\,\mathrm{d}t \right| \leqslant C\|\varphi\|_\infty \tag{3.3}$$

for any $\varphi \in C^\infty_{\mathrm{c}}$, where $\|\varphi\|_\infty = \max\{|\varphi(t)|;\ t \in (a,b)\}$. We have the following fractional variant of this result

Theorem 3.4. *Let $\alpha \in (0,1]$. A function $x \in L^1$ has finite α-variation if and only if there exists a constant $C > 0$ such that*

$$\left| \int_a^b x(t)D^\alpha_{b-}\varphi(t)\,\mathrm{d}t \right| \leqslant C\|\varphi\|_\infty \tag{3.4}$$

for any $\varphi \in C^\infty_{\mathrm{c}}$.

Proof. Case of $\alpha = 1$ is obvious because $D^\alpha_{b-}\varphi = -\varphi'$. So, assume that $\alpha \in (0,1)$ and $x \in L^1$ has finite α-variation. We have

$$\int_a^b x(t)D^\alpha_{b-}\varphi(t)\,\mathrm{d}t = -\int_a^b x(t)I^{1-\alpha}_{b-}(\varphi')(t)\,\mathrm{d}t = -\int_a^b I^{1-\alpha}_{a+}x(t)\varphi'(t)\,\mathrm{d}t \tag{3.5}$$

for any $\varphi \in C^\infty_{\mathrm{c}}$. To obtain the first equality we used the fact that if $\varphi\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $\varphi(b) = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}I^{1-\alpha}_{b-}\varphi = \frac{\mathrm{d}}{\mathrm{d}t}I^{1-\alpha}_{b-}(-I^1_{b-}\varphi') = -\frac{\mathrm{d}}{\mathrm{d}t}I^1_{b-}(I^{1-\alpha}_{b-}\varphi') = I^{1-\alpha}_{b-}\varphi'. \tag{3.6}$$

So, the classical result implies

$$\left| \int_a^b x(t)D^\alpha_{b-}\varphi(t)\,\mathrm{d}t \right| = \left| \int_a^b I^{1-\alpha}_{a+}x(t)\varphi'(t)\,\mathrm{d}t \right| \leqslant C\|\varphi\|_\infty.$$

Now, let us assume that (3.4) is satisfied. Using (3.5) we obtain

$$\left|\int_a^b I_{a+}^{1-\alpha}x(t)\varphi'(t)\,\mathrm{d}t\right| = \left|\int_a^b x(t)D_{b-}^{\alpha}\varphi(t)\,\mathrm{d}t\right| \leqslant C\|\varphi\|_\infty$$

for any $\varphi \in C_{\mathrm{c}}^\infty$. □

The next theorem describes some imbedding property.

Theorem 3.5. *If $0 < \beta < \alpha \leqslant 1$ and a function $x \in L^1$ has finite α-variation, then it has finite β-variation.*

Proof. First, assume that $\alpha = 1$. Let a function $x \in L^1$ have finite variation and let φ belong to C_{c}^∞. We have

$$\int_a^b x(t)D_{b-}^{\beta}\varphi(t)\,\mathrm{d}t = -\int_a^b x(t)\frac{\mathrm{d}}{\mathrm{d}t}I_{b-}^{1-\beta}\varphi(t)\,\mathrm{d}t = -\int_a^b x(t)\,\mathrm{d}(I_{b-}^{1-\beta}\varphi)(t)$$

where the last integral is the Riemann-Stieltjes integral of x with respect to $I_{b-}^{1-\beta}\varphi$ (to obtain second equality we used the fact that if $f\colon [a,b] \to \mathbb{R}$ is integrable in the Riemann-Stieltjes sense with respect to an absolutely continuous function $g\colon [a,b] \to \mathbb{R}$, then $\int_a^b f(t)\,\mathrm{d}g(t) = \int_a^b f(t)g'(t)\,\mathrm{d}t$ (cf. ([20], Theorem VII.4.10)). Using the theorem on integration by parts for the Riemann-Stieltjes integral ([20], Theorem I.6.7) we obtain

$$\begin{aligned} (3.7)\qquad & -\int_a^b x(t)\,\mathrm{d}(I_{b-}^{1-\beta}\varphi)(t) \\ &= -x(b)I_{b-}^{1-\beta}\varphi(b) + x(a)I_{b-}^{1-\beta}\varphi(a)x + \int_a^b I_{b-}^{1-\beta}\varphi(t)\,\mathrm{d}x(t) \\ &= x(a)I_{b-}^{1-\beta}\varphi(a) + \int_a^b I_{b-}^{1-\beta}\varphi(t)\,\mathrm{d}x(t) \end{aligned}$$

(to obtain the last equality we used the fact that $I_{b-}^{1-\beta}\varphi(b) = 0$ following from [6], Property 4). Consequently,

$$\left|\int_a^b x(t)D_{b-}^{\beta}\varphi(t)\,\mathrm{d}t\right| \leqslant (|x(a)| + \operatorname{var} x)\|I_{b-}^{1-\beta}\varphi\|_\infty$$

where $\operatorname{var} x$ is the variation of x on $[a,b]$. Cited Property 4 from [6] implies also the inequality

$$\|I_{b-}^{1-\beta}\varphi\|_\infty \leqslant \frac{(b-a)^{1-\beta-1/p}}{\Gamma(1-\beta)(-\beta q+1)^{1/q}}\|\varphi\|_{L^p}$$

where p is any number such that $0 < 1/p < 1 - \beta$ and $q = p/(p-1)$. So,

$$\left|\int_a^b x(t) D^{\beta}_{b-}\varphi(t)\,\mathrm{d}t\right| \leqslant (|x(a)| + \operatorname{var} x)\frac{(b-a)^{1-\beta-1/p}}{\Gamma(1-\beta)(-\beta q+1)^{1/q}}(b-a)^{1/p}\|\varphi\|_{\infty}.$$

From Theorem 3.4 it follows that x has finite β-variation.

Now, assume that $0 < \beta < \alpha < 1$ and $x \in L^1$ has finite α-variation. We shall show that the function $I^{1-\beta}_{a+}x$ satisfies (3.3). Indeed, since $I^{1-\beta}_{a+}x = I^{\alpha-\beta}_{a+}I^{1-\alpha}_{a+}x$, therefore (cf. (3.6))

$$\begin{aligned}\int_a^b I^{1-\beta}_{a+}x(t)\varphi'(t)\,\mathrm{d}t &= \int_a^b I^{\alpha-\beta}_{a+}I^{1-\alpha}_{a+}x(t)\varphi'(t)\,\mathrm{d}t = \int_a^b I^{1-\alpha}_{a+}x(t)I^{\alpha-\beta}_{b-}(\varphi')(t)\,\mathrm{d}t\\ &= \int_a^b I^{1-\alpha}_{a+}x(t)\frac{\mathrm{d}}{\mathrm{d}t}(I^{\alpha-\beta}_{b-}\varphi)(t)\,\mathrm{d}t\end{aligned}$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}$. In the same way as in the first case (cf. (3.7)),

$$\int_a^b I^{1-\alpha}_{a+}x(t)\frac{\mathrm{d}}{\mathrm{d}t}(I^{\alpha-\beta}_{b-}\varphi)(t)\,\mathrm{d}t = -I^{1-\alpha}_{a+}x(a)I^{\alpha-\beta}_{b-}\varphi(a) - \int_a^b I^{\alpha-\beta}_{b-}\varphi(t)\,\mathrm{d}(I^{1-\alpha}_{a+}x)(t).$$

So,

$$\begin{aligned}&\left|\int_a^b I^{1-\beta}_{a+}x(t)\varphi'(t)\,\mathrm{d}t\right|\\ &\qquad\leqslant (|I^{1-\alpha}_{a+}x(a)| + \operatorname{var} I^{1-\alpha}_{a+}x)\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)((\alpha-\beta-1)q+1)^{1/q}}\|\varphi\|_{\infty}\end{aligned}$$

where $q = p/(p-1)$ with $0 < 1/p < \alpha - \beta$. The proof is completed. □

4. Weak σ-additive fractional derivative

Now, we shall introduce the notion of a weak σ-additive derivative of order $\alpha \in (0,1]$ of a function $x \in L^1$:

Definition 4.1. We shall say that a function $x \in L^1$ has weak σ-additive derivative of order $\alpha \in (0,1]$ if there exists a σ-additive function $\lambda\colon \mathcal{B}((a,b)) \to \mathbb{R}$ such that

$$\int_a^b x(t)D^{\alpha}_{b-}\varphi(t)\,\mathrm{d}t = \int_{(a,b)} \varphi(t)\,\mathrm{d}\lambda(t) \tag{4.1}$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}$.

In the limit case of $\alpha = 1$ the above notion reduces to the definition of σ-additive (often called signed measure) distributional derivative of the distribution determined by a function x.

From Theorem 8.10 it follows that the σ-additive function $\lambda\colon \mathcal{B}((a,b)) \to \mathbb{R}$ satisfying (4.1) is unique. It is called the weak σ-additive derivative of order α of the function x and denoted by $\mathcal{D}^{\alpha}_{a+}x$.

Below, we characterize the class of functions possessing σ-additive weak derivative of order $\alpha \in (0,1]$.

Theorem 4.2. *Let $\alpha \in (0,1]$ and $x \in L^1$. Then x has weak σ-additive derivative $\mathcal{D}^{\alpha}_{a+}x$ if and only if x has finite α-variation. In such a case $\mathcal{D}^{\alpha}_{a+}x = F^{*}_{I^{1-\alpha}_{a+}x}$ where $F_{I^{1-\alpha}_{a+}x}$ is a function of an interval associated with $I^{1-\alpha}_{a+}x$ (cf. Section 8.1) and $F^{*}_{I^{1-\alpha}_{a+}x}$ is a σ-additive function associated with $F_{I^{1-\alpha}_{a+}x}$ (cf. Section 8.3).*

Proof. First, assume that x has finite α-variation, i.e., $I^{1-\alpha}_{a+}x$ has finite variation. Then

$$
\begin{aligned}
(4.2)\quad \int_a^b x(t) D^{\alpha}_{b-}\varphi(t)\,\mathrm{d}t &= -\int_a^b x(t)\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}_{b-}\varphi(t)\,\mathrm{d}t = -\int_a^b x(t) I^{1-\alpha}_{b-}(\varphi')(t)\,\mathrm{d}t \\
&= -\int_a^b I^{1-\alpha}_{a+}x(t)\varphi'(t)\,\mathrm{d}t = -\int_a^b I^{1-\alpha}_{a+}x(t)\,\mathrm{d}\varphi(t) \\
&= -I^{1-\alpha}_{a+}x(b)\varphi(b) + I^{1-\alpha}_{a+}x(a)\varphi(a) + \int_a^b \varphi(t)\,\mathrm{d}I^{1-\alpha}_{a+}x(t)) \\
&= \int_a^b \varphi(t)\,\mathrm{d}I^{1-\alpha}_{a+}x(t) = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{I^{1-\alpha}_{a+}x}(t)
\end{aligned}
$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}$ (to obtain the last equality we used (8.3)). This means that x has the weak σ-additive derivative $\mathcal{D}^{\alpha}_{a+}x$ and $\mathcal{D}^{\alpha}_{a+}x = F^{*}_{I^{1-\alpha}_{a+}x}$.

Now, let us assume that there exists a σ-additive function $\lambda\colon \mathcal{B}((a,b)) \to \mathbb{R}$ such that

$$
\int_a^b x(t) D^{\alpha}_{b-}\varphi(t)\,\mathrm{d}t = \int_{(a,b)} \varphi(t)\,\mathrm{d}\lambda(t)
$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}$. Then (cf. (8.3)) we have (below, the function $f_{F_\lambda}\colon [a,b] \to \mathbb{R}$ of a point argument is defined by $f_{F_\lambda}(t) = F_\lambda([a,t])$ where F_λ is given by (8.2) and $F^{*}_{\lambda} = \lambda$; since F_λ has finite variation, f_{F_λ} satisfies (3.1); moreover, $F_\lambda = F_{f_{F_\lambda}}$)

$$
\begin{aligned}
\int_{(a,b)} \varphi(t)\,\mathrm{d}\lambda(t) &= \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{\lambda}(t) = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{f_{F_\lambda}}(t) \\
&= \int_a^b \varphi(t)\,\mathrm{d}f_{F_\lambda}(t) = -\int_a^b f_{F_\lambda}(t)\,\mathrm{d}\varphi(t) = -\int_a^b f_{F_\lambda}(t)\varphi'(t)\,\mathrm{d}t.
\end{aligned}
$$

(to obtain the third equality we used formula (8.3); to obtain the fourth equality we used integration by parts for the Riemann-Stieltjes integral (cf. [20], Theorem I.6.7)). On the other hand (cf. (4.2)),

$$\int_a^b x(t) D_{b-}^{\alpha} \varphi(t)\,\mathrm{d}t = -\int_a^b I_{a+}^{1-\alpha} x(t) \varphi'(t)\,\mathrm{d}t.$$

So,

$$\int_a^b f_{F_\lambda}(t) \varphi'(t)\,\mathrm{d}t = \int_a^b I_{a+}^{1-\alpha} x(t) \varphi'(t)\,\mathrm{d}t$$

for any $\varphi \in C_{\mathrm{c}}^{\infty}$. Consequently, there exists a constant $C \in \mathbb{R}$ such that

$$f_{F_\lambda} = I_{a+}^{1-\alpha} x + C \quad \text{a.e. on } [a,b].$$

This means that $I_{a+}^{1-\alpha} x$ has finite variation, i.e. x has finite α-variation. □

In the limit case of $\alpha = 1$ the above theorem reduces to the characterization of functions possessing the distributional derivatives determined by σ-additive set functions (cf. [10] for functions of locally finite variation).

Remark 4.3. As we said, if x is α-absolutely continuous, then it has the left weak integrable derivative g of order α. In such a case $g = D_{a+}^{\alpha} x$, i.e.

$$\int_a^b x(t) D_{b-}^{\alpha} \varphi(t)\,\mathrm{d}t = \int_a^b \varphi(t) D_{a+}^{\alpha} x(t)\,\mathrm{d}t$$

for any $\varphi \in C_{\mathrm{c}}^{\infty}$. Of course, x has also finite α-variation and, according to the above theorem, it has the weak σ-additive derivative $\mathcal{D}_{a+}^{\alpha} x = F^*_{I_{a+}^{1-\alpha} x}$, i.e.

$$\int_a^b x(t) D_{b-}^{\alpha} \varphi(t)\,\mathrm{d}t = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^*_{I_{a+}^{1-\alpha} x}(t)$$

So, we see that the derivatives $D_{a+}^{\alpha} x$ (weak integrable) and $F^*_{I_{a+}^{1-\alpha} x}$ (weak σ-additive) are equal in the following "distributional" sense:

$$\int_a^b \varphi(t) D_{a+}^{\alpha} x(t)\,\mathrm{d}t = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^*_{I_{a+}^{1-\alpha} x}(t)$$

for any $\varphi \in C_{\mathrm{c}}^{\infty}$.

Remark 4.4. Saying that an $\mathbb{R}^n$-valued function $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, of a point- (set-) argument has a property "p" when coordinate functions x_i (λ_i) have such property, and putting

$$\int \varphi(t)x(t)\,\mathrm{d}t = \sum_{i=1}^{n} \int \varphi_i(t)x_i(t)\,\mathrm{d}t,$$

$$\Big(\int \varphi(t)\,\mathrm{d}\lambda(t) = \sum_{i=1}^{n} \int \varphi_i(t)\,\mathrm{d}\lambda_i(t)\Big)$$

for an $\mathbb{R}^n$-valued function $\varphi = (\varphi_1, \ldots, \varphi_n)$ of a point-argument, we see that all results of this section and the preceding ones remain true with $\mathbb{R}$ replaced by $\mathbb{R}^n$.

5. A fractional equation containing weak σ-additive derivative of order α

Let us consider the fractional Cauchy problem

$$\begin{cases} \mathcal{D}^{\alpha}_{a+}x = Ax + \mathcal{D}^{\alpha}_{a+}u \quad \text{on } (a,b), \\ I^{1-\alpha}_{a+}x(a) = c \end{cases} \tag{5.1}$$

where x is an unknown function, $u \in L^1([a,b],\mathbb{R}^n)$—a given function possessing finite α-variation, $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$—given constants.

By a solution of (5.1) we mean a function $x \in L^1([a,b],\mathbb{R}^n)$ possessing finite α-variation, satisfying the above equation in a "distributional" sense:

$$\int_{(a,b)} \varphi(t)\,\mathrm{d}\mathcal{D}^{\alpha}_{a+}x(t) = \int_a^b Ax(t)\varphi(t)\,\mathrm{d}t + \int_{(a,b)} \varphi(t)\,\mathrm{d}\mathcal{D}^{\alpha}_{a+}u(t) \tag{5.2}$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}((a,b),\mathbb{R}^n)$, and the initial condition

$$I^{1-\alpha}_{a+}x(a) = c.$$

Since by (8.3)

$$\begin{aligned} \int_a^b Ax(t)\varphi(t)\,\mathrm{d}t &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} I^1_{a+}(Ax)(t)\varphi(t)\,\mathrm{d}t \\ &= \int_a^b \varphi(t)\,\mathrm{d}I^1_{a+}(Ax)(t) = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^*_{I^1_{a+}(Ax)}(t), \end{aligned}$$

the equality (5.2) can be written in the form

$$\int_{(a,b)} \varphi(t)\,\mathrm{d}\mathcal{D}^{\alpha}_{a+}x(t) = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{I^{1}_{a+}(Ax)}(t) + \int_{(a,b)} \varphi(t)\,\mathrm{d}\mathcal{D}^{\alpha}_{a+}u(t).$$

So, $x \in L^1([a,b],\mathbb{R}^n)$ is a solution to (5.1) if and only if it has finite α-variation,

$$\int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{I^{1-\alpha}_{a+}x}(t) = \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{I^{1}_{a+}(Ax)}(t) + \int_{(a,b)} \varphi(t)\,\mathrm{d}F^{*}_{I^{1-\alpha}_{a+}u}(t) \tag{5.3}$$

for any $\varphi \in C^{\infty}_{\mathrm{c}}((a,b),\mathbb{R}^n)$ and

$$I^{1-\alpha}_{a+}x(a) = c.$$

Since condition (5.3) is equivalent (cf. Theorem 8.10) to the equality

$$F^{*}_{I^{1-\alpha}_{a+}x} = F^{*}_{I^{1}_{a+}(Ax)} + F^{*}_{I^{1-\alpha}_{a+}u} \quad \text{on } \mathcal{B}((a,b)),$$

therefore, finally, $x \in L^1([a,b],\mathbb{R}^n)$ is a solution to (5.1) if and only if it has finite α-variation and

$$\begin{cases} F^{*}_{I^{1-\alpha}_{a+}x} = F^{*}_{I^{1}_{a+}(Ax)} + F^{*}_{I^{1-\alpha}_{a+}u} \quad \text{on } \mathcal{B}((a,b)),\\ I^{1-\alpha}_{a+}x(a) = c. \end{cases} \tag{5.4}$$

In view of Corollary 8.7, the equality

$$F^{*}_{I^{1-\alpha}_{a+}x} = F^{*}_{I^{1}_{a+}(Ax)} + F^{*}_{I^{1-\alpha}_{a+}u} \quad \text{on } \mathcal{B}((a,b))$$

is equivalent to

$$F_{I^{1-\alpha}_{a+}x}([t_1,t_2]) = F_{I^{1}_{a+}(Ax)}([t_1,t_2]) + F_{I^{1-\alpha}_{a+}u}([t_1,t_2]) \tag{5.5}$$

for any $[t_1,t_2] \subset \Pi_{F_{I^{1-\alpha}_{a+}x}} \cap \Pi_{F_{I^{1-\alpha}_{a+}u}}$. Let (τ_n) be a sequence of continuity points for $I^{1-\alpha}_{a+}x$ and $I^{1-\alpha}_{a+}u$ such that $\lim \tau_n = a$. Then, it follows from the above condition that

$$F_{I^{1-\alpha}_{a+}x}([\tau_n,t]) = F_{I^{1}_{a+}(Ax)}([\tau_n,t]) + F_{I^{1-\alpha}_{a+}u}([\tau_n,t]), \quad n \in \mathbb{N}$$

for any point t being a continuity point for $I^{1-\alpha}_{a+}x$ and $I^{1-\alpha}_{a+}u$. Consequently, by the definition of the function of interval associated with a function of point argument,

$$F_{I^{1-\alpha}_{a+}x}([a,t]) = F_{I^{1}_{a+}(Ax)}([a,t]) + F_{I^{1-\alpha}_{a+}u}([a,t]) \tag{5.6}$$

for any point t being a continuity point for $I_{a+}^{1-\alpha}x$ and $I_{a+}^{1-\alpha}u$. Conversely, the above condition implies condition (5.5) (because the associated functions of interval are additive). Of course, condition (5.6) is equivalent to the condition

$$I_{a+}^{1-\alpha}x(t) - I_{a+}^{1-\alpha}x(a) = AI_{a+}^{1}x(t) + I_{a+}^{1-\alpha}u(t) - I_{a+}^{1-\alpha}u(a)$$

for any point t being a continuity point for $I_{a+}^{1-\alpha}x$ and $I_{a+}^{1-\alpha}u$. Thus, we see that a function $x \in L^1([a,b],\mathbb{R}^n)$ has finite α-variation and satisfies (5.4) if and only if it satisfies

$$(5.7) \qquad I_{a+}^{1-\alpha}x(t) = AI_{a+}^{1}x(t) + I_{a+}^{1-\alpha}u(t) + c - I_{a+}^{1-\alpha}u(a) \quad \text{for } t \in [a,b] \text{ a.e.}$$

If we write the term $c - I_{a+}^{1-\alpha}u(a)$ in the form $I_{a+}^{1-\alpha}\big(\frac{1}{\Gamma(\alpha)}\frac{c-I_{a+}^{1-\alpha}u(a)}{(t-a)^{1-\alpha}}\big)$, then the above equation can be written as

$$(5.8) \qquad I_{a+}^{1-\alpha}x(t) = I_{a+}^{1-\alpha}(AI_{a+}^{\alpha}x)(t) + I_{a+}^{1-\alpha}u(t) + I_{a+}^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\frac{c - I_{a+}^{1-\alpha}u(a)}{(\cdot - a)^{1-\alpha}}\Big)(t)$$

for $t \in [a,b]$ a.e., or, equivalently,

$$x(t) = AI_{a+}^{\alpha}x(t) + u(t) + \frac{1}{\Gamma(\alpha)}\frac{c - I_{a+}^{1-\alpha}u(a)}{(t-a)^{1-\alpha}} \quad \text{for } t \in [a,b] \text{ a.e.}$$

More precisely, a function $x \in L^1([a,b],\mathbb{R}^n)$ is a solution of problem (5.1) if and only if

$$x(t) = AI_{a+}^{\alpha}x(t) + u(t) + \frac{1}{\Gamma(\alpha)}\frac{c - I_{a+}^{1-\alpha}u(a)}{(t-a)^{1-\alpha}} \quad \text{for } t \in [a,b] \text{ a.e.}$$

In other words, x is a unique solution to Cauchy problem (5.1) if and only if x is a unique fixed point of the operator

$$\Phi\colon\ L^1([a,b],\mathbb{R}^n) \to L^1([a,b],\mathbb{R}^n),$$
$$\Phi(x)(t) = AI_{a+}^{\alpha}x(t) + b(t), \quad t \in [a,b] \text{ a.e.},$$

where $b(t) = u(t) + \Gamma(\alpha)^{-1}(c - I_{a+}^{1-\alpha}u(a))/(t-a)^{1-\alpha}$ (of course, $b \in L^1([a,b],\mathbb{R}^n)$).

Existence of such a point x_* (for any fixed $b \in L^1([a,b],\mathbb{R}^n)$) follows from the proof of Theorem 3.1 in [14] where it is also shown that

$$\begin{aligned} I_{a+}^{\alpha}x_*(t) &= A^m(I_{a+}^{(m+1)\alpha}x_*)(t) + A^{m-1}(I_{a+}^{m\alpha}b)(t) + \ldots + A(I_{a+}^{2\alpha}b)(t) + I_{a+}^{\alpha}b(t) \\ &= R_m(t) + \sum_{k=0}^{m-1} A^k(I_{a+}^{(k+1)\alpha}b)(t), \quad t \in [a,b] \text{ a.e.,} \end{aligned}$$

for any $m \in \mathbb{N}$ where

$$R_m(t) = A^m (I_{a+}^{(m+1)\alpha} x_*)(t)$$

and

$$R_m(t) \to 0, \quad t \in [a,b] \text{ a.e.}$$

Consequently,

$$x_*(t) = Q_m(t) + \sum_{k=0}^{m-1} A^k (I_{a+}^{k\alpha} b)(t), \quad t \in [a,b] \text{ a.e.}, \tag{5.9}$$

where

$$Q_m(t) = A^m (I_{a+}^{m\alpha} x_*)(t).$$

In the same way as in [14] we check that

$$Q_m(t) \to 0, \quad t \in [a,b] \text{ a.e.} \tag{5.10}$$

So, for any $b \in L^1([a,b], \mathbb{R}^n)$, the series

$$\sum_{k=0}^{m-1} A^k (I_{a+}^{k\alpha} b)(t) \tag{5.11}$$

is convergent a.e. on $[a,b]$. Formula (5.9) with

$$b(t) = u(t) + \frac{1}{\Gamma(\alpha)} \frac{c - I_{a+}^{1-\alpha} u(a)}{(t-a)^{1-\alpha}}$$

takes the form

$$\begin{aligned} x_*(t) &= Q_m(t) + \sum_{k=0}^{m-1} A^k (I_{a+}^{k\alpha} u)(t) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{m-1} A^k \Big(I_{a+}^{k\alpha} \Big(\frac{1}{(\cdot - a)^{1-\alpha}} \Big) \Big)(t)(c - I_{a+}^{1-\alpha} u(a)) \\ &= Q_m(t) + \sum_{k=0}^{m-1} A^k (I_{a+}^{k\alpha} u)(t) + \sum_{k=0}^{m-1} A^k \frac{(t-a)^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)} (c - I_{a+}^{1-\alpha} u(a)) \end{aligned}$$

for $t \in [a,b]$ a.e. (to obtain the second equality we used the formula

$$I_{a+}^{\nu}((\cdot - a)^{\delta - 1})(t) = \frac{\Gamma(\delta)}{\Gamma(\nu + \delta)} (t-a)^{\nu + \delta - 1}, \quad t \in [a,b] \text{ a.e.}$$

for $\nu > 0$, $\delta > 0$). From the convergence (5.10) and the convergence of the series (5.11) for integrable functions $u(t)$ and $(t-a)^{-1+\alpha}$, we obtain

$$\begin{aligned} x_*(t) &= \sum_{k=0}^{\infty} A^k (I_{a+}^{k\alpha} u)(t) + \sum_{k=0}^{\infty} A^k \frac{(t-a)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (c - I_{a+}^{1-\alpha} u(a)) \\ &= \sum_{k=1}^{\infty} A^k (I_{a+}^{k\alpha} u)(t) + u(t) + \sum_{k=0}^{\infty} A^k \frac{(t-a)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (c - I_{a+}^{1-\alpha} u(a)) \\ &= A \sum_{k=1}^{\infty} A^{k-1} (I_{a+}^{k\alpha} u)(t) + u(t) + \Phi_\alpha(t-a)(c - I_{a+}^{1-\alpha} u(a)) \\ &= A \sum_{k=0}^{\infty} A^{k} (I_{a+}^{(k+1)\alpha} u)(t) + u(t) + \Phi_\alpha(t-a)(c - I_{a+}^{1-\alpha} u(a)) \\ &= \Phi_\alpha(t-a)(c - I_{a+}^{1-\alpha} u(a)) + A \int_a^t \Phi_\alpha(t-\tau) u(\tau)\, \mathrm{d}\tau + u(t) \end{aligned}$$

for $t \in [a,b]$ a.e., where $\Phi_\alpha(t) = \sum_{k=0}^{\infty} A^k t^{(k+1)\alpha-1}/\Gamma((k+1)\alpha)$ is the Mittag-Leffler function.

Thus, we have proved

Theorem 5.1. *If $u \in L^1([a,b],\mathbb{R}^n)$ has finite α-variation, then there exists a unique solution x_* of Cauchy problem* (5.1) *and*

$$x_*(t) = \Phi_\alpha(t-a)(c - I_{a+}^{1-\alpha} u(a)) + A \int_a^t \Phi_\alpha(t-\tau) u(\tau)\, \mathrm{d}\tau + u(t), \quad t \in [a,b] \text{ a.e.} \tag{5.12}$$

Remark 5.2. When $\alpha = 1$, the formula (5.12) takes the form

$$x_*(t) = \mathrm{e}^{A(t-a)}(c - u(a)) + A \int_a^t \mathrm{e}^{A(t-\tau)} u(\tau)\, \mathrm{d}\tau + u(t). \tag{5.13}$$

In [10], the authors derived the following formula for the solution to an impulsive linear system of the first order:

$$x_*(t) = \mathrm{e}^{A(t-a)} c + \int_{(a,t)} \mathrm{e}^{A(t-\tau)}\, \mathrm{d}(\mathcal{D}_{a+}^{\alpha} u)(\tau) \tag{5.14}$$

(in fact, in [10], functions of locally finite variation are considered and matrix A may depend on t). Let us observe that the two formulas are the same. Indeed, using our notation, the right hand side of formula (5.14) can be written in the form

$$
\begin{aligned}
e^{A(t-a)}c + \int_{(a,t)} e^{A(t-\tau)}\,d(\mathcal{D}^{\alpha}_{a+}u)(\tau) &= e^{A(t-a)}c + \int_{(a,t)} e^{A(t-\tau)}\,d(F^{*}_{u})(\tau) \\
&= e^{A(t-a)}c + \int_{a}^{t} e^{A(t-\tau)}\,du(\tau)
\end{aligned}
$$

(to obtain the last equality we used (8.3)). The right hand side of formula (5.13) can be written in the form

$$
\begin{aligned}
& e^{A(t-a)}(c-u(a)) + A\int_{a}^{t} e^{A(t-\tau)}u(\tau)\,d\tau + u(t) \\
&= e^{A(t-a)}(c-u(a)) - \int_{a}^{t} \frac{d}{dt}(e^{A(t-\tau)})u(\tau)\,d\tau + u(t) \\
&= e^{A(t-a)}(c-u(a)) - \int_{a}^{t} u(\tau)\,d(e^{A(t-\tau)}) + u(t) \\
&= e^{A(t-a)}(c-u(a)) + \int_{a}^{t} e^{A(t-\tau)}\,du(\tau) - e^{A(t-t)}u(t) + e^{A(t-a)}u(a) + u(t) \\
&= e^{A(t-a)}c + \int_{a}^{t} e^{A(t-\tau)}\,du(\tau)
\end{aligned}
$$

(to obtain the third equality we used the integration by parts for Riemann-Stieltjes integral ([20], Theorem I.6.7)).

6. Example—impulsive equation with one impulse

Let us consider problem

$$
\begin{cases} \mathcal{D}^{\alpha}_{a+}x = Ax + \Delta^{1}_{d} & \text{on } (a,b), \\ I^{1-\alpha}_{a+}x(a) = c \end{cases} \tag{6.1}
$$

where A, $c \in \mathbb{R}$, $\Delta^{1}_{d}\colon \mathcal{B}((a,b)) \to \mathbb{R}$ is the Dirac measure centered in a fixed point $d \in (a,b)$, i.e.

$$
\Delta^{1}_{d}(E) = \begin{cases} 1 & \text{if } d \in E, \\ 0 & \text{otherwise} \end{cases}
$$

for $E \in \mathcal{B}((a,b))$.

Observe that if

$$
u\colon [a,b] \ni t \longmapsto \begin{cases} 0 & \text{for } t \in [a,d], \\ \frac{1}{\Gamma(\alpha)}(t-d)^{\alpha-1} & \text{for } t \in (d,b], \end{cases}
$$

then $I^{1-\alpha}_{a+}u$ is the Heaviside function, i.e.

$$I^{1-\alpha}_{a+}u\colon\ [a,b]\ni t\longmapsto\begin{cases}0 & \text{for } t\in[a,d],\\ 1 & \text{for } t\in(d,b].\end{cases}$$

Indeed, if $t\in[a,d]$, then

$$I^{1-\alpha}_{a+}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{u(s)}{(t-s)^\alpha}\,\mathrm{d}s=0.$$

If $t\in(d,b]$, then

$$\begin{aligned}I^{1-\alpha}_{a+}u(t)&=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{u(s)}{(t-s)^\alpha}\,\mathrm{d}s=\frac{1}{\Gamma(1-\alpha)}\int_d^t\frac{u(s)}{(t-s)^\alpha}\,\mathrm{d}s\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\int_d^t(t-s)^{-\alpha}(s-d)^{\alpha-1}\,\mathrm{d}s\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{-\alpha}\,\mathrm{d}s=\frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\beta(\alpha,1-\alpha)\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}=1.\end{aligned}$$

Of course, we identify the function $I^{1-\alpha}_{a+}u$ with the function

$$[a,b]\ni t\longmapsto\begin{cases}0 & \text{for } t\in[a,d),\\ 1 & \text{for } t\in[d,b].\end{cases}$$

It is clear that $\mathcal{D}^\alpha_{a+}u=F^*_{I^{1-\alpha}_{a+}u}\colon\ \mathcal{B}((a,b))\to\mathbb{R}$ is the Dirac measure Δ^1_d, i.e. $\mathcal{D}^\alpha_{a+}u=\Delta^1_d$. Indeed,

$$\begin{aligned}F^*_{I^{1-\alpha}_{a+}u}((a,b))&=I^{1-\alpha}_{a+}u(b-)-I^{1-\alpha}_{a+}u(a+)=I^{1-\alpha}_{a+}u(b)-I^{1-\alpha}_{a+}u(a)=1,\\ F^*_{I^{1-\alpha}_{a+}u}(\{d\})&=I^{1-\alpha}_{a+}u(d+)-I^{1-\alpha}_{a+}u(d-)=I^{1-\alpha}_{a+}u(d)-I^{1-\alpha}_{a+}u(d-)=1\end{aligned}$$

and, consequently,

$$\begin{aligned}1&\leqslant F^*_{I^{1-\alpha}_{a+}u}(E)+1=F^*_{I^{1-\alpha}_{a+}u}(E)+F^*_{I^{1-\alpha}_{a+}u}(\{d\})=F^*_{I^{1-\alpha}_{a+}u}(E\cup\{d\})\\ &\leqslant F^*_{I^{1-\alpha}_{a+}u}((a,b))=1\end{aligned}$$

when $d \notin E$, i.e., $F^*_{I^{1-\alpha}_{a+}u}(E) = 0$ when $d \notin E$. So,

$$1 = F^*_{I^{1-\alpha}_{a+}u}(\{d\}) \leqslant F^*_{I^{1-\alpha}_{a+}u}(E) \leqslant F^*_{I^{1-\alpha}_{a+}u}((a,b)) = 1$$

when $d \in E$, i.e., $F^*_{I^{1-\alpha}_{a+}u}(E) = 1$ when $d \in E$. From Theorem 5.1 it follows that the Cauchy problem (6.1) has a unique solution (defined a.e. on $[a,b]$)

$$x_*(t) = \begin{cases} \Phi_\alpha(t-a)c, \quad t \in [a,d], \\ \Phi_\alpha(t-a)c + \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_d^t \Phi_\alpha(t-\tau)(\tau-d)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad + \dfrac{1}{\Gamma(\alpha)}(t-d)^{\alpha-1}, \quad t \in (d,b]. \end{cases}$$

7. Example—impulsive equation with k impulses

Similarly, for any fixed points $a < d_1 < \ldots < d_k < b$ and numbers $\delta_1, \ldots, \delta_k \in \mathbb{R} \setminus \{0\}$ ($k \in \mathbb{N}$) there exists a unique solution of the problem

$$\text{(7.1)} \qquad \begin{cases} \mathcal{D}^\alpha_{a+}x = Ax + \Delta^{\delta_1,\ldots,\delta_k}_{d_1,\ldots,d_k} \text{ on } (a,b), \\ I^{1-\alpha}_{a+}x(a) = c \end{cases}$$

where $\Delta^{\delta_1,\ldots,\delta_k}_{d_1,\ldots,d_k} \colon \mathcal{B}((a,b)) \to \mathbb{R}$ is a Dirac type σ-additive function given by

$$\Delta^{\delta_1,\ldots,\delta_k}_{d_1,\ldots,d_k}(E) = \begin{cases} 0 & \text{if } \{d_1,\ldots,d_k\} \cap E = \emptyset, \\ \delta_i + \ldots + \delta_j & \text{if } d_i,\ldots,d_j \in E \end{cases}$$

for $E \in \mathcal{B}((a,b))$. In this case, if

$$u_s \colon [a,b] \ni t \longmapsto \begin{cases} 0 & \text{for } t \in [a,d_s], \\ \dfrac{1}{\Gamma(\alpha)}(t-d_s)^{\alpha-1} & \text{for } t \in (d_s,b] \end{cases}$$

for $s = 1,\ldots,k$, then

$$\begin{aligned} \mathcal{D}^\alpha_{a+}\left(\sum_{s=1}^k \delta_s u_s\right) &= F^*_{I^{1-\alpha}_{a+}(\sum_{s=1}^k \delta_s u_s)} = F^*_{\sum_{s=1}^k \delta_s I^{1-\alpha}_{a+}(u_s)} \\ &= \left(\sum_{s=1}^k \delta_s F_{I^{1-\alpha}_{a+}(u_s)}\right)^* = \sum_{s=1}^k \delta_s F^*_{I^{1-\alpha}_{a+}(u_s)} = \Delta^{\delta_1,\ldots,\delta_k}_{d_1,\ldots,d_k} \end{aligned}$$

on $\mathcal{B}((a,b))$. So, the Cauchy problem (7.1) has a unique solution (defined a.e. on $[a,b]$)

$$x_*(t) = \begin{cases} \Phi_\alpha(t-a)c, \quad t \in [a, d_1], \\ \Phi_\alpha(t-a)c + \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_{d_1}^{t} \Phi_\alpha(t-\tau)\delta_1(\tau-d_1)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad\qquad + \dfrac{\delta_1}{\Gamma(\alpha)}(t-d_1)^{\alpha-1}, \quad t \in (d_1, d_2], \\ \Phi_\alpha(t-a)c + \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_{d_1}^{d_2} \Phi_\alpha(t-\tau)\delta_1(\tau-d_1)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad\qquad + \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_{d_2}^{t} \Phi_\alpha(t-\tau)\sum_{s=1}^{2}\delta_s(\tau-d_s)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad\qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=1}^{2}\delta_s(t-d_s)^{\alpha-1}, \quad t \in (d_2, d_3], \\ \vdots \\ \Phi_\alpha(t-a)c + \dfrac{A}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k-1}\int_{d_i}^{d_{i+1}} \Phi_\alpha(t-\tau)\sum_{s=1}^{i}\delta_s(\tau-d_s)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad\qquad + \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_{d_k}^{t} \Phi_\alpha(t-\tau)\sum_{s=1}^{k}\delta_s(\tau-d_s)^{\alpha-1}\,\mathrm{d}\tau \\ \qquad\qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=1}^{k}\delta_s(t-d_s)^{\alpha-1}, \quad t \in (d_k, b]. \end{cases}$$

8. Appendix

8.1. Function of an interval associated with a function of a point argument. Let $\Omega \subset \mathbb{R}$ be a nonempty set. By an additive function of an interval we mean the real valued function F defined on the set of closed bounded intervals contained in Ω (we write in such a case $F\colon \Omega \to \mathbb{R}$) and such that

$$F([t_1,t_3]) = F([t_1,t_2]) + F([t_2,t_3])$$

for any closed bounded intervals $[t_1,t_2]$, $[t_2,t_3] \subset \Omega$. If the values of F are nonnegative, then it is called a nonnegative function of an interval.

If $g\colon [a,b] \to \mathbb{R}$ is a function of a point argument, then the function of an interval $F_g\colon [a,b] \to \mathbb{R}$ given by

$$F_g([c,d]) = g(d) - g(c)$$

for $[c,d] \subset [a,b]$ is called the function of an interval associated with g.

8.2. Measure associated with a nonnegative function of an interval. A set function $\lambda\colon \mathcal{S} \to \mathbb{R} \cup \{-\infty, \infty\}$ defined on a σ-algebra of sets $\mathcal{S}$ is called σ-additive, if

▷ $\lambda(E) > -\infty$ for any $E \in \mathcal{S}$ or $\lambda(E) < \infty$ for any $E \in \mathcal{S}$,

▷ $\lambda(\emptyset) = 0$,

▷ $\lambda\Big(\bigcup\limits_{i=1}^{\infty} E_i\Big) = \sum\limits_{i=1}^{\infty} \lambda(E_i)$ for any denumerable collection of disjoint sets $E_i \in S$.

If, additionally, λ is nonnegative, then it is called a measure.

If λ, $\mu\colon \mathcal{S} \to \mathbb{R} \cup \{-\infty, \infty\}$ are σ-additive and α, $\beta \in \mathbb{R}$, then $\alpha\lambda + \beta\mu$ is σ-additive provided that $\alpha\lambda(E) > -\infty$ and $\beta\mu(E) > -\infty$ for any $E \in \mathcal{S}$, or $\alpha\lambda(E) < \infty$ and $\beta\mu(E) < \infty$ for any $E \in \mathcal{S}$.

Let $(a, b) \subset \mathbb{R}$ be a bounded open interval, $\mathcal{B}((a, b))$ the σ-algebra of Borel subsets of (a, b). We have ([20], Theorem VII.5.5)

Theorem 8.1. *If $F\colon (a, b) \to \mathbb{R}$ is an additive nonnegative function of an interval, then there exists a unique measure $F^*\colon \mathcal{B}((a, b)) \to \mathbb{R} \cup \{\infty\}$ finite on closed intervals and such that*

$$F^*((c, d)) \leqslant F([c, d]) \leqslant F^*([c, d])$$

for any $[c, d] \subset (a, b)$.

F^* is called the measure associated with the additive nonnegative function of an interval F.

Each measure $\mu\colon \mathcal{B}((a, b)) \to \mathbb{R} \cup \{\infty\}$ finite on closed intervals is associated with a nonnegative additive function of an interval, defined on the set of closed intervals contained in (a, b), for example with the function

$$F_\mu([c, d]) = \mu([c, d))$$

for $[c, d] \subset (a, b)$, i.e. $\mu = F_\mu^*$. Indeed, F_μ is a nonnegative additive function of an interval and

$$\mu((c, d)) \leqslant F_\mu([c, d]) \leqslant \mu([c, d])$$

for $[c, d] \subset (a, b)$. So, from the uniqueness of the associated measure it follows that $(F_\mu)^* = \mu$.

It is clear that

$$(\alpha F + \beta G)^* = \alpha F^* + \beta G^*$$

provided that $\alpha, \beta > 0$.

We shall use the notion of a continuity interval of an additive nonnegative function of an interval $F\colon (a,b)\to\mathbb{R}$. Namely (cf. [20], Part VII.5), we say that a closed interval $[c,d]\subset(a,b)$ is a continuity interval of F if

$$F^*(\{c\}) = F^*(\{d\}) = 0$$

or equivalently,

$$F^*((c,d)) = F([c,d]) = F^*([c,d]).$$

Any point $e\in(a,b)$ such that

$$F^*(\{e\}) = 0$$

is called the continuity hyperplane of F. In the opposite case, e is called the discontinuity hyperplane of F. One proves that there exists at most denumerable number of discontinuity hyperplanes of an additive nonnegative function of an interval F. Consequently, we have

Lemma 8.2. *If $F\colon (a,b)\to\mathbb{R}$ is an additive nonnegative function of an interval, then there exists an increasing sequence $([c_i,d_i])$ of continuity intervals of F, such that* $\lim[c_i,d_i]=(a,b)$.

8.3. σ-additive function associated with a function of an interval of finite variation. If $[a,b]\subset\mathbb{R}$ is a closed bounded interval, then the additive function of an interval $F\colon [a,b]\to\mathbb{R}$, satisfying the condition

$$\sup\left\{\sum_{i=1}^{m}|F([t_i,t_{i-1}])|;\ a=t_0<\ldots<t_m=b,\ m\in\mathbb{N}\right\}<\infty,$$

is called the function of an interval of finite variation.

Theorem 8.3 (Jordan decomposition)**.** *If $F\colon [a,b]\to\mathbb{R}$ is an additive function of an interval of finite variation, then there exist two additive nonnegative functions of an interval $F_1\colon [a,b]\to\mathbb{R}$, $F_2\colon [a,b]\to\mathbb{R}$ such that*

$$F = F_1 - F_2.$$

Now, let $F\colon [a,b]\to\mathbb{R}$ be an additive function of an interval of finite variation and $F=F_1-F_2$ its Jordan decomposition. Denote $F_1^*=(F_1\big|_{(a,b)})^*\colon \mathcal{B}((a,b))\to\mathbb{R}\cup\{\infty\}$, $F_2^*=(F_2\big|_{(a,b)})^*\colon \mathcal{B}((a,b))\to\mathbb{R}\cup\{\infty\}$.

Lemma 8.4. *The measures F_1^* and F_2^* take finite values.*

Proof. Of course, it is sufficient to show that $F_j^*((a,b)) < \infty$ for $j = 1,2$. Let $([c_i, d_i])$ be an increasing sequence of continuity intervals of $F_j\big|_{(a,b)}$, such that $\lim[c_i, d_i] = (a,b)$ (cf. Lemma 8.2). Then we have

$$F_j^*((a,b)) = F_j^*(\lim[c_i,d_i]) = \lim F_j^*([c_i,d_i]) = \lim(F_j\big|_{(a,b)})([c_i,d_i])$$
$$= \lim F_j([c_i,d_i]) \leqslant F_j([a,b]) < \infty$$

and the proof is completed. □

The function $F^* := F_1^* - F_2^* \colon \mathcal{B}((a,b)) \to \mathbb{R}$ is σ-additive. It is called the σ-additive function associated with F and does not depend on the Jordan decomposition of F (if $F = G_1 - G_2$, then $F_1^* + G_2^* = G_1^* + F_2^*$ and, consequently, $F_1^* - F_2^* = G_1^* - G_2^*$).

One can show that

$$(\alpha F + \beta G)^* = \alpha F^* + \beta G^* \tag{8.1}$$

for any α, $\beta \in \mathbb{R}$.

It is known (cf. [20], Part VII.5) that if $g\colon [a,b] \to \mathbb{R}$ satisfies (3.1), then $F_g^*((c,d)) = g(d^-) - g(c^+)$ for any $(c,d) \subset (a,b)$ and $F_g^*(\{e\}) = g(e^+) - g(e^-)$ for any $e \in (a,b)$.

Each σ-additive finite function $\lambda\colon \mathcal{B}((a,b)) \to \mathbb{R}$ is associated with a function of an interval of finite variation on $[a,b]$, for example with the function

$$F_\lambda([c,d]) = \lambda([c,d) \cap (a,b)) \tag{8.2}$$

for $[c,d] \subset [a,b]$, i.e. $\lambda = F_\lambda^*$. Indeed, it is known that $\lambda = \lambda^+ - \lambda^-$ where $\lambda^+\colon \mathcal{B}((a,b)) \to \mathbb{R}$, $\lambda^-\colon \mathcal{B}((a,b)) \to \mathbb{R}$ are finite measures (cf. [20], Theorem VII.5.2). Of course, $F_\lambda = F_{\lambda^+} - F_{\lambda^-}$ where F_{λ^+}, F_{λ^-} are defined by (8.2) for $[c,d] \subset [a,b]$. Consequently, F_λ has finite variation and

$$(F_\lambda)^* = (F_{\lambda^+}\big|_{(a,b)})^* - (F_{\lambda^-}\big|_{(a,b)})^* = \lambda^+ - \lambda^- = \lambda.$$

A closed interval $R \subset (a,b)$ is called the continuity interval of an additive function of an interval $F\colon [a,b] \to \mathbb{R}$ of finite variation if it is a continuity interval for restrictions $F^+\big|_{(a,b)}$, $F^-\big|_{(a,b)}$ of upper and lower variations F^+, F^- of F that are nonnegative additive functions of an interval given by

$$F^+(Q) = \sup_{\bigcup R_i \subset Q} \sum F(R_i),$$
$$F^-(Q) = \sup_{\bigcup R_i \subset Q} \sum -F(R_i)$$

for $Q \subset (a,b)$, where $\{R_i\}$ is a system of closed nonoverlaping intervals, finite or empty (in the last case $\bigcup R_i = \emptyset$ and $\sum F(R_i) = 0$). Similarly, as in the case of a nonnegative additive function of an interval,

$$F^*(\operatorname{Int} R) = F(R) = F^*(R)$$

for any continuity interval $R \subset (a,b)$ of F.

Remark 8.5. If $g\colon [a,b] \to \mathbb{R}$ satisfies (3.1), then an interval $[c,d] \subset (a,b)$ is a continuity interval of the function of an interval F_g associated with g, if and only if g is continuous at $t = c$ and $t = d$ (cf. [20], Theorem VII.5.2). So, the set Π_{F_g} of all continuity intervals of F_g is dense in (a,b), i.e. each closed interval in (a,b) is the limit of a decreasing sequence of intervals belonging to Π_{F_g}.

In [20], Lemma VII.5.2, the following lemma has been proved

Lemma 8.6. *If two σ-additive functions in $\mathcal{B}((a,b))$ are finite and equal in a dense set of subintervals of (a,b), then these functions are equal in $\mathcal{B}((a,b))$.*

Using the above lemma we obtain

Corollary 8.7. *If two additive functions of an interval of finite variation $F\colon [a,b] \to \mathbb{R}$, $G\colon [a,b] \to \mathbb{R}$ are equal in a set of continuity intervals for F and G and this set is dense in (a,b), then the σ-additive functions F^*, G^* are equal in $\mathcal{B}((a,b))$.*

Proof. Let us denote by K the set of continuity intervals of F and G, given in the formulation of the corollary. It is sufficient to observe that for any interval $R \in K$ we have

$$F^*(R) = F(R) = G(R) = G^*(R)$$

and to use Lemma 8.6. □

This corollary implies

Corollary 8.8. *If functions f, $g\colon [a,b] \to \mathbb{R}$ satisfy (3.1) and are equal a.e. on $[a,b]$, then the σ-additive functions F_f^*, F_g^* are equal on $\mathcal{B}((a,b))$.*

Proof. Let $A = [a,b] \setminus (Z_f \cup Z_g \cup Z)$ where Z_f, Z_g are the sets of discontinuity points of f, g, respectively, and $Z = \{t \in [a,b];\ f(t) \neq g(t)\}$. Of course, A is of full measure in $[a,b]$. Consequently, the set of intervals $K = \{[x,y] \subset (a,b);\ x,y \in A\}$ is dense in (a,b), each interval belonging to K is a continuity interval for F_f and F_g and

$$F_f([c,d]) = f(d) - f(c) = g(d) - g(c) = F_g([c,d])$$

for $[c,d] \in K$. So, Corollary 8.8 completes the proof. □

We also use the following useful theorems (for the first of them see [20], Formula VII.5.20)

Theorem 8.9. *If functions* $f\colon [a,b] \to \mathbb{R}$, $g\colon [a,b] \to \mathbb{R}$ *satisfy condition* (3.1) *and do not have common discontinuity points, then (for the integral with respect to a* σ*-additive function we use the symbol* $\int_{(a,b)}$ *and for the Riemann-Stieltjes integral the symbol* $\int_a^b$*)*

$$(8.3)\qquad \int_a^b f\,\mathrm{d}g = (g(a^+) - g(a))f(a) + \int_{(a,b)} f\,\mathrm{d}F_g^* + (g(b) - g(b^-))f(b).$$

Theorem 8.10. *If* λ, $\mu\colon \mathcal{B}((a,b)) \to \mathbb{R}$ *are two* σ*-additive functions and*

$$\int_{(a,b)} \varphi\,\mathrm{d}\lambda = \int_{(a,b)} \varphi\,\mathrm{d}\mu$$

for any $\varphi \in C_{\mathrm{c}}^{\infty}((a,b),\mathbb{R})$, *then* $\lambda = \mu$.

Proof. We know (cf. [24], Theoreme II.1.1) that any continuous function $f\colon (a,b) \to \mathbb{R}$ with compact support (the set of all such functions is denoted by $C_{\mathrm{c}}((a,b),\mathbb{R})$) can be approximated uniformly by functions from $C_{\mathrm{c}}^{\infty}((a,b),\mathbb{R})$. More precisely, for any continuous function $f\colon (a,b) \to \mathbb{R}$ with compact support $\operatorname{supp} f$ such that $\operatorname{supp} f \subset V \subset \overline{V} \subset (a,b)$ for some open interval V, there exists a sequence $(\varphi_n) \subset C_{\mathrm{c}}^{\infty}((a,b),\mathbb{R})$ converging uniformly on (a,b) to f and $\operatorname{supp} \varphi_n \subset V$ for any $n \in \mathbb{N}$.

Denote $I(\varphi) = \int_{(a,b)} \varphi\,\mathrm{d}\lambda$, $J(\varphi) = \int_{(a,b)} \varphi\,\mathrm{d}\mu$ for $\varphi \in C_{\mathrm{c}}((a,b),\mathbb{R})$. These functionals are finite and continuous on $C_{\mathrm{c}}((a,b),\mathbb{R})$ in the following sense: $I(\varphi_n) \to 0$ and $J(\varphi_n) \to 0$ provided that $\varphi_n \to 0$ uniformly on (a,b) and $\operatorname{supp} \varphi_n \subset K$ for all $n \in \mathbb{N}$, where $K \subset (a,b)$ is a compact set (indeed, $|I(\varphi_n)| = \left|\int_{(a,b)} \varphi_n\,\mathrm{d}\lambda\right| \leqslant \int_{(a,b)} |\varphi_n|\,\mathrm{d}|\lambda| \to 0$ where $|\lambda|$ is the variation of the σ-additive set function (cf. [20], Formula VII.5.10).

So, $I(f) = J(f)$ for any $f \in C_{\mathrm{c}}((a,b),\mathbb{R})$. From the Riesz heorem (cf. [20] Theorem VII.5.4) it follows that $\lambda = \mu$ on $\mathcal{B}((a,b))$. □

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